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On a certain subclass of analytic univalent function defined by using Komatu integral operator

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Abstract. In this paper a certain class of analytic univalent functions in the open unit disk is defined. Some interesting results including inclusion relations argument properties and the effect of the certain integral operator to the elements of this class are investigated.

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1. Introduction

Let A denote the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Also let S denotes the subclass of A consisting of univalent functions in U. A function $f \in A$ is said to be starlike of order γ $(0 \leq \gamma < 1)$ in U if

$$\text{Re}\, \frac{zf'(z)}{f(z)} > \gamma.$$

We denote by $S^*(\gamma)$, the class of all such functions. A function $f \in A$ is said to be convex of order γ $(0 \leq \gamma < 1)$ in U if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma.$$

Let $K(\gamma)$ denote the class of all those functions $f \in A$ which are convex of order γ in U. We have

$$f \in K(\gamma) \quad \text{if and only if} \quad zf'(z) \in S^*(\gamma).$$

Recently, Komatu [4] has introduced a certain integral operator L_a^λ $(a>0,\ \lambda>0)$

$$L_a^\lambda f(z)=\frac{a^\lambda}{\Gamma(\lambda)}\int_0^1 t^{a-2}\left(\log\frac{1}{t}\right)^{\lambda-1} f(zt)dt,\quad z\in U,\ a>0,\ \lambda>0. \tag{1.1}$$

Thus, if $f\in A$ is of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, it is easily seen form (1.1) that

$$L_a^\lambda f(z)=z+\sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^\lambda a_nz^n,\quad a>0,\ \lambda>0.$$

According to the above series expansion for L_a^λ one can define L_a^λ for all real λ. Using the above relation, it is easy to verify that

$$z(L_a^{\lambda+1}f(z))'=aL_a^\lambda f(z)-(a-1)L_a^{\lambda+1}f(z),\quad a>0,\ \lambda\geq 0. \tag{1.2}$$

We note that

(i) For $a=1$, $\lambda=k$ (k is any integer number), the multiplier transformation $L_1^\lambda=I^k$, was studied by Flet [2] and Sălăgean [9];

(ii) For $a=1$, $\lambda=-k$ ($k\in\mathbb{N}_0=\{0,1,2,\dots\}$), the differential operator $L_1^{-k}=D^k$, was studied by Sălăgean [9];

(iii) For $a=2$, $\lambda=k$ (k is any integer number), the operator $L_2^k=L^k$, was studied by Uralegddi and Somantha [10];

(iv) For $a=2$, the multiplier transformation $L_2^\lambda=I^\lambda$, was studied by Jung et all [3].

If $f\in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\eta\right)\right|<\frac{\pi}{2}\beta,\quad z\in U,\ 0\leq\eta<1,\ 0<\beta\leq 1,$$

then f is said to be strongly starlike of order β and type η in U. If $f\in A$ satisfies

$$\left|\arg\left(\frac{1+zf''(z)}{f'(z)}-\eta\right)\right|<\frac{\pi}{2}\beta,\quad z\in U,\ 0\leq\eta<1,\ 0<\beta\leq 1,$$

then f is said to be strongly convex of order β and type η in U. We denote by $S^*(\beta,\eta)$ and $K(\beta,\eta)$, respectively, the subclasses of a consisting of all strongly starlike and strongly convex of order β and type η in U. We also note that $S^*(1,\eta)=S^*(\eta)$ and $K(1,\eta)=K(\eta)$. We shall use $S^*(\beta)$ and $K(\beta)$ to denote $S^*(\beta,0)$ and $K(\beta,0)$, respectively, which are the classes of univalent starlike and univalent convex functions of order β $(0\leq\beta<1)$.

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n,$$

which are analytic in U and satisfy the condition $\operatorname{Re} p(z)>0$. For two functions f and g, analytic in U, we say that the function f is subordinate to g, and write $f(z)\prec g(z)$, if there exists a Schwarz function w in U, such that $f(z)=g(w(z))$.

For $a>0$, let $S^{\lambda}(a,\eta,h)$ be the class of functions $f\in A$ satisfying the condition

$$\frac{1}{1-\eta}\left(\frac{z(L_a^{\lambda}f(z))'}{L_a^{\lambda}f(z)}-\eta\right)\prec h(z),\quad 0\le\eta<1,\ h\in\mathcal{P}.$$

For simplicity we write

$$S^{\lambda}\left(a,\eta,\frac{1+Az}{1+Bz}\right)=S^{\lambda}(a,\eta,A,B),\quad -1\le B<A\le 1.$$

2. Preliminaries

Lemma 2.1. [1] *For $\beta,\gamma\in\mathbb{C}$ let h be convex univalent in U with $h(0)=1$ and* $\operatorname{Re}(\beta h(z)+\gamma)>0$, *if p is analytic in U with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z),$$

implies that $p(z)\prec h(z)$.

Lemma 2.2. [5] *Let h be convex univalent in U and w be analytic in U with* $\operatorname{Re}w(z)>0$. *If p is analytic in U and $p(0)=h(0)$, and*

$$p(z)+zw(z)p'(z)\prec h(z),$$

then $p(z)\prec h(z)$.

Lemma 2.3. [7] *Let p be analytic in U with $p(0)=1$ and $p(z)\neq 0$ for all $z\in U$. Suppose that there exists a point $z_0\in U$ such that*

$$|\arg p(z)|<\frac{\pi}{2}\alpha,\quad |z|<|z_0|,$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}\alpha,\quad 0<\alpha\le 1,$$

then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\alpha,$$

where

$$k\ge\frac{1}{2}\left(a+\frac{1}{a}\right),\quad \textit{when}\quad \arg p(z_0)=\frac{\pi}{2}\alpha,$$

$$k\le-\frac{1}{2}\left(a+\frac{1}{a}\right),\quad \textit{when}\quad \arg p(z_0)=-\frac{\pi}{2}\alpha,$$

and

$$p(z_0)^{\frac{1}{\alpha}}=\pm ia.$$

Lemma 2.4. [8] *The function*

$$(1-z)^{\gamma}\equiv\exp(\gamma\log(1-z)),\quad \gamma\neq 0,$$

is univalent if only if γ is either in the closed disk $|\gamma-1|\le 1$ or in the closed disk $|\gamma+1|\le 1$.

Lemma 2.5. [6] *Let q be analytic in U and let Θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)),\ h(z) = \Theta(q(z)) + Q(z),$$

and suppose that

(1) *Q is starlike; either*

(2) *h is convex;*

(3) $\mathrm{Re}\, \dfrac{zh'(z)}{Q(z)} = \mathrm{Re}\left(\dfrac{\Theta'(q(z)}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right) > 0.$

If p is analytic in U with $p(0) = q(0)$ and $p(U) \subset \mathbb{D}$, and

$$\Theta(p(z)) + zp'(z)\phi(p(z)) \prec \Theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

3. Main results

Theorem 3.1. *$S^{\lambda}(a, \eta, h) \subset S^{\lambda+1}(a, \eta, h)$, where*

$$\mathrm{Re}\ ((1-\eta)h(z) + \eta + (a-1)) > 0.$$

Proof. Suppose that $f \in S^{\lambda}(a, \eta, h)$, set

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} f(z)} - \eta\right), \quad z \in U,\ 0 \leq \eta < 1,$$

where p is analytic function with $p(0) = 1$. By using the equation

$$z(L_a^{\lambda+1} f(z))' = aL_a^{\lambda} f(z) - (a-1)L_a^{\lambda+1} f(z), \quad a > 0,\ \lambda > 0, \tag{3.1}$$

we have

$$(a-1) + \eta + (1-\eta)p(z) = (a-1) + \frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} f(z)}. \tag{3.2}$$

Hence from (3.1) and (3.2) we have

$$(a-1) + \eta + (1-\eta)p(z) = a\frac{L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)}. \tag{3.3}$$

Differentiating logarithmically derivatives in both sides of (3.3) and using (3.1) we have

$$\frac{1}{1-\eta}\left(\frac{z(L_a^{\lambda} f(z))'}{L_a^{\lambda} f(z)} - \eta\right) = \frac{zp'(z)}{(a-1) + \eta + (1-\eta)p(z)} + p(z),\ 0 \leq \eta < 1,\ z \in U.$$

Since $\mathrm{Re}\ ((1-\eta)h(z) + \eta + (a-1)) > 0$, applying Lemma 2.1, it follows that $p(z) \prec h(z)$, that is

$$\frac{1}{1-\eta}\left(\frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} f(z)} - \eta\right) \prec h(z),$$

and $f \in S^{\lambda+1}(a, \eta, h)$. □

Taking $h(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, we have the following Corollary:

Corollary 3.2. *The inclusion relation* $S^{\lambda}(a, \eta, A, B) \subset S^{\lambda+1}(a, \eta, A, B)$ *holds for any* $a > 0$.

Letting $a = 1$, $\lambda = 0$ and $h(z) = (\frac{1+z}{1-z})^{\beta}$ in Theorem 3.1 and using $S^{\lambda-1}(a, \eta, h) \subset S^{\lambda}(a, \eta, h)$ we have the following inclusion relation:

Corollary 3.3. $K(\beta, \eta) \subset S^{*}(\beta, \eta)$.

Theorem 3.4. *Let* $0 < \rho < 1$, $\gamma \neq 1$ *and* $a \geq 1$ *be a real number satisfying either* $|2a\gamma\rho - 1| \leq 1$ *or* $|2a\gamma\rho + 1| \leq 1$. *If* $f \in A$ *satisfies the condition*

$$\operatorname{Re}\left(1 + \frac{L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)}\right) > 1 - \rho, \quad z \in U, \tag{3.4}$$

then

$$\left(z^{a-1} L_a^{\lambda+1} f(z)\right)^{\gamma} \prec q_1(z) = \frac{1}{(1-z)^{2a\gamma\rho}},$$

where q_1 *is the best dominant.*

Proof. Denoting $p(z) = (z^{a-1} L_a^{\lambda+1} f(z))^{\gamma}$, it follows that

$$\frac{zp'(z)}{p(z)} = \gamma a \frac{L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)}. \tag{3.5}$$

Combing (3.4) and (3.5), we find that

$$1 + \frac{zp'(z)}{a\gamma p(z)} \prec \frac{1 + (2\rho - 1)z}{1 - z}, \tag{3.6}$$

and if we set $\Theta(w) = 1$, $\phi(w) = \frac{1}{\gamma a w}$, and $q_1(z) = \frac{1}{(1-z)^{2a\gamma\rho}}$, then by the assumption of the theorem and making use of Lemma 2.5, we know that q_1 is univalent in U. It follows that

$$Q(z) = zq_1'(z)\phi(q_1(z)) = \frac{2\rho z}{1 - z},$$

and

$$h(z) = \Theta(q_1(z)) + Q(z) = \frac{1 + (2\rho - 1)z}{1 - z}.$$

If we consider D such that

$$q(U) = \left\{w : |w^{\frac{1}{\xi}} - 1| < |w^{\frac{1}{\xi}}|,\ \xi = 2\gamma\rho a\right\} \subset D,$$

then it is easy to check that the conditions (i) and (ii) of Lemma 2.5 hold true. Thus, the desired result of Theorem 3.4 follows from (3.6). □

Theorem 3.5. *Let* h *be convex univalent function in* U *and*

$$\operatorname{Re}\left(c + \eta + (1 - \eta)h(z)\right) > 0, \quad z \in U.$$

If $f \in A$ *satisfies the condition*

$$\frac{1}{1 - \eta}\left(\frac{z(L_a^{\lambda} f(z))'}{L_a^{\lambda} f(z)} - \eta\right) \prec h(z), \quad 0 \leq \eta < 1,$$

then

$$\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(f)(z)}-\eta\right)\prec h(z), \quad 0\le\eta<1,$$

where F_c is the integral operator defined by

$$F_c(f)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{3.7}$$

Proof. From (3.7), we have

$$z(L_a^\lambda F_c(f)(z))'=(c+1)L_a^\lambda f(z)-cL_a^\lambda F_c(f)(z). \tag{3.8}$$

Let

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(f)(z)}-\eta\right), \tag{3.9}$$

where p is analytic function with $p(0)=1$. Then, using (3.8) we get

$$c+\eta+(1-\eta)p(z)=(c+1)\frac{L_a^\lambda f(z)}{L_a^\lambda F_c(f)(z)}. \tag{3.10}$$

Differentiating logarithmically in both sides of (3.10) and multiplying by z, we have

$$p(z)+\frac{zp'(z)}{c+\eta+(1-\eta)p(z)}=\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda f(z)}-\eta\right).$$

Since Re $(c+\eta+(1-\eta)p(z))>0$ thus by Lemma 2.1, we have

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(f)(z)}-\eta\right)\prec h(z).$$

□

Letting $h(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in the Theorem 3.5, we have the following Corollary.

Corollary 3.6. *If $f\in S^\lambda(a,\eta,A,B)$, then $F_c(f)\in S^\lambda(a,\eta,A,B)$, where $F_c(f)$ is the integral operator defined by* (3.7).

Theorem 3.7. *Let $f\in A$, $0<\delta\le 1$, $a\ge 1$ and $0\le\gamma<1$. If*

$$\left|\arg\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

for some $g\in S^\lambda(a,\eta,A,B)$. Then

$$\left|\arg\left(\frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha.$$

where α $(0<\alpha\le 1)$ is the solution of the equation

$$\delta=\begin{cases}\alpha+\frac{2}{\pi}+\dfrac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+(a-1)+\alpha\sin\frac{\pi}{2}t_1}, & B\ne -1,\\ \alpha, & B=-1,\end{cases} \tag{3.11}$$

and

$$t_1 = \frac{2}{\pi}\arcsin\left(\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB)+(\eta+a-1)(1-B^2)}\right). \tag{3.12}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^{\lambda+1}f(z))'}{L_a^{\lambda+1}g(z)} - \gamma\right).$$

Using (1.2), it is easy to see that

$$((1-\gamma)p(z)+\gamma)\,L_a^{\lambda+1}g(z) = aL_a^{\lambda}f(z) - (a-1)L_a^{\lambda+1}f(z). \tag{3.13}$$

Differentiating (3.13) and multiplying by z, we obtain

$$\begin{aligned}(1-\gamma)zp'(z)L_a^{\lambda+1}g(z) + ((1-\gamma)p(z)+\gamma)\,z(L_a^{\lambda+1}g(z))' \\ = az(L_a^{\lambda}f(z))' - (a-1)z(L_a^{\lambda+1}f(z))'.\end{aligned} \tag{3.14}$$

Since $g \in S^{\lambda}(a,\eta,A,B)$, by Theorem 3.1, we have $g \in S^{\lambda+1}(a,\eta,A,B)$. Let

$$q(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^{\lambda+1}g(z))'}{L_a^{\lambda+1}g(z)} - \eta\right).$$

Then by using (1.2) once again, we have

$$q(z)(1-\eta)+\eta+(a-1) = a\frac{L_a^{\lambda}g(z))'}{L_a^{\lambda+1}g(z)}. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$\frac{zp'(z)}{q(z)(1-\eta)+\eta+(a-1)} + p(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^{\lambda+1}f(z))'}{L_a^{\lambda+1}g(z)} - \eta\right).$$

Since $q(z) \prec \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, we have

$$\left|q(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \quad z \in U,\ B \neq -1, \tag{3.16}$$

and

$$\frac{1-A}{2} \le \text{Re } q(z), \quad z \in U,\ B \neq -1. \tag{3.17}$$

Therefore, from (3.16) and (3.17), for $B \neq -1$, we obtain

$$\left|q(z)(1-\eta)+\eta+(a-1) - \frac{(1-\eta)(1-AB)}{1-B^2} - \eta - (a-1)\right| < \frac{(1-\eta)(A-B)}{1-B^2}.$$

For $B \neq -1$, we have

$$\text{Re }(q(z)(1-\eta)+\eta+(a-1)) > \frac{(1-\eta)(1-A)}{2} + \eta + (a-1).$$

Let

$$q(z)(1-\eta)+\eta+(a-1) = r\exp\left(i\frac{\Phi}{2}\right),$$

where

$$\frac{(1-\eta)(1-A)}{1-B} + \eta + (a-1) < r < \frac{(1-\eta)(1+A)}{1+B} + \eta + (a-1), \quad -t_1 < \Phi < t_1,$$

and t_1 is given by (3.12), and

$$\frac{(1-\eta)(1-A)}{2}+\eta+(a-1)<r<\infty.$$

We note that p is analytic in U with $p(0)=1$, so by applying the assumption and Lemma 2.2 with

$$w(z)=\frac{1}{q(z)(1-\eta)+\eta+(a-1)},$$

we have Re $w(z)>0$. Set

$$Q(z)=\frac{1}{1-\gamma}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)}-\gamma\right),\quad 0\leq\gamma<1.$$

At first, suppose that $p(z_0)^{\frac{1}{\alpha}}=ia(a>0)$. For $B\neq-1$ we have

$$\begin{aligned}\arg\ Q(z_0)&=\arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\\&=\frac{\pi}{2}\alpha+\arg\left(1+ik\alpha\left(r\exp\left(\frac{i\pi}{2}\Phi\right)\right)^{-1}\right)\\&=\frac{\pi}{2}\alpha+\arg\left(1+\frac{ik\alpha}{r}\left(\exp\left(\frac{-i\pi}{2}\Phi\right)\right)\right)\\&\geq\frac{\pi}{2}\alpha+\arctan\left(\frac{k\alpha\sin\frac{\pi}{2}(1-\Phi)}{r+k\alpha\cos\frac{i\pi}{2}(1-\Phi)}\right)\\&\geq\frac{\pi}{2}\alpha+\arctan\left(\frac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+(a-1)+\alpha\sin\frac{\pi}{2}t_1}\right)\\&=\frac{\pi}{2}\delta,\end{aligned}$$

where δ and t_1 are given by (3.11) and (3.12), respectively.

Similarly, for the case $B=-1$, we have

$$\arg\ Q(z)=\arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\geq\frac{\pi}{2}\alpha.$$

These results obviously contradict the assumption.
Next, suppose that $p(z_0)^{\frac{1}{\alpha}}=-ia$ $(a>0)$, $B=-1$ and $z_0\in U$. Applying the same

method we have

$$\begin{aligned}\arg\ Q(z_0) &= \arg\left(\frac{z_0 p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\\ &= \frac{-\pi}{2}\alpha+\arg\left(1-ik\alpha\left(r\exp\left(\frac{i\pi}{2}\Phi\right)\right)^{-1}\right)\\ &\le \frac{-\pi}{2}\alpha-\arctan\left(\frac{k\alpha\sin\frac{\pi}{2}(1-\Phi)}{r+k\alpha\cos\frac{i\pi}{2}(1-\Phi)}\right)\\ &\le \frac{-\pi}{2}\alpha-\arctan\left(\frac{\alpha\cos\frac{i\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+(a-1)+\alpha\sin\frac{\pi}{2}t_1}\right)\\ &= \frac{-\pi}{2}\delta,\end{aligned}$$

where δ and t_1 are given by (3.11) and (3.12) respectively.
Similarly, for the case $B=-1$, we have

$$\arg\ Q(z) = \arg\left(\frac{z_0 p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right) \le \frac{-\pi}{2}\alpha,$$

which contradicts the assumption of Theorem 3.7. Therefore, the proof of Theorem 3.7 is completed. □

Theorem 3.8. *Let* $f \in A$, $0<\delta\le 1$, $a\ge 1$, $0\le\gamma<1$ *and* $\operatorname{Re}\,(c+\eta(1-\eta)h(z))>0$. *If*

$$\left|\arg\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha,$$

for some $g\in S^\lambda(a,\eta,A,B)$. *Then*

$$\left|\arg\left(\frac{z(L_a^{\lambda+1}F_c(f)(z))'}{L_a^{\lambda+1}F_c(g)(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F_c *is defined by* (3.8), *and* α $(0<\alpha\le 1)$ *is the solution of the equation given by* (3.11).

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(g)(z)}-\gamma\right).$$

Since $g\in S^\lambda(a,\eta,A,B)$, so Theorem 3.5 implies that $F_c(g)\in S^\lambda(a,\eta,A,B)$. Using (3.9) we have

$$((1-\gamma)p(z)+\gamma)L_a^\lambda F_c(g)(z) = z(L_a^\lambda F_c(f)(z))'.$$

Now, by a simple calculation, we get

$$(1-\gamma)zp(z)'+((1-\gamma)p(z)+\gamma)\,(c+\eta+(1-\eta)q(z)) = (c+1)\frac{z(L_a^\lambda f(z))'}{L_a^\lambda F_c(g)(z)},$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(g)(z))'}{L_a^\lambda F_c(g)(z)} - \eta\right).$$

Hence we have

$$\frac{1}{1-\gamma}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)(1-\eta)+\eta+c}.$$

The remaining part of the proof in Theorem 3.8 is similar to that Theorem 3.7 and so we omit it. □

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