

MODIFIED KANTOROVICH-STANCU OPERATORS (II)

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ABSTRACT. In this paper, we introduce a new kind of Bernstein-Kantorovich-Stancu operators. These operators generalize the operators introduced in the paper [2] by V. Gupta, G. Tachev and A.M. Acu.

1. INTRODUCTION

For $f \in C([0, 1])$, the Bernstein operator of degree n is defined by

$$B_n(f; x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1]$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n$$

and

$$p_{n,k}(x) = 0 \text{ if } k < 0 \text{ or } k > n.$$

In [3], H. Khosravian-Arab, M. Delghan and M.R. Eslahchi, starting from well-known equalities

$$p_{n,k}(x) = (1-x)p_{n-1,k}(x) + xp_{n-1,k-1}(x)$$

and

$$p_{n,k}(x) = (1-x)^2 p_{n-2,k}(x) + 2x(1-x)p_{n-2,k-1}(x) + x^2 p_{n-2,k-2}(x), \quad 0 < k < n$$

have introduced modified Bernstein operators:

(i) $B_n^{M,1}$ defined by

$$B_n^{M,1}(f; x) = \sum_{k=0}^n p_{n,k}^{M,1}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1] \quad (1.1)$$

where

$$p_{n,k}^{M,1} = a(x, n)p_{n-1,k}(x) + a(1-x, n)p_{n-1,k-1}(x)$$

and

$$a(x, n) = a_1(n)x + a_0(n), \quad n = 0, 1, \dots$$

(ii) $B_n^{M,2}$ defined by

$$B_n^{M,2}(f; x) = \sum_{k=0}^n p_{n,k}^{M,2}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1] \quad (1.2)$$

where

$$p_{n,k}^{M,2}(x) = b(x, n)p_{n-2,k}(x) + d(x, n)p_{n-2,k-1}(x) + b(1-x, n)p_{n-2,k-2}(x)$$

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and

$$\begin{aligned} b(x, n) &= b_2(n)x^2 + b_1(n)x + b_0(n), \\ d(x, n) &= d_0(n)x(1-x), \quad n = 0, 1, \dots \end{aligned}$$

$a_0(n)$, $a_1(n)$, $b_0(n)$, $b_1(n)$, $b_2(n)$ and $d_0(n)$ are the unknown sequences which are determined in appropriate way for each forms.

V. Gupta, G. Tachev and A.M. Acu ([2]) have considered the operators:

$$K_n^{M,1}(f; x) = (n+1) \sum_{k=0}^n p_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s) ds \quad (1.3)$$

and

$$K_n^{M,2}(f; x) = (n+1) \sum_{k=0}^n p_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s) ds. \quad (1.4)$$

Here, they have discussed a uniform convergence estimate for these modified operators. In 1968, D.D. Stancu ([5]) has introduced the linear positive operators

$$P_n^{(\alpha, \beta)} : C([0, 1]) \rightarrow C([0, 1])$$

defined by

$$P_n^{(\alpha, \beta)}(f; x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right)$$

where α, β are two fixed real numbers such that $0 \leq \alpha \leq \beta$.

In 2004, D. Bărbosu ([1]) has introduced Kantorovich-Stancu operators

$$K_n^{(\alpha, \beta)} : L_1([0, 1]) \rightarrow C([0, 1])$$

defined by

$$K_n^{(\alpha, \beta)}(f; x) = (n+\beta+1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(s) ds.$$

Regarding the previously modified operators, we note the following:

(a) The operators $B_n^{M,1}$ and $B_n^{M,2}$ are linear combinations of the operators $P_{n-1}^{(0,1)}$ and $P_{n-1}^{(1,1)}$, respectively of the operators $P_{n-2}^{(0,2)}$, $P_{n-2}^{(1,2)}$ and $P_{n-2}^{(2,2)}$, more precisely

$$B_n^{M,1}(f; x) = a(x, n)P_{n-1}^{(0,1)}(f; x) + a(1-x, n)P_{n-1}^{(1,1)}(f; x)$$

and

$$B_n^{M,2}(f; x) = b(x, n)P_{n-2}^{(0,2)}(f; x) + d(x, n)P_{n-2}^{(1,2)}(f; x) + b(1-x, n)P_{n-2}^{(2,2)}(f; x);$$

(b) The operators $K_n^{M,1}$ and $K_n^{M,2}$ are linear combinations of the operators $K_{n-1}^{(0,1)}$ and $K_{n-1}^{(1,1)}$, respectively of the operators $K_{n-2}^{(0,2)}$, $K_{n-2}^{(1,2)}$ and $K_{n-2}^{(2,2)}$, therefore

$$K_n^{M,1}(f; x) = a(x, n)K_{n-1}^{(0,1)}(f; x) + a(1-x, n)K_{n-1}^{(1,1)}(f; x)$$

and

$$K_n^{M,2}(f; x) = b(x, n)K_{n-2}^{(0,2)}(f; x) + d(x, n)K_{n-2}^{(1,2)}(f; x) + b(1-x, n)K_{n-2}^{(2,2)}(f; x).$$

From the above reasons, in this paper we introduce, for any $\alpha, \beta \in \mathbb{R}$, $0 \leq \alpha \leq \beta$ the operators

$$\overline{K}_n^{(\alpha, \beta)} = (n + \beta + 1) \sum_{k=0}^n p_{m,k}^{M,1}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(s) ds \quad (1.5)$$

and

$$\overline{\overline{K}}_n^{(\alpha, \beta)} = (n + \beta + 1) \sum_{k=0}^n p_{m,k}^{M,2}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(s) ds. \quad (1.6)$$

We mention that the Kantorovich-Stancu type operators $\overline{K}_n^{(\alpha, \beta)}$ was studied in a recent paper submitted for publication ([4]).

2. AUXILIARY RESULTS

Lemma 2.1. *The central moments of $K_n^{(\alpha, \beta)}$ are given by:*

$$\begin{aligned} K_n^{(\alpha, \beta)}((t-x)^s; x) &= \frac{1}{s+1} \sum_{k=0}^n p_{n,k}(x) \left\{ \sum_{i=0}^{s+1} \binom{s+1}{i} \left(\frac{k+1}{n+1} - x \right)^i \frac{1}{(n+\beta+1)^{s-i}} \right. \\ &\quad \left. \times \left[\sum_{j=1}^{s+1-i} (-1)^{j+1} \binom{s+1-i}{j} \left(\alpha - \beta \frac{k+1}{n+1} \right)^{s+1-i-j} \right] \right\}. \end{aligned} \quad (2.1)$$

Proof.

$$\begin{aligned} K_n^{(\alpha, \beta)}((t-x)^s; x) &= (n + \beta + 1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} (t-x)^s dt \\ &= (n + \beta + 1) \frac{1}{s+1} \sum_{k=0}^n p_{n,k}(x) \left[\left(\frac{k+\alpha+1}{n+\beta+1} - x \right)^{s+1} - \left(\frac{k+\alpha}{n+\beta+1} - x \right)^{s+1} \right] \end{aligned} \quad (2.2)$$

Because

$$\begin{aligned} \frac{k+\alpha+1}{n+\beta+1} - x &= \frac{k+1}{n+1} - x + \frac{k+\alpha+1}{n+\beta+1} - \frac{k+1}{n+1} \\ &= \frac{k+1}{n+1} - x + \left(\alpha - \beta \frac{k+1}{n+1} \right) \frac{1}{n+\beta+1}, \end{aligned}$$

we have

$$\begin{aligned} &\left(\frac{k+\alpha+1}{n+\beta+1} - x \right)^{s+1} \\ &= \sum_{i=0}^{s+1} \binom{s+1}{i} \left(\frac{k+1}{n+1} - x \right)^i \left(\alpha - \beta \frac{k+1}{n+1} \right)^{s+1-i} \frac{1}{(n+\beta+1)^{s+1-i}} \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{k+\alpha}{n+\beta+1} - x \right)^{s+1} \\ &= \sum_{i=0}^{s+1} \binom{s+1}{i} \left(\frac{k+1}{n+1} - x \right)^i \left(\alpha - 1 - \beta \frac{k+1}{n+1} \right)^{s+1-i} \frac{1}{(n+\beta+1)^{s+1-i}}. \end{aligned}$$

So, (2.2) becomes

$$\begin{aligned}
K_n^{(\alpha, \beta)}((t-x)^s; x) &= \frac{1}{s+1} \sum_{k=0}^n p_{n,k}(x) \left\{ \sum_{i=0}^{s+1} \binom{s+1}{i} \left(\frac{k+1}{n+1} - x \right)^i \frac{1}{(n+\beta+1)^{s-i}} \right. \\
&\quad \times \left[\left(\alpha - \beta \frac{k+1}{n+1} \right)^{s+1-i} - \left(\alpha - 1 - \beta \frac{k+1}{n+1} \right)^{s+1-i} \right] \Big\} \\
&= \frac{1}{s+1} \sum_{k=0}^n p_{n,k}(x) \left\{ \sum_{i=0}^{s+1} \binom{s+1}{i} \left(\frac{k+1}{n+1} - x \right)^i \frac{1}{(n+\beta+1)^{s-i}} \right. \\
&\quad \times \left[\sum_{j=1}^{s+1-i} (-1)^{j+1} \binom{s+1-i}{j} \left(\alpha - \beta \frac{k+1}{n+1} \right)^{s+1-i-j} \right] \Big\}.
\end{aligned}$$

Remark 2.2. For $s = \overline{1, 6}$ we have

$$\begin{aligned}
K_n^{(\alpha, \beta)}(t-x; x) &= -\frac{\beta+1}{n+\beta+1}x + \frac{2\alpha+1}{2(n+\beta+1)}, \\
K_n^{(\alpha, \beta)}((t-x)^2; x) &= \frac{n-(2\alpha+1)(\beta+1)}{(n+\beta+1)^2}x(1-x) \\
&\quad + \frac{(\beta-2\alpha)(\beta+1)}{(n+\beta+1)^2}x^2 + \frac{3\alpha^2+3\alpha+1}{3(n+\beta+1)^2}, \\
K_n^{(\alpha, \beta)}((t-x)^3; x) &= -\frac{(3\beta+5)n-(\beta+1)^3}{(n+\beta+1)^3}x^2(1-x) \\
&\quad + \frac{(12\alpha+10)n-6(2\alpha+1)(\beta+1)^2+4(\beta+1)^3}{4(n+\beta+1)^3}x(1-x) \\
&\quad - \frac{4(3\alpha^2+3\alpha+1)(\beta+1)-6(2\alpha+1)(\beta+1)^2+4(\beta+1)^3}{4(n+\beta+1)^3}x \\
&\quad + \frac{4\alpha^3+6\alpha^2+4\alpha+1}{4(n+\beta+1)^3}, \\
K_n^{(\alpha, \beta)}((t-x)^4; x) &= \frac{3n^2-2(3+4(\beta+1)+3(\beta+1)^2)n}{(n+\beta+1)^4}(x(1-x))^2 \\
&\quad - \frac{[4(2\alpha+1)+2(6\alpha+1)(\beta+1)-6(\beta+1)^2]n-2(2\alpha+1)(\beta+1)^3}{(n+\beta+1)^4}x^2(1-x) \\
&\quad + \frac{(6\alpha^2+10\alpha+5)n+2(2\alpha+1)(\beta+1)^3-2(3\alpha^2+3\alpha+1)(\beta+1)^2}{(n+\beta+1)^4}x(1-x) \\
&\quad - \frac{2(2\alpha+1)(\beta+1)^3-2(3\alpha^2+3\alpha+1)(\beta+1)^2+(4\alpha^3+6\alpha^2+4\alpha+1)(\beta+1)}{(n+\beta+1)^4}x \\
&\quad + \frac{5\alpha^4+10\alpha^3+10\alpha^2+5\alpha+1}{5(n+\beta+1)^4}, \\
K_n^{(\alpha, \beta)}((t-x)^5; x) &= \frac{[(30\beta+70)x^2(1-x)^3+(30\alpha-30\beta-35)x^2(1-x)^2]}{2(n+\beta+1)^5}n^2 \\
&\quad + \left[-\frac{30(\beta+1)^3+60(\beta+1)^2+90(\beta+1)+72}{3(n+\beta+1)^5}x^2(1-x)^3 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{60(\beta+1)^3 - 45(2\alpha-1)(\beta+1)^2 - 30(4\alpha-1)(\beta+1) - 9(10\alpha+1)}{3(n+\beta+1)^5} x^2(1-x)^2 \\
& - \frac{30(\beta+1)^3 - 15(6\alpha+1)(\beta+1)^2 + 15(6\alpha^2+2\alpha+1)(\beta+1) + 2(30\alpha^2+30\alpha+13)}{3(n+\beta+1)^5} x^2(1-x) \\
& + \frac{30\alpha^3 + 75\alpha^2 + 75\alpha + 28}{3(n+\beta+1)^5} x(1-x) \Bigg] n + O\left(\frac{1}{n^5}\right), \\
K_n^{(\alpha,\beta)}((t-x)^6; x) &= \frac{15x^3(1-x)^3}{(n+\beta+1)^6} n^3 + \left[\frac{45(\beta+1)^2 x^4(1-x)^2}{(n+\beta+1)^6} \right. \\
& - \frac{(120x^3(1-x)^3 + 15(6\alpha-1)x^3(1-x)^2)(\beta+1)}{(n+\beta+1)^6} \\
& \left. + \frac{130x^2(1-x)^4 + 10(12\alpha-7)x^2(1-x)^3 + 5(9\alpha^2-3\alpha+2)x^2(1-x)^2}{(n+\beta+1)^6} \right] n^2 + O\left(\frac{1}{n^5}\right).
\end{aligned}$$

3. MAIN RESULTS

Here, we will extend the results from [4] for modified operators $\overline{\overline{K}}_n^{(\alpha,\beta)}$ defined by (1.6).

It is easy to see that

$$\begin{aligned}
\overline{\overline{K}}_n^{(\alpha,\beta)}(f; x) &= b(x; n) K_{n-2}^{(\alpha,\beta+2)}(f; x) + d(x; n) K_{n-2}^{(\alpha+1,\beta+2)}(f; x) \\
&+ b(1-x; n) K_{n-2}^{(\alpha+2,\beta+2)}(f; x).
\end{aligned} \tag{3.1}$$

Lemma 3.1. *For $i = 0, 1, 2$, the moments of $\overline{\overline{K}}_n(t^i; x)$ are given by:*

$$\begin{aligned}
\overline{\overline{K}}_n^{(\alpha,\beta)}(1; x) &= (2b_2(n) - d_0(n))x^2 - (2b_2(n) - d_0(n))x + b_2(n) + b_1(n) + 2b_0(n), \\
\overline{\overline{K}}_n^{(\alpha,\beta)}(t; x) &= \frac{(2(n-2)x + 2\alpha + 3)(2b_2(n) - d_0(n))}{2(n+\beta+1)} x(x-1) \\
&+ \frac{(n-4)(b_2(n) + b_1(n)) + 2(n-2)b_0(n)}{n+\beta+1} x \\
&+ \frac{(2\alpha+5)(b_2(n) + b_1(n)) + 2(2\alpha+3)b_0(n)}{2(n+\beta+1)}, \\
\overline{\overline{K}}_n^{(\alpha,\beta)}(t^2; x) &= [(2b_2(n) - d_0(n))x(x-1) + b_2(n) + b_1(n) + 2b_0(n)] \\
&\times \left[\frac{(n-2)(n-3)}{(n+\beta+1)^2} x^2 + \frac{2(\alpha+1)(n-2)}{(n+\beta+1)^2} x + \frac{3\alpha^2+3\alpha+1}{3(n+\beta+1)^2} \right] \\
&+ \left[\frac{2(2b_2(n) - d_0(n))x^3}{(n+\beta+1)^2} + \frac{(2(2b_2(n) - d_0(n)) + 4(b_2(n) + b_1(n)))x^2}{(n+\beta+1)^2} \right] (n-2) \\
&+ \frac{2(\alpha+1)(2b_2(n) - d_0(n)) + 2b_2(n)}{(n+\beta+1)^2} x^2 \\
&- \left[\frac{2(\alpha+1)(2b_2(n) - d_0(n))}{(n+\beta+1)^2} + \frac{4(\alpha+2)b_2(n) + 2(2\alpha+3)b_1(n)}{(n+\beta+1)^2} \right] x \\
&+ \frac{2(2\alpha+3)(b_2(n) + b_1(n) + b_0(n))}{(n+\beta+1)^2}.
\end{aligned}$$

We want to demonstrate the uniform convergence of the sequence $(\overline{\overline{K}}_n^{(\alpha, \beta)} f)_{n \geq 2}$. For this purpose, we will consider that

$$\begin{aligned} \overline{\overline{K}}_n^{(\alpha, \beta)}(e_0; x) &= 1 \\ \Leftrightarrow 2b_2(m) - d_0(m) &= 0 \text{ and } b_2(m) + b_1(m) + 2b_0(m) = 1. \end{aligned} \quad (3.2)$$

Using these, we obtain that

$$\begin{aligned} \overline{\overline{K}}_n^{(\alpha, \beta)}(t; x) &= x + \frac{4b_0(n) - \beta - 5}{n + \beta + 1} + \frac{2\alpha + 5 - 4b_0(n)}{2(n + \beta + 1)} \\ \overline{\overline{K}}_n^{(\alpha, \beta)}(t^2; x) &= \frac{n^2 - (9 - 8b_0(n))n + 16 - 2b_1(n) - 20b_0(n)}{(n + \beta + 1)^2} x^2 \\ &\quad + \frac{2(\alpha + 3 - 2b_0(n))n + 2b_1(n) + 8(\alpha + 3)b_0(n) - 4(2\alpha + 5)}{(n + \beta + 1)^2} x \\ &\quad + \frac{3\alpha^2 + 15\alpha + 19 - 6(2\alpha + 3)b_0(n)}{3(n + \beta + 1)^2}. \end{aligned}$$

Assume that $\beta = 2\alpha$, for $b_0(n) = \frac{\beta + 5}{4}$ the above expressions become

$$\begin{aligned} \tilde{K}_n^{(\alpha, \beta)}(t; x) &= x \\ \tilde{K}_n^{(\alpha, \beta)}(t^2; x) &= x^2 + \frac{n + 2b_1(n) + (\beta + 2)(\beta + 5)}{(n + \beta + 1)^2} x(1 - x) - \frac{3\beta^2 + 18\beta + 14}{12(n + \beta + 1)^2}. \end{aligned}$$

Taking $b_1(n) = -\frac{n + (\beta + 2)(\beta + 5)}{2}$ we have that

$$\tilde{K}_n^{(\alpha, \beta)}(t^2; x) = x^2 - \frac{3\beta^2 + 18\beta + 14}{12(n + \beta + 1)^2}.$$

By (3.2) we obtain

$$b_2(n) = \frac{n + 2 + (\beta + 1)(\beta + 5)}{2}$$

and

$$d_0(n) = n + 2 + (\beta + 1)(\beta + 5).$$

In this situation, we can give other expressions for the first six central moments.

Lemma 3.2.

$$\begin{aligned} \tilde{K}_n^{(\alpha, \beta)}(t - x; x) &= 0, \\ \tilde{K}_n^{(\alpha, \beta)}((t - x)^2; x) &= -\frac{3\beta^2 + 18\beta + 14}{12(n + \beta + 1)^2}, \\ \tilde{K}_n^{(\alpha, \beta)}((t - x)^3; x) &= -\frac{1 - 2x}{4(n + \beta + 1)^3} [2(3\beta + 7)nx(1 - x) \\ &\quad + 2(\beta^3 + 9\beta^2 + 21\beta + 13)x(1 - x) + \beta^3 + 9\beta^2 + 23\beta + 15], \\ \tilde{K}_n^{(\alpha, \beta)}((t - x)^4; x) &= \frac{-3n^2}{(n + \beta + 1)^4} x^2(1 - x)^2 + O\left(\frac{1}{n^3}\right), \\ \tilde{K}_n^{(\alpha, \beta)}((t - x)^5; x) &= \frac{15(\beta + 3)n^2}{(n + \beta + 1)^5} x^2(1 - x)^2(2x - 1) + O\left(\frac{1}{n^4}\right), \\ \tilde{K}_n^{(\alpha, \beta)}((t - x)^6; x) &= \frac{-30n^3}{(n + \beta + 1)^6} x^3(1 - x)^3 + O\left(\frac{1}{n^4}\right). \end{aligned}$$

Using this, we will prove the following result:

Theorem 3.3. *For $x \in [0, 1]$, if $f \in C^{(6)}([0, 1])$, we have*

$$\tilde{K}_n^{(\alpha, \beta)}(f; x) - f(x) = O\left(\frac{1}{n^2}\right), \quad (3.3)$$

for sufficient large n .

Proof. Applying the Taylor's formula to the operators $\tilde{K}_n^{(\alpha, \beta)}$ we have

$$\begin{aligned} \tilde{K}_n^{(\alpha, \beta)}(f; x) &= f(x) + \sum_{k=1}^6 \frac{1}{k!} \tilde{K}_n^{(\alpha, \beta)}((t-x)^k; x) f^{(k)}(x) \\ &\quad + \tilde{K}_n^{(\alpha, \beta)}(\rho(t; x)(t-x)^6; x), \end{aligned}$$

where $\rho \in C([0, 1])$ and $\lim_{t \rightarrow x} \rho(t; x) = 0$.

It is sufficient to prove that

$$|\tilde{K}_n^{(\alpha, \beta)}(\rho(t; x)(t-x)^6; x)| = O\left(\frac{1}{n^2}\right). \quad (3.4)$$

We know that operators are not positive, so we rewrite them like this

$$\tilde{K}_n^{(\alpha, \beta)}(f; x) = \tilde{K}_{n,1}^{(\alpha, \beta)}(f; x) - \tilde{K}_{n,2}^{(\alpha, \beta)}(f; x)$$

where

$$\begin{aligned} \tilde{K}_{n,1}^{(\alpha, \beta)}(f; x) &= (b_2(n)x^2 + b_0(n)) \cdot K_{n-2}^{(\alpha, \beta+2)}(f; x) + d_0(n)x \cdot K_{n-2}^{(\alpha+1, \beta+2)}(f; x) \\ &\quad + b_2(n)x^2 \cdot K_{n-2}^{(\alpha+2, \beta+2)}(f; x) \end{aligned}$$

and

$$\begin{aligned} \tilde{K}_{n,2}^{(\alpha, \beta)}(f; x) &= -b_1(n)x \cdot K_{n-2}^{(\alpha, \beta+2)}(f; x) + d_0(n)x^2 \cdot K_{n-2}^{(\alpha+1, \beta+2)}(f; x) \\ &\quad + ((2b_2(n) + b_1(n))x - (b_2(n) + b_1(n) + b_0(n))) \cdot K_{n-2}^{(\alpha+2, \beta+2)}(f; x). \end{aligned}$$

We note that $\tilde{K}_{n,1}^{(\alpha, \beta)}$ and $\tilde{K}_{n,2}^{(\alpha, \beta)}$ are linear and positive operators.

$$\begin{aligned} |\tilde{K}_n^{(\alpha, \beta)}(\rho(t; x)(t-x)^6; x)| &\leq |\tilde{K}_{n,1}^{(\alpha, \beta)}(\rho(t; x)(t-x)^6; x)| \\ &\quad + |\tilde{K}_{n,2}^{(\alpha, \beta)}(\rho(t; x)(t-x)^6; x)|. \end{aligned} \quad (3.5)$$

Computing $\tilde{K}_{n,i}^{(\alpha, \beta)}((t-x)^6; x)$, $i = 1, 2$ we obtain the following expressions

$$\tilde{K}_{n,i}^{(\alpha, \beta)}((t-x)^6; x) = \frac{15x^4(1-x)^3(1+x)n^4}{(n+\beta+1)^6} + A_i(\alpha, \beta, x) \frac{n^3}{(n+\beta+1)^6}, \quad i = 1, 2$$

where

$$\begin{aligned} A_1(\alpha, \beta, x) &= 15(\beta^2 + 6\beta + 7)x^4(1-x)^3(1+x) \\ &\quad - [120x^4(1-x)^3(1+x) + 15(6\alpha + 5)x^4(1-x)^2(1+x)](\beta + 3) \\ &\quad + 45x^5(1-x)^2(1+x)(\beta + 3)^2 + 130x^3(1-x)^4(1+x) \\ &\quad + 10(12\alpha + 5)x^3(1-x)^3(1+x) + 5(9\alpha^2 + 15\alpha + 8)x^3(1-x)^2(1+x) \\ &\quad + 45x^4(1-x)^2 + \frac{15(\beta + 5)x^3(1-x)^3}{4} \end{aligned}$$

and

$$\begin{aligned} A_2(\alpha, \beta, x) &= 15(\beta^2 + 6\beta + 7)x^4(1-x)^3(1+x) \\ &\quad - [120x^4(1-x)^3(1+x) + 15(6\alpha + 5)x^4(1-x)^2(1+x)](\beta + 3) \end{aligned}$$

$$\begin{aligned}
& +45x^5(1-x)^2(1+x)(\beta+3)^2 + 130x^3(1-x)^4(1+x) \\
& +10(12\alpha+5)x^3(1-x)^3(1+x) + 5(9\alpha^2+15\alpha+8)x^3(1-x)^2(1+x) \\
& +45x^3(1-x)^2 - \frac{15(\beta+1)x^3(1-x)^3}{4}.
\end{aligned}$$

For $\varepsilon > 0$ be given, by $\rho(\cdot; x) \in C([0, 1])$ there exist a $\delta > 0$ such that if $|t - x| < \delta$ then $|\rho(t; x)| < \varepsilon$. Using the above results for $\tilde{K}_{n,i}^{(\alpha,\beta)}((t-x)^6; x)$, we obtain

$$\begin{aligned}
|\tilde{K}_{n,i}^{(\alpha,\beta)}(\rho(t; x)(t-x)^6; x)| & \leq \varepsilon \left| \frac{15x^4(1-x)^3(1+x)n^4}{(n+\beta+1)^6} + O\left(\frac{1}{n^3}\right) \right| \\
& = O\left(\frac{1}{n^2}\right), \quad i = 1, 2.
\end{aligned}$$

So, (3.4) is proved.

Combining this with Lemma 3.2 we complete the proof of theorem.

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