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# The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain $q$-integral operator

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**Abstract.** In our present investigation, we first introduce several new subclasses of analytic and bi-univalent functions by using a certain $q$-integral operator in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{and} \, |z| < 1\}$. By applying the Faber polynomial expansion method as well as the $q$-analysis, we then determine bounds for the $n$th coefficient in the Taylor-Maclaurin series expansion for functions in each of these newly-defined analytic and bi-univalent function classes subject to a gap series condition. We also highlight some known consequences of our main results.

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## 1. Introduction and definitions

Let $\mathcal{A}$ be the class of all functions $f$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by

$$f(0) = 0 = f'(0) - 1.$$

Thus, clearly, the function $f \in \mathcal{A}$ has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Further, by $\mathcal{S} \subset \mathcal{A}$ we shall denote the class of all functions which are univalent in $\mathbb{U}$.

For two functions $f, g \in \mathcal{A}$, the function $f$ is said to be subordinate to the function $g$ in $\mathbb{U}$, denoted by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a function

$$w \in \mathbb{B}_0 := \{w : w \in \mathcal{A}, \quad w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U})\}$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

In the case when the function $g$ is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next, for a function $f \in \mathcal{A}$ given by (1.1) and another function $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{U}),$$

the convolution (or the Hadamard product) of the functions $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

It is well known that every univalent function $f$ has an inverse $f^{-1}$, defined by

$$f^{-1}\big(f(z)\big) = z = f\left(f^{-1}(z)\right) \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. We denote the class of all such functions by $\Sigma$. In recent years, the pioneering work of Srivastava *et al.* [22] essentially revived the investigation of various subclasses of the analytic and bi-univalent function class $\Sigma$. In fact, in a remarkably large number of sequels to the pioneering work of Srivastava *et al.* [22], several different subclasses of the analytic and bi-univalent function class $\Sigma$ were introduced and studied analogously by the many authors (see, for example, [5], [7], [9], [23], [24], [25], [28] and [29]). However, only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were obtained in these recent papers.

The Faber polynomials introduced by Faber [11] play an important rôle in various areas of mathematical sciences, especially in Geometric Function Theory of Complex Analysis (see, for details, [27]). Recently, several authors (see, for example, [13] and [26]; see also [6], [8], [12] and [20]) investigated some interesting and useful properties for analytic functions by applying the Faber polynomial expansion method. Motivated by these and other recent works (see, for example, [1], [14] and [30]), here we make use of the $q$-analysis in order to define new subclasses of analytic and bi-univalent functions in $\mathbb{U}$ and (by means of the Faber polynomial expansion method) we determine estimates for the general coefficient $|a_n|$ $(n \geqq 3)$ in the Taylor-Maclaurin series expansion (1.1) of functions in each of these subclasses.

We begin by recalling here some basic definitions and other concept details of the $q$-calculus $(0 < q < 1)$, which will be used in this paper.

**Definition 1.1.** Let $q \in (0,1)$ and define the $q$-number $[\kappa]_q$ by

$$[\kappa]_q = \begin{cases} \dfrac{1-q^\kappa}{1-q} & (\kappa \in \mathbb{C}) \\[2ex] \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\kappa = n \in \mathbb{N}), \end{cases}$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

**Definition 1.2.** Let $q \in (0,1)$ and define the $q$-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\[2ex] \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

**Definition 1.3.** (see [15] and [16]) The $q$-derivative (or the $q$-difference) $D_q f$ of a function $f$ is defined, in a given subset of $\mathbb{C}$, by

$$\big(D_q f\big)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\[2ex] f'(0) & (z = 0), \end{cases} \tag{1.4}$$

provided that $f'(0)$ exists.

We note from Definition 1.3 that

$$\lim_{q \to 1-} \big(D_q f\big)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a function $f$ which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.4) that

$$\big(D_q f\big)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}.$$

**Definition 1.4.** The $q$-Pochhammer symbol $[\kappa]_{n,q}$ $(\kappa \in \mathbb{C};\ n \in \mathbb{N}_0)$ is defined as follows:

$$[\kappa]_{n,q} = \frac{(q^\kappa; q)_n}{(1-q)^n} := \begin{cases} 1 & (n = 0) \\[2ex] [\kappa]_q [\kappa+1]_q [\kappa+2]_q \cdots [\kappa+n-1]_q & (n \in \mathbb{N}). \end{cases}$$

Moreover, the $q$-gamma function $\Gamma_q(z)$ is defined by the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \, \Gamma_q(z) \qquad \text{and} \qquad \Gamma_q(1) = 1.$$

**Definition 1.5.** [17] For $f \in \mathcal{A}$, let the Ruscheweyh $q$-derivative operator be defined as follows:

$$\mathcal{I}_q^\lambda f(z) = f(z) * \mathcal{F}_{q,\lambda+1}(z) \qquad (z \in \mathbb{U};\ \lambda > -1),$$

where

$$\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda+n)}{[n-1]_q!\Gamma_q(\lambda+1)}\, z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!}\, z^n$$

in terms the Hadamard product (or convolution) given by (1.2).

We next define a certain $q$-integral operator by using the same technique as that used by Noor [19].

**Definition 1.6.** For $f \in \mathcal{A}$, let the $q$-integral operator $\mathcal{F}_{q,\lambda}$ be defined by

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) * \mathcal{F}_{q,\lambda+1}(z) = zD_q f(z).$$

Then

$$\begin{aligned} \mathcal{I}_q^\lambda f(z) &= f(z) * \mathcal{F}_{q,\lambda+1}^{-1}(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_{n-1}\, a_n z^n \qquad (z \in \mathbb{U};\ \lambda > -1), \end{aligned} \tag{1.5}$$

where

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) = z + \sum_{n=2}^{\infty} \Psi_{n-1}\, z^n$$

and

$$\Psi_{n-1} = \frac{[n]_q!\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+n)} = \frac{[n]_q!}{[\lambda+1]_{q,n-1}}.$$

Clearly, we have

$$\mathcal{I}_q^0 f(z) = zD_q f(z) \qquad \text{and} \qquad \mathcal{I}_q^1 f(z) = f(z).$$

We note also that, in the limit case when $q \to 1-$, the $q$-integral operator $\mathcal{F}_{q,\lambda}$ given by Definition 1.6 would reduce to the integral operator which was studied by Noor [18].

The following identity can be easily verified:

$$zD_q\left(\mathcal{I}_q^{\lambda+1} f(z)\right) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right)\mathcal{I}_q^\lambda f(z) - \frac{[\lambda]_q}{q^\lambda}\mathcal{I}_q^{\lambda+1} f(z). \tag{1.6}$$

When $q \to 1-$, this last identity (1.6) implies that

$$z\left(\mathcal{I}^{\lambda+1} f(z)\right)' = (1+\lambda)\mathcal{I}^\lambda f(z) - \lambda\mathcal{I}^{\lambda+1} f(z),$$

which is the well-known recurrence relation for the above-mentioned integral operator which studied by Noor [18].

The above-defined $q$-calculus provides valuable tools that have been extensively used in order to examine several subclasses of $\mathcal{A}$. Even though Ismail *et al.* [14] were the first to use the $q$-derivative operator $D_q$ in order to study a certain $q$-analogue of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, yet a rather significant usage of the $q$-calculus in the context of Geometric Function Theory of Complex Analysis was basically furnished and the basic (or $q$-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [21, pp. 347 *et seq.*]; see also [23]).

We now introduce the following subclasses of the analytic and bi-univalent function class $\Sigma$.

**Definition 1.7.** For a function $f \in \Sigma$, we say that

$$f \in \mathcal{R}_q(\Sigma, \alpha, \gamma) \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0)$$

if and only if

$$\left| D_q f(z) + \gamma z D_q^2 f(z) - \frac{1-\alpha q}{1-q} \right| < \frac{1-\alpha}{1-q} \qquad (z \in \mathbb{U})$$

and

$$\left| D_q g(w) + \gamma w D_q^2 g(w) - \frac{1-\alpha q}{1-q} \right| < \frac{1-\alpha}{1-q} \qquad (w \in \mathbb{U}).$$

Equivalently, by using the principle of subordination between analytic functions, we can write the above conditions as follows (see, for details, [30]):

$$D_q f(z) + \gamma z D_q^2 f(z) \prec \frac{1 + [1 - \alpha(1+q)]\ z}{1 - qz} \qquad (z \in \mathbb{U})$$

and

$$D_q g(w) + \gamma w D_q^2 g(w) \prec \frac{1 + [1 - \alpha(1+q)]\, w}{1 - qw} \qquad (w \in \mathbb{U}),$$

respectively, where $g(w) = f^{-1}(w)$ is given by (1.3).

**Definition 1.8.** For a function $f \in \Sigma$, we say that

$$f \in \mathcal{R}_q(\Sigma, \alpha, \gamma, \lambda) \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0;\ \lambda \geqq 0)$$

if and only if

$$D_q \mathcal{I}_q^\lambda f(z) + \gamma z D_q^2 \mathcal{I}_q^\lambda f(z) \prec \frac{1 + [1 - \alpha(1+q)]\, z}{1 - qz} \qquad (z \in \mathbb{U})$$

and

$$D_q \mathcal{I}_q^\lambda g(w) + \lambda w D_q^2 \mathcal{I}_q^\lambda g(w) \prec \frac{1 + [1 - \alpha(1+q)]\, w}{1 - qw} \qquad (w \in \mathbb{U}),$$

where $g(w) = f^{-1}(w)$ is given by (1.3).

## 2. The Faber polynomial expansion method and its applications

In this section, by using the Faber polynomial expansion of a function $f \in \mathcal{A}$ of the form (1.1), we observe that the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [4]; see also [13] and [26]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\, w^n, \tag{2.1}$$

where

$$\begin{aligned}K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\ & + \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] \\ & + \sum_{j \geqq 7} a_2^{n-j}V_j. \end{aligned} \tag{2.2}$$

Here, and in what follows, such expressions as (for example) $(-n)!$ occurring in (2.2) are to be interpreted *symbolically* by

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots \qquad \big(n \in \mathbb{N}_0\big)$$

and $V_j$ $(7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$. In particular, the first three terms of $K_{n-1}^{-n}$ are given below:

$$K_1^{-2} = -2a_2, \qquad K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, an expansion of $K_n^{\mathfrak{p}}$ is given by (see, for details, [3])

$$K_n^{\mathfrak{p}} = \mathfrak{p}a_n + \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}\,E_n^2 + \frac{\mathfrak{p}!}{(-3)!3!}\,E_n^3 + \cdots + \frac{\mathfrak{p}!}{(\mathfrak{p}-n)!n!}\,E_n^n \qquad (\mathfrak{p} \in \mathbb{Z}),$$

where $\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$ and

$$E_n^{\mathfrak{p}} = E_n^{\mathfrak{p}}\left(a_2, a_3, \cdots\right).$$

It is clearly seen that

$$E_n^n(a_1, a_2, \cdots, a_n) = a_1^n.$$

and

$$E_{n-1}^m(a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}\cdots(a_n)^{\mu_{n-1}}}{\mu_{1!}, \cdots, \mu_{n-1}!} \qquad (m \leqq n).$$

We also have (see [2])

$$E_{n-1}^{n-1}\left(a_2, \cdots, a_n\right) = a_2^{n-1}$$

and

$$E_n^m(a_1, a_2, \cdots, a_n) = \sum \left(\frac{m!}{\mu_1! \cdots \mu_n!}\right) a_1^{\mu_1} \cdots a_n^{\mu_n} \qquad (m \leqq n),$$

where $a_1 = 1$ and the sum is taken over all non-negative integers $\mu_1, \cdots, \mu_n$ satisfying the following conditions:

$$\mu_1 + \mu_2 + \cdots + \mu_n = m$$

and

$$\mu_1 + 2\mu_2 + \cdots + n\mu_n = n.$$

By a similar argument, we note that

$$E_n^n(a_1,\cdots,a_n)=E_1^n$$

and that the first and the last polynomials are given by

$$E_n^n=a_1^n \qquad \text{and} \qquad E_n^1=a_n.$$

We now state and prove our main results. Throughout our discussion, the parameters $\mathcal{L}$ and $\mathcal{M}$ are given by

$$\mathcal{L}:=[1-\alpha(1+q)] \qquad \text{and} \qquad \mathcal{M}:=-q.$$

**Theorem 2.1.** *For* $0 \leqq \alpha < 1$ *and* $\gamma \geqq 0$, *let* $f \in \mathcal{R}_q(\Sigma,\alpha,\gamma)$. *If*

$$a_m=0 \qquad (2 \leqq m \leqq n-1),$$

*then*

$$|a_n| \leqq \frac{|1-\alpha+q(1-\alpha)|}{[n]_q+\gamma\,[n]_q\,[n-1]_q} \qquad (n \geqq 3). \tag{2.3}$$

*Proof.* For the function $f \in \mathcal{R}_q(\Sigma,\alpha,\gamma)$ of the form (1.1), we have

$$D_q f(z)+\gamma z D_q^2 f(z)=1+\sum_{n=2}^{\infty}\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)a_n z^{n-1} \tag{2.4}$$

and, for its inverse map $g=f^{-1}$, we get

$$D_q g(w)+\gamma w D_q^2 g(w)=1+\sum_{n=2}^{\infty}\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)b_n w^{n-1}, \tag{2.5}$$

where

$$b_n=\frac{1}{[n]_q}\,K_{n-1}^{-n}(a_2,a_3,\cdots,a_n).$$

Since both the function $f$ and its inverse map $g=f^{-1}$ are in $\mathcal{R}_q(\Sigma,\alpha,\gamma)$, by the definition of subordination, there exist two Schwarz functions $p(z)$ and $q(w)$ given by

$$p(z)=\sum_{n=1}^{\infty}c_n z^n \qquad \text{and} \qquad q(w)=\sum_{n=1}^{\infty}d_n w^n \qquad (z,w\in\mathbb{U}),$$

so that we have

$$\begin{aligned} D_q f(z)+\gamma z D_q^2 f(z) &= \frac{1+\mathcal{L}p(z)}{1+\mathcal{M}p(z)} \\ &= 1-\sum_{n=1}^{\infty}(\mathcal{L}-\mathcal{M})K_n^{-1}(c_1,c_2,\cdots,c_n,\mathcal{M})\,z^n \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} D_q g(w)+\gamma w D_q^2 g(w) &= \frac{1+\mathcal{L}q(w)}{1+\mathcal{M}q(w)} \\ &= 1-\sum_{n=1}^{\infty}(\mathcal{L}-\mathcal{M})K_n^{-1}(d_1,d_2,\cdots,d_n,\mathcal{M})\,w^n. \end{aligned} \tag{2.7}$$

In general, for any $\mathfrak{p} \in \mathbb{N}$ and $n \geqq 2$, we have the following expansion of $K_n^{\mathfrak{p}}(k_1, k_2, \cdots, k_n, \mathcal{M})$ (see [3] and [4]):

$$\begin{aligned}
&K_n^{\mathfrak{p}}(k_1, k_{2,} \cdots, k_n, \mathcal{M}) \\
&\quad = \frac{\mathfrak{p}!}{(\mathfrak{p}-n)!n!}\, k_1^n \mathcal{M}^{n-1} + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+1)!(n-2)!}\, k_1^{n-2} k_2 \mathcal{M}^{n-2} \\
&\qquad + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+2)!(n-3)!} \cdot k_1^{n-3} k_3 \mathcal{M}^{n-3} \\
&\qquad + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+3)!(n-4)!}\, k_1^{n-4} \left[ k_4 \mathcal{M}^{n-4} + \frac{\mathfrak{p}-n+3}{2}\, k_3^2 \mathcal{M} \right] \\
&\qquad + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+4)!(n-5)!}\, k_1^{n-5} \left[ k_5 \mathcal{M}^{n-5} + (\mathfrak{p}-n+4) k_3 k_4 \mathcal{M} \right] \\
&\qquad + \sum_{j \geqq 6} k_1^{n-1} X_j, 
\end{aligned} \tag{2.8}$$

where $X_j$ is a homogeneous polynomial of degree $j$ in the variables $k_1, k_2, \cdots, k_n$. For the coefficients of the Schwarz functions $p(z)$ and $q(w)$, we have (see [10])

$$|c_n| \leqq 1 \qquad \text{and} \qquad |d_n| \leqq 1.$$

Thus, upon comparing with the corresponding coefficients in (2.4) and (2.6), we find that

$$\left( [n]_q + \gamma\, [n]_q\, [n-1]_q \right) a_n = -(\mathcal{L} - \mathcal{M}) K_{n-1}^{-1}(c_1, c_2, \cdots, c_{n-1}, \mathcal{M}). \tag{2.9}$$

Similarly, in view of the corresponding coefficients in (2.5) and (2.7), we have

$$\left( [n]_q + \gamma\, [n]_q\, [n-1]_q \right) b_n = -(\mathcal{L} - \mathcal{M}) K_n^{-1}(d_1, d_2, \cdots, d_n, \mathcal{M}). \tag{2.10}$$

We note for

$$a_m = 0 \quad (2 \leqq m \leqq n-1) \qquad \text{and} \qquad b_n = -a_n,$$

that

$$\left( [n]_q + \gamma\, [n]_q\, [n-1]_q \right) a_n = -(\mathcal{L} - \mathcal{M}) c_{n-1} \tag{2.11}$$

and

$$-\left( [n]_q + \gamma\, [n]_q\, [n-1]_q \right) a_n = -(\mathcal{L} - \mathcal{M}) d_{n-1}. \tag{2.12}$$

Taking the moduli in (2.11) and (2.12), we thus obtain

$$\begin{aligned}
|a_n| &\leqq \frac{|\mathcal{L} - \mathcal{M}|}{[n]_q + \gamma\, [n]_q\, [n-1]_q}\, |c_{n-1}| \\
&= \frac{|\mathcal{L} - \mathcal{M}|}{[n]_q + \gamma\, [n]_q\, [n-1]_q}\, |d_{n-1}|\,.
\end{aligned}$$

Therefore, we have

$$|a_n| \leqq \frac{|1 - \alpha + q(1-\alpha)|}{[n]_q + \gamma\, [n]_q\, [n-1]_q} \qquad (n \geqq 3),$$

which completes the proof of the assertion (2.3) of Theorem 2.1. □

If we let $q \to 1-$ in Theorem 2.1 above, we obtain the following known result given by Srivastava *et al.* [26].

**Corollary 2.2.** (see [26]) Let $f$ given by (1.1) be in the class

$$\mathcal{R}_{\Sigma}^{\alpha,\gamma} \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0).$$

If

$$a_m = 0 \qquad (2 \leqq m \leqq n-1),$$

then

$$|a_n| \leqq \frac{2\,(1-\alpha)}{n\,[1+\gamma(n-1)]} \qquad (n \in \mathbb{N} \setminus \{1,2\}).$$

**Theorem 2.3.** *For $0 \leqq \alpha < 1$ and $0 \leqq \gamma$, let $f \in \mathcal{R}_q(\Sigma,\alpha,\gamma)$. Then*

$$|a_2| \leqq \min\left\{ \frac{|1-\alpha+q(1-\alpha)|}{[2]_q + \gamma\,[2]_q\,[1]_q},\ \sqrt{\frac{2(1+q)\,|1-\alpha+q(1-\alpha)|}{[2]_q\left([3]_q + \gamma\,[3]_q\,[2]_q\right)}} \right\},$$

$$|a_3| \leqq \min\left\{ \frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q}\left(\frac{[2]_q\,|1-\alpha+q(1-\alpha)|}{\left([2]_q + \gamma\,[2]_q\,[1]_q\right)^2} + \frac{2}{[3]_q+\gamma\,[3]_q\,[2]_q}\right),\right.$$
$$\left. \frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|}{\left([1]_q+[1]_q\right)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)} \right\},$$

$$\left|a_3 - [2]_q\,a_2^2\right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|}{[3]_q+\gamma\,[3]_q\,[2]_q}$$

*and*

$$\left|a_3 - \frac{[2]_q}{[1]_q+[1]_q}a_2^2\right| \leqq \frac{2\,|1-\alpha+q(1-\alpha)|}{\left|\left([1]_q+[1]_q\right)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\right|}.$$

*Proof.* Upon setting $n=2$ and $n=3$ in (2.9) and (2.10), respectively, we get

$$\left([2]_q + \gamma\,[2]_q\,[1]_q\right) a_2 = -(\mathcal{L}-\mathcal{M})c_1, \tag{2.13}$$

$$\left([3]_q + \gamma\,[3]_q\,[2]_q\right) a_3 = -(\mathcal{L}-\mathcal{M})(\mathcal{M}c_1^2 - c_2), \tag{2.14}$$

$$-\left([2]_q + \gamma\,[2]_q\,[1]_q\right) a_2 = -(\mathcal{L}-\mathcal{M})d_1 \tag{2.15}$$

and

$$\left([3]_q + \gamma\,[3]_q\,[2]_q\right)\left([2]_q\,a_2^2 - a_3\right) = -(\mathcal{L}-\mathcal{M})(\mathcal{M}d_1^2 - d_2). \tag{2.16}$$

From (2.13) and (2.15), we have

$$\begin{aligned} |a_2| &\leqq \frac{|\mathcal{L}-\mathcal{M}|}{[2]_q+\gamma\,[2]_q\,[1]_q}\,|c_1| \\ &= \frac{|\mathcal{L}-\mathcal{M}|}{[2]_q+\gamma\,[2]_q\,[1]_q}\,|d_1| \\ &\leqq \frac{|1-\alpha+q(1-\alpha)|}{[2]_q+\gamma\,[2]_q\,[1]_q}. \end{aligned} \tag{2.17}$$

Adding (2.14) and (2.16), we find that

$$[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)a_2^2=-(\mathcal{L}-\mathcal{M})\left[\mathcal{M}\left(c_1^2+d_1^2\right)-(c_2+d_2)\right], \tag{2.18}$$

which, upon taking the moduli on both sides, yields

$$|a_2|^2=\frac{2\,|\mathcal{L}-\mathcal{M}|\,(|\mathcal{M}|+1)}{[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}.$$

This last equation can be written as follows:

$$|a_2|\leqq\sqrt{\frac{2(1+q)\,|1-\alpha+q(1-\alpha)|}{[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}}. \tag{2.19}$$

Now, in order to find $|a_3|$, by subtracting (2.16) from (2.14), we obtain

$$a_3=\frac{(\mathcal{L}-\mathcal{M})\left[\mathcal{M}\left(d_1^2-c_1^2\right)-(c_2-d_2)\right]}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}+\frac{[2]_q}{([1]_q+[1]_q)}\,a_2^2. \tag{2.20}$$

Taking the moduli in (2.20) and using the fact that $d_1^2=c_1^2$, we have

$$|a_3|\leqq\frac{2\,|\mathcal{L}-\mathcal{M}|}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}+\frac{[2]_q}{[1]_q+[1]_q}\;|a_2|^2. \tag{2.21}$$

Using (2.17) in (2.21), we obtain

$$\begin{aligned} |a_3| \leqq &\frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q} \\ &\cdot\left(\frac{[2]_q\,|1-\alpha+q(1-\alpha)|}{\left([2]_q+\gamma\,[2]_q\,[1]_q\right)^2}+\frac{2}{[3]_q+\gamma\,[3]_q\,[2]_q}\right). \end{aligned} \tag{2.22}$$

Again, by using the equation (2.19) in (2.21), we have

$$|a_3|\leqq\frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}. \tag{2.23}$$

We also find from (2.16) that

$$\left|a_3-[2]_q\,a_2^2\right|\leqq\frac{(1+q)\,|1-\alpha+q(1-\alpha)|}{[3]_q+\gamma\,[3]_q\,[2]_q}.$$

From (2.20) and using the fact that $d_1^2 = c_1^2$, we have

$$a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 = \frac{(\mathcal{L} - \mathcal{M})(c_2 - d_2)}{([1]_q + [1]_q)\left([3]_q + \gamma [3]_q [2]_q\right)}. \tag{2.24}$$

Finally, by taking the moduli in (2.24), we finally obtain

$$\left| a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 \right| \leqq \frac{2\,|1 - \alpha + q(1-\alpha)|}{\left|([1]_q + [1]_q)\left([3]_q + \gamma [3]_q [2]_q\right)\right|}.$$

The proof of Theorem 2.3 is thus completed. □

In the limit case when $q \to 1-$, Theorem 2.3 yields the following bounds on $|a_2|$ and $|a_3|$ given by Srivastava *et al.* [26].

**Corollary 2.4.** (see [26]) Let $f$ given by (1.1) be in the class

$$\mathcal{R}_{\Sigma}^{\alpha,\gamma} \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0).$$

Then

$$a_2 \leqq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{3(1+2\gamma)}} & \left(0 \leqq \alpha \leqq \dfrac{1+2\gamma-2\gamma^2}{3(1+2\gamma)}\right) \\[2ex] \dfrac{1-\alpha}{1+\gamma} & \left(\dfrac{1+2\gamma-2\gamma^2}{3(1+2\gamma)} \leqq \alpha < 1\right) \end{cases}$$

and

$$a_3 \leqq \frac{2(1-\alpha)}{3(1+2\gamma)}.$$

**Theorem 2.5.** *For* $0 \leqq \alpha < 1$ *and* $0 \leqq \gamma$, *let* $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma, \lambda)$. *If*

$$a_m = 0 \qquad (2 \leqq m \leqq n-1),$$

*then*

$$|a_n| \leqq \frac{|1 - \alpha + q(1-\alpha)|\,[\lambda + 1]_{q,n-1}}{\left([n]_q + \gamma [n]_q [n-1]_q\right)[n]_q!} \qquad (n \geqq 3). \tag{2.25}$$

*Proof.* For the function $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma, \lambda)$ of the form (1.1), we have

$$\begin{aligned} &D_q \mathcal{I}_q^{\lambda} f(z) + \gamma z D_q^2 \mathcal{I}_q^{\lambda} f(z) \\ &\qquad = 1 + \sum_{n=2}^{\infty} \left([n]_q + \gamma [n]_q [n-1]_q\right) \Psi_{n-1} a_n z^{n-1}. \end{aligned} \tag{2.26}$$

Also, for its inverse mapping $g = f^{-1}$, we have

$$\begin{aligned} &D_q \mathcal{I}_q^{\lambda} g(w) + \gamma w D_q^2 \mathcal{I}_q^{\lambda} g(w) \\ &\qquad = 1 + \sum_{n=2}^{\infty} \left([n]_q + \gamma [n]_q [n-1]_q\right) \Psi_{n-1} b_n w^{n-1}, \end{aligned} \tag{2.27}$$

where

$$b_n = \frac{1}{[n]_q} K_{n-1}^{-n}\left(a_2, a_3, \cdots, a_n\right).$$

Since, both $f$ and its inverse $g = f^{-!}$ are in the function class $\mathcal{R}_q\left(\Sigma, \alpha, \gamma, \lambda\right)$, by the definition of subordination, there exist two Schwarz functions $p(z)$ and $q(w)$ given by

$$p(z) = \sum_{n=1}^{\infty} c_n z^n \qquad \text{and} \qquad q(w) = \sum_{n=1}^{\infty} d_n w^n \qquad (z, w \in \mathbb{U}),$$

so that we have

$$\begin{aligned} &D_q \mathcal{I}_q^{\lambda} f\left(z\right) + \gamma z D_q^2 \mathcal{I}_q^{\lambda} f\left(z\right) \\ &\qquad = \frac{1 + \mathcal{L}p(z)}{1 + \mathcal{M}p(z)} \\ &\qquad = 1 - \sum_{n=1}^{\infty} (\mathcal{L} - \mathcal{M}) K_n^{-1}\left(c_1, c_2, \cdots, c_n, \mathcal{M}\right) z^n \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &D_q \mathcal{I}_q^{\lambda} g(w) + \gamma w D_q^2 \mathcal{I}_q^{\lambda} g(w) \\ &\qquad = \frac{1 + \mathcal{L}q(w)}{1 + \mathcal{M}q(w)} \\ &\qquad = 1 - \sum_{n=1}^{\infty} (\mathcal{L} - \mathcal{M}) K_n^{-1}\left(d_1, d_2, \cdots, d_n, \mathcal{M}\right) w^n. \end{aligned} \tag{2.29}$$

In general, for any $\mathfrak{p} \in \mathbb{N}$ and $n \geqq 2$, an expansion of

$$K_n^{\mathfrak{p}}\left(k_1, k_{2,} \cdots, k_n, \mathcal{M}\right)$$

is given by (2.8) (see [3] and [4]). Moreover, the coefficients of the Schwarz functions $p(z)$ and $q(w)$ are constrained by (see [10])

$$|c_n| \leqq 1 \qquad \text{and} \qquad |d_n| \leqq 1.$$

Thus, upon comparing the corresponding coefficients in (2.26) and (2.28), we find that

$$\begin{aligned} &\left([n]_q + \gamma\,[n]_q\,[n-1]_q\right) \Psi_{n-1} a_n \\ &\qquad = -(\mathcal{L} - \mathcal{M}) K_{n-1}^{-1}\left(c_1, c_2, \cdots, c_{n-1}, \mathcal{M}\right). \end{aligned} \tag{2.30}$$

Similarly, by comparing the corresponding coefficients in (2.27) and (2.29), we have

$$\begin{aligned} &\left([n]_q + \gamma\,[n]_q\,[n-1]_q\right) \Psi_{n-1} b_n \\ &\qquad = -(\mathcal{L} - \mathcal{M}) K_n^{-1}\left(d_1, d_2, \cdots, d_n, \mathcal{M}\right). \end{aligned} \tag{2.31}$$

We note also that, for

$$a_m = 0 \quad (2 \leqq m \leqq n-1) \qquad \text{and} \qquad b_n = -a_n,$$

we have

$$\left([n]_q + \gamma\,[n]_q\,[n-1]_q\right) \Psi_{n-1} a_n = -(\mathcal{L} - \mathcal{M}) c_{n-1} \tag{2.32}$$

and

$$-\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)\Psi_{n-1}a_n=-(\mathcal{L}-\mathcal{M})d_{n-1}. \tag{2.33}$$

Finally, by taking the moduli in (2.32) and (2.33), we obtain

$$\begin{aligned}|a_n| &\leqq \frac{|\mathcal{L}-\mathcal{M}|}{\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)\Psi_{n-1}}\,|c_{n-1}| \\ &= \frac{|\mathcal{L}-\mathcal{M}|}{\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)\Psi_{n-1}}\,|d_{n-1}|\,.\end{aligned}$$

Consequently, we have

$$|a_n| \leqq \frac{|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,n-1}}{\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)[n]_q!} \qquad (n \geqq 3),$$

which completes the proof of the assertion (2.25) of Theorem 2.5. □

**Theorem 2.6.** *For $0 \leqq \alpha < 1$ and $\gamma \geqq 0$, let $f \in \mathcal{R}_q\left(\Sigma,\alpha,\gamma,\lambda\right)$. Then*

$$|a_2| \leqq \min\left\{\frac{|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,1}}{\left([2]_q+\gamma\,[2]_q\,[1]_q\right)[2]_q!},\ \sqrt{\frac{2(1+q)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}}\right\}, \tag{2.34}$$

$$\begin{aligned}|a_3| \leqq \min\Bigg\{&\frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q}\left(\frac{([\lambda+1]_{q,1})^2\,[2]_q\,|1-\alpha+q(1-\alpha)|}{([2]_q!)^2\left([2]_q+\gamma\,[2]_q\,[1]_q\right)^2}\right. \\ &\qquad\left.+\frac{2[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}\right), \\ &\frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}\Bigg\},\end{aligned} \tag{2.35}$$

$$\left|a_3-[2]_q\,a_2^2\right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!} \tag{2.36}$$

*and*

$$\left|a_3-\left(\frac{[2]_q}{[1]_q+[1]_q}\right)a_2^2\right| \leqq \frac{2\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left|([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\right|[3]_q!}. \tag{2.37}$$

*Proof.* Upon setting $n = 2$ and $n = 3$ in (2.30) and (2.31), respectively, we have

$$\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1 a_2 = -(\mathcal{L} - \mathcal{M}) c_1, \tag{2.38}$$

$$\left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2 a_3 = -(\mathcal{L} - \mathcal{M})(\mathcal{M} c_1^2 - c_2), \tag{2.39}$$

$$-\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1 a_2 = -(\mathcal{L} - \mathcal{M}) d_1 \tag{2.40}$$

and

$$\left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2 \left([2]_q a_2^2 - a_3\right) = -(\mathcal{L} - \mathcal{M})(\mathcal{M} d_1^2 - d_2). \tag{2.41}$$

Making use of (2.38) and (2.40), we find that

$$\begin{aligned} |a_2| &\leqq \frac{|\mathcal{L} - \mathcal{M}|}{\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1} |c_1| \\ &= \frac{|\mathcal{L} - \mathcal{M}|}{\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1} |d_1| \\ &\leqq \frac{|1 - \alpha + q(1 - \alpha)| \, [\lambda + 1]_{q,1}}{\left([2]_q + \gamma [2]_q [1]_q\right) [2]_q!}. \end{aligned} \tag{2.42}$$

Also, by adding (2.39) and (2.41), we have

$$[2]_q \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2 a_2^2 = -(\mathcal{L} - \mathcal{M}) \left[\mathcal{M}\left(c_1^2 + d_1^2\right) - (c_2 + d_2)\right]. \tag{2.43}$$

Now, if we take the moduli in both sides of (2.43), we obtain

$$|a_2|^2 = \frac{2 |\mathcal{L} - \mathcal{M}| \, (|\mathcal{M}| + 1)}{[2]_q \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2},$$

so that

$$|a_2| \leqq \sqrt{\frac{2(1 + q) \, |1 - \alpha + q(1 - \alpha)| \, [\lambda + 1]_{q,2}}{[2]_q \left([3]_q + \gamma [3]_q [2]_q\right) [3]_q!}}. \tag{2.44}$$

In order to find $|a_3|$, we subtract (2.41 ) from (2.39), We thus obtain

$$a_3 = \frac{(\mathcal{L} - \mathcal{M}) \left[\mathcal{M}\left(d_1^2 - c_1^2\right) - (c_2 - d_2)\right]}{([1]_q + [1]_q) \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2} + \left(\frac{[2]_q}{([1]_q + [1]_q)}\right) a_2^2, \tag{2.45}$$

which, after taking the moduli and using the fact that

$$d_1^2 = c_1^2,$$

yields

$$|a_3| \leqq \frac{2 |\mathcal{L} - \mathcal{M}|}{([1]_q + [1]_q) \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2} + \left(\frac{[2]_q}{[1]_q + [1]_q}\right) |a_2|^2. \tag{2.46}$$

Using (2.42) in (2.46), we have

$$|a_3| \leqq \frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q}\left(\frac{\left([\lambda+1]_{q,1}\right)^2[2]_q\,|1-\alpha+q(1-\alpha)|}{([2]_q!)^2\left([2]_q+\gamma\,[2]_q\,[1]_q\right)^2} + \frac{2[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}\right). \tag{2.47}$$

Again, by using (2.44) in (2.46), we get

$$|a_3| \leqq \frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}.$$

It follows from (2.41) that

$$\left|a_3-[2]_q\,a_2^2\right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}.$$

Using the fact that

$$d_1^2=c_1^2$$

in (2.45), we have

$$a_3-\left(\frac{[2]_q}{[1]_q+[1]_q}\right)a_2^2=\frac{(\mathcal{L}-\mathcal{M})\,(c_2-d_2)}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\Psi_2}. \tag{2.48}$$

By taking the moduli on both sides of (2.48), we finally obtain

$$\left|a_3-\left(\frac{[2]_q}{([1]_q+[1]_q)}\right)a_2^2\right| \leqq \frac{2\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left|([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\right|[3]_q!},$$

which completes the proof of Theorem 2.6. □

## 3. Concluding remarks and observations

Here, in our present investigation, we have successfully applied the Faber polynomial expansion method as well as the $q$-analysis in our study of several new subclasses of analytic and bi-univalent functions by using a certain $q$-integral operator in the open unit disk $\mathbb{U}$. We have derived bounds for the $n$th coefficient in the Taylor-Maclaurin series expansion for functions in each of these newly-defined analytic and bi-univalent function classes subject to a gap series condition. By means of corollaries of our main theorems, we have also highlighted some known consequences of our main results, which were given recently by Srivastava *et al.* [26].

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