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On some coefficient estimates for a class of p-valent functions

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Abstract. In this paper, we consider a class of p-valent functions. For functions in this class we find sharp estimates for their first three coefficients. Upper bound for the second order Hankel determinant is also obtained.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

defined on the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Note that for $p = 1$ we obtain $\mathcal{A}(1) = \mathcal{A}$ which is the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

Let $\mathcal{P}$ be the the well known Carathéodory class of functions consisting of functions q such that

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.3}$$

which are analytic in the unit disc $\mathcal{U}$ and satisfy $\Re q(z) > 0, z \in \mathcal{U}$ (see [2]).

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The Hankel determinant of a function f, for $q \geq 1, n \geq 1$ was defined by Pommerenke ([12]), [13]), as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

For our discussion in this paper , we consider the second order Hankel determinant for the case $q = 2$ and $n = p + 1$

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2.$$

Bounds for this determinant, for different classes of p-valent functions, has been investigated by several authors, see [1], [4], [5], [10] to mention only a few.

In a recent paper, Gupta et al. [3] extended Marx-Strohhäcker result [9], [14], to multivalent functions $f \in \mathcal{A}(p)$ $(p \geq 2)$, by finding β and γ such that

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \Longrightarrow \Re\sqrt{\frac{f'(z)}{pz^{p-1}}} > \beta \Longrightarrow \Re\frac{f(z)}{z^p} > \gamma, z \in \mathcal{U}. \tag{1.4}$$

Starting from Marx-Strohhäcker implication (1.4), we consider the following class of p-valent functions.

Definition 1.1. A function $f \in \mathcal{A}(p)$ $(p \geq 1)$ is said to be in the class $\mathcal{SQ}(p)$ if and only if

$$\Re\sqrt{\frac{f'(z)}{pz^{p-1}}} > 0, z \in \mathcal{U}. \tag{1.5}$$

In this paper, for the class $\mathcal{SQ}(p)$, we obtain sharp estimates for the coefficients $a_{p+1}, a_{p+2}, a_{p+3}$. We also find an upper bound for the second Hankel determinant $H_2(p+1)$.

In order to obtain our results we will need the next two lemmas.

Lemma 1.2. *[[6], [7]] If the function $p \in \mathcal{P}$ is given by (1.3), then*

$$|c_n| \leq 2, n \geq 1$$

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.6}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y \tag{1.7}$$

for some x, y with $|x| \leq 1$ and $|y| \leq 1$.

The second lemma is a special case of a more general result due to Ohno and Sugawa [11] (see also [8]).

Lemma 1.3. *For some given real numbers A, B, C, let*

$$Y(A, B, C) = \max_{z \in \overline{\mathbb{U}}}(|A + Bz + Cz^2| + 1 - |z|^2).$$

If $AC \geq 0$, *then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|\,, & |B| \geq 2(1-|C|) \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}\,, & |B| < 2(1-|C|). \end{cases}$$

If $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \ \ and \ \ |B| < 2(1-|C|) \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\} \\ R(A,B,C), & otherwise \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| - |C|\,, & |C|(|B| + 4|A|) \leq |AB| \\ -|A| + |B| + |C|\,, & |AB| \leq |C|(|B| - 4|A|) \\ (|C| + |A|)\sqrt{1 - \dfrac{B^2}{4AC}}\,, & otherwise. \end{cases}$$

2. Coefficient estimates

In this section we obtain sharp inequalities for the coefficients a_{p+1}, a_{p+2} and a_{p+3}.

Theorem 2.1. *Let* $f \in \mathcal{SQ}(p)$ *be given be (1.1). Then*

$$|a_{p+1}| \leq \frac{4p}{p+1},$$

$$|a_{p+2}| \leq \frac{8p}{p+2},$$

$$|a_{p+3}| \leq \frac{12p}{p+3}.$$

Proof. Since $f \in \mathcal{SQ}(p)$, we have that $\sqrt{\frac{f'(z)}{pz^{p-1}}} \in \mathcal{P}$. It results that there exists a function $q \in \mathcal{P}$ such that

$$\sqrt{\frac{f'(z)}{pz^{p-1}}} = q(z), z \in \mathcal{U}. \tag{2.1}$$

Equating the coefficients in (2.1), we obtain

$$a_{p+1} = \frac{2p}{p+1}c_1,$$

$$a_{p+2} = \frac{2p}{p+2}(c_2 + \frac{c_1^2}{2}),$$

$$a_{p+3} = \frac{2p}{p+3}(c_3 + c_1c_2).$$

Since $q \in \mathcal{P}$ we have $|c_1| \leq 2$ and thus $|a_{p+1}| \leq \dfrac{4p}{p+1}$. The inequality is sharp for $c_1 = 2$. In order to obtain $|a_{p+2}|$, making use of Lemma 1.2 , we replace the coefficient c_2 from (1.6) and we get

$$a_{p+2} = \frac{p}{p+2}(2c_1^2 + (4-c_1^2)x), |x| \leq 1.$$

Suppose now that $c_1 = c$ and $0 \leq c \leq 2$. Then

$$|a_{p+2}| = \frac{p}{p+2}|2c_1^2 + (4-c_1^2)x| \leq \frac{p}{p+2}(2c^2 + 4 - c^2) \leq \frac{8p}{p+2}.$$

The inequality is sharp for $c = 2$.
Since $a_{p+3} = \dfrac{2p}{p+3}(c_3 + c_1 c_2)$, making use of Lemma 1.2 and replacing the coefficients c_2 and c_3, given by (1.6) and (1.7) respectively, we have

$$a_{p+3} = \frac{p}{p+3}\left[\frac{3c^3}{2} + 2cx(4-c^2) - (4-c^2)\frac{cx^2}{2} + (4-c^2)(1-|x^2|)y\right].$$

In view of triangle inequality, after some calculations, we obtain

$$|a_{p+3}| \leq \frac{p(4-c^2)}{p+3}\left[|\frac{3c^3}{2(4-c^2)} + 2cx - \frac{cx^2}{2}| + (1-|x^2|)\right].$$

To obtain the upper bound of $|a_{p+3}|$ we use Lemma 1.3 with

$$A = \frac{3c^3}{2(4-c^2)},\ B = 2c,\ C = -\frac{c}{2}.$$

It is easy to see that $AC > 0$ and $-4AC(C^{-2} - 1) \leq B^2$.
The inequality $|B| < 2(1 - |C|)$ holds true for $c < \frac{2}{3}$.
Thus, for the case $c \in [0, \frac{2}{3})$, we have

$$|a_{p+3}| \leq \frac{p(4-c^2)}{p+3} Y(A,B,C) \ \text{ where } Y(A,B,C) = 1 - |A| + \frac{B^2}{4(1-|C|)}.$$

By replacing A, B and C we obtain

$$Y(A,B,C) = \frac{c^3 + 6c^2 + 8}{2(4-c^2)},$$

which implies

$$|a_{p+3}| \leq \frac{p}{2(p+3)}(c^3 + 6c^2 + 8).$$

Let $\varphi(c) = c^3 + 6c^2 + 8, c \in [0, \frac{2}{3})$ with $\varphi'(c) = 3c(c+4)$. Since, $\varphi'(c) \geq 0, c \in [0, \frac{2}{3})$ we get $\varphi(c) < \frac{296}{27}$.
Therefore, if $c \in [0, \frac{2}{3})$, we have $|a_{p+3}| \leq \dfrac{148p}{27(p+3)}$.
We consider now the case $\frac{2}{3} \leq c \leq 2$ and we check the condition

$$B^2 < \min\left\{4(1+|C|^2); -4AC(C^{-2} - 1)\right\} \tag{2.2}$$

from Lemma 1.3, which is equivalent to

$$4c^2 < \min\left\{4(1+c+\frac{c^2}{4}), 3c^2\right\}.$$

Hence, for $c \in [\frac{2}{3}, 2]$ the inequality (2.2) is not satisfied. We check now the conditions for $R(A,B,C)$ from the same Lemma 1.3.

It is easy to obtain that $|AB| \leq |C|(|B| - 4|A|)$ for $c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$. For $c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$ we have $Y(A,B,C) = R(A,B,C)$, where

$$R(A,B,C) = \frac{10c - 4c^3}{4 - c^2}.$$

In this case,

$$|a_{p+3}| \leq \frac{p}{p+3}(10c - 4c^3).$$

Let $\mu(c) = 10c - 4c^3, c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$. Then $\mu^{'}(c) = 10 - 12c^2$. It follows that $\mu(c)$ is an increasing function, so $\mu(c) \leq \mu(\frac{2}{\sqrt{7}}) = \frac{108\sqrt{7}}{49}, c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$. We obtain

$$|a_{p+3}| \leq \frac{p}{p+3}\frac{108\sqrt{7}}{49}.$$

Now, for $c \in (\frac{2}{\sqrt{7}}, 2]$ we get, $|a_{p+3}| \leq \frac{p(4-c^2)}{p+3}R(A,B,C)$, where

$$R(A,B,C) = (|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = \frac{2+c^2}{4-c^2}\frac{\sqrt{16-c^2}}{\sqrt{3}}.$$

Then,

$$|a_{p+3}| \leq \frac{p}{p+3}(2+c^2)\frac{\sqrt{16-c^2}}{\sqrt{3}}.$$

We denote by $\eta(c) = (c^2+2)\sqrt{16-c^2}, c \in \left(\frac{2}{\sqrt{7}}, 2\right]$. Then

$$\eta'(c) = \frac{3c(10-c^2)}{\sqrt{16-c^2}} \geq 0,\ c \in \left(\frac{2}{\sqrt{7}}; 2\right],$$

which shows that $\eta(c)$ is an increasing function on $\left(\frac{2}{\sqrt{7}}; 2\right]$ and $\eta(c) \leq \eta(2) = 12\sqrt{3}$. Thus

$$|a_{p+3}| \leq \frac{12p}{p+3}.$$

Finally, we get

$$|a_{p+3}| \leq \max\left\{\frac{148p}{27(p+3)}; \frac{108\sqrt{7}}{49}\frac{p}{p+3}; \frac{12p}{p+3}\right\}, p \geq 1, c \in [0; 2]$$

which implies

$$|a_{p+3}| \leq \frac{12p}{p+3}.$$

The last inequality is sharp for $c = 2$.
Now, the proof of our theorem is completed. □

3. Second Hankel determinant

In this section we find an upper bound for the second order Hankel determinant

$$H_2(p+1) = a_{p+1}a_{p+3} - a_{p+2}^2.$$

Theorem 3.1. *Let $f \in \mathcal{SQ}(p)$ be given be (1.1). Then*

$$|H_2(p+1)| \le \frac{16p^2}{(p+1)(p+3)}.$$

Proof. Since $f \in \mathcal{SQ}(p)$, from the proof of Theorem 2.1, we have

$$a_{p+1} = \frac{2p}{p+1}c,$$

$$a_{p+2} = \frac{2p}{p+2}(c_2 + \frac{c^2}{2}),$$

$$a_{p+3} = \frac{2p}{p+3}(c_3 + c_2 c).$$

Then

$$H_2(p+1) = \frac{4p^2}{(p+1)(p+3)}c(c_3 + c_2 c) - \frac{4p^2}{(p+2)^2}(c_2 + \frac{c^2}{2})$$

$$= \frac{p^2}{(p+1)(p+2)^2(p+3)}[4c^2c_2 - c^4 - c^4(p+1)(p+3) + 4(p+2)^2]cc_3 - 4(p+1)(p+3)c_2^2].$$

Making use of Lemma 1.2, we get

$$\begin{aligned}
&4c^2c_2 = 2c^4 + 2c^2(4-c^2)x\\
&4c_2^2 = c^4 + 2c^2(4-c^2)x + (4-c^2)^2x^2\\
&4cc_3 = c^4 + 2c^2(4-c^2)x - c^2(4-c^2)x^2 + 2(4-c^2)c(1-|x|^2)y,
\end{aligned}$$

where $c \in [0,2]$, and $|x| \le 1, |y| \le 1$.
After lengthy calculations, we obtain

$$|H_2(p+1)| \le \frac{p^2}{(p+1)(p+3)}2c(4-c^2)\left\{A + Bx + Cx^2 + (1-|x|^2)\right\},$$

where

$$A = \frac{-c^3(p^2+2p)}{2(p+2)^2(4-c^2)} < 0$$

$$B = \frac{2c}{(p+2)^2} > 0$$

$$C = -\frac{c^2 + 4(p+1)(p+3)}{2c(p+2)^2} < 0.$$

In order to obtain the upper bound of $|H_2(p+1)|$, we use Lemma 1.3 for the case $AC > 0$. Since the inequality $|B| < 2(1-|C|)$ is satisfied, then we have

$$Y(A,B,C) = 1 + |A| + \frac{B^2}{4(1-|C|)}$$

$$= 1 + \frac{c^3}{2(p+2)^2(4-c^2)} \frac{c^2(p+2)^2 - 2c(p^2+4p)(p+2)^2 + 4(p^2+4p)(p+1)(p+3) - 16}{c^2 - 2c(p+2)^2 + 4(p+1)(p+3)}.$$

It follows that

$$|H_2(p+1)| \leq \frac{p^2}{(p+1)(p+3)} 2c(4-c^2) Y(A,B,C)$$

$$= \frac{2p^2(4-c^2)c}{(p+1)(p+3)} + \frac{p^2c^4}{(p+1)(p+2)^2(p+3)} \frac{u(c)}{v(c)},$$

where

$$u(c) = c^2(p+2)^2 - 2c(p^2+4p)(p+2)^2 + 4(p^2+4p)(p+1)(p+3) - 16$$

and

$$v(c) = c^2 - 2c(p+2)^2 + 4(p+1)(p+3), \;\; c \in [0,2], \;\; p \geq 1.$$

We observe that $u(2) = 0$ and $u(c) = (c-2)[c-2(p^2+4p-1)](p+2)^2$. Also $v(2) = 0$ and $v(c) = (c-2)[c-2(p^2+4p+3)]$.
It follows that

$$\begin{aligned}
|H_2(p+1)| &\leq \frac{2p^2(4-c^2)c}{(p+1)(p+3)} + \frac{p^2c^4}{(p+1)(p+3)} \frac{c-2(p^2+4p-1)}{c-2(p^2+4p+3)} \\
&= \frac{p^2}{(p+1)(p+3)} c \left[2(4-c^2) + c^3 \frac{c-2(p^2+4p-1)}{c-2(p^2+4p+3)} \right] \\
&= \frac{p^2}{(p+1)(p+3)} \left\{ 2c(4-c^2) + c^4 \left[1 + \frac{8}{c-2(p^2+4p+3)} \right] \right\} \\
&= \frac{p^2}{(p+1)(p+3)} \left[f_1(c) + 8f_2(c) \right],
\end{aligned}$$

where $f_1(c) = 2c(4-c^2) + c^4$ and $f_2(c) = \dfrac{c^4}{c-2(p^2+4p+3)}, c \in [0,2]$.
Since $f_1'(c) = 2(2c^3 - 3c^2 + 4)$ for $c \in [0,2]$, we have that $f_1(c)$ is an increasing function and $f_1(c) \leq f_1(2) = 16$.
Further $f_2{}'(c) = \dfrac{c^3[3c-8(p^2+4p+3)]}{[c-2(p^2+4p+3)]^2} \leq 0$, which shows that $f_2(c)$ is a decreasing function on $[0,2]$ and $f_2(c) \leq f_2(0) = 0$, $c \in [0,2]$.
Therefore

$$|H_2(p+1)| \leq \frac{16p^2}{(p+1)(p+3)}.$$

The proof of theorem is now completed. □

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