Abstract

In this paper we consider an integral operator for analytic functions in the open unit disk U and we obtain sufficient conditions for univalence of this integral operator.

30C45.

Integral operator; univalence; unit disk.

1 Introduction

Let $\mathcal{A}$ be the class of the functions f which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} :\mid z \mid < 1\}$ and $f(0) = f^{'}(0) - 1 = 0$.

We denote by S the subclass of A consisting of functions $f \in A$, which are univalent in $\mathcal{U}$.

We consider the integral operator

$$\mathcal{T}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}}, \tag{1}$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$ and the complex numbers $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i$, with $\delta \neq 0$, $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$.

2 Preliminary rezults

We need the following lemmas.

Lemma 2.1. [6] *Let γ, δ be complex numbers, $Re\gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left| \frac{zf''(z))}{f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function F_δ defined by

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t)\, dt \right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathcal{U}$.

Lemma 2.2. [8] *Let* δ *be complex number,* $Re\delta > 0$ *and* c *a complex number,* $|c| \leq 1$, $c \neq -1$, *and* $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$.

If

$$\left| c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zf''(z))}{\delta f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, *then the function* F_δ *defined by*

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t)\, dt \right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathcal{U}$.

Lemma 2.3. [5] *Let* $f \in \mathcal{A}$, *satisfy the condition*

$$\left| \frac{z^2 f'(z))}{[f(z)]^2} - 1 \right| < 1, \tag{2}$$

for all $z \in \mathcal{U}$, *then* f *is regular and univalent in* $\mathcal{U}$.

Lemma 2.4. [10] *Let* $g \in \mathcal{A}$, α *a real number, and* c *a complex number,* $|c| \leq \frac{1}{\alpha}$, $c \neq -1$. *If*

$$\left| \frac{g''(z))}{g'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, *then the function*

$$G_\alpha(z) = \left(\alpha \int_0^z \left[t^{\alpha-1} g'(t) \right]^{\alpha-1} dt \right)^{\frac{1}{\alpha}},$$

is in the class in $\mathcal{S}$.

Lemma 2.5. [10] *Let the function* g, *satisfy (2),* M *a positive real number fixed, and* c *a complex number. If* $\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M} \right]$

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (2M + 1), \qquad c \neq -1,$$

$$|g(z)| \leq M,$$

for all $z \in \mathcal{U}$, *then the function*

$$G_\alpha(z) = \left(\alpha \int_0^z [g(t)]^{\alpha-1} dt \right)^{\frac{1}{\alpha}},$$

is in the class in $\mathcal{S}$.

Lemma 2.6. [4] *Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$,* M *fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} z^m,$$

the equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

3 Main results

Theorem 3.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = Re\gamma > 0$, $i = \overline{1,n}$, $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers, $i = \overline{1,n}$, and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{3}$$

then for all δ complex numbers, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{T}_n$, given by (1) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t)}{k_i{}'(t)}\right)^{\delta_i}\right] \mathrm{dt},$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$, $i = \overline{1,n}$.

The function H_n is regular in $\mathcal{U}$ and satisfy the following usual normalization conditions $H_n(0) = H_n^{'}(0) - 1 = 0$.

Now

$$H_n{}'(z) = \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(z)}{k_i(z)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(z)}{k_i'(z)} \right)^{\delta_i} \right],$$

We have

$$\frac{zH_n''(z)}{H_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right] +$$

$$+ \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right],$$

for all $z \in \mathcal{U}$.

Thus, we have

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| = \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right] +$$

$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right],$$

for all $z \in \mathcal{U}$.

Therefore

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \le \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \right] +$$

$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[|\gamma_i| \left(\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| + \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \right) \right] +$$

$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[|\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right],$$

for all $z \in \mathcal{U}$.

By applying the General Schwarz Lemma we obtain

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i\,|z|\,, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le N_i\,|z|\,, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le P_i\,|z|\,,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \le Q_i\,|z|\,, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le R_i\,|z|\,, \quad \left|\frac{zK_i''(z)}{K_i'(z)}\right| \le S_i\,|z|\,,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$.

Using these inequalities we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le$$

$$\le \frac{1-|z|^{2c}}{c}\,|z|\sum_{i=1}^{n}\left[|\alpha_i - 1|\,M_i + |\beta_i|\,N_i + |\gamma_i|\,(P_i + Q_i) + |\delta_i|\,(R_i + S_i)\right], \tag{4}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\le 1}\frac{\left(1-|z|^{2c}\right)|z|}{c} = \frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

from (4) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le$$

$$\le \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\alpha_i - 1|\,M_i + |\beta_i|\,N_i + |\gamma_i|\,(P_i + Q_i) + |\delta_i|\,(R_i + S_i)\right],$$

and hence, by (3) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\cdot\frac{(2c+1)^{\frac{2c+1}{2c}}}{2} = 1,$$

for all $z \in \mathcal{U}$.

So,

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 1. \tag{5}$$

and using (5), by Lemma 2.1, it results that the integral operator $\mathcal{T}_n$, given by (1) is in the class $\mathcal{S}$.

□

If we consider $\delta = 1$ in Theorem 3.1, obtain the next corollary:

Corollary 3.1.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers, $i = \overline{1,n}$, and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$.*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

$$\sum_{i=1}^{n}\left[|\alpha_i - 1|\, M_i + |\beta_i|\, N_i + |\gamma_i|\,(P_i + Q_i) + |\delta_i|\,(R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{F}_n$ defined by

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{6}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 3.1.2. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, M_i, N_i, P_i, Q_i real positive numbers, $i = \overline{1,n}$, and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$.*

If

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\le M_i,\quad \left|\frac{zg_i''(z)}{g_i'(z)}\right|\le N_i,$$
$$\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\le P_i,\quad \left|\frac{zk_i'(z)}{k_i(z)}-1\right|\le Q_i,$$

for all $z\in\mathcal{U}$, $i=\overline{1,n}$ *and*

$$\sum_{i=1}^{n}\left[|\alpha_i-1|\,M_i+|\beta_i|\,N_i+|\gamma_i|\,(P_i+Q_i)\right]\le\frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{S}_n$ *defined by*

$$\mathcal{S}_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left(g_i(t)'\right)^{\beta_i}\cdot\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\right]dt, \tag{7}$$

is in the class $\mathcal{S}$.

If we consider $\delta=1$ and $\beta_1=\beta_2=...=\beta_n=0$ in Theorem 3.1, obtain the next corollary:

Corollary 3.1.3. *Let* $\gamma,\alpha_i,\gamma_i,\delta_i$ *be complex numbers,* $0<Re\gamma\le1$, $c=Re\gamma$, $i=\overline{1,n}$, M_i,P_i,Q_i,R_i,S_i *real positive numbers,* $i=\overline{1,n}$, *and* $f_i,h_i,k_i\in\mathcal{A}$, $f_i(z)=z+a_{2i}z^2+a_{3i}z^3+....$, $h_i(z)=z+c_{2i}z^2+c_{3i}z^3+....$, $k_i(z)=z+d_{2i}z^2+d_{3i}z^3+....$, $i=\overline{1,n}$.

If

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\le M_i,\quad \left|\frac{zh_i'(z)}{h_i(z)}-1\right|\le P_i,\quad \left|\frac{zk_i'(z)}{k_i(z)}-1\right|\le Q_i,$$
$$\left|\frac{zh_i''(z)}{h_i'(z)}\right|\le R_i,\quad \left|\frac{zk_i''(z)}{k_i'(z)}\right|\le S_i,$$

for all $z\in\mathcal{U}$, $i=\overline{1,n}$ *and*

$$\sum_{i=1}^{n}\left[|\alpha_i-1|\,M_i+|\gamma_i|\,(P_i+Q_i)+|\delta_i|\,(R_i+S_i)\right]\le\frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{X}_n$ *defined by*

$$\mathcal{X}_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\cdot\left(\frac{{h_i}'(t))}{{k_i}'(t)}\right)^{\delta_i}\right]dt, \tag{8}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 3.1.4. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, N_i, P_i, Q_i, R_i, S_i real positive numbers, $i = \overline{1,n}$, and $g_i, h_i, k_i \in \mathcal{A}$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$.*

If

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{D}_n$ defined by

$$\mathcal{D}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{9}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 3.1.5. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, M_i, N_i, R_i, S_i real positive numbers, $i = \overline{1,n}$, and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$.*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathcal{U}$*,* $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \, M_i + |\beta_i| \, N_i + |\delta_i| \, (R_i + S_i) \right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{Y}_n$ *defined by*

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)} \right)^{\delta_i} \right] dt, \qquad (10)$$

is in the class $\mathcal{S}$*.*

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 3.1, obtain the next corollary:

Corollary 3.1.6. *Let* α *be complex number,* $Re\alpha > 0$*,* M,N,P,Q,R,S *real positive numbers, and* $f, g, h, k \in \mathcal{A}$*,* $f(z) = z + a_2 z^2 + a_3 z^3 + ...$*,* $g(z) = z + b_2 z^2 + b_3 z^3 + ...$*,* $h(z) = z + c_2 z^2 + c_3 z^3 + ...$*,* $k(z) = z + d_2 z^2 + d_3 z^3 + ...$

If

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le M, \quad \left| \frac{zg''(z)}{g(z)'} \right| \le N, \quad \left| \frac{zh'(z)}{h(z)} - 1 \right| \le P,$$

$$\left| \frac{zk'(z)}{k(z)} - 1 \right| \le Q, \quad \left| \frac{zh''(z)}{h'(z)} \right| \le R, \quad \left| \frac{zk''(z)}{k'(z)} \right| \le S,$$

for all $z \in \mathcal{U}$*, and*

$$|\alpha - 1| \, (M + N + P + Q + R + S) \le \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{T}$ *defined by*

$$\mathcal{T}(z) = \left[\alpha \int_0^z t^{\alpha - 1} \left(f(t) \cdot g'(t) \cdot \frac{h(t)}{k(t)} \cdot \frac{h'(t))}{k'(t)} \right)^{\alpha - 1} \; dt \right]^{\frac{1}{\alpha}}, \qquad (11)$$

is in the class $\mathcal{S}$*.*

Theorem 3.2. *Let* $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $i = \overline{1,n}$*,* $c = Re\gamma > 0$ *and* $f_i, h_i, k_i \in \mathcal{S}$*,* $g_i{}', h_i{}', k_i{}' \in \mathcal{P}$*,* $f_i(z) = z + a_{2i} z^2 + a_{3i} z^3 + ...$*,* $g_i(z) = z + b_{2i} z^2 + b_{3i} z^3 + ...$*,* $h_i(z) = z + c_{2i} z^2 + c_{3i} z^3 + ...$*,* $k_i(z) = z + d_{2i} z^2 + d_{3i} z^3 + ...$*,* $i = \overline{1,n}$

If

$$4\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+8\sum_{i=1}^{n}|\gamma_i|+4\sum_{i=1}^{n}|\delta_i|\le\frac{c}{2},\quad for\quad 0<c<1 \tag{12}$$

or

$$4\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+8\sum_{i=1}^{n}|\gamma_i|+4\sum_{i=1}^{n}|\delta_i|\le\frac{1}{2},\quad for\quad c\ge 1 \tag{13}$$

then for any complex numbers δ, $Re\delta\ge c$, the integral operator $\mathcal{T}_n$ defined in (1) is in the class $\mathcal{S}$.

Proof. We consider the function

$$H_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left(g_i(t)'\right)^{\beta_i}\cdot\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\cdot\left(\frac{h_i{}'(t)}{k_i{}'(t)}\right)^{\delta_i}\right]\mathrm{dt},$$

for $f_i,h_i,k_i\in\mathcal{S}$, $g_i{}',h_i{}',k_i{}'\in\mathcal{P}$, $i=\overline{1.n}$.

The function H_n is regular in $\mathcal{U}$ and satisfy the following usual normalization conditions $H_n(0)=H_n'(0)-1=0$.

We obtain

$$\frac{zH_n''(z)}{H_n'(z)}=\sum_{i=1}^{n}\left[(\alpha_i-1)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\beta_i\frac{zg_i''(z)}{g_i'(z)}\right]+$$

$$+\sum_{i=1}^{n}\left[\gamma_i\left(\frac{zh_i'(z)}{h_i(z)}-\frac{zk_i'(z)}{k_i(z)}\right)+\delta_i\left(\frac{zh_i''(z)}{h_i'(z)}-\frac{zk_i''(z)}{k_i'(z)}\right)\right],$$

for all $z\in\mathcal{U}$.

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\le\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right]$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\gamma_i|\left(\left|\frac{zh_i'(z)}{h_i(z)}\right|+1+\left|\frac{zk_i'(z)}{k_i(z)}\right|+1\right)\right]+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right],$$

for all $z \in \mathcal{U}$.

Since $f_i, h_i, k_i \in \mathcal{S}$ we have

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \le \frac{1+|z|}{1-|z|}, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \le \frac{1+|z|}{1-|z|}, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \le \frac{1+|z|}{1-|z|},$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$.

For $g_i{}', h_i{}', k_i{}' \in \mathcal{P}$ we have

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \le \frac{2|z|}{1-|z|^2}, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le \frac{2|z|}{1-|z|^2}, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le \frac{2|z|}{1-|z|^2},$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$.

Using these relations we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\left(\frac{1+|z|}{1-|z|}+1\right)\sum_{i=1}^{n}|\alpha_i-1| +$$

$$+\frac{1-|z|^{2c}}{c}\cdot\frac{2|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i| + \frac{1-|z|^{2c}}{c}\left(\frac{1+|z|}{1-|z|}+1+\frac{1+|z|}{1-|z|}+1\right)\sum_{i=1}^{n}|\gamma_i| +$$

$$+\frac{1-|z|^{2c}}{c}\left(\frac{2|z|}{1-|z|^2}+\frac{2|z|}{1-|z|^2}\right)\sum_{i=1}^{n}|\delta_i|$$

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\cdot\frac{2}{1-|z|}\sum_{i=1}^{n}|\alpha_i-1| +$$

$$+\frac{1-|z|^{2c}}{c}\cdot\frac{2|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i| + \frac{1-|z|^{2c}}{c}\cdot\frac{4}{1-|z|}\sum_{i=1}^{n}|\gamma_i| + \frac{1-|z|^{2c}}{c}\cdot\frac{4|z|}{1-|z|^2}\sum_{i=1}^{n}|\delta_i| \tag{14}$$

for all $z \in \mathcal{U}$.

For $0 < c < 1$, we have $1-|z|^{2c} \le 1-|z|^2$, $z \in \mathcal{U}$ and by (14) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{4}{c}\sum_{i=1}^{n}|\alpha_i-1| + \frac{2}{c}\sum_{i=1}^{n}|\beta_i| + \frac{8}{c}\sum_{i=1}^{n}|\gamma_i| + \frac{4}{c}\sum_{i=1}^{n}|\delta_i|, \tag{15}$$

for all $z \in \mathcal{U}$.

From (12) and (15) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 1. \tag{16}$$

for all $z \in \mathcal{U}$ and $0 < c < 1$.

For $c \geq 1$ we have $\frac{1-|z|^{2c}}{c} \leq 1-|z|^2$, for all $z \in \mathcal{U}$ and by (14) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 4\sum_{i=1}^{n}|\alpha_i-1| + 2\sum_{i=1}^{n}|\beta_i| + 8\sum_{i=1}^{n}|\gamma_i| + 4\sum_{i=1}^{n}|\delta_i|\,, \tag{17}$$

for all $z \in \mathcal{U}$ and $c \geq 1$.

From (13) and (17) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 1. \tag{18}$$

for all $z \in \mathcal{U}$ and $c \geq 1$.

And by (16), (18) and Lemma 1 it results that the integral operator $\mathcal{T}_n$, defined by (1) is in the class $\mathcal{S}$.

□

If we consider $\delta = 1$ in Theorem 3.2, we obtain the next corollary:

Corollary 3.2.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $i = \overline{1,n}$, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, $g_i{}', h_i{}', k_i{}' \in \mathcal{P}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$4\sum_{i=1}^{n}|\alpha_i-1| + 2\sum_{i=1}^{n}|\beta_i| + 8\sum_{i=1}^{n}|\gamma_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \;\; 0 < c < 1$$

then the integral operator $\mathcal{F}_n$ defined by (6) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = ... = \beta_n = 0$ in Theorem 3.2, we obtain the next corollary:

Corollary 3.2.2. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $i = \overline{1, n}$, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, $h_i{}', k_i{}' \in \mathcal{P}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1, n}$*

If

$$4\sum_{i=1}^{n}|\alpha_i - 1| + 8\sum_{i=1}^{n}|\gamma_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{X}_n$ defined by (8) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 3.2, we obtain the next corollary:

Corollary 3.2.3. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $i = \overline{1, n}$, $0 < Re\gamma \leq 1$ and $f_i \in \mathcal{S}$, $g_i{}', h_i{}', k_i{}' \in \mathcal{P}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1, n}$*

If

$$4\sum_{i=1}^{n}|\alpha_i - 1| + 2\sum_{i=1}^{n}|\beta_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{Y}_n$ defined by (10) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 3.2, we obtain the next corollary:

Corollary 3.2.4. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $i = \overline{1, n}$, $0 < Re\gamma \leq 1$ and $h_i, k_i \in \mathcal{S}$, $g_i{}', h_i{}', k_i{}' \in \mathcal{P}$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1, n}$*

If

$$2\sum_{i=1}^{n}|\beta_i| + 8\sum_{i=1}^{n}|\gamma_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{D}_n$ defined by (9) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 3.2, we obtain the next corollary:

Corollary 3.2.5. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $i = \overline{1,n}$, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, $g_i' \in \mathcal{P}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$4\sum_{i=1}^{n}|\alpha_i - 1| + 2\sum_{i=1}^{n}|\beta_i| + 8\sum_{i=1}^{n}|\gamma_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{S}_n$ defined by (7) belongs to the class $\mathcal{S}$.

Theorem 3.3. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $Re\gamma > 0$, $i = \overline{1,n}$, M_i, N_i, P_i, real positive numbers, $i = \overline{1,n}$, and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z+a_{2i}z^2+a_{3i}z^3+....$, $g_i(z) = z+b_{2i}z^2+b_{3i}z^3+....$, $h_i(z) = z+c_{2i}z^2+c_{3i}z^3+....$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq 1, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq N_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ and

$$|c| \leq 1 - \frac{1}{|\delta|}\left[(2 + M_i)\sum_{i-1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + (N_i + P_i + 4)\sum_{i=}^{n}|\gamma_i| + 2\sum_{i=1}^{n}|\delta_i|\right], \tag{19}$$

where $c \in \mathbb{C}$, $c \neq -1$, then the integral operator $\mathcal{T}_n$, defined by (1) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i'(t)}{k_i'(t)}\right)^{\delta_i}\right] \mathrm{dt},$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$, $i = \overline{1,n}$.

The function H_n is regular in $\mathcal{U}$ and satisfy the following usual normalization conditions $H_n(0) = H_n'(0) - 1 = 0$.

Also, a simple computation yields

$$\left|c\,|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\right|\left|\sum_{i=1}^{n}\left[(\alpha_i-1)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\right.\right.$$

$$\left.+\beta_i\frac{zg_i''(z)}{g_i'(z)}+\gamma_i\left(\frac{zh_i'(z)}{h_i(z)}-\frac{zk_i'(z)}{k_i(z)}\right)+\delta_i\left(\frac{zh_i''(z)}{h_i'(z)}-\frac{zk_i''(z)}{k_i'(z)}\right),\right.$$

for all $z\in\mathcal{U}$.

Then, we obtain

$$\frac{zH_n''(z)}{H_n'(z)}=\left||c\,|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right|\leq$$

$$\leq|c|+\frac{1}{|\delta|}\sum_{i-1}^{n}|\alpha_i-1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+\frac{1}{|\delta|}\sum_{i-1}^{n}|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|+$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}|\gamma_i|\left[\left(\left|\frac{zh_i'(z)}{h_i(z)}\right|+1\right)+\left(\left|\frac{zk_i'(z)}{k_i(z)}\right|+1\right)\right]+$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right), \tag{20}$$

for all $z\in\mathcal{U}$.

Using these inequalities from hypotesis we have

$$\left||c\,|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right|\leq$$

$$\leq|c|+\frac{1}{|\delta|}\left[(2+M_i)\sum_{i-1}^{n}|\alpha_i-1|+\sum_{i=1}^{n}|\beta_i|\right]+$$

$$+\frac{1}{|\delta|}\left[(N_i+P_i+4)\sum_{i=}^{n}|\gamma_i|+2\sum_{i=1}^{n}|\delta_i|\right],$$

for all $z\in\mathcal{U}$. and hence, by inequality (20) we have

$$\left||c\,|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right|\leq 1, \tag{21}$$

for all $z\in\mathcal{U}$.

Applying Lemma 2.2, we conclude that the integral operator $\mathcal{T}_n$, given by (1) is in the class $\mathcal{S}$.

$\square$

If we consider $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ and $n = 1$ in Theorem 3.3, we obtain the next corollary:

Corollary 3.3.1. *Let α be complex number, $Re\alpha > 0$ M, N, P real positive numbers, and $f, g, h, k \in \mathcal{A}$, $f(z) = z + a_2 z^2 + a_3 z^3 + ...$, $g(z) = z + b_2 z^2 + b_3 z^3 + ...$, $h(z) = z + c_2 z^2 + c_3 z^3 + ...$, $k(z) = z + d_2 z^2 + d_3 z^3 + ...$*

If

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq M, \quad \left| \frac{zg''(z)}{g'(z)} \right| \leq 1, \quad \left| \frac{zh'(z)}{h(z)} - 1 \right| \leq N,$$

$$\left| \frac{zk'(z)}{k(z)} - 1 \right| \leq P, \quad \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \quad \left| \frac{zk''(z)}{k'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$ and

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M_i + N_i + P_i + 8), \qquad c \in \mathbb{C}, \qquad c \neq -1,$$

then the integral operator $\mathcal{T}$, given by (11) is in the class $\mathcal{S}$.

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