Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 4, 567–572
DOI: 10.24193/subbmath.2025.4.02

On nilpotent matrices that are unit-regular

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Abstract. In this paper, we characterize regular nilpotent 2×2 matrices over Bézout domains and prove that they are unit-regular. We also demonstrate that nilpotent $n \times n$ matrices over division rings are unit-regular.

Mathematics Subject Classification (2010): 15B99, 15B33, 16U10, 16U90.

Keywords: von Neumann regular; nilpotent; matrix; Bezout domain; exchange ring; shift matrix; block diagonal matrix.

1. Introduction

The genuine ring-theoretic definition of exchange elements and exchange rings (also referred to as "suitable" in [4]) is as follows: An element $a \in R$ is called *left exchange* if there exists an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1 - a)$. This definition is left-right symmetric, and a ring is called *exchange* if every element satisfies this property.

Nicholson ([4], Theorem 2.1) proved that a ring R is exchange if and only if the full $n \times n$ matrix ring $\mathbb{M}_n(R)$ is exchange.

A remarkable result first proved by P. Ara (1996), with a simplified proof by D. Khurana (2016), states that the regular nilpotents in any exchange ring are unit-regular.

This result motivated us to search for other classes of rings where this property holds even if the ring itself is not exchange.

In this note, we show that this property holds in the matrix ring $\mathbb{M}_2(R)$ for any Bézout domain R. Specifically, we determine all the regular nilpotents in such matrix rings. Since Bézout domains need not be exchange rings (for example, $\mathbb{Z}$ is not exchange), our result identifies a new class of rings in which regular nilpotents are unit-regular.

Received 13 August 2025; Accepted 02 October 2025.

Recall that an element a in a ring R is (von Neumann) *regular* if there exists $b \in R$ such that $a = aba$. Such an element b is called an *inner inverse* of a. If b is a unit, then a is said to be *unit-regular*, and b is called a *unit inner inverse* of a.

Throughout this note, we assume that all rings are associative, nonzero, and possess a multiplicative identity. We use the well-known abbreviations UFD (unique factorization domain) and GCD (greatest common divisor), assuming that GCD exists.

2. Preliminaries

Since we intend to work with 2×2 matrices over commutative domains, we begin by recalling some well-known results.

Lemma 2.1. *Over any ring, both nilpotent and regular properties are invariant under conjugation.*

Since this will be used in the proof of Theorem 3.4, we supply a proof for the next special case.

Lemma 2.2. *If a is conjugate to b and b is unit-regular then a is also unit-regular.*

Proof. Assume $b = u^{-1}au$ and $b = bvb$ for $u, v \in U(R)$. Then $a(uvu^{-1})a = a$. □

Lemma 2.3. *Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any matrix over a commutative ring R.*

(i) If $\det(T) = Tr(T) = 0$ then $T^2 = 0_2$.
(ii) If R is a domain, the converse holds.

Proof. For a 2×2 matrix over a commutative ring, if $\det(T) = Tr(T) = 0$, by Cayley-Hamilton theorem, it follows that $T^2 = 0_2$.

Conversely, also by Cayley-Hamilton theorem, if $T^2 = 0_2$ then $Tr(T)T = \det(T)I_2$ and so, denoting $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a^2 = -bc = d^2$ and $b(a+d) = c(a+d) = 0$.

In the absence of zero divisors, we have $Tr(T) = a + d = 0$ or $b = c = 0$.

In the first case, $\det(T) = 0$ and in the second $T = 0_2$. □

The nonzero nilpotent $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ (with $a^2 + bc = 0$), is also regular if there is $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with $T = TXT$. This equality is equivalent to the linear system

$$(S)\begin{cases} a^2x + acy + abz + bcw & = & a \\ abx - a^2y + b^2z - abw & = & b \\ acx + c^2y - a^2z - acw & = & c \\ bcx - acy - abz + a^2w & = & -a \end{cases}.$$

As T is supposed to be nonzero, there are two cases to consider: $a = 0$, then T can be either $T = bE_{12}$ or $T = cE_{21}$ (with b or c nonzero), or else $a, b, c \neq 0$. In the first case, since cE_{21} is similar to cE_{12}, the case is settled as follows.

Lemma 2.4. *Over any commutative domain, a nonzero nilpotent* $rE_{12} = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ *is regular if and only if* $r \in U(R)$. *In this case,* rE_{12} *is unit-regular.*

Proof. One way, $(rE_{12}) \begin{bmatrix} x & y \\ z & w \end{bmatrix} (rE_{12}) = rE_{12}$ is equivalent to $r^2 z = r$. Hence r^2 and r are associates, whence $r \in U(R)$. Conversely, if r is a unit, $r^{-1}(E_{12} + E_{21})$ is a unit inner inverse for rE_{12}. □

In the remaining case, we focus on matrices with only nonzero entries. Recall that elements a, b, c in a ring R are said to form a *unimodular row* if $aR+bR+cR = R$.

Theorem 2.5. *Over any commutative domain, the nilpotent matrix* $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ *with nonzero entries is regular if and only if* a, b, c *form a unimodular row.*

Proof. In the linear system (S) above, proceed as follows. In the first three equations, replace a^2 with $-bc$. In the last equation, replace bc with $-a^2$. Then, in each of the four equations, cancel a, b, c, and $-a$ respectively. As a result, all equations reduce to $a(x - w) + cy + bz = 1$. Hence the system is solvable if and only if a, b, c form a unimodular row. □

Over Bézout domains, the precise form of these regular nilpotents is given in Theorem 3.3.

Remarks. 1) In a UFD (such domains are GCD), if $a^2 + bc = 0$ then a, b, c cannot be pairwise coprime, unless $a^2 = 1$.

Indeed, since a^2 divides $-bc$, *any prime* dividing a (if any !) divides bc. But then $\gcd(a, b) = \gcd(a, c) = 1$ fails and so a, b, c cannot be pairwise coprime.

As mentioned, the $a^2 = 1$ case is excepted. For example,

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} E_{11} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

2) There are many triples (a, b, c) that satisfy $\gcd(a, b, c) = 1$ and $a^2 + bc = 0$.

A trivial example is $(0, 1, 0)$, but easy examples are found taking $b = 1$ (i.e., $(a, 1, -a^2)$) or $c = 1$ (i.e., $(a, -a^2, 1)$).

3) The determinant Δ of the system matrix is

$$\Delta = \begin{vmatrix} a^2 & ac & ab & bc \\ ab & -a^2 & b^2 & -ab \\ ac & c^2 & -a^2 & -ac \\ bc & -ac & -ab & a^2 \end{vmatrix} = (a^2 + bc)^4 = 0, \tag{2.1}$$

and the other four minors of the augmented matrix also vanish. However, applying the Kronecker–Capelli theorem (also known as the Rouché–Capelli theorem) to analyze the solvability of a system of linear equations can be quite challenging.

Example. The nonzero nilpotent $A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$ is regular only if $4 \mid 2$.

Indeed, for $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $A = AXA$ reduces to $4[a(x-w)+cy+bz] = 2$. Over any commutative domain r, this amounts to $2 \in U(R)$.

Since division rings are (trivially) clean, and hence exchange, it follows from Ara's result that regular nilpotent matrices over division rings are unit-regular. In closing this section, we show that this result still holds even when the regularity hypothesis is dropped.

Recall (a Jordan canonical form for matrices) that every nilpotent matrix over a field is similar to a block diagonal matrix $\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{bmatrix}$, where each block B_i is a shift matrix (possibly of different sizes). A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as $n \times n$ matrix. When $n = 1$, $S = 0$.

Proposition 2.6. *Over any ring, the shift matrices are unit-regular.*

Proof. The $n \times n$ shift matrix is $S = E_{12} + E_{23} + ... + E_{n-1,n}$. Consider

$$U = E_{21} + E_{32} + ... + E_{n,n-1} + E_{1n},$$

that is, the nonzero entries are on the subdiagonal and in the NE corner and all are equal to 1. Clearly U is invertible and $SUS = S$. $\square$

Theorem 2.7. *Every block diagonal matrix where each block is a shift matrix (possibly of different sizes) is unit-regular.*

Proof. Such block diagonal matrices have nonzero entries (equal to 1) only on the superdiagonal, but some entries on the superdiagonal may be zero (if there are at least two blocks). We can use the same unit inner inverse as in the proof of the previous proposition. To be more specific, assume $S' = E_{12} + 0_{23} + E_{34} + .. + E_{n-1,n}$ (i.e., the first block has size two). Then $S'U = E_{11} + 0_{22} + E_{33} + ... + E_{n-1,n-1}$ and so $S'US' = S'$. $\square$

Using Theorem 3.3 in [2], it follows that *over any division ring R, the nilpotent matrices of $\mathbb{M}_n(R)$ are unit-regular.*

3. Over Bézout domains

Proposition 3.1. *Over any Bézout domain R, every nonzero nilpotent 2×2 matrix is similar to rE_{12}, for some $r \in R$.*

Proof. Take $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 0$. We construct an invertible matrix $U = (u_{ij})$ such that $TU = U(rE_{12})$ with a suitable $r \in R$.

Let $d = \gcd(a,b)$ and denote $a = da_1$, $b = db_1$ with $\gcd(a_1,b_1) = 1$. Then $d^2a_1^2 = -db_1c$ and since $\gcd(a_1,b_1) = 1$ implies $\gcd(a_1^2,b_1) = 1$, it follows b_1 divides d. Set $d = b_1u_2$ and so $T = \begin{bmatrix} a_1b_1u_2 & b_1^2u_2 \\ -a_1^2u_2 & -a_1b_1u_2 \end{bmatrix} = u_2 \begin{bmatrix} a_1b_1 & b_1^2 \\ -a_1^2 & -a_1b_1 \end{bmatrix} = u_2T'$.

Since $\gcd(a_1,b_1) = 1$ there exist $s,t \in R$ such that $sa_1 + tb_1 = 1$. If we take $U = \begin{bmatrix} b_1 & s \\ -a_1 & t \end{bmatrix}$, which is invertible (indeed, $U^{-1} = \begin{bmatrix} t & -s \\ a_1 & b_1 \end{bmatrix}$), one can check $T'U = \begin{bmatrix} 0 & b_1 \\ 0 & -a_1 \end{bmatrix} = UE_{12}$, so $r = u_2$. □

Corollary 3.2. *Over Bézout domains, the nonzero nilpotents* 2×2 *matrices that are regular are those similar to unit ring multiples of* E_{12}.

As this result is not explicit enough, we will elaborate further.

For the first main result, we revisit the proof of the Proposition 3.1 and use Lemma 2.4.

Theorem 3.3. *Up to association, over Bézout domains, the nonzero nilpotent matrices that are regular are of form* $\begin{bmatrix} b_1a_1 & b_1^2 \\ -a_1^2 & -b_1a_1 \end{bmatrix}$ *with* $\gcd(a_1,b_1) = 1$.

Proof. Following the proof of the Proposition 3.1, for $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ we take $d = \gcd(a,b)$ and write $a = da_1$, $b = db_1$. From $a^2 = -bc$ (i.e., $d^2a_1^2 = -db_1c$) using $\gcd(a_1,b_1) = 1$ (and so $\gcd(a_1^2,b_1) = 1$) we get $b_1 \mid d$. If $d = b_1u_2$ then for $r = u_2$ we have T similar to rE_{12}.

According to Lemma 2.4, for T to be regular, u_2 should be a unit, that is, b_1 and d are associates. Then up to association, $b = b_1^2$ and $a = b_1a_1$, that is, b is a square $(= b_1^2)$ and $b_1 \mid a$.

These are precisely the nilpotent matrices $\begin{bmatrix} b_1a_1 & b_1^2 \\ -a_1^2 & -b_1a_1 \end{bmatrix}$. If $sa_1 + tb_1 = 1$ then for $U = \begin{bmatrix} b_1 & s \\ -a_1 & t \end{bmatrix}$ we have $TU = UE_{12}$ (here $r = u_2 = 1$). □

Interestingly (see Theorem 4.4, [2]), over GCD domains, these coincide with the nonzero nilpotents that are *fine* - namely, those that can be expressed as the sum of a unit and a nilpotent. It would be worthwhile to explore whether any relationship exists between regular nilpotents and fine nilpotents.

We can now state and prove the second main result of this section.

Theorem 3.4. *Over any Bézout domain, the regular nilpotents* 2×2 *matrices are unit-regular.*

Proof. This follows at once from Proposition 3.1, Lemma 2.2 and the unit inner inverse $E_{12} + E_{21}$ for E_{12}. Indeed, $E_{12} = E_{12}(E_{12} + E_{21})E_{12}$.

An *explicit* unit inner inverse for $T = \begin{bmatrix} b_1a_1 & b_1^2 \\ -a_1^2 & -b_1a_1 \end{bmatrix}$ (see Theorem 3.3) with $sa_1 + tb_1 = 1$ is $U(E_{12} + E_{21})U^{-1} = \begin{bmatrix} st + a_1b_1 & -s^2 + b_1^2 \\ t^2 - a_1^2 & -st - a_1b_1 \end{bmatrix}$ (see the proof of Lemma 2.2) where $U = \begin{bmatrix} b_1 & s \\ -a_1 & t \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} t & -s \\ a_1 & b_1 \end{bmatrix}$. □

Example. For $T = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$, an unit inner inverse is $\begin{bmatrix} 5 & 8 \\ -3 & -5 \end{bmatrix}$.

Acknowledgments: Thanks are due to the referee for suggestions that improved the presentation.

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