Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 2, 301–319
DOI: 10.24193/subbmath.2025.2.09

Study on interval Volterra integral equations via parametric approach of intervals

Subhajit Das, Md Sadikur Rahman,
Ali Akbar Shaikh and Asoke Kumar Bhunia

Abstract. This work investigates the interval Volterra integral equation (IVIE) and its solution techniques through the parametric representation of intervals. First, the general form of the second-kind IVIE is expressed in both lower-upper bound format and its equivalent parametric form. Next, the methods of successive approximations and resolvent kernel are developed to solve the IVIE, utilizing parametric approaches and interval arithmetic. The solutions are presented in both parametric and lower-upper bound representations. Lastly, a series of numerical examples are provided to illustrate the application of these methods.

Mathematics Subject Classification (2010): 45G10, 45D05,45N05.

Keywords: Interval IVP, interval integral equation, parametric approach, successive approximation method, resolvent kernel.

1. Introduction

Integral equations play a crucial role in various fields of applied mathematics, with numerous applications in real-world problems such as radioactive decay, diffusion and heat transfer analysis, energy systems, web security, and population growth models. In these cases, the parameters involved are often not fixed but fluctuate within certain ranges due to randomness or uncertainty, making the problem imprecise. Based on the nature of these problems, the theory of integral equations can be categorized into two types:

- Precise integral equations
- Imprecise integral equations

Received 15 September 2024; Accepted 15 October 2024.

Historically, integral equations have been studied in a crisp environment, where all variables (dependent and independent) are deterministic. Numerous works have focused on solving these crisp integral equations. However, as uncertainty in areas such as technology, energy, communications, and financial security continues to grow exponentially, scientists, engineers, and system analysts face increasing challenges in solving decision-making problems under inexact conditions. To address this uncertainty, researchers have introduced various approaches, such as stochastic, fuzzy, and interval methods. Imprecise integral equations, therefore, can be classified as:

- Stochastic integral equations
- Fuzzy integral equations
- Interval integral equations

In stochastic integral equation, all the imprecise known and unknown functions are represented by the random variables with suitable distribution functions. In this sector, many researchers and mathematicians contributed their works, among those some excellent pieces are reported here. Mao [15] investigated the results on existence of the solutions of a stochastic delay integral equation. Ogawa [20, 21]studied Fredholm stochastic integral equation in random environment whereas Mirzaee et al. [16], Yong et al. [30] derived computational method and backword method respectively for solving non-linear Volterra integral equations in stochastic environment. Recently, Mohammadi [17], Samadyar and Mirzaee [24, 25] contributed their works on stochastic integral equation.

In fuzzy integral equation, the imprecise functions are presented precisely by using fuzzy set having appropriate membership functions or fuzzy numbers. In this area, Subrahmanyam et al. [27], Agarwal et al. [2], Attari et al. [6] established different methods of solving Volterra integral equations in fuzzy environment. Later Mordeson and Newman [18] studied the different solution approaches of fuzzy integral equation. Babolian et al. [7], Abbasbandy et al. [1] established some numerical technique for solving Fredholm integral equation in fuzzy environment. Also, Bica and Popescu [8] together developed a methodology for approximate solution of nonlinear Hammerstein fuzzy integral equation. Recently, Zakeri et al. [31], Ziari et al. [32], Agheli and Firozja [3]and Noeiaghdam et al. [19] accomplished their works on different types of fuzzy integral equations.

Alternatively, if the known and unknown functions of an imprecise integral equation are presented in the form of intervals, then that imprecise integral equation is called as interval integral equation (or IIE). The general form of an interval integral equation (or IIE) is given below:

$$[g_L,\, g_U]\,(u)\,[y_L,\, y_U]\,(u) = [f_L,\, f_U]\,(u) + \lambda \int\limits_{u_0}^{u \text{ or } u_1} K\,(u,z)\,[y_L,\, y_U]\,(z)\,dz$$

where $[f_L,\, f_U]\,,[g_L,\, g_U] : [u_0, u_1] \to K_c$ defined by

$$[f_L,\, f_U]\,(u) = [f_L\,(u)\,, f_U\,(u)]\ ,$$
$$[g_L,\, g_U]\,(u) = [g_L\,(u)\,, g_U\,(u)]$$

and

$$K : [u_0, u_1] \times [u_0, u_1] \to \mathbb{R}$$

which are known function whereas,

$$[y_L, y_U] (u) = [y_L (u), y_U (u)]$$

is unknown function.

In the above-mentioned integral equation, if the upper limit is fixed or variable, then the integral equation is called interval Fredholm or interval Volterra integral equation respectively. With the help of this concept of interval environment in the mathematical lingua franca of these variables/parameters, the concepts of interval differential equation or interval system of differential equations have been formulated mathematically for those real-life problems. The interval differential equations have been studied by several researchers representing the imprecise parameters by random variables/fuzzy sets/intervals. Among of them, the contributions of Kaleva et al. [11], Buckley et al. [9], Vorobiev et al. [28], Stefanini et al. [26], Malinowski [13], Ramezanadeh et al. [23], Wang et al. [29], de Costa et al. [10] and Ahmady [4] are noteworthy. On the other hand, very few works on integral equations in interval environment were accomplished among which works of An et al. [5], Otadi and Mosleh [22], Lupulescu and Van [12] are worth mentioning. An et al. [5] in their work, studied the Fredholm integral equation in interval environment using interval arithmetic and Hukuhara differentiation. Otadi and Mosleh [22] developed simulation technique for the evaluation of linear fuzzy Fredholm type integral equations. Further, Lupulescu and Van [12] extended the theory of RiemannLiouville fractional integral to develop the theory of the Abel integral equation in interval environment.

In this work, the theory of interval Volterra integral equation is studied using parametric representation of intervals and parametric differentiation. To navigate the derivation of the theorems properly, the concepts of set of parameterizations of intervals, continuous parametric interval-valued functions along with metric with respect to which their different analytical properties (like continuity, differentiability, integrability etc.) are discussed. After that the class of all parametric interval valued L^2-functions is defined, over which all the discussions of interval Volterra interval equations have been performed. Beside these, two types of solution methodologies of interval Volterra integral equations named as successive approximation and Resolvent kernel theorems are developed in the parametric form of intervals.

2. Basic notations and definitions

Let $K_c = \{[\alpha_L, \alpha_U] : \alpha_L, \alpha_U \in \mathbb{R}\}$ the set of closed and bounded intervals.

Definition 2.1. Parametric representations of $[\alpha_L, \alpha_U]$ can be defined in the following manner:

- Increasing form (IF):

$$[\alpha_L, \alpha_U] = \{\alpha (\zeta) = \alpha_L + \zeta (\alpha_U - \alpha_L) : \ 0 \leqslant \zeta \leqslant 1\}$$

- Decreasing form (DF):

$$[\alpha_L, \alpha_U] = \{\alpha(\zeta) = \alpha_U + \zeta(\alpha_L - \alpha_U) : 0 \leqslant \zeta \leqslant 1\}$$

Therefore, the set of all parametric intervals K_P is defined as follows:

$$K_P = \{\alpha(\zeta) : \alpha(\zeta) \text{ is parametric form of the interval } [\alpha_L, \alpha_U] \in K_c\}.$$

Definition 2.2. Let $I_1 = \{\alpha(\zeta_1) : \zeta_1 \in [0,1]\}, I_2 = \{\beta(\zeta_2) : \zeta_2 \in [0,1]\} \in K_p$ be the parametric forms of two intervals $[\alpha_L, \alpha_U]$ and $[\beta_L, \beta_U]$ respectively and $\lambda \in \mathbb{R}$. Then,

- Addition:

$$I_1 + I_2 = \{\alpha(\zeta_1) + \beta(\zeta_2) : \zeta_1, \zeta_2 \in [0,1]\}$$

- Subtraction:

$$I_1 - I_2 = \{\alpha(\zeta_1) - \beta(\zeta_2) : \zeta_1, \zeta_2 \in [0,1]\}$$

- Parametric difference:

$$I_1 \ominus_p I_2 = \{\alpha(\zeta) - \beta(\zeta) : \zeta \in [0,1]\}$$

- Multiplication:

$$I_1 I_2 = \{\alpha(\zeta_1)\beta(\zeta_2) : \zeta_1, \zeta_2 \in [0,1]\}$$

- Division:

$$I_1/I_2 = \left\{\frac{\alpha(\zeta_1)}{\beta(\zeta_2)} : \zeta_1, \zeta_2 \in [0,1]\right\}$$

- Scalar multiplication:

$$\lambda I_1 = \{\lambda\alpha(\zeta) : \zeta \in [0,1]\}$$

- Equality of two intervals:

$$I_1 = I_2 \Leftrightarrow \alpha(\zeta) = \beta(\zeta), \ \forall \zeta \in [0,1]$$

Proposition 2.3. *Let $I_1 = [\alpha_L, \alpha_U], I_2 = [\beta_L, \beta_U] \in K_c$, $\lambda \in \mathbb{R}$ and their parametric representations be $I_1 = \{\alpha(\zeta_1) : \zeta_1 \in [0,1]\}$ and $I_2 = \{\beta(\zeta_2) : \zeta_2 \in [0,1]\}$. The different arithmetic operations on the set K_c can be obtained as follows:*

- *Addition:*

$$I_1 + I_2 = \left[\min_{\zeta_1,\zeta_2 \in [0,1]} (\alpha(\zeta_1) + \beta(\zeta_2)), \max_{\zeta_1,\zeta_2 \in [0,1]} (\alpha(\zeta_1) + \beta(\zeta_2))\right]$$

- *Subtraction:*

$$I_1 - I_2 = \left[\min_{\zeta_1,\zeta_2 \in [0,1]} (\alpha(\zeta_1) - \beta(\zeta_2)), \max_{\zeta_1,\zeta_2 \in [0,1]} (\alpha(\zeta_1) - \beta(\zeta_2))\right]$$

- *Parametric difference:*

$$I_1 \ominus_p I_2 = \left[\min_{\zeta \in [0,1]} (\alpha(\zeta) - \beta(\zeta)), \max_{\zeta \in [0,1]} (\alpha(\zeta) - \beta(\zeta))\right]$$

- *Multiplication:*

$$I_1 I_2 = \left[\min_{\zeta_1,\zeta_2 \in [0,1]} (\alpha(\zeta_1)\beta(\zeta_2)), \max_{\zeta_1,\zeta_2 \in [0,1]} (\alpha(\zeta_1)\beta(\zeta_2))\right]$$

- *Division:*

$$^{I_1}/_{I_2} = \left[\min_{\zeta_1,\zeta_2 \in [0,1]} \left(\frac{\alpha(\zeta_1)}{\beta(\zeta_2)} \right), \max_{\zeta_1,\zeta_2 \in [0,1]} \left(\frac{\alpha(\zeta_1)}{\beta(\zeta_2)} \right) \right], 0 \notin I_2$$

- *Scalar multiplication:*

$$\lambda I_1 = \left[\min_{\zeta \in [0,1]} (\lambda \alpha(\zeta)), \max_{\zeta \in [0,1]} (\lambda \alpha(\zeta)) \right]$$

Definition 2.4. The distance function on K_p is a function $\rho_p : K_p \times K_p \to \mathbb{R}^+ \cup \{0\}$ be a function defined by

$$\rho_p(\alpha(\zeta), \beta(\zeta)) = \sup_{\zeta \in [0,1]} |\alpha(\zeta) - \beta(\zeta)|, \ \forall \alpha(\zeta), \beta(\zeta) \in K_p.$$

Clearly ρ_p is a metric on K_p.

Proposition 2.5. *Let $\alpha(\zeta), \beta(\zeta) \in K_p$, then*

$$\sup_{\zeta \in [0,1]} |\alpha(\zeta) - \beta(\zeta)| = \max_{\zeta \in \{0,1\}} |\alpha(\zeta) - \beta(\zeta)|.$$

Corollary 2.6. *Let $\rho_p^1 : K_p \times K_p \to \mathbb{R}^+ \cup \{0\}$ be defined by*

$$\rho_p^1(\alpha(\zeta), \beta(\zeta)) = \max_{\zeta \in \{0,1\}} |\alpha(\zeta) - \beta(\zeta)|, \ \forall \alpha(\zeta), \beta(\zeta) \in K_p.$$

Then ρ_p^1 is a metric on K_p.

Corollary 2.7. *Let $\rho_c : K_c \times K_c \to \mathbb{R}^+ \cup \{0\}$ be defined by*

$$\rho_c([\alpha_L, \alpha_U], [\beta_L, \beta_U]) = \max\{|\alpha_L - \beta_L|, |\alpha_U - \beta_U|\}, \ \forall [\alpha_L, \alpha_U], [\beta_L, \beta_U] \in K_c.$$

Then ρ_c is a metric on K_c.

Corollary 2.8. *The metrics ρ_p^1 and ρ_c are equivalent.*

2.1. Parametric form of interval valued functions (IVF)

Definition 2.9. An IVF is a function $[f_L, f_U] : I \subseteq \mathbb{R} \to K_c$ given by $[f_L, f_U](u) = [f_L(u), f_U(u)]$, where $f_L, f_U : I \to \mathbb{R}$ are real valued functions with $f_L(u) \leqslant f_U(u), \forall u \in I$.

Definition 2.10. The parametric form (in IF) of IVF $[f_L, f_U](u)$ is denoted as $f_{\zeta \in [0,1]} : I \to K_p$ and it is defined by

$$f_{\zeta \in [0,1]}(u) = \{f_L(u) + \zeta(f_U(u) - f_L(u)) : \ \zeta \in [0,1]\}, \ \forall u \in I.$$

Let us consider an IVF in parametric form $f_{\zeta \in [0,1]} : I \to K_p$ defined by

$$f_{\zeta \in [0,1]}(u) = \{f_L(u) + \zeta(f_U(u) - f_L(u)) : \ \zeta \in [0,1]\}, \ \forall u \in I.$$

Definition 2.11. The IVF in parametric form $f_{\zeta \in [0,1]} : I \to K_p$ is called continuous at $u_0 \in I$ if the real valued function $\tilde{f} : I \times [0,1] \to \mathbb{R}$ defined by

$$\tilde{f}(u, \zeta) = f_L(u) + \zeta(f_U(u) - f_L(u))$$

is continuous at $(u_0, \zeta), \ \forall \zeta \in [0,1]$.

Definition 2.12. The IVF in parametric form $f_{\zeta\in[0,1]} : I \to K_p$ is called differentiable at $u_0 \in I$ if the real valued function $\tilde{f} : I \times [0,1] \to \mathbb{R}$ defined by

$$\tilde{f}\left(u,\zeta\right) = f_L\left(u\right) + \zeta\left(f_U\left(u\right) - f_L\left(u\right)\right)$$

is differentiable at (u_0, ζ), $\forall \zeta \in [0,1]$. And the derivative is obtained by the following limit:

$$\left.\frac{\partial \tilde{f}\left(u,\zeta\right)}{\partial u}\right|_{u=u_0} = \tilde{f}_u\left(u_0,\zeta\right) = \lim_{u\to u_0} \frac{\tilde{f}\left(u,\zeta\right) - \tilde{f}\left(u_0,\zeta\right)}{u-u_0}.$$

The parametric derivative of $f_{\zeta\in[0,1]}$ at u_0 is denoted by $f'_{\zeta\in[0,1]}\left(u_0\right)$.

Proposition 2.13.
1. The IVF $f_{\zeta\in[0,1]}$ is continuous at u_0 iff both the bounds f_L and f_U are continuous at u_0.
2. The IVF $f_{\zeta\in[0,1]}$ is differentiable at u_0 iff both the bounds f_L and f_U are differentiable at u_0.

Proposition 2.14. *If the IVF in parametric form $f_{\zeta\in[0,1]}$ is differentiable at u_0, then*

$$f'_{\zeta\in[0,1]}\left(u_0\right) = \left\{f_L{}'\left(u_0\right) + \zeta\left(f_U{}'\left(u_0\right) - f'{}_L\left(u_0\right)\right) : \zeta \in [0,1]\right\}.$$

Definition 2.15. Let I be a Lebesgue measurable set. The IVF in parametric form $f_{\zeta\in[0,1]}$ is said to be a Lebesgue measurable parametric interval valued function over I if for every fixed $\zeta^* \in [0,1]$, the function $\tilde{f}\left(u,\zeta^*\right)$ is a measurable function.

Definition 2.16. The IVF in parametric form $f_{\zeta\in[0,1]}$ is said to be integrable over I if for every fixed $\zeta^* \in [0,1]$, the function $\tilde{f}\left(u,\zeta^*\right)$ is integrable in over I and

$$\int_I f_{\zeta\in[0,1]}\left(u\right) du = \left\{I_L + \zeta\left(I_U - I_L\right) : \forall\zeta \in [0,1]\right\},$$

where

$$I_L = \int_I f_L\left(u\right) du\,,\; I_U = \int_I f_U\left(u\right) du.$$

Proposition 2.17. *The IVF $f_{\zeta\in[0,1]}$ is integrable over I iff both the bounds f_L and f_U are integrable over I.*

Definition 2.18. The IVF $f_{\zeta\in[0,1]}$ is said to be a parametric L^2- function over I if

$$\rho_p\left(\int_I f_{\zeta\in[0,1]}\left(u\right)\, du,\, 0\right) < \infty.$$

Proposition 2.19. *The IVF $f_{\zeta\in[0,1]}$ is L^2- function over I iff both the bounds f_L and f_U are L^2- functions over I.*

Remark 2.20. The set of all parametric L^2- function over I is denoted by $L_p^2\left(I\right)$.

2.2. Interval initial value problem (Interval IVP)

Let $[y_L, y_U] : [u_0, u_1] \to K_c$ be a p-differentiable function and the IVF $[f_L, f_U] : [u_0, u_1] \to K_c$ be a continuous, then a second order interval valued initial value problem can be defined as follows:

$$\frac{d^2}{du^2}\left(\left[y_L(u), y_U(u)\right]\right) + a_1(u)\frac{d}{du}\left(\left[y_L(u), y_U(u)\right]\right) + a_2(u)\left[y_L(u), y_U(u)\right] \quad (2.1)$$
$$= \left[f_L(u), f_U(u)\right]$$

with $[y_L(u_0), y_U(u_0)] = [y_{L0}, y_{U0}]$ and $\left.\frac{d}{du}[y_L(u), y_U(u)]\right|_{u=u_0} = [y_{L1}, y_{U1}]$

where $a_1(u)$, $a_2(u)$ are real valued continious functions over $[u_0, u_1]$.

The interval initial value problem (2.1) can be represented in parametric form as follows:

$y''_{\zeta\in[0,1]}(u) + a_1(u)\, y'_{\zeta\in[0,1]}(u) + a_2(u)\, y_{\zeta\in[0,1]}(u) = f_{\zeta\in[0,1]}(u)$

with $y_{\zeta\in[0,1]}(u_0) = \{y_0(\zeta) : \zeta \in [0,1]\}$ and $y'_{\zeta\in[0,1]}(u_0) = \{y_1(\zeta) : \zeta \in [0,1]\}$

where $y_0(\zeta) = y_{0L} + \zeta\,(y_{0U} - y_{0L})$ and $y_1(\zeta) = y_{1L} + \zeta\,(y_{1U} - y_{1L})$.

3. Interval Volterra integral equation (IVIE)

In this section, we have presented some theoretical aspects regarding an IVIE of second kind. Also, the different solution approaches viz. general solution method, method of series solutions and resolvent kernel for solving an IVIE of second kind are discussed.

The general form of an IVIE is

$$[g_L, g_U](u)\,[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)\,[y_L, y_U](z)\,dz \quad (3.1)$$

where $[f_L, f_U], [g_L, g_U] : [u_0, u_1] \to K_c$ are known functions. However, $[y_L, y_U](u)$ is an unknown function and λ is a non-zero real number. Here we discuss IVIE of second kind only.

An IVIE of second kind is defined as

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)\,[y_L, y_U](z)\,dz \quad (3.2)$$

The parametric form of (3.2) is of the following form:

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} K(u,z)\, y_{\zeta\in[0,1]}(z)\,dz \quad (3.3)$$

where $y_{\zeta\in[0,1]}$ and $f_{\zeta_1\in[0,1]}$ are respectively parametric forms of $[y_L, y_U]$ and $[f_L, f_U]$.

Remark 3.1. Here the real valued function $K(u,z)$ is a L^2- function and the parametric interval valued functions $y_{\zeta\in[0,1]}$ and $f_{\zeta_1\in[0,1]}$ are taken from $L_p^2[u_0,u_1]$.

Remark 3.2. For, $y_{\zeta\in[0,1]}$ and $f_{\zeta_1\in[0,1]}$ the parametric forms are given by

$$\tilde{y}(u,\zeta) = y_L(u) + \zeta(y_U(u) - y_L(u)) \text{ and } \tilde{f}(u,\zeta_1) = f_L(u) + \zeta_1(f_U(u) - f_L(u)).$$

Proposition 3.3. *The interval Volterra integral equation (3.2) is equivalent to its parametric form (3.3).*

Definition 3.4. The interval valued function $[y_L(u), y_U(u)]$ is called a solution of (3.2) if it satisfies the equation (3.2). Similarly, the solution of (3.3) can be defined.

Proposition 3.5. *The solutions of (3.2) and (3.3) are equivalent.*

Proof. Proof follows from the equality of two intervals in parametric form. □

Before to discuss the solution procedures of the IVIE, an important formula for converting multiple integrals into a single integral for integrable interval valued functions is presented in the next subsection.

3.1. Conversion of multiple integrals into a single integral for interval integrals

Theorem 3.6. *Let $[y_L, y_U] : [u_0,u_1] \to K_c$ be given by*

$$[f_L, f_U](u) = [f_L(u), f_U(u)], \quad \forall u \in [u_0,u_1]$$

be an integrable interval valued function. Then it satisfies the following integral formula:

$$\int_{u_0}^{u} [f_L(z), f_U(z)]\, dz^n = \int_{u_0}^{u} \frac{(u-z)^{n-1}}{(n-1)!} [f_L(z), f_U(z)]\, dz \tag{3.4}$$

To prove this theorem, we have required the following Lemma:

Lemma 3.7. *Let $g, h : [u_0,u_1] \to \mathbb{R}$ be two differentiable functions with non-negative derivatives over $[u_0,u_1]$. Then,*

$$\frac{d}{du}\left(\int_{g(u)}^{h(u)} [f_L(z), f_U(z)]\, dz\right) = [f_L(h(u)), f_U(h(u))]\frac{dh(u)}{du} \ominus_p [f_L(g(u)), f_U(g(u))]\frac{dg(u)}{du} \tag{3.5}$$

Proof. From the parametric representation of $[f_L, f_U]$, it can written as

$$f_{\zeta\in[0,1]}(u) = \left\{\tilde{f}(u,\zeta) = f_L(u) + \zeta(f_U(u) - f_L(u)) : \ \zeta \in [0,1]\right\}, \ \forall u \in I.$$

From the Leibnitz's rule of differentiation under the sign of integration for real valued functions, it follows that

$$\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) = \tilde{f}(h(u),\zeta)\frac{dh(u)}{du} - \tilde{f}(g(u),\zeta)\frac{dg(u)}{du},\ \ \forall\zeta\in[0,1]$$

$$\Rightarrow \left\{\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) :\ \zeta\in[0,1]\right\}$$

$$= \left\{\tilde{f}(h(u),\zeta)\frac{dh(u)}{du} - \tilde{f}(g(u),\zeta)\frac{dg(u)}{du} :\ \zeta\in[0,1]\right\}$$

$$\Rightarrow \left\{\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) :\ \zeta\in[0,1]\right\}$$

$$= \left\{\tilde{f}(h(u),\zeta)\frac{dh(u)}{du} :\ \zeta\in[0,1]\right\} \ominus_p \left\{\tilde{f}(g(u),\zeta)\frac{dg(u)}{du}\ :\zeta\in[0,1]\right\}$$

$$\Rightarrow \left\{\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) :\ \zeta\in[0,1]\right\}$$

$$= \left\{\tilde{f}(h(u),\zeta) :\ \zeta\in[0,1]\right\}\frac{dh(u)}{du} \ominus_p \left\{\tilde{f}(g(u),\zeta)\ :\zeta\in[0,1]\right\}\frac{dg(u)}{du}$$

since, g', h' are non - negative

$$\Rightarrow \frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} [f_L(z), f_U(z)]\ dz\right)$$

$$= [f_L(h(u)), f_U(h(u))]\frac{dh(u)}{du} \ominus_p [f_L(g(u)), f_U(g(u))]\frac{dg(u)}{du}.$$

This completes the proof. □

Now, we have proved the Theorem 3.6.

Proof of Theorem 3.6

Proof. Let us consider the interval integral

$$[J_{Ln}, J_{Un}](u) = \int\limits_{u_0}^{u} (u-z)^{n-1}[f_L(z), f_U(z)]\ dz \tag{3.6}$$

Differentiating (3.6) successively with respect to u for k times and using Lemma 3.7, we get

$$\frac{d^k\left[J_{Ln}(u), J_{Un}(u)\right]}{du^k} = (n-1)(n-2)\cdots(n-k)\left[J_{Ln-k}(u), J_{Un-k}(u)\right], \text{ for } n > k. \tag{3.7}$$

Therefore from (3.6) and (3.7), it follows that:

$$\frac{d^n\left[J_{Ln}(u), J_{Un}(u)\right]}{du^n} = (n-1)!\left[f_L(u), f_U(u)\right] \tag{3.8}$$

From (3.8), we get the following recurring integrals:

$$\left[J_{L1}(u), J_{U1}(u)\right] = \int_{u_0}^{u}\left[f_L(z_1), f_U(z_1)\right]dz_1$$

$$\left[J_{L2}(u), J_{U2}(u)\right] = \int_{u_0}^{u}\int_{u_0}^{z}\left[f_L(z_1), f_U(z_1)\right]dz_1\,dz$$

Proceeding similarly, one can get the following relation:

$$\left[J_{Ln}(u), J_{Un}(u)\right] = (n-1)!\int_{u_0}^{u}\int_{u_0}^{z}\cdots\int_{u_0}^{z_{n-1}}\left[f_L(z_{n-1}), f_U(z_{n-1})\right]\,dz_{n-1}dz_{n-2}\cdots dz \tag{3.9}$$

$$\Longrightarrow \left[J_{Ln}(u), J_{Un}(u)\right] = (n-1)!\int_{u_0}^{u}\left[f_L(u), f_U(u)\right]dz^n, \tag{3.10}$$

which is the required relation. □

3.2. General solution procedure for solving IVIE

Since the equations (3.2) and (3.3) are equivalent, to get the solution of (3.2), it is sufficient to solve (3.3). Also, since the equation (3.3) represents the crisp Volterra integral equation for each fixed $\zeta, \zeta_1 \in [0,1]$, (3.3) can be solved by using any existing method for solving the Volterra integral equation. Let $\tilde{y}(u,\zeta)$ be the solutions of (3.3). Then it satisfies (3.3). Thus,

$$\therefore \tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda\int_{u_0}^{u} K(u,z)\,\tilde{y}(z,\zeta)\,dz$$

Therefore, from IPF of intervals, it follows that

$$y_L(u) = \min_{\zeta\in[0,1]}\{\tilde{y}(u,\zeta)\} \quad\text{and}\quad y_U(u) = \max_{\zeta\in[0,1]}\{\tilde{y}(u,\zeta)\}$$

So, from Proposition 2.5, it follows that,

$$y_L(u) = \min_{\zeta\in\{0,1\}}\{\tilde{y}(u,\zeta)\} \quad\text{and}\quad y_U(u) = \max_{\zeta\in\{0,1\}}\{\tilde{y}(u,\zeta)\} \tag{3.11}$$

and $y(u) = [y_L(u), y_U(u)]$ is the desired solution of (3.2).

3.3. Solution of interval Volterra integral equation of second kind by iterative method

Theorem 3.8. *Let us consider an IVIE of second kind of the form*

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[y_L, y_U](z)\,dz \tag{3.12}$$

satisfying the following conditions:

a) kernel K be a non-negative real valued continuous function on $[u_0, u_1] \times [u_0, u_1]$ and $\exists \alpha > 0$ such that

$$|K(u,z)| \leqslant \alpha, \ \ \forall (u,z) \in [u_0, u_1] \times [u_0, u_1]. \tag{3.13}$$

b)$[f_L, f_U]$ is an interval valued continuous function over $[u_0, u_1]$ and $\exists \beta > 0$ such that

$$\rho_c(f(u), 0) \leqslant \beta, \ \forall u \in [u_0, u_1]. \tag{3.14}$$

c) $\lambda > 0$ be a non-negative constant.

Then the IVIE (3.12) has a series solution as follows:

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[f_L, f_U](z)\,dz \tag{3.15}$$

$$+\lambda^2 \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1)[f_L, f_U](z_1)\, dz_1 dz + \cdots$$

Proof. From Proposition 3.5, the given IVIE is equivalent to its parametric form

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} K(u,z)\, y_{\zeta\in[0,1]}(z)\,dz \tag{3.16}$$

Therefore for fixed $\zeta, \zeta_1 \in [0,1]$

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{y}(z,\zeta)\,dz \tag{3.17}$$

After nth substitution, the equation (3.17) gives

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{f}(z,\zeta_1)\,dz + \lambda^2 \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1)\tilde{f}(z_1,\zeta_1)\, dz_1 dz$$

$$+ \cdots + \lambda^n \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1) \cdots \int_{u_0}^{z_{n-2}} K(z_{n-2}, z_{n-1})\tilde{f}(z_{n-1},\zeta_1)\, dz_{n-1} \ldots dz_1 dz$$

$$+ \tilde{R}_{n+1}(u,\zeta) \tag{3.18}$$

where,

$$\tilde{R}_{n+1}(u,\zeta) = \lambda^{n+1}\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\cdots\int_{u_0}^{z_{n-1}} K(z_{n-1},z_n)\tilde{y}(z_n,\zeta)\ dz_n\ldots dz_1dz.$$

Let

$$\tilde{M}_n(u,\zeta_1)=\lambda^n\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\cdots\int_{u_0}^{z_{n-2}} K(z_{n-2},z_{n-1})\tilde{f}(z_{n-1},\zeta_1)\ dz_{n-1}\ldots dz_1dz,$$

then from the conditions (a) and (b) and by the equivalency of the metrics $\rho_c$and ρ_p, we get

$$\left|\tilde{M}_n(z,\zeta_1)\right| \leqslant |\lambda|^n\alpha^n\frac{(b-a)^n}{n!}\beta,\ \ \forall\zeta_1\in[0,1],\ \forall u\in[u_0,u_1] \tag{3.19}$$

Now $\sum_n |\lambda|^n\alpha^n\frac{(b-a)^n}{n!}\beta$ is convergent and hence $\sum_n \tilde{M}_n(u,\zeta_1)$ is uniformly convergent over $[u_0,u_1]$, for every choice of $\zeta_1\in[0,1]$.
So, if (3.17) has a solution, clearly it can be expressed by (3.19). Therefore $\tilde{y}(u,\zeta)$ is continuous over $[u_0,u_1]$ and hence bounded.
Thus, let

$$|\tilde{y}(u,\zeta)| \leqslant \gamma(\zeta),\ \forall\zeta\in[0,1]. \tag{3.20}$$

Now,

$$\left|\tilde{R}_{n+1}(u,\zeta)\right| = \left|\lambda^{n+1}\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\cdots\int_{u_0}^{z_{n-1}} K(z_{n-1},z_n)\tilde{y}(z_n,\zeta)\ dz_n\ldots dz_1dz\right|$$

$$\leqslant |\lambda|^{n+1}\alpha^{n+1}\frac{(u_1-u_0)^{n+1}}{(n+1)!}\max_{\zeta\in\{0,1\}}\gamma(\zeta)\to 0\ \text{ as } n\to\infty.$$

Hence (3.17) has a series solution

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda\int_{u_0}^{u} K(u,z)\,\tilde{f}(z,\zeta_1)\,dz$$

$$+\lambda^2\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\tilde{f}(z_1,\zeta_1)\ dz_1dz + \cdots$$

Therefore,

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda\int_{u_0}^{u} K(u,z)\,f_{\zeta_1\in[0,1]}(z)\,dz$$

$$+\lambda^2\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)f_{\zeta_1\in[0,1]}(z_1)\ dz_1dz + \cdots.$$

Hence,

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[f_L, f_U](z)\, dz$$
$$+ \lambda^2 \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1)\ [f_L, f_U](z_1)\, dz_1 dz + \cdots .$$

□

3.3.1. Solution of interval Volterra integral equation by the method of Resolvent Kernel.

Theorem 3.9. *Consider an IVIE of the form*

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[y_L, y_U](z)\, dz. \tag{3.21}$$

Then it has a solution of the form

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} R(u,z;\lambda)[y_L(z), y_U(z)]\, dz, \tag{3.22}$$

where, $R(u,z;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,z)$ *is the resolvent kernel.*

Proof. Here the iterated kernel $K_n(u,z)$ is defined as

$$K_1(u,z) = K(u,z),\ \ K_n(u,z) = \int_{z}^{u} K(u,z_1) K_{n-1}(z_1,z)\ dt.$$

From Proposition 3.5, (3.21) is equivalent to

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} K(u,z)\, y_{\zeta\in[0,1]}(z)\, dz$$

Therefore for fixed $\zeta, \zeta_1 \in [0,1]$

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\, \tilde{y}(z,\zeta)\, dz$$

Let

$$\tilde{y}_0(u,\zeta) = \tilde{f}(u,\zeta_1). \tag{3.23}$$

Then,

$$\tilde{y}_1(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\, \tilde{y}_0(z,\zeta)\, dz$$
$$= \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\, \tilde{f}(z,\zeta_1)\, dz$$

Proceeding in this way and using (3.23), we get

$$\tilde{y}_n(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\, \tilde{f}(z,\zeta_1)\, dz + \lambda^2 \int_{u_0}^{u} K_2(u,z)\, \tilde{f}(z,\zeta_1)\, dz + \cdots$$
$$+ \lambda^n \int_{u_0}^{u} K_n(u,z)\, \tilde{f}(z,\zeta_1)\, dz.$$

Therefore, $\forall \zeta \in [0,1]$,

$$\tilde{y}(u,\zeta) = \lim_{n\to\infty} \tilde{y}_n(u,\zeta) = \tilde{f}(u,\zeta_1) + \int_{u_0}^{u} \left(\sum_{n=1}^{\infty} \lambda^n K_n(u,z) \right) \tilde{f}(z,\zeta_1)\, dz$$
$$= \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} R(u,z;\lambda) \tilde{f}(z,\zeta_1)\, dz$$

This gives,

$$\{\tilde{y}(u,\zeta) : \zeta \in [0,1]\} = \left\{ \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} R(u,z;\lambda) \tilde{f}(z,\zeta_1)\, dz : \zeta_1 \in [0,1] \right\}.$$

i.e.,

$$\tilde{y}_{\zeta\in[0,1]}(u) = \tilde{f}_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} R(u,z;\lambda) \tilde{f}_{\zeta_1\in[0,1]}(z)\, dz.$$

Hence, by the equivalency of parametric form, we have

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} R(u,z;\lambda)\, [y_L(z), y_U(z)]\, dz.$$

This completes the proof. □

4. Illustrative examples

To validate all the methods, three numerical examples are considered and solved.

Example 4.1. Let us consider the interval IVP:

$$\frac{d^2}{du^2}\left[y_L\left(u\right),y_U\left(u\right)\right]+u\frac{d}{du}\left[y_L\left(u\right),y_U\left(u\right)\right]+\left[y_L\left(u\right),y_U\left(u\right)\right]=0$$

with the initial conditions

$$\left[y_L\left(0\right),y_U\left(0\right)\right]=\left[1,2\right]\ \text{ and }\ \left[{y'}_L\left(0\right),{y'}_U\left(0\right)\right]=\left[0,1\right]. \tag{4.1}$$

Solution. The parametric form of (4.1) is

$$y''_{\zeta\in[0,1]}\left(u\right)+uy'_{\zeta\in[0,1]}\left(u\right)+y_{\zeta\in[0,1]}\left(u\right)=0$$

with the initial conditions

$$y_{\zeta\in[0,1]}\left(0\right)=\left\{1+\zeta:\zeta\in\left[0,1\right]\right\}\ \text{ and }\ y'_{\zeta\in[0,1]}\left(0\right)=\left\{\zeta:\zeta\in\left[0,1\right]\right\}. \tag{4.2}$$

Therefore, for a fixed $\zeta\in[0,1]$, we have

$$y''\left(u,\zeta\right)+uy'\left(u,\zeta\right)+y\left(u,\zeta\right)=0.$$

Let us take,

$$v\left(u,\zeta_1\right)=y''\left(u,\zeta\right). \tag{4.3}$$

Integrating (4.3) from 0 to u and using the second initial condition of (4.2), we obtain

$$y'\left(u,\zeta\right)=\int_0^u v\left(z,\zeta_1\right)dz+\zeta. \tag{4.4}$$

Again, integrating (4.4) from 0 to u and using (4.2), it gives

$$y\left(u,\zeta\right)=\int_0^u\left(u-z\right)v\left(z,\zeta_1\right)dz+\zeta u+1+\zeta. \tag{4.5}$$

Now, multiplying (4.3) by 1, (4.4) by u and (4.5) by 1 and adding, we get,

$$v\left(u,\zeta_1\right)=-\int_0^u\left(2u-z\right)v\left(z,\zeta_1\right)dz-\left(1+\zeta\right)-2\zeta u.$$

Therefore, the required interval integral equation is

$$\left[v_L\left(u\right),v_U\left(u\right)\right]=-\left[1,2u+2\right]-\int_0^u\left(2u-z\right)\left[v_L\left(z\right),v_U\left(z\right)\right]dz.$$

This is the required interval Volterra integral equation.

Example 4.2. Consider the following interval Volterra integral equation:

$$\left[y_L,y_U\right]\left(u\right)=\left[e^{u^2},3e^{u^2}\right]+\int_0^u e^{u^2-z^2}\left[y_L\left(z\right),y_U\left(z\right)\right]\,dz \tag{4.6}$$

Solution. The parametric representation of the equation (4.6) is

$$\tilde{y}\left(u,\zeta\right)=\tilde{f}\left(u,\zeta_1\right)+\int_0^u e^{u^2-z^2}\tilde{y}\left(z,\zeta\right)\,dz\,,\quad\forall\zeta,\zeta_1\in\left[0,1\right] \tag{4.7}$$

where $\tilde{f}\left(u,\zeta_1\right)=\left(1+2\zeta_1\right)e^{u^2}$ and $\tilde{y}\left(u,\zeta\right)=y_L\left(u\right)+\zeta\left(y_U\left(u\right)-y_L\left(u\right)\right)$.

Therefore, by the method of successive approximation, the solution of (4.7) is

$$\tilde{y}(u,\zeta)=\tilde{f}(u,\zeta_1)+\int_0^u K(u,z)\,\tilde{f}(z,\zeta_1)\,dz+\int_0^u K(u,z)\int_0^z K(z,z_1)\tilde{f}(z_1,\zeta_1)\,dz_1dz+\cdots$$

$$\Rightarrow \tilde{y}(u,\zeta)=(1+2\zeta_1)\,e^{u^2}+(1+2\zeta_1)\,e^{u^2}u+(1+2\zeta_1)\,e^{u^2}\frac{u^2}{2!}+\cdots$$

$$\Rightarrow \tilde{y}(u,\zeta)=(1+2\zeta_1)\,e^{u^2+u}$$

Therefore, the solution of the equation (4.6) is

$$[y_L(u),\,y_U(u)]=\left[e^{u^2+u},\;3e^{u^2+u}\right].$$

Example 4.3. Consider the following interval Volterra integral equation:

$$[y_L,y_U](u)=[1,2]+\int_0^u [y_L(z),y_U(z)]\,dz. \tag{4.8}$$

Solution. The parametric representation of the equation (4.8) is

$$\tilde{y}(u,\zeta)=\tilde{f}(u,\zeta_1)+\int_0^u \tilde{y}(z,\zeta)\,dz\,,\quad \forall\zeta,\zeta_1\in[0,1]$$
$$\text{where } \tilde{f}(u,\zeta_1)=1+\zeta_1 \text{ and } \tilde{y}(u,\zeta)=y_L(u)+\zeta\,(y_U(u)-y_L(u)) \tag{4.9}$$

Here,

$$K_n(u,z)=\int_z^u K_1(u,z_1)\,K_{n-1}(z_1,z)\,dz_1=\frac{(u-z)^{n-1}}{(n-1)!},\quad n=1,2,3,\ldots$$

Therefore, the resolvent kernel for this problem is of the form

$$R(u,z;1)=\sum_{n=1}^{\infty}K_n(u,z)=\sum_{n=1}^{\infty}\frac{(u-z)^{n-1}}{(n-1)!}=e^{(u-z)}.$$

Therefore, the solution of the equation (4.9) is of the form

$$\tilde{y}(u,\zeta)=1+\zeta_1+\int_0^u (1+\zeta)\,e^{u-z}dz\quad\forall\zeta,\zeta_1\in[0,1].$$

Hence, the required solution of the equation (4.8) is

$$[y_L(u),\,y_U(u)]=[e^u,2e^u].$$

5. Conclusion

In this work, the concept imprecise Volterra integral equation is introduced in the interval form with a brief motivation. Then the solution procedure of interval Volterra integral equation is derived in parametric form in a simple way. Then all the results including solution procedure regarding interval Volterra integral equation are derived in a simple way. In these derivations, all the results are presented in parametric form of intervals. After that, a set of examples have been solved for the illustration of the solution procedure. This concept of imprecise Volterra integral equation can be implemented in various real-life problems viz. analyses of diffusion and heat transferring, power sector, web-security problems in which fluctuation of parameters is occurred due to the uncertainty.

For future research, one may develop the same for nonlinear interval Volterra type integral equations. One can develop the numerical methods for solving an interval Volterra integral equation. Also, this concept can be extended by introducing interval-valued kernels etc.

References

[1] Abbasbandy, S., Babolian, E., Alavi, M., *Numerical method for solving linear Fredholm fuzzy integral equations of the second kind*, Chaos Solitons Fractals, **31**(2007), no. 1, 138-146.

[2] Agarwal, R.P., O'Regan, D., Lakshmikantham, V., *Fuzzy Volterra integral equations: A stacking theorem approach*, Appl. Anal., **83**(2004), no. 5, 521-532.

[3] Agheli, B., Firozja, M.A., *A fuzzy transform method for numerical solution of fractional Volterra integral equations*, Int. J. Appl. Comput. Math., **6**(2020), no. 1, 1-13.

[4] Ahmady, N., *A numerical method for solving fuzzy differential equations with fractional order*, Int. J. Ind. Math., **11**(2019), no. 2, 71-77.

[5] An, T.V., Phu, N.D., Van Hoa, N., *A note on solutions of interval-valued Volterra integral equations*, J. Integral Equ. Appl., (2014), 1-14.

[6] Attari, H., Yazdani, A., *A computational method for fuzzy Volterra-Fredholm integral equations*, Fuzzy Inf. Eng., **3**(2011), no. 2, 147.

[7] Babolian, E., Goghary, H.S., Abbasbandy, S., *Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method*, Appl. Math. Comput., **161**(2005), no. 3, 733-744.

[8] Bica, A.M., Popescu, C., *Approximating the solution of nonlinear Hammerstein fuzzy integral equations*, Fuzzy Sets Syst., **245**(2014), 1-17.

[9] Buckley, J.J., Feuring, T., *Fuzzy differential equations*, Fuzzy Sets Syst., **110**(2000), no. 1, 43-54.

[10] da Costa, T.M., Chalco-Cano, Y., Lodwick, W.A., Silva, G.N., *A new approach to linear interval differential equations as a first step toward solving fuzzy differential*, Fuzzy Sets Syst., **347**(2018), 129-141.

[11] Kaleva, O., *Fuzzy differential equations*, Fuzzy Sets Syst., **24**(1987), no. 3, 301-317.

[12] Lupulescu, V., Van Hoa, N., *Interval Abel integral equation*, Soft Comput., **21**(2017), no. 10, 2777-2784.

[13] Malinowski, M.T., *Interval differential equations with a second type Hukuhara derivative*, Appl. Math. Lett., **24**(2011), no. 12, 2118-2123.

[14] Mao, X., *Existence and uniqueness of the solutions of delay stochastic integral equations*, Stochastic Anal. Appl., **7**(1989), no. 1, 59-74.

[15] Mao, X., Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[16] Mirzaee, F., Hamzeh, A., *A computational method for solving nonlinear stochastic Volterra integral equations*, J. Comput. Appl. Math., **306**(2016), 166-178.

[17] Mohammadi, F., *Second kind Chebyshev wavelet Galerkin method for stochastic Ito-Volterra integral equations*, Mediterr. J. Math., **13**(2016), no. 5, 2613-2631.

[18] Mordeson, J., Newman, W., *Fuzzy integral equations*, Inf. Sci., **87**(1995), no. 4, 215-229.

[19] Noeiaghdam, S., Araghi, M.A.F., Abbasbandy, S., *Valid implementation of Sinc-collocation method to solve the fuzzy Fredholm integral equation*, J. Comput. Appl. Math., **370**(2020), 112632.

[20] Ogawa, S., *Stochastic integral equations for the random fields*, Sémin. Probab. Strasbourg, **25**(1991), 324-329.

[21] Ogawa, S., *Stochastic integral equations of Fredholm type*, in "Harmonic, Wavelet and P-Adic Analysis", (2007), 331-342.

[22] Otadi, M., Mosleh, M., *Simulation and evaluation of interval-valued fuzzy linear Fredholm integral equations with interval-valued fuzzy neural network*, Neurocomputing, **205**(2016), 519-528.

[23] Ramezanzadeh, M., Heidari, M., Fard, O., Borzabadi, A., *On the interval differential equation: Novel solution methodology, Adv. Differ. Equ.*, (2015), 10.1186/s13662-015-0671-8.

[24] Samadyar, N., Mirzaee, F., *Numerical solution of two-dimensional weakly singular stochastic integral equations on non-rectangular domains via radial basis functions*, Eng. Anal. Bound. Elem., **101**(2019), 27-36.

[25] Samadyar, N., Mirzaee, F., *Orthonormal Bernoulli polynomials collocation approach for solving stochastic Ito-Volterra integral equations of Abel type*, Int. J. Numer. Model., **33**(2020), no. 1, e2688.

[26] Stefanini, L., Bede, B., *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*, Nonlinear Anal. Theory Methods Appl., **71**(2009), no. 3-4, 1311-1328.

[27] Subrahmanyam, P.V., Sudarsanam, S.K., *A note on fuzzy Volterra integral equations*, Fuzzy Sets Syst., **81**(1996), no. 2, 237-240.

[28] Vorobiev, D., Seikkala, S., *Towards the theory of fuzzy differential equations*, Fuzzy Sets Syst., **125**(2002), no. 2, 231-237.

[29] Wang, B., Zhu, Q., *Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems*, Syst. Control Lett., **105**(2017), 55-61.

[30] Yong, J.M., *Backward stochastic Volterra integral equations – A brief survey*, Appl. Math. J. Chin. Univ., 28 (2013), no. 4, 383-394.

[31] Zakeri, K.A., Ziari, S., Araghi, M.A.F., Perfilieva, I., *Efficient numerical solution to a bivariate nonlinear fuzzy Fredholm integral equation*, IEEE Trans. Fuzzy Syst., (2019).

[32] Ziari, S., Perfilieva, I., Abbasbandy, S., *Block-pulse functions in the method of successive approximations for nonlinear fuzzy Fredholm integral equations*, Differ. Equ. Dyn. Syst., (2019), 1-15.

Subhajit Das
Bhagalpur College of Engineering, Sabour,
Bhagalpur-813210, Bihar, India
e-mail: mathsubhajitdas@gmail.com

Md Sadikur Rahman
Khalisani Mahavidyalaya, Department of Mathematics,
Khalisani, Chandannagar Hooghly - 712138, West Bengal, India
(Corresponding author)
e-mail: `mdsadikur.95@gmail.com`

Ali Akbar Shaikh
University of Burdwan, Department of Mathematics,
Burdwan-713104, West Bengal, India
e-mail: `aliashaikh@math.buruniv.ac.in`

Asoke Kumar Bhunia
University of Burdwan, Department of Mathematics,
Burdwan-713104, West Bengal, India
e-mail: `akbhunia@math.buruniv.ac.in`