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Symmetric Toeplitz Determinants for Classes Defined by Post Quantum Operators Subordinated to the Limaçon Function

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Dedicated to the memory of Professor Bhaskaran Adolf Stephen (1966–2017)

Abstract. The present extensive study is focused to find estimates for the upper bounds of the Toeplitz determinants. The logarithmic coefficients of univalent functions play an important role in different estimates in the theory of univalent functions, and in the this paper we derive the estimates of Toeplitz determinants and Toeplitz determinants of the logarithmic coefficients for the subclasses $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$, and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$, $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, $0 < q \leq p \leq 1$, respectively, defined by post quantum operators, which map the open unit disc $\mathbb{D}$ onto the domain bounded by the limaçon curve defined by $\partial\mathcal{D}_s := \left\{u + iv \in \mathbb{C} : \left[(u-1)^2 + v^2 - s^4\right]^2 = 4s^2 \left[(u-1+s^2)^2 + v^2\right]\right\}$, where $s \in [-1, 1] \setminus \{0\}$.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathcal{A}$, then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathbb{D}, \tag{1.1}$$

and denotes by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$ (see [6] for details).

For two functions f and g analytic in $\mathbb{D}$, we say that the function f is *subordinate* to g in $\mathbb{D}$, and write $f(z) \prec g(z)$, if there exists an analytic function in $\mathbb{D}$ denoted by w, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

We recall that $\mathcal{B}$ denote the class of analytic self-mappings of the unit disc, that maps the origin onto the origin [13], that is

$$\mathcal{B} := \left\{ w(z) = \sum_{n=1}^{\infty} w_n z^n : |w(z)| < 1,\ z \in \mathbb{D} \right\}, \tag{1.2}$$

and the class $\mathcal{B}$ is known as the class of *Schwarz functions*.

In 2018, Yunus et. al. [21] studied the subclass of starlike functions associated with a limaçon domain. The limaçon of Pascal also known as limaçon is a curve that in polar coordinates has the form $r = b + a\cos\theta$, where a and b are real positive real and $\theta \in (0, 2\pi)$. If $b \geq 2a$ the limaçon is a convex curve and if $2a > b > a$ it has an indentation bounded by two inflection points. For $b = a$ the limaçon degenerates to a cardioid.

Recently, Kanas et. al. [13] introduced subclasses $ST_L(s)$ and $CV_L(s)$ of starlike and convex function respectively. Geometrically, they consist of functions $f \in \mathcal{A}$ such that $\dfrac{zf'(z)}{f(z)}$ and $\dfrac{\left(zf'(z)\right)'}{f'(z)}$ lie in the region bounded by the limaçon curve defined as

$$\partial\mathcal{D}_s := \left\{ u + iv \in \mathbb{C} : \left[(u-1)^2 + v^2 - s^4\right]^2 = 4s^2 \left[\left(u - 1 + s^2\right)^2 + v^2\right] \right\},$$

where $s \in [-1, 1] \setminus \{0\}$. If we define the *limaçon function*

$$\mathbb{L}_s(z) := (1 + sz)^2,\ s \in [-1, 1] \setminus \{0\}, \tag{1.3}$$

then the analytic characterization of the *limaçon domain* $\mathbb{L}_s(\mathbb{D})$ is given by the inclusion relation (see [13] inclusions (9) and (10))

$$\left\{ w \in \mathbb{C} : |w - 1| < 1 - (1 - |s|)^2 \right\} \subset \mathbb{L}_s(\mathbb{D}) \\ \subset \left\{ w \in \mathbb{C} : |w - 1| < (1 + |s|)^2 - 1 \right\}.$$

In 1991 Chakrabarti and Jagannathan [5] introduced the concept of (p, q)–calculus in order to generalize or unify several forms of q–oscillator algebras. In the last three decades, applications of the q–calculus have been studied and investigated extensively. Inspired and motivated by these applications many researchers (for example [1], [4]) have developed the theory of quantum calculus based on two-parameter (p, q)–integer which is used efficiently in many fields such as difference equations, Lie group, hypergeometric series, physical sciences, etc.

The (p,q)–bracket or twin basic number $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \begin{cases} \dfrac{p^n - q^n}{p-q}, & \text{if} \quad q \neq p, \\ np^{n-1}, & \text{if} \quad q = p, \end{cases}$$

where $0 < q \leq p < 1$.

For $0 < q < 1$, the q-bracket $[n]_q$ for $n = 0, 1, 2, \ldots$ is given by $[n]_q := [n]_{1,q}$. The (p,q)*–derivative* of a function f is defined by

$$\mathrm{D}_{p,q} f(z) := \begin{cases} \dfrac{f(pz) - f(qz)}{(p-q)z}, & \text{if} \quad q \neq p,\ z \neq 0, \\ 1, & \text{if} \quad p \neq q,\ z = 0, \\ f'(z), & \text{if} \quad p = q. \end{cases}$$

In particular, $\mathrm{D}_{p,q} z^n = [n]_{p,q} z^{n-1}$, therefore, for a function $f \in \mathcal{A}$ of the form (1.1) the (p,q)–derivative operator is given by

$$\mathrm{D}_{p,q} f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},\ z \in \mathbb{D}.$$

In the univalent function theory many extensive studies were given to estimate the upper bounds of the *Hankel determinants*, and for further reading one may refer to [15], [16], [18]. The closer connection with the Hankel determinants are the *Toeplitz determinants.* A Toeplitz determinant can be thought of as an "upside-down" Hankel determinant, in that Hankel determinant have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. In recent past, many researchers have focussed on finding sharp estimates for second and third order Toeplitz determinants [10], [7], etc.

Thomas and Halim [19] defined the *symmetric Toeplitz determinant* $\mathrm{T}_m(n)$ by

$$\mathrm{T}_m(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix},$$

and in particular

$$\mathrm{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad \mathrm{T}_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}.$$

For a good summary of the applications of Toeplitz matrices to the wide range of areas of pure and applied mathematics, one can refer to [20].

The *logarithmic coefficients* $\gamma_n := \gamma_n(f)$, $n \geq 1$, for a function $f \in \mathcal{S}$ of the form (1.1) play an important role in Milin's conjecture [14] and Brennan's conjecture [12], and can also be used to find estimations for the coefficients of an inverse function. It is given by the power series representation (see [14, p. 53])

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n,\ z \in \mathbb{D}, \tag{1.4}$$

where the function "log" is considered to the main branch, i.e. $\log 1 = 0$. Differentiating the definition relation (1.4) and then equating the coefficients of z^n, the logarithmic coefficients γ_1 and γ_2 will be given by

$$\gamma_1 = \frac{a_2}{2}, \tag{1.5}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right). \tag{1.6}$$

In the theory of univalent functions the problem of finding the sharp estimates for the logarithmic coefficients for various significant classes have gained a high importance (see, for details, [2], [3]). Recently, S. Giri and S. Kumar [8] initiated the study of Toeplitz determinants whose elements are logarithmic coefficients of $f \in \mathcal{S}$ which is given by

$$\mathcal{T}_{m,n}(\gamma_f) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+m-1} \\ \gamma_{n+1} & \gamma_n & \cdots & \gamma_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+m-1} & \gamma_{n+m-2} & \cdots & \gamma_n \end{vmatrix},$$

thus

$$\mathcal{T}_{2,1}(\gamma_f) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{vmatrix}.$$

In this paper we obtained the estimates of Toeplitz determinants and Toeplitz determinanats of logarithmic coefficients for the subclasses $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$, and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$, $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, $0 < q \le p \le 1$, respectively, defined by post quantum operators which map the open unit disc $\mathbb{D}$ in a domain included in the limaçon domain.

2. The Subclasses $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$ and Preliminary Results

The new subclasses of $\mathcal{A}$ we will define and investigate extend and are connected with the below subclass functions:

Definition 2.1. [17] *Denote by* S_S^* *the subclass of* $\mathcal{A}$ *consisting of functions given by* (1.1) *and satisfying*

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0,\ z \in \mathbb{D}.$$

These functions introduced by Sakaguchi are called *functions starlike with respect to symmetric points*, and for a function $f \in \mathcal{A}$ the above inequality is a necessary and sufficient condition for f to b e univalent and starlike with respect to symmetrical points in $\mathbb{D}$ (see [17, Theorem 1]).

Like we can see in [13, Lemma 2], the function $\mathbb{L}_s$ defined by (1.3) is starlike with respect to the point $z_0 = 1$ for all $s \in [-1, 1] \setminus \{0\}$, hence is univalent in $\mathbb{D}$. Moreover, if $0 < s \leq 1/\sqrt{2}$ then $\mathbb{L}_s$ has real positive part in $\mathbb{D}$, i.e. $\mathbb{L}_s$ is a *Carathéodory function* (see [13, p. 10]).

Now we define the classes $\mathrm{L}_s\mathcal{S}_p^q$ and $\mathrm{L}_s\mathcal{C}_p^q$ which maps the open unit disc onto the region included in the limaçon domain $\mathbb{L}_s(\mathbb{D})$ as follows:

Definition 2.2. *Let* $\mathrm{L}_s\mathcal{S}_p^q$ *be the subclass of function* $f \in \mathcal{A}$ *of the form* (1.1) *and satisfying the condition*

$$\frac{2z\mathrm{D}_{p,q}f(z)}{f(z) - f(-z)} \prec \mathbb{L}_s(z),\ 0 < s \leq \frac{1}{\sqrt{2}}.$$

Definition 2.3. *Let* $\mathrm{L}_s\mathcal{C}_p^q$ *be the subclass of* $\mathcal{A}$ *consisting of the function* f *of the form* (1.1) *such that*

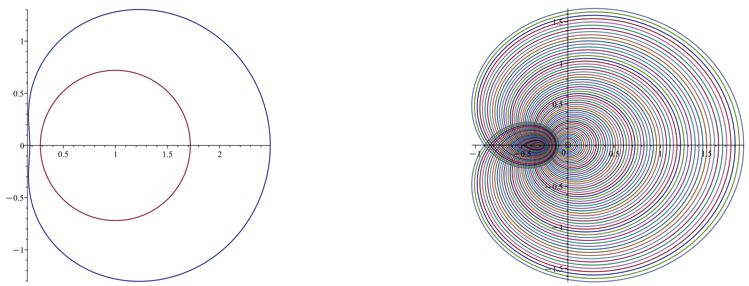
$$\frac{\left(2z\mathrm{D}_{p,q}f(z)\right)'}{\left(f(z) - f(-z)\right)'} \prec \mathbb{L}_s(z),\ 0 < s \leq \frac{1}{\sqrt{2}}.$$

Remark 2.4. The above mentioned classes are not empty, as we will show in the below examples.

(i) Taking $f_*(z) = z + az^2$, $a \in \mathbb{C}$, then

$$\Phi_*(z) := \frac{2z\mathrm{D}_{p,q}f_*(z)}{f_*(z) - f_*(-z)} = 1 + (p+q)az,\ z \in \mathbb{D}.$$

For the values $q = 0.3$, $p = 0.5$, $a = 0.9$, and $s = 1/\sqrt{3}$, like we see in the below Figure 1(A) made with MAPLE™ computer software we have $\Phi_*(\mathbb{D}) \subset \mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$, and because $\Phi_*(0) = \mathbb{L}_{1/\sqrt{3}}(0)$ from the univalence of $\mathbb{L}_{1/\sqrt{3}}$ it follows that $\Phi_*(z) \prec \mathbb{L}_{1/\sqrt{3}}(z)$, i.e. $f_* \in \mathrm{L}_s\mathcal{S}_p^q$ for the previous parameters. Also, the Figure 1(B) shows that the function f_* is not univalent in $\mathbb{D}$ because $f_*(\mathbb{D})$ twice overlaps a subset of $\mathbb{C}$.



(A) The images of $\Phi_*(\partial\mathbb{D})$ (red color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (blue color)

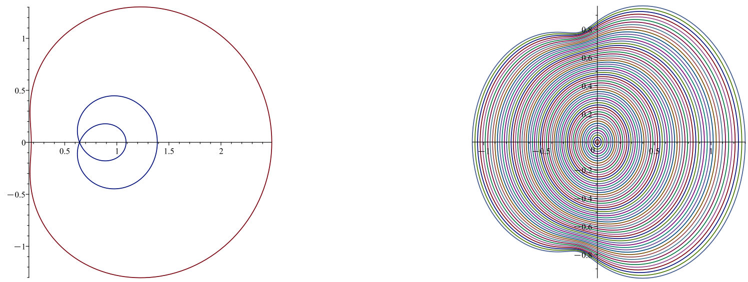
(B) The domain $f_*(\mathbb{D})$

Figure 1. Figures for the Remark 2.4(i)

(ii) For $\widehat{f}(z)=z+az^2+bz^3$, $a,b\in\mathbb{C}$, we get

$$\widehat{\Phi}(z):=\frac{2z\mathrm{D}_{p,q}\widehat{f}(z)}{\widehat{f}(z)-\widehat{f}(-z)}=\frac{1+(p+q)az+\left(p^2+pq+q^2\right)bz^2}{1+bz^2},\ z\in\mathbb{D}.$$

If $q=0.85$, $p=0.95$, $a=0.1$, $b=0.2$, and $s=1/\sqrt{3}$, we see in the Figure 2(A) made with MAPLE™ that $\widehat{\Phi}(\mathbb{D})\subset\mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$, and from $\widehat{\Phi}(0)=\mathbb{L}_{1/\sqrt{3}}(0)$ and the univalence of $\mathbb{L}_{1/\sqrt{3}}$ we have $\widehat{\Phi}(z)\prec\mathbb{L}_{1/\sqrt{3}}(z)$, that is $\widehat{\Phi}\in\mathrm{L}_s\mathcal{S}_p^q$ for this choice of the parameters. Moreover, from this figure we wee that $\widehat{\Phi}$ is not univalent in $\mathbb{D}$, while the Figure 2(B) shows that $\widehat{f}$ is univalent in $\mathbb{D}$.



(A) The images of $\widehat{\Phi}(\partial\mathbb{D})$ (blue color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (red color)

(B) The domain $\widehat{f}(\mathbb{D})$

Figure 2. Figures for the Remark 2.4(ii)

(iii) Using the above notations, and

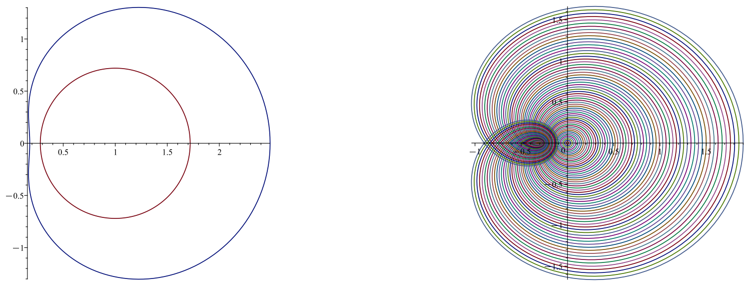
$$\Psi_*(z):=\frac{\left(2z\mathrm{D}_{p,q}f(z)\right)'}{\left(f(z)-f(-z)\right)'}=1+2(p+q)az,\ z\in\mathbb{D}.$$

for $q=0.15$, $p=0.25$, $a=0.9$, and $s=1/\sqrt{3}$, the Figure 3(A) made with MAPLE™ computer software shows that $\Psi_*(\mathbb{D})\subset\mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$, and because $\Psi_*(0)=\mathbb{L}_{1/\sqrt{3}}(0)$ from the univalence of $\mathbb{L}_{1/\sqrt{3}}$ it follows $\Psi_*(z)\prec\mathbb{L}_{1/\sqrt{3}}(z)$, i.e. $f_*\in\mathrm{L}_s\mathcal{C}_p^q$ for these values of the parameters. The Figure 3(B) shows that the function f_* is not univalent in $\mathbb{D}$ since there exists a subset of $\mathbb{C}$ that's twice overlapped by $f_*(\mathbb{D})$.

(iv) Considering the function $\widehat{f}(z)=z+az^2+bz^3$, $a,b\in\mathbb{C}$, we get

$$\widehat{\Psi}(z):=\frac{\left(2z\mathrm{D}_{p,q}f(z)\right)'}{\left(f(z)-f(-z)\right)'}=\frac{1+2(p+q)az+3\left(p^2+pq+q^2\right)bz^2}{1+3bz^2},\ z\in\mathbb{D}.$$

For $q=0.4$, $p=0.5$, $a=0.25$, $b=0.2$, and $s=1/\sqrt{3}$, we see in the Figure 4(A) made with MAPLE™ that $\widehat{\Psi}(\mathbb{D})\subset\mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$. Using that $\widehat{\Psi}(0)=\mathbb{L}_{1/\sqrt{3}}(0)$ together with the univalence of $\mathbb{L}_{1/\sqrt{3}}$ we have $\widehat{\Psi}(z)\prec\mathbb{L}_{1/\sqrt{3}}(z)$,

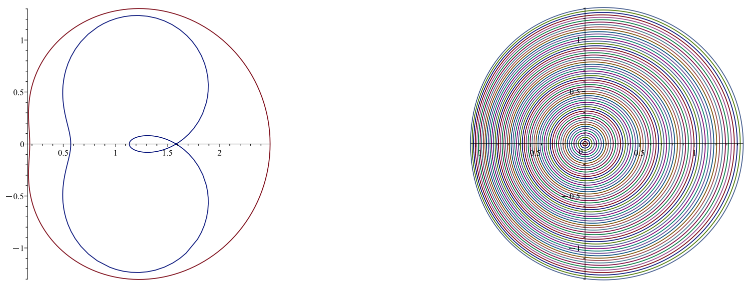


(A) The images of $\Phi_*(\partial\mathbb{D})$ (red color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (blue color)

(B) The domain $f_*(\mathbb{D})$

FIGURE 3. Figures for the Remark 2.4(iii)

that is $\widehat{\Psi} \in \mathrm{L}_s\mathcal{S}_p^q$ for these choice of the parameters. Moreover, from this figure we wee that $\widehat{\Psi}$ is not univalent in $\mathbb{D}$, and from the Figure 4(B) we see that $\widehat{f}$ is univalent in $\mathbb{D}$.



(A) The images of $\widehat{\Psi}(\partial\mathbb{D})$ (blue color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (red color)

(B) The domain $\widehat{f}(\mathbb{D})$

FIGURE 4. Figures for the Remark 2.4(iv)

(v) Concluding, the examples given in the Remark 2.4(i)–(iv) show that $\mathrm{L}_s\mathcal{S}_p^q \neq \emptyset$ and $\mathrm{L}_s\mathcal{C}_p^q \neq \emptyset$. From the examples of the Remark 2.4(i) and (iii) it follows that $\mathrm{L}_s\mathcal{S}_p^q \not\subset \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \not\subset \mathcal{S}$. In addition, the examples of the Remark 2.4(ii) and (iv) show that the corresponding functions of the form f_* and $\widehat{f}$ belong to $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, respectively, i.e. $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S} \neq \emptyset$ and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{C} \neq \emptyset$. These above comments are very important for the motivations of the results presented in the Sections 3 and 4.

In our investigations we will use the next lemmas:

Lemma 2.5. [11, Lemma 2.1] *If the function $w \in \mathcal{B}$ is of the form* (1.2), *then for some complex numbers ξ and ζ such that $|\xi| \leq 1$ and $|\zeta| \leq 1$, we have*

$$\begin{aligned} w_2 &= \xi \left(1 - w_1^2\right), \text{ and} \\ w_3 &= \left(1 - w_1^2\right)\left(1 - |\xi|^2\right)\zeta - w_1 \left(1 - w_1^2\right)\xi^2. \end{aligned}$$

Lemma 2.6. [9, p. 3, Lemma 1], [6] *If the function $w \in \mathcal{B}$ is of the form* (1.2), *then the sharp estimate $|w_n| \leq 1$ holds for $n \geq 1$.*

3. Symmetric Toeplitz Determinants of the Coefficients for the Classes $\mathrm{L}_s\mathcal{S}_p^q$ and $\mathrm{L}_s\mathcal{C}_p^q$

Now we will give upper bounds for some symmetric Toeplitz determinants for the functions belonging to the above defined classes $\mathrm{L}_s\mathcal{S}_p^q$ and $\mathrm{L}_s\mathcal{C}_p^q$, emphasizing that for $|\mathrm{T}_2(2)|$ the results are sharp.

Theorem 3.1. *If the function $f \in \mathrm{L}_s\mathcal{S}_p^q$ has the form* (1.1), *then*

$$|\mathrm{T}_2(2)| \leq \frac{s^2(s+4)^2}{\left([3]_{p,q} - 1\right)^2} + \frac{4s^2}{\left([2]_{p,q}\right)^2},$$

and this inequality is sharp (i.e. the best possible).

Proof. Assuming that $f \in \mathrm{L}_s\mathcal{S}_p^q$, according to the definition of the subordination there exists a function $w \in \mathcal{B}$ of the form (1.2) such that

$$\frac{2z\mathrm{D}_{p,q}f(z)}{f(z) - f(-z)} = \left(1 + sw(z)\right)^2, \ z \in \mathbb{D}. \tag{3.1}$$

Since (3.1) is equivalent to

$$2z\mathrm{D}_{p,q}f(z) = \left(f(z) - f(-z)\right)\left(1 + sw(z)\right)^2, \ z \in \mathbb{D},$$

expanding in Taylor series the both sides of the above relation and equating the corresponding terms we have

$$\begin{aligned} &z + z^2[2]_{p,q}a_2 + z^3a_3[3]_{p,q} + z^4a_4[4]_{p,q} + \cdots = \\ &z + 2sw_1z^2 + z^3\left(a_3 + 2sw_2 + s^2w_1^2\right) + 2z^4\left(sw_1a_3 + sw_3 + w_1w_2\right) + \ldots, \end{aligned}$$

thus

$$a_2 = \frac{2sw_1}{[2]_{p,q}} = \frac{2sw_1}{t_2}, \tag{3.2}$$

$$a_3 = \frac{2sw_2 + s^2w_1^2}{[3]_{p,q} - 1} = \frac{2sw_2 + s^2w_1^2}{t_3 - 1}, \tag{3.3}$$

where, for simplicity, we use the notation $t_n{:=}[n]_{p,q}$.

It follows that

$$|\mathrm{T}_2(2)| = \left|a_3^2 - a_2^2\right| = \left|\left(\frac{2sw_2 + s^2w_1^2}{t_3 - 1}\right)^2 - \left(\frac{2sw_1}{t_2}\right)^2\right|, \tag{3.4}$$

and rewriting w_2 in terms of w_1 from Lemma 2.5, we get

$$|\mathrm{T}_2(2)| = \left| \left(\frac{2s\left(1-w_1^2\right)\xi + s^2 w_1^2}{t_3-1} \right)^2 - \left(\frac{2sw_1}{t_2} \right)^2 \right|. \tag{3.5}$$

From the relation (3.5), using the triangle's inequality and the fact that $s>0$ we get first that

$$\begin{aligned} |\mathrm{T}_2(2)| &= \left| \frac{4s^2\left(1-w_1^2\right)^2\xi^2 + s^4 w_1^4 + 4s^3\left(1-w_1^2\right)\xi\, w_1^2}{\left(t_3-1\right)^2} - \frac{4s^2 w_1^2}{t_2^2} \right| \\ &\le \frac{4s^2\left|1-w_1^2\right|^2|\xi|^2 + s^4|w_1|^4 + 4s^3\left|1-w_1^2\right|\,|\xi|\,|w_1|^2}{\left(t_3-1\right)^2} + \frac{4s^2|w_1|^2}{t_2^2}. \end{aligned} \tag{3.6}$$

Denoting $x := |w_1|$ and $y := |\xi|$, then $x,\, y \in [0,1]$, and

$$\left|1-w_1^2\right| \le 1+x^2, \quad \left|1-w_1^2\right|^2 \le \left(1+x^2\right)^2, \tag{3.7}$$

if we combine the inequalities (3.7) with (3.6) it follows

$$|\mathrm{T}_2(2)| \le \frac{4s^2\left(1+x^2\right)^2 y^2 + s^4 x^4 + 4s^3\left(1+x^2\right) y\, x^2}{\left(t_3-1\right)^2} + \frac{4s^2 x^2}{t_2^2} =: h(x,y). \tag{3.8}$$

Since

$$\frac{\partial}{\partial y}h(x,y) = \frac{8s^2\left(x^2+1\right)^2 y + 4s^3\left(x^2+1\right)x^2}{\left(t_3-1\right)^2} \ge 0,\ (x,y)\in[0,1]\times[0,1],$$

we obtain that for any $x\in[0,1]$ we have

$$\max\left\{h(x,y) : y\in[0,1]\right\} = h(x,1) =: g(x).$$

and consequently, from (3.8) we get

$$|\mathrm{T}_2(2)| \le \frac{4s^2\left(1+x^2\right)^2 + s^4 x^4 + 4s^3\left(1+x^2\right) x^2}{\left(t_3-1\right)^2} + \frac{4s^2 x^2}{t_2^2} = g(x). \tag{3.9}$$

Using the fact that

$$g'(x) = \frac{8x\left[\dfrac{(s+2)\left(s\,x^2+2x^2+2\right)t_2^2}{2} + \left(t_3-1\right)^2\right]s^2}{\left(t_3-1\right)^2 t_2^2} \ge 0,\ x\in[0,1],$$

we have that g is an increasing function on $[0,1]$. Therefore, the inequality (3.9) leads us to

$$|\mathrm{T}_2(2)| \le g(1) = \frac{s^2(s+4)^2}{\left(t_3-1\right)^2} + \frac{4s^2}{t_2^2},\ x\in[0,1],$$

that proves the required inequality.

To prove the sharpness of our result, let consider the function $f \in \mathcal{A}$ given by (3.1) with $w(z) = iz - 2z^2$. Since $w_1 = i$, $w_2 = -2$, using the relation (3.4) we have

$$|\mathrm{T}_2(2)| = \left|\left(\frac{-4s-s^2}{t_3-1}\right)^2 + \left(\frac{2s}{t_2}\right)^2\right| = \frac{s^2(s+4)^2}{(t_3-1)^2} + \frac{4s^2}{t_2^2},$$

which shows the sharpness of our inequality. □

Theorem 3.2. *If the function* $f \in \mathrm{L}_s\mathcal{S}_p^q$ *has the form* (1.1)*, then*

$$|\mathrm{T}_3(1)| \leq 1 + \frac{8s^2}{\left([2]_{p,q}\right)^2} + \frac{8s^3(s+4)}{\left([2]_{p,q}\right)^2\left|[3]_{p,q}-1\right|} + \frac{s^2(s+4)^2}{\left([3]_{p,q}-1\right)^2}.$$

Proof. Using the same techniques and notations like in the proof of Theorem 3.1 we have

$$\begin{aligned}|\mathrm{T}_3(1)| &= \left|1 - 2a_2^2 + 2a_2^2a_3 - a_3^2\right| \\ &= \left|1 - 2\left(\frac{2sw_1}{t_2}\right)^2 + 2\left(\frac{2sw_1}{t_2}\right)^2 \cdot \frac{2sw_2 + s^2w_1^2}{t_3-1} - \left(\frac{2sw_2+s^2w_1^2}{t_3-1}\right)^2\right|.\end{aligned}$$

From Lemma 2.5, rewriting the expression w_2 in terms of w_1 the above relation leads to

$$\begin{aligned}|\mathrm{T}_3(1)| = \Bigg|1 - 2\left(\frac{2sw_1}{t_2}\right)^2 + 2\left(\frac{2sw_1}{t_2}\right)^2 \cdot \frac{2s\left(1-w_1^2\right)\xi + s^2w_1^2}{t_3-1} \\ - \frac{4s^2\left(1-w_1^2\right)^2\xi^2 + s^4w_1^4 + 4s^3w_1^2\left(1-w_1^2\right)\xi}{(t_3-1)^2}\Bigg|. \qquad (3.10)\end{aligned}$$

Letting $x := |w_1|$ and $y := |\xi|$, then $x, y \in [0,1]$, and applying the triangle's inequality in the right hand side of (3.10), since $s > 0$ we obtain

$$\begin{aligned}|\mathrm{T}_3(1)| \leq 1 + \frac{8s^2x^2}{t_2^2} + \frac{8s^2x^2\left[2s\left(1+x^2\right)y + s^2x^2\right]}{t_2^2\,|t_3-1|} \\ + \frac{4s^2\left(1+x^2\right)^2y^2 + s^4x^4 + 4s^3x^2\left(1+x^2\right)y}{(t_3-1)^2} =: q(x,y). \qquad (3.11)\end{aligned}$$

A simple computation shows that for all $(x,y) \in [0,1] \times [0,1]$ we have

$$\frac{\partial}{\partial y}q(x,y) = \frac{16s^3x^2\left(x^2+1\right)}{t_2^2|t_3-1|} + \frac{8s^2\left(x^2+1\right)^2y + 4s^3x^2\left(x^2+1\right)}{(t_3-1)^3} \geq 0,$$

therefore, for any $x \in [0,1]$ we have

$$\begin{aligned}\max\left\{q(x,y) : y \in [0,1]\right\} = q(x,1) = 1 + \frac{8s^2x^2}{t_2^2} + \frac{8s^2x^2\left[2s\left(1+x^2\right) + s^2x^2\right]}{t_2^2\,|t_3-1|} \\ + \frac{4s^2\left(1+x^2\right)^2 + s^4x^4 + 4s^3x^2\left(1+x^2\right)}{(t_3-1)^2} =: t(x),\end{aligned}$$

hence, from (3.11) it follows

$$|\mathrm{T}_3(1)| \leq t(x),\ x \in [0,1]. \tag{3.12}$$

Moreover, since

$$t'(x) = \frac{16s^2x}{t_2^2} + \frac{16s^2x\left[2s\left(x^2+1\right)+s^2x^2\right]}{t_2^2|t_3-1|} + \frac{8s^2x^2\left(2s^2x+4sx\right)}{t_2^2\,|t_3-1|}$$
$$+\frac{16s^2\left(x^2+1\right)x+4s^4x^3+8s^3x\left(x^2+1\right)+8s^3x^3}{\left(t_3-1\right)^3} \geq 0,\ x\in[0,1],$$

the function t is increasing on $[0,1]$, and from (3.12) we deduce that

$$|\mathrm{T}_3(1)| \leq t(1) = 1 + \frac{8s^2}{t_2^2} + \frac{8s^3(s+4)}{t_2^2\,|t_3-1|} + \frac{s^2(s+4)^2}{\left(t_3-1\right)^2},$$

which represents the required inequality. ☐

Theorem 3.3. *If the function* $f \in \mathrm{L}_s\mathcal{C}_p^q$ *has the form* (1.1), *then*

$$|\mathrm{T}_2(2)| \leq \frac{s^2(s+4)^2}{9\big([3]_{p,q}-1\big)^2} + \frac{s^2}{\big([2]_{p,q}\big)^2},$$

and this inequality is sharp (i.e. the best possible).

Proof. For the function $f \in \mathrm{L}_s\mathcal{C}_p^q$, using the definition of the subordination there exists a function $w(z) = w_1z + w_2z^2 + \cdots \in \mathcal{B}$, $z \in \mathbb{D}$, such that

$$\frac{\big(2z\mathrm{D}_{p,q}f(z)\big)'}{\big(f(z)-f(-z)\big)'} = \big(1+sw(z)\big)^2,\ z\in\mathbb{D}. \tag{3.13}$$

The relation (3.13) could be written in the form

$$\big(2z\mathrm{D}_{p,q}f(z)\big)' = \big(f(z)-f(-z)\big)'\big(1+sw(z)\big)^2,\ z\in\mathbb{D},$$

and expanding in Taylor series both sides of this equality we get

$$1 + 2z[2]_{p,q}a_2 + 3z^2a_3[3]_{p,q} + 4z^3a_4[4]_{p,q} + \cdots =$$
$$1 + 2sw_1z + z^2\left(s^2w_1^2 + 2sw_2 + 3a_3\right) + z^3\left(2s^2w_1w_2 + 2sw_3 + 6sw_1a_3\right) + \ldots.$$

Equating the corresponding coefficients it follows that

$$a_2 = \frac{sw_1}{[2]_{p,q}}, \tag{3.14}$$

$$a_3 = \frac{2sw_2 + s^2w_1^2}{3\big([3]_{p,q}-1\big)}. \tag{3.15}$$

Using Lemma 2.5 it's easy to check that

$$|\mathrm{T}_2(2)| = \left|a_3^2 - a_2^2\right| = \left|\frac{4s^2w_2^2 + s^4w_1^4 + 4s^3w_1^2w_2}{9\left(t_3-1\right)^2} - \frac{s^2w_1^2}{t_2^2}\right|$$
$$= \left|\frac{4s^2\left(1-w_1^2\right)^2\xi^2 + s^4w_1^4 + 4s^3w_1^2\left(1-w_1^2\right)\xi}{9\left(t_3-1\right)^2} - \frac{s^2w_1^2}{t_2^2}\right|, \tag{3.16}$$

where we use the previous notation $t_n:=[n]_{p,q}$.

Denoting $x := |w_1|$ and $y := |\xi|$, then $x, y \in [0,1]$, and using the triangle's inequality in the right hand side of the above relation, since $s > 0$ we have

$$|\mathrm{T}_2(2)| \leq \frac{4s^2\left(1+x^2\right)^2 y^2 + s^4x^4 + 4s^3\left(1+x^2\right)x^2y}{9\left(t_3-1\right)^2} + \frac{s^2x^2}{t_2^2} =: h(x,y). \tag{3.17}$$

It is easy to see that

$$\frac{\partial}{\partial y}h(x,y) = \frac{4s^2\left(x^2+1\right)\left[(s+2y)\,x^2+2y\right]}{9\left(t_3-1\right)^2} \geq 0,\ (x,y) \in [0,1]\times[0,1],$$

consequently, for each $x \in [0,1]$ we have

$$\max\left\{h(x,y) : y \in [0,1]\right\} = h(x,1)$$
$$= \frac{4s^2\left(1+x^2\right)^2 + s^4x^4 + 4s^3\left(1+x^2\right)x^2}{9\left(t_3-1\right)^2} + \frac{s^2x^2}{t_2^2} =: g(x).$$

Combining this last relation with the inequality (3.17) we obtain

$$|\mathrm{T}_2(2)| \leq g(x). \tag{3.18}$$

Since for all $x \in [0,1]$ we have

$$g'(x) = \frac{16s^2\left(x^2+1\right)x + 4s^4x^3 + 8s^3x^3 + 8s^3\left(x^2+1\right)x}{9\left(t_3-1\right)^2} + \frac{2s^2x}{t_2^2} \geq 0,$$

the function g is increasing on $[0,1]$, therefore the inequality (3.18) leads to

$$|\mathrm{T}_2(2)| \leq g(1) = \frac{s^2(s+4)^2}{9\left(t_3-1\right)^2} + \frac{s^2}{t_2^2},$$

and our conclusion is proved.

The inequality is sharp for the function $f \in \mathcal{A}$ given by (3.1) with $w(z) = iz - 2z^2$. In this case $w_1 = i$, $w_2 = -2$, and from the relation (3.16) we get

$$|\mathrm{T}_2(2)| = \frac{s^2(s+4)^2}{\left(t_3-1\right)^2} + \frac{4s^2}{t_2^2},$$

which proves the sharpness of our inequality □

Using the same techniques as in the previous theorem, we obtain the next upper bound for $|\mathrm{T}_3(1)|$ if $f \in \mathrm{L}_s\mathcal{C}_p^q$.

Theorem 3.4. *If the function $f \in \mathrm{L}_s\mathcal{C}_p^q$ has the form* (1.1), *then*

$$|\mathrm{T}_3(1)| \leq 1 + \frac{2s^2}{\left([2]_{p,q}\right)^2} + \frac{2s^3(s+4)}{3\left([2]_{p,q}\right)^2\left|[3]_{p,q}-1\right|} + \frac{s^2(s+4)^2}{9\left([3]_{p,q}-1\right)^2}.$$

Proof. With the same techniques and notations as in the proof of the previous theorem we have

$$|\mathrm{T}_3(1)| = \left|1 - 2\,\frac{s^2w_1^2}{t_2^2} + 2\,\frac{s^2w_1^2}{t_2^2}\cdot\frac{s^2w_1^2+2sw_2}{3\,(t_3-1)} - \frac{s^4w_1^4+4s^2w_2^2+4s^3w_1^2w_2}{9\,(t_3-1)^2}\right|.$$

Rewriting the expression w_2 in terms of w_1 like in Lemma 2.5, applying the triangle's inequality, denoting $x = |w_1| \le 1$, $y = |\xi| \le 1$, and using that $s > 0$ we get

$$|\mathrm{T}_3(1)| \le 1 + \frac{2s^2x^2}{t_2^2} + \frac{2s^2x^2\left[s^2x^2+2s\left(1+x^2\right)y\right]}{3t_2^2\,|t_3-1|}$$
$$+\frac{s^4x^4+4s^2\left(1+x^2\right)^2y^2+4s^3x^2\left(1+x^2\right)y}{9\,(t_3-1)^2} =: p(x,y). \tag{3.19}$$

It follows that

$$\frac{\partial}{\partial y}p(x,y) = \frac{4s^3x^2\left(x^2+1\right)}{3t_2^2|t_3-1|} + \frac{4s\left[2sy\left(x^2+1\right)+s^2x^2\right]\left(x^2+1\right)}{9\,(t_3-1)^2} \ge 0,$$
$$(x,y)\in[0,1]\times[0,1],$$

hence, for each $x \in [0,1]$ we have

$$\max\left\{p(x,y) : y\in[0,1]\right\} = p(x,1) = 1 + \frac{2s^2x^2}{t_2^2} + \frac{2s^2x^2\left[s^2x^2+2s\left(1+x^2\right)\right]}{3t_2^2\,|t_3-1|}$$
$$+\frac{s^4x^4+4s^2\left(1+x^2\right)^2+4s^3x^2\left(1+x^2\right)}{9\,(t_3-1)^2} =: q(x). \tag{3.20}$$

Using that

$$q'(x) = \frac{8s^3x\left(x^2+1\right)}{3t_2^2|t_3-1|} + \frac{8s^3x^3}{3t_2^2|t_3-1|} + \frac{\left(2s^2x+4syx\right)s\left(x^2+1\right)}{9\,(t_3-1)^2}$$
$$+\frac{8\left[2sy\left(x^2+1\right)+s^2x^2\right]sx}{9\,(t_3-1)^2} \ge 0,\ x\in[0,1],$$

the function q is increasing on $[0,1]$, and from the inequalities (3.19) and (3.20) we conclude that

$$|\mathrm{T}_3(1)| \le q(1) = 1 + \frac{2s^2}{t_2^2} + \frac{2s^3(s+4)}{3t_2^2\,|t_3-1|} + \frac{s^2(s+4)^2}{9\,(t_3-1)^2}.$$

□

4. Symmetric Toeplitz Determinants of the Logarithmic Coefficients for the Classes $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$

In this section we find the estimates of initial two logarithimic coefficients and then the estimate of symmetric Toeplitz determinants $\mathcal{T}_{2,1}(\gamma_f)$ of logarithmic coefficients for the subclasses $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$.

Theorem 4.1. *If the function* $f \in \mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ *has the form* (1.1) *and the logarithmic coefficients are given by* (1.4), *then*

$$|\gamma_1| \le \frac{s}{[2]_{p,q}} \quad \text{and} \quad |\gamma_2| \le \frac{s(s+4)}{2\left|[3]_{p,q}-1\right|} + \frac{s^2}{\left([2]_{p,q}\right)^2}.$$

Proof. Replacing the values of a_2 and a_3 given by (3.2) and (3.3) in (1.5) and (1.6), using the notation $t_n{:=}[n]_{p,q}$, from Lemma 2.6 we obtain

$$|\gamma_1| = \left|\frac{sw_1}{t_2}\right| \le \frac{s}{t_2} = \frac{s}{[2]_{p,q}}.$$

In addition, using Lemma 2.5 we get

$$|\gamma_2| = \frac{1}{2}\left|\frac{2sw_2+s^2w_1^2}{t_3-1} - \frac{2s^2w_1^2}{t_2^2}\right| = \frac{1}{2}\left|\frac{2s\left(1-w_1^2\right)\xi+s^2w_1^2}{t_3-1} - \frac{2s^2w_1^2}{t_2^2}\right|,$$

where $|\xi| \le 1$. Letting $x := |w_1|$ and $y := |\xi|$, then $x, \xi \in [0,1]$ and using the triangle's inequality in the above relation together with $s > 0$ we obtain

$$|\gamma_2| \le \frac{2s\left(1+x^2\right)y+s^2x^2}{2\left|t_3-1\right|} + \frac{s^2x^2}{t_2^2} =: F(x,y). \tag{4.1}$$

It follows that

$$\frac{\partial}{\partial y}F(x,y) = \frac{s\left(1+x^2\right)}{|t_3-1|} > 0,\ (x,y) \in [0,1]\times[0,1],$$

hence, for each $x \in [0,1]$ we have

$$\max\left\{F(x,y) : y \in [0,1]\right\} = F(x,1) = \frac{2s\left(1+x^2\right)+s^2x^2}{2\left|t_3-1\right|} + \frac{s^2x^2}{t_2^2} =: r(x). \tag{4.2}$$

From the fact

$$r'(x) = \frac{sx(s+2)}{|t_3-1|} + \frac{2s^2x}{t_2^2} \ge 0,\ x \in [0,1],$$

the function r is increasing on $[0,1]$, and from (4.1) and (4.2) we conclude that

$$|\gamma_2| \le r(1) = \frac{4s+s^2}{2\left|t_3-1\right|} + \frac{s^2}{t_2^2},$$

which proves our second inequality. □

Theorem 4.2. *If the function* $f \in \mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$ *has the form* (1.1) *and the logarithmic coefficients are given by* (1.4), *then*

$$|\gamma_1| \le \frac{s}{2\,[2]_{p,q}} \quad \text{and} \quad |\gamma_2| \le \frac{s(s+4)}{6\left|[3]_{p,q}-1\right|} + \frac{s^2}{4\left([2]_{p,q}\right)^2}.$$

Proof. Using the values of a_2 and a_3 given by (3.14) and (3.15), from (1.5) and (1.6), using Lemma 2.6 we obtain

$$|\gamma_1| = \left|\frac{sw_1}{2t_2}\right| \le \frac{s}{2|t_2|} \quad \text{and} \quad |\gamma_2| = \frac{1}{2}\left|\frac{2sw_2+s^2w_1^2}{3\left(t_3-1\right)} - \frac{s^2w_1^2}{2t_2^2}\right|.$$

Rewriting the expression of w_2 in terms of w_1 according to Lemma 2.5, using the triangle's inequality in the above last relation, and the notations $x := |w_1|$, $y := |\xi|$, with $x, \xi \in [0,1]$, since $s > 0$ we obtain

$$|\gamma_2| \leq \frac{2s\left(1+x^2\right)y + s^2x^2}{6\,|t_3-1|} + \frac{s^2x^2}{4t_2^2} =: G(x,y). \tag{4.3}$$

Therefore

$$\frac{\partial}{\partial y}G(x,y) = \frac{s\left(1+x^2\right)}{3\,|t_3-1|} > 0,\ (x,y) \in [0,1]\times[0,1],$$

hence, for each $x \in [0,1]$ we have

$$\max\left\{G(x,y) : y \in [0,1]\right\} = G(x,1) = \frac{2s\left(1+x^2\right)+s^2x^2}{6\,|t_3-1|} + \frac{s^2x^2}{4t_2^2} =: k(x). \tag{4.4}$$

Since

$$k'(x) = \frac{sx(sx+2)}{3\,|t_3-1|} + \frac{s^2x}{2t_2^2} \geq 0,\ x \in [0,1],$$

the function k is increasing on $[0,1]$, and combining (4.3) with (4.4) it follows

$$|\gamma_2| \leq k(1) = \frac{s(s+4)}{6\,|t_3-1|} + \frac{s^2}{4t_2^2},$$

and the proof is complete. □

The following two results, where we determined the upper bounds for the Toeplitz determinant $|\mathcal{T}_{2,1}(\gamma_f)|$ for the classes $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$ are immediately consequences of the previous two theorems.

Corollary 4.3. *For the class* $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ *the next inequality holds:*

$$|\mathcal{T}_{2,1}(\gamma_f)| \leq \left(\frac{s}{[2]_{p,q}}\right)^2 + \left(\frac{s(s+4)}{2\,|[3]_{p,q}-1|} + \frac{s^2}{\left([2]_{p,q}\right)^2}\right)^2.$$

Proof. Since

$$|\mathcal{T}_{2,1}(\gamma_f)| = |\gamma_1^2 - \gamma_2^2| \leq |\gamma_1^2| + |\gamma_2^2|$$

from the inequalities of Theorem 4.1 we get

$$|\mathcal{T}_{2,1}(\gamma_f)| \leq \left(\frac{s}{t_2}\right)^2 + \left(\frac{s(s+4)}{2\,|t_3-1|} + \frac{s^2}{t_2^2}\right)^2.$$

□

Similarly, using the inequalities obtained in Theorem 4.2 it's easy to prove the next result:

Corollary 4.4. *For the class* $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$ *the next inequality holds:*

$$|\mathcal{T}_{2,1}(\gamma_f)| \leq \left(\frac{s}{2\,[2]_{p,q}}\right)^2 + \left(\frac{s(s+4)}{6\big|[3]_{p,q}-1\big|} + \frac{s^2}{4\big([2]_{p,q}\big)^2}\right)^2.$$

5. Concluding Remarks

The quantum calculus is one of the important tools in many area of mathematics, physics and in the areas of ordinary fractional calculus, optimal control problems, quantum physics, operator theory, and q–transform analysis, and in this paper we made a connection with some subclasses of analytic functions.

In addition, the logarithmic coefficients play an important role for different estimates in the theory of univalent functions. Many researchers have found the upper bounds for the second and third order Toeplitz determinants and logarithmic coefficients for various subclasses of analytic function. The present investigation deals with the subclasses of symmetric function using the (p, q)–calculus for some functions defined by subordinations to the limaçon domain, and we determined upper bounds for some special symmetric Toeplitz determinants containing the coefficients and the logarithmic coefficients of the functions belonging to these classes. We obtained bounds for the second and third order Toeplitz determinants and Toeplitz determinants for logarithmic coefficients for the classes $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$, and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$, $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, respectively, defined by the post-quantum operators and subordinated to $\mathbb{L}_s$ function.

We hope that these results could be important in several fields related to mathematics, engineering, science and technology, and we encourage the researchers to find the sharp estimates for third order Toeplitz determinants and Toeplitz determinants for logarithmic coefficients.

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