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Results on $\phi-$like functions involving Hadamard product

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Abstract. In this paper, we derive a differential subordination theorem involving convolution of normalized analytic functions. By selecting different dominants to our main result, we find certain sufficient conditions for $\phi-$likeness and parabolic $\phi-$likeness of functions in class $\mathcal{A}$.

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1. Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighbourhood of the point z. A function f is said to be analytic in a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Let $\mathcal{A}$ be the class of functions f, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots.$$

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Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1,\ \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.

Let $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum\limits_{k=0}^{\infty} b_k z^k$ be two analytic functions, then the Hadamard product or convolution of f and g, written as $f * g$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Ronning [8] and Ma and Minda [6] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Let ϕ be analytic in a domain containing $f(\mathbb{E}),\ \phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0,\ z \in \mathbb{E}.$$

This concept was introduced by Brickman [4]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$ like for some analytic function ϕ. Later, Ruscheweyh [9] investigated the following general class of $\phi-$like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0,\ \phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function $q,\ q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|,\ z \in \mathbb{E}. \tag{1.2}$$

Equivalently, condition (1.2) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2007, Shanmugham et al. [10] proved the following result for $\phi-$like functions.

Theorem 1.1. *Let $q(z) \neq 0$ be analytic and univalent in $\mathbb{E}$ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{E}$. Let $q(z)$ satisfy*

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] > 0.$$

Let

$$\Psi(\alpha,\gamma,g;z) := \alpha\left\{\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi(f*g)(z))'}{\phi(f*g)(z)}\right\}.$$

If q satisfies

$$\Psi(\alpha,\gamma,g;z) \prec \alpha q(z) + \frac{\gamma zq'(z)}{q(z)},$$

then

$$\frac{z(f*g)'(z)}{\phi(f*g)(z)} \prec q(z)$$

and q is the best dominant.

Later in 2018, Brar and Billing [3] obtained the following result.

Theorem 1.2. *Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + (\gamma-1)\frac{zq'(z)}{q(z)}\right] > 0$ *and*

(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + (\gamma-1)\frac{zq'(z)}{q(z)} + \frac{\beta(1-\alpha)}{\alpha}(q(z))^{\beta-\gamma} + \gamma\right] > 0.$

If f and $g \in \mathcal{A}$ satisfy

$$(1-\alpha)\left[\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right]^\beta + \alpha\left[\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right]^\gamma\left[2 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi((f*g)(z)))'}{\phi((f*g)(z))}\right]$$
$$\prec (1-\alpha)(q(z))^\beta + \alpha(q(z))^\gamma\left(1 + \frac{zq'(z)}{q(z)}\right),$$

then

$$\frac{z(f*g)'(z)}{\phi(f*g)(z)} \prec q(z),\ z \in \mathbb{E},$$

where $\alpha,\ \beta,\ \gamma$ are complex numbers such that $\alpha \neq 0$, and $q(z)$ is the best dominant.

In 2019, Adegani et al. [1] established sufficient subordination conditions for functions to be close-to-convex.
Moreover, this study is also motivated by the findings of Cho et al. [5] and Adegani et al. [2] who explored subordination conditions in geometric function theory.
The aim of the present investigation is to find sufficient conditions for parabolic $\phi-$likeness and $\phi-$likeness of analytic functions.
To prove our main result, we shall use the following lemma of Miller and Mocanu.

Lemma 1.3. *([7], Theorem 3.4h, p.132). Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

1. *h is convex, or*
2. *Q is starlike.*
   In addition, assume that
3. $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for all $z \in \mathbb{E}$.*

If p is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)],\ z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. A subordination theorem

In what follows, all the powers taken are principal ones.

Theorem 2.1. *Let β and γ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
$(i)\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\gamma}{\beta} - 1\right)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*
$(ii)\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \frac{a}{c}\left(\frac{\gamma}{\beta} + 1\right)q(z) + \frac{b}{c}\left(\frac{\gamma}{\beta} + 2\right)q^2(z)\right] > 0,$ *where a, b and c are real numbers with $c \neq 0$. Let ϕ be analytic function in the domain containing $(f*g)(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in (f*g)(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{z(f*g)'(z)}{\phi((f*g)(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right]^{\gamma}\cdot$$
$$\cdot\left\{a\frac{z(f*g)'(z)}{\phi((f*g)(z))} + b\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right]^2 + c\left[1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi((f*g)(z)))'}{\phi((f*g)(z))}\right]\right\}^{\beta}$$
$$\prec [q(z)]^{\gamma}\left[aq(z) + bq^2(z) + c\frac{zq'(z)}{q(z)}\right]^{\beta}, \tag{2.1}$$

then

$$\frac{z(f*g)'(z)}{\phi((f*g)(z))} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

Proof. Define the function p by

$$p(z)=\frac{z(f*g)'(z)}{\phi((f*g)(z))},\ z\in\mathbb{E}.$$

Then the function p is analytic in $\mathbb{E}$ and $p(0)=1$.
Therefore, from equation (2.1), we get:

$$[p(z)]^{\gamma}\left[ap(z)+bp^2(z)+c\frac{zp'(z)}{p(z)}\right]^{\beta}\prec[q(z)]^{\gamma}\left[aq(z)+bq^2(z)+c\frac{zq'(z)}{q(z)}\right]^{\beta}$$

or

$$a\,[p(z)]^{\frac{\gamma}{\beta}+1}+b\,[p(z)]^{\frac{\gamma}{\beta}+2}+c\,[p(z)]^{\frac{\gamma}{\beta}-1}\,zp'(z)$$
$$\prec a\,[q(z)]^{\frac{\gamma}{\beta}+1}+b\,[q(z)]^{\frac{\gamma}{\beta}+2}+c\,[q(z)]^{\frac{\gamma}{\beta}-1}\,zq'(z)$$

Let the functions θ and φ be defined as:

$$\theta(w)=aw^{\frac{\gamma}{\beta}+1}+bw^{\frac{\gamma}{\beta}+2}\ \textit{and}\ \varphi(w)=cw^{\frac{\gamma}{\beta}-1}$$

Clearly, the functions θ and φ are analytic in domain $\mathbb{D}=\mathbb{C}\backslash\{0\}$ and $\varphi(w)\neq 0$ in $\mathbb{D}$.
Therefore,

$$Q(z)=\varphi[q(z)]zq'(z)=c[q(z)]^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z)=\theta[q(z)]+Q(z)=a[q(z)]^{\frac{\gamma}{\beta}+1}+b[q(z)]^{\frac{\gamma}{\beta}+2}+c[q(z)]^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we get

$$\frac{zQ'(z)}{Q(z)}=1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)}=1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\frac{a}{c}\left(\frac{\gamma}{\beta}+1\right)q(z)+\frac{b}{c}\left(\frac{\gamma}{\beta}+2\right)q^2(z).$$

In view of the given conditions (i) and (ii), we see that Q is starlike and
$\Re\left(\frac{zh'(z)}{Q(z)}\right)>0.$
Therefore, the proof, now follows from Lemma [1.3]. □

For $g(z)=\dfrac{z}{1-z}$ in Theorem 2.1, we have

Theorem 2.2. *Let β and γ be complex numbers such that $\beta\neq 0$. Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2.1. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$\left\{\frac{zf'(z)}{\phi(f(z))}\right\}^{\gamma}\left\{a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right\}^{\beta}$$
$$\prec(q(z))^{\gamma}\left\{aq(z)+bq^2(z)+c\frac{zq'(z)}{q(z)}\right\}^{\beta},$$

where $a,\ b$ and c are real numbers with $c \neq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

3. Applications to parabolic $\phi-$like functions

Remark 3.1. Selecting $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, $\beta = \gamma = 1$ in Theorem 2.2, then after having some calculations,

$$q'(z) = \frac{4}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

$$\frac{q'(z)}{q(z)} = \frac{\frac{4}{\pi^2\sqrt{z}(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1+\frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

$$\frac{q''(z)}{q'(z)} = \frac{3z-1}{2z(1-z)} + \frac{1}{\sqrt{z}(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}.$$

Thus the conditions (i) and (ii) of Theorem 2.1 becomes

$$1+\frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)} = 1+\frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

and

$$\begin{aligned}
&1+\frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)} + \frac{a}{c}\left(\frac{\gamma}{\beta}+1\right)q(z) + \frac{b}{c}\left(\frac{\gamma}{\beta}+2\right)q^2(z)\\
&= 1+\frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z)\\
&= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} + \frac{2a}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]\\
&\quad +\frac{3b}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2.
\end{aligned}$$

Therefore, for real numbers $a,\ b,\ c$ with $c \neq 0$ *and* $\frac{a}{c},\ \frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we derive the following result from Theorem 2.2.

Theorem 3.2. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$

$$\prec a\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + b\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0$ and $\frac{a}{c},\ \frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ \ z \in \mathbb{E}.$$

Hence f is parabolic ϕ-like.

Remark 3.3. Selecting $q(z) = 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, then after having some calculations, we have

$$q'(z) = \frac{4}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

$$\frac{q'(z)}{q(z)} = \frac{\frac{4}{\pi^2\sqrt{z}(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

$$\frac{q''(z)}{q'(z)} = \frac{3z-1}{2z(1-z)} + \frac{1}{\sqrt{z}(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}.$$

Thus the conditions (i) and (ii) of Theorem 2.1 becomes

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$

$$= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

and
$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\frac{a}{c}\left(\frac{\gamma}{\beta}+1\right)q(z)+\frac{b}{c}\left(\frac{\gamma}{\beta}+2\right)q^2(z)$$
$$=1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}q^2(z)$$
$$=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1+\frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$
$$+\frac{a}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]+\frac{2b}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2.$$
Therefore, for real numbers $a,\ b,\ c$ with $c\ \neq 0$ *and* $\frac{a}{c},\ \frac{b}{c}\geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we derive the following result from Theorem 2.2.

Theorem 3.4. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*
$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec a\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]+b\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$
$$+\frac{\frac{4c\sqrt{z}}{\pi^2(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1+\frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2},$$
where $a,\ b,\ c$ are real numbers such that $c\neq 0$ and $\frac{a}{c},\ \frac{b}{c}\geq 0$, then
$$\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$
Hence f is parabolic ϕ-like.

4. Applications to $\phi-$like functions

Remark 4.1. By taking $q(z)=1+tz,\ 0<t\leq 1,\ \beta=\gamma=1$ in Theorem 2.2, then after having some calculations we have
$$1+\frac{zq''(z)}{q'(z)}=1$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\ q^2(z)=1+\frac{2a}{c}(1+tz)+\frac{3b}{c}(1+tz)^2.$$

Thus for real numbers $a,\ b$ *and* $c\ (\neq 0)$ such that $0\leq \frac{a}{c}\leq 1$,
$0\leq \frac{b}{c}\leq 1$, we observe that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately, arrive at the following result from Theorem 2.2.

Theorem 4.2. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 +cz\left\{\frac{[\phi(f(z))]\,[zf''(z)+f'(z)]-zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec a(1+tz)^2+b(1+tz)^3+ctz,$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0$, $0\leq\frac{a}{c}\leq 1$ and $0\leq\frac{b}{c}\leq 1$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+tz,\ 0<t\leq 1,\ z\in\mathbb{E}.$$

Therefore, f is ϕ-like in $\mathbb{E}$.

Remark 4.3. When we select $q(z)=e^z,\ \beta=\gamma=1$ in Theorem 2.2, a little calculation yields that

$$1+\frac{zq''(z)}{q'(z)}=1+z$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\ q^2(z)=1+z+\frac{2a}{c}e^z+\frac{3b}{c}e^{2z}.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0$, $\frac{a}{c}\geq 0.4$ and $\frac{b}{c}=1$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.4. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 +cz\left\{\frac{[\phi(f(z))]\,[zf''(z)+f'(z)]-zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec ae^{2z}+be^{3z}+cze^z,$$

where $a,\ b,\ c$ *are real numbers such that* $c \neq 0,\ \dfrac{a}{c} \geq 0.4$ *and* $\dfrac{b}{c} = 1$, *then*

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z,\ z \in \mathbb{E},$$

i.e. f is ϕ*-like.*

Remark 4.5. By selecting $q(z) = 1 + \dfrac{2}{3}z^2,\ \beta = \gamma = 1$ in Theorem 2.2, we have

$$1 + \frac{zq''(z)}{q'(z)} = 2$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\ q^2(z) = 2 + \frac{2a}{c}\left(1 + \frac{2}{3}z^2\right) + \frac{3b}{c}\left(1 + \frac{2}{3}z^2\right)^2.$$

For real numbers $a,\ b,\ c$ such that $c\ \neq 0,\ \dfrac{a}{c} \geq -0.6$ and $\dfrac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.6. *Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E})\backslash\{0\}$. *If* $f \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec a\left(1 + \frac{2}{3}z^2\right)^2 + b\left(1 + \frac{2}{3}z^2\right)^3 + \frac{4}{3}cz^2,$$

where $a,\ b,\ c$ *are real numbers such that* $c \neq 0,\ \dfrac{a}{c} \geq -0.6$ *and* $\dfrac{b}{c} \geq 0$, *then*

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{3}z^2,\ z \in \mathbb{E}.$$

Thus f is ϕ*-like.*

Remark 4.7. By taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^\delta;\ 0 < \delta \leq 1,\ \beta = \gamma = 1$ in Theorem 2.2, we get

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1 + 2\delta z + z^2}{1 - z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\ q^2(z) = \frac{1 + 2\delta z + z^2}{1 - z^2} + \frac{2a}{c}\left(\frac{1+z}{1-z}\right)^\delta + \frac{3b}{c}\left(\frac{1+z}{1-z}\right)^{2\delta}.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0,\ b = 0$ *and* $\frac{a}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.8. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec a\left(\frac{1+z}{1-z}\right)^{2\delta} + b\left(\frac{1+z}{1-z}\right)^{3\delta} + cz\left(\frac{2\delta}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\delta},$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0,\ b = 0$ and $\frac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta};\ 0 < \delta \leq 1,\ z \in \mathbb{E}.$$

Remark 4.9. When we put $q(z) = \frac{1+(1-2\eta)z}{1-z};\ 0 \leq \eta < 1,\ \beta = \gamma = 1$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\ q^2(z) = \frac{1+z}{1-z} + \frac{2a}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right] + \frac{3b}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c\ \neq 0,\ b = 0$ *and* $\frac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we obtain the following result from Theorem 2.2.

Theorem 4.10. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$

$$\prec a\left[\frac{1+(1-2\eta)z}{1-z}\right]^2+b\left[\frac{1+(1-2\eta)z}{1-z}\right]^3+cz\left[\frac{2(1-\eta)}{(1-z)^2}\right],$$

where $a,\ b,\ c$ *are real numbers such that* $c\neq 0,\ b=0$ *and* $\dfrac{a}{c}\geq 0$, *then*

$$\frac{zf'(z)}{\phi(f(z))}\prec\frac{1+(1-2\eta)z}{1-z},\ z\in\mathbb{E},\ 0\leq\eta<1,$$

i.e. f *is* ϕ*-like in* $\mathbb{E}$.

Remark 4.11. When we select $q(z)=\dfrac{\alpha'(1-z)}{\alpha'-z};\ \alpha'>1,\ \beta=\gamma=1$ in Theorem 2.2, after a little calculation, we obtain

$$1+\frac{zq''(z)}{q'(z)}=\frac{\alpha'+z}{\alpha'-z}$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\ q^2(z)=\frac{\alpha'+z}{\alpha'-z}+\frac{2a}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]$$
$$+\frac{3b}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0,\ b=0$ *and* $\dfrac{a}{c}\geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we get the following Theorem from Theorem 2.2.

Theorem 4.12. *Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0)=0=\phi'(0)-1$ *and* $\phi(w)\neq 0$ *for* $w\in f(\mathbb{E})\backslash\{0\}$. *If* $f\in\mathcal{A},\ \dfrac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, *satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3$$
$$+cz\left\{\frac{[\phi(f(z))]\,[zf''(z)+f'(z)]-zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$
$$\prec a\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2+b\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^3+cz\left[\frac{\alpha'(1-\alpha')}{(\alpha'-z)^2}\right],$$

where $a,\ b,\ c$ *are real numbers such that* $c\neq 0,\ b=0$ *and* $\dfrac{a}{c}\geq 0$, *then*

$$\frac{zf'(z)}{\phi(f(z))}\prec\frac{\alpha'(1-z)}{\alpha'-z},\ z\in\mathbb{E},\ \alpha'>1,$$

i.e. f *is* ϕ*-like.*

Remark 4.13. By taking $q(z)=1+tz,\ 0<t\leq 0.8,\ \beta=1$ and $\gamma=0$ in Theorem 2.2, then after having some calculations we have

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1}{1+tz}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}\,q^2(z)=\frac{1}{1+tz}+\frac{a}{c}\,(1+tz)+\frac{2b}{c}\,(1+tz)^2\,.$$

Thus for real numbers $a,\ b,\ c$ such that $c\neq 0$ *and* $\frac{a}{c},\ \frac{b}{c}\geq 0$, we observe that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately, arrive at the following result from Theorem 2.2.

Theorem 4.14. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec a\,(1+tz)+b\,(1+tz)^2+\frac{ctz}{1+tz},$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0$ and $\frac{a}{c},\ \frac{b}{c}\geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+tz,\ \ 0<t\leq 0.8,\ \ z\in\mathbb{E}.$$

Therefore, f is ϕ-like.

Remark 4.15. When we select $q(z)=e^z,\ \beta=1$ and $\gamma=0$ in Theorem 2.2, a little calculation yields that

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=1$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}\,q^2(z)=1+\frac{a}{c}e^z+\frac{2b}{c}e^{2z}.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0,\ \frac{a}{c}\geq 0$ and $0\leq\frac{b}{c}\leq 0.8$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.16. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec ae^z+be^{2z}+cz,$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0,\ \frac{a}{c}\geq 0$ and $0\leq\frac{b}{c}\leq 0.8$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec e^z,\ \ z\in\mathbb{E},$$

i.e. f is ϕ-like.

Remark 4.17. By selecting $q(z) = 1 + \frac{2}{3}z^2$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, we have

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{6}{3+2z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}\ q^2(z) = \frac{6}{3+2z^2} + \frac{a}{c}\left(1 + \frac{2}{3}z^2\right) + \frac{2b}{c}\left(1 + \frac{2}{3}z^2\right)^2.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0,\ \frac{a}{c} \geq 0.6$ and $0 \leq \frac{b}{c} \leq 0.7$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.18. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left(1 + \frac{2}{3}z^2\right) + b\left(1 + \frac{2}{3}z^2\right)^2 + \frac{4cz^2}{3+2z^2},$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0,\ \frac{a}{c} \geq 0.6$ and $0 \leq \frac{b}{c} \leq 0.7$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{3}z^2,\ z \in \mathbb{E}.$$

Thus f is ϕ-like.

Remark 4.19. By taking $q(z) = \left(\frac{1+z}{1-z}\right)^\delta$; $0 < \delta \leq 0.5$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, we get

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}\ q^2(z) = \frac{1+z^2}{1-z^2} + \frac{a}{c}\left(\frac{1+z}{1-z}\right)^\delta + \frac{2b}{c}\left(\frac{1+z}{1-z}\right)^{2\delta}.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0$ *and* $\frac{a}{c},\ \frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.20. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec a\left(\frac{1+z}{1-z}\right)^{\delta} + b\left(\frac{1+z}{1-z}\right)^{2\delta} + \frac{2\delta cz}{1-z^2},$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0$ and $\dfrac{a}{c},\ \dfrac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta};\ 0 < \delta \leq 0.5,\ z \in \mathbb{E}.$$

Remark 4.21. When we put $q(z) = \dfrac{1+(1-2\eta)z}{1-z}$; $0 \leq \eta < 1$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z} - \frac{2z(1-\eta)}{(1-z)\left[1+(1-2\eta)z\right]}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}\ q^2(z) = \frac{1+z}{1-z} - \frac{2z(1-\eta)}{(1-z)\left[1+(1-2\eta)z\right]}$$
$$+\frac{a}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right] + \frac{2b}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0,\ b = 0$ *and* $\dfrac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we obtain the following result from Theorem 2.2.

Theorem 4.22. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec a\left[\frac{1+(1-2\eta)z}{1-z}\right] + b\left[\frac{1+(1-2\eta)z}{1-z}\right]^2 + cz\left[\frac{2(1-\eta)}{(1-z)(1+(1-2\eta)z)}\right],$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0,\ b = 0$ and $\dfrac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\eta)z}{1-z},\ z \in \mathbb{E},\ 0 \leq \eta < 1,$$

i.e. f is ϕ-like.

Remark 4.23. When we select $q(z) = \dfrac{\alpha'(1-z)}{\alpha'-z}$; $\alpha' > 1$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, after a little calculation, we obtain

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{\alpha' - z^2}{(1-z)(\alpha'-z)}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}\ q^2(z) = \frac{\alpha' - z^2}{(1-z)(\alpha'-z)} + \frac{a}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right] + \frac{2b}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0$, $\dfrac{a}{c} \geq 0$ and $\dfrac{b}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we get the following Theorem from Theorem 2.2.

Theorem 4.24. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left[\frac{\alpha'(1-z)}{\alpha'-z}\right] + b\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2 + \frac{(1-\alpha')cz}{(1-z)(\alpha'-z)},$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0$, $\dfrac{a}{c} \geq 0$ and $\dfrac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\alpha'(1-z)}{\alpha'-z},\ z \in \mathbb{E},\ \alpha' > 1,$$

i.e. f is ϕ-like.

5. Conclusion

Using the differential subordination technique involving convolution, we derived new conditions under which normalized analytic functions exhibit ϕ-likeness and parabolic ϕ-likeness. These results contribute to a deeper understanding of geometric function theory and open pathways for further applications.

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