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A modified inertial shrinking projection algorithm with adaptive step size for solving split generalized equilibrium, monotone inclusion and fixed point problems

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Abstract. In this paper, we study the common solution problem of split generalized equilibrium problem, monotone inclusion problem and common fixed point problem for a countable family of strict pseudocontractive multivalued mappings. We propose a modified shrinking projection algorithm of inertial form with self-adaptive step sizes for finding a common solution of the aforementioned problem. The self-adaptive step size eliminates the difficulty of computing the operator norm while the inertial term accelerates the rate of convergence of the proposed algorithm. Moreover, unlike several of the existing results in the literature, the monotone inclusion problem considered is a more general problem involving the sum of Lipschitz continuous monotone operators and maximal monotone operators, and knowledge of the Lipschitz constant is not required to implement our algorithm. Under some mild conditions, we establish strong convergence result for the proposed method. Finally, we present some applications and numerical experiments to illustrate the usefulness and applicability of our algorithm as well as comparing it with some related methods. Our results improve and extend corresponding results in the literature.

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1. Introduction

Let H be a real Hilbert space with induced norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let C be a nonempty closed convex subset of a real Hilbert space and let

$F : C \times C \to \mathbb{R}$ be A bifunction. The *equilibrium problem* (shortly, (EP)) in the sense of Blum and Oettli [8] is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall\ y \in C. \tag{1.1}$$

The set of all solutions of EP (1.1) is denoted by $EP(F)$. The EP attracts considerable research efforts and serves as a unifying framework for studying many well-known problems, such as the Nonlinear Complementarity Problems (NCPs), Optimization Problems (OPs), Variational Inequality Problems (VIPs), Saddle Point Problems (SPPs), the Fixed Point Problem (FPP), the Nash equilibria and many others, and has many applications in physics and economics, (see, for example [1, 11, 12, 31, 32, 49] and the references therein). On the other hand, the *generalized equilibrium problem* (GEP) is defined as finding a point $x \in C$ such that

$$F(x, y) + \phi(x, y) \geq 0, \forall y \in C, \tag{1.2}$$

where $F, \phi : C \times C \to \mathbb{R}$ are bifunctions. We denote the solution set of GEP (1.2) by $GEP(F, \phi)$. If $\phi = 0$, then the GEP (1.2) reduces to the equilibrium problem (1.1).
Let $C \subseteq H_1$ and $Q \subseteq H_2$ where H_1 and H_2 are real Hilbert spaces. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The *split generalized equilibrium problem* (SGEP) introduced by Kazmi and Rizvi [23] is defined as follows: Find $\bar{x} \in C$ such that

$$F_1(\bar{x}, x) + \phi_1(\bar{x}, x) \geq 0, \forall x \in C, \tag{1.3}$$

and such that

$$\bar{y} = A\bar{x} \in Q \text{ solves } F_2(\bar{y}, y) + \phi_2(\bar{y}, y) \geq 0,\ \forall y \in Q. \tag{1.4}$$

The solution set of the split generalized equilibrium problem is denoted by

$$SGEP(F_1, \phi_1, F_2, \phi_2) = \{\bar{x} \in C : \bar{x} \in GEP(F_1, \phi_1) \text{ and } A\bar{x} \in GEP(F_2, \phi_2)\}. \tag{1.5}$$

If $\phi_1 = 0$ and $\phi_2 = 0$, we obtain a special case of the split generalized equilibrium problem (1.3)-(1.4) called the *split equilibrium problem* (SEP) which is defined as follows:

$$F_1(\bar{x}, x) \geq 0, \forall x \in C, \tag{1.6}$$

and such that

$$\bar{y} = A\bar{x} \in Q \text{ solves } F_2(\bar{y}, y) \geq 0,\ \forall y \in Q. \tag{1.7}$$

We denote the solution set of the SEP (1.6)-(1.7) by $\Omega := \{\bar{x} \in EP(F_1) : A\bar{x} \in EP(F_2)\}$. The split generalized equilibrium problem has been studied by numerous authors and several iterative algorithms have been proposed by many authors for solving the problem (see, [39, 42]).
Another important problem that we consider is the *monotone inclusion problem* (MIP), which is defined as finding a point $z \in H$ such that

$$0 \in (B + D)z, \tag{1.8}$$

where $B : H \to H$ is a nonlinear operator and $D : H \to 2^H$ is a set-valued operator. We denote the set of solutions of (1.8) by $(B+D)^{-1}(0)$. The MIP (1.8) and related optimization problems have been studied by several authors with various iterative algorithms proposed for approximating their solutions in Hilbert spaces and Banach spaces (see, for instance [4, 28, 30, 47, 48, 50]). One of the most efficient methods for solving the MIP is the forward-backward splitting method (see [6, 9, 14, 17, 18, 26]).
Martinez [27] first introduced the Proximal Point Algorithm (PPA) for finding the zero point of a maximal monotone operator B. The sequence generated by PPA is defined as follows:

$$x_{n+1} = J_{r_n}^D x_n,$$

where $0 < r_n < \infty$, $J_{r_n}^D = (I + r_n D)^{-1}$ is the resolvent operator of D and I is the identity mapping. This algorithm was eventually modified by Rockafellar [40] to the following PPA with errors:

$$x_{n+1} = J_{r_n}^D x_n + e_n,$$

where $\{e_n\}$ is an error sequence. It was proved that if $e_n \to 0$ such that $\sum\limits_{n=1}^{\infty} \|e_n\| < +\infty$, and the solution set $D^{-1}(0) \neq \emptyset$ and $\liminf\limits_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges weakly to a zero point of D.
Also, Moudafi and Théra [28] introduced the following iterative algorithm for solving MIP (1.8):

$$\begin{cases} x_n = J_r^D v_n, \\ v_{n+1} = t v_n + (1-t)x_n - \mu(1-t)Bx_n, \end{cases} \tag{1.9}$$

where $t \in (0,1)$, $r > 0$, B is Lipschitz continuous and strongly monotone and D is maximal monotone. They proved that the sequence $\{x_n\}$ generated by the iterative algorithm converges weakly to an element in $(B+D)^{-1}(0)$.
Alvarez and Attouch [5] proposed the following modified PPA of inertial form:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^D y_n, \qquad n \geq 1, \end{cases} \tag{1.10}$$

where $\{\mu_n\} \subset [0,1)$, $\{\lambda_n\}$ is non-decreasing and

$$\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty, \quad \forall \mu_n < \frac{1}{3}. \tag{1.11}$$

It was proved that Algorithm (1.10) converges weakly to a zero of D.
Recently, Moudafi and Oliny [29] introduced the following inertial PPA for approximating the zero point problem of the sum of two monotone operators:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^D (y_n - \lambda_n B x_n), n \geq 1, \end{cases} \tag{1.12}$$

where $D : H \to 2^H$ is maximal monotone and B is Lipschitz continuous. They proved that the sequence generated by Algorithm (1.12) converges weakly if $\lambda_n < \frac{2}{L}$, where L is the Lipschitz constant of B.
Moreover, the following inertial forward-backward algorithm was introduced by Lorenz and Pock [26]:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J^D_{\lambda_n}(y_n - \lambda_n B y_n), n \geq 1, \end{cases} \tag{1.13}$$

where $\{\lambda_n\}$ is a positive real sequence. Algorithm (1.13) differs from Algorithm (1.12) since the operator B is evaluated as the inertial extrapolate y_n. The proposed algorithm was also proved to converge weakly to a solution of the MIP (1.8).
In 2016, Deepho [16] introduced the general Cesáro mean iterative method for approximating a common solution of split generalized equilibrium, fixed point of nonexpansive mappings T_j and variational inequality problems:

$$\begin{cases} z_n = T_{r_n}^{(F_1,\phi_1)}(x_n + \gamma A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n), \\ u_n = P_C(z_n - \lambda_n G z_n), \\ x_{n+1} = \alpha_n \eta f(x_n) + \beta x_n + ((1-\beta_n)I - \alpha_n K)\frac{1}{n+1}\sum_{j=0}^{n} T_j u_n, \ \forall n \geq 0, \end{cases} \tag{1.14}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \in [a,b] \subset (0, 2\beta)$ and $\{r_n\} \subset (0,\alpha)$ and $\gamma \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Under the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(C4) $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

the authors proved that the sequence $\{x_n\}$ converges strongly to an element q in the solution set Ω, where $q = P_\Omega(I - K + \gamma f)(q)$ is the unique solution of the variational inequality problem

$$\langle (K - \gamma f)q, x - q \rangle \geq 0, \ \forall x \in \Omega.$$

Also, in 2017, Sitthithakerngkiet [42] proposed and studied the following iterative method for approximating a common solution of split generalized equilibrium, variational inequality for an inverse-strongly monotone mapping and fixed point problems of nonexpansive mappings in Hilbert spaces:

$$\begin{cases} z_n = T_{r_n}^{(F_1,\phi_1)}(x_n + \gamma A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \xi_n T[\sigma_n v + (1-\sigma_n)P_C(z_n - \lambda_n G z_n)], \end{cases} \tag{1.15}$$

where $v \in C$ is a fixed point, $r_n \in (0, \infty), \mu \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of the operator A^*A, A^* is the adjoint of A, sequences $\{\alpha_n\}, \{\beta_n\}, \{\xi_n\}$ and $\{\sigma_n\}$ are in $(0,1)$ and satisfy $\alpha_n + \beta_n + \xi_n = 1, \lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta_n$ and $\{\gamma_n\} \subset [c, 1]$ for some $c \in (0,1)$. Assume that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \sigma_n = 0$;

(C3) $0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C4) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(C5) $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

the authors proved that the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega f(z)$.

Recently, Phuengrattana and Lerkchaiyaphum [39] introduced the following shrinking projection method for solving SGEP and FPP for a countable family of nonexpansive multivalued mappings: For $x_1 \in C$ and $C_1 = C$, then

$$\begin{cases} z_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2,\phi_2)})A)x_n, \\ y_n = \delta_{n,0}x_n + \sum_{j=1}^{n} \delta_{n,j}u_{n,j}, \qquad u_{n,j} \in P_j z_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\|^2 \leq \|x_n - p\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \qquad n \in \mathbb{N}. \end{cases} \tag{1.16}$$

They proved that if

(i) $\liminf_{n\to\infty} r_n > 0$,
(ii) The limits $\lim_{n\to\infty} \delta_{n,j} \in (0,1)$ exist for all $j \geq 0$,

then the sequence $\{x_n\}$ generated by (1.16) converges strongly to $P_\Gamma x_1$, where $\Gamma = \bigcap_{j=1}^{\infty} F(P_j) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$, $F(P_j)$ is the set of fixed points of P_j and P_j is a countable family of nonexpansive multivalued mappings.

In 2021, Olona et al. [35] proposed an inertial shrinking projection defined as follows for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings : for $x_0, x_1 \in C$ with $C_1 = C$, then

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)w_n, \\ z_n = \delta_{n,0}u_n + \sum_{i=1}^{n} \delta_{n,j}y_{n,j}, \ y_{n,j} \in P_j u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 \\ -2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{1.17}$$

$$\gamma_n = \begin{cases} \frac{\tau_n \|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}), \end{cases}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $0 < a \leq \tau_n \leq b < 1, \{\theta_n\} \subset \mathbb{R}$, $\{\delta_{n,j}\} \subset (0,1)$, such that $\sum_{j=0}^{n} \delta_{n,j} = 1$, and $\{r_n\} \subset (0,\infty)$. $\{P_j\}$ is a countable family of nonexpansive multivalued mappings, $F_1, \phi_1 : C \times C \to \mathbb{R}$, $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ are bifunctions. Under some appropriate conditions, it was proved that the sequence $\{x_n\}$ converges strongly to $P_\Omega x_1$, where $\Omega = \bigcap_{j=1}^{\infty} F(P_j) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$.
Motivated by the above results and the current research interest in this direction, in this paper, we propose a new iterative algorithm of inertial type with self-adaptive step size for approximating the common solution of SGEP (1.3)-(1.4), MIP (1.8) and FPP of strictly pseudo-contractive multivalued mappings. We prove that the sequence generated by our algorithm converges strongly to a solution of the investigated problem. Finally, we present some applications and numerical examples to illustrate the usefulness and efficiency of the proposed method in comparison with some related methods. Our proposed method uses self-adaptive step size and employs inertial technique to accelerate the rate of convergence of the proposed method. The implementation of our proposed algorithm does not require a prior knowledge of the norm of the bounded linear operator.
Subsequent sections of this paper are organised as follows: In Section 2, we recall some basic definitions and lemmas that are relevant in establishing our main results. In Section 3, we present our proposed algorithm and highlight some of its features. In Section 4, we prove some lemmas that are useful in establishing the strong convergence of our proposed algorithm and also prove the strong convergence theorem for the algorithm. In Section 5, we apply our result to study some optimization problems while in Section 6, we present some numerical experiments to illustrate the performance of our method and compare it with some related methods in the literature. Finally, in Section 7 we give a concluding remark.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote $x_n \to x$ to mean that sequence $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. We write $w_\omega(x_n)$ to denote set of weak limits of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

The nearest point projection of H onto C denoted by P_C is defined for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall \ y \in C. \tag{2.1}$$

It is well known that P_C is nonexpansive and has the following characteristics (see [3, 21]:

$$\|P_Cx - P_Cy\|^2 \leq \langle x - y, P_Cx - P_Cy\rangle, \ \forall \ x, y \in H_1, \tag{2.2}$$

$$\langle x - P_Cx, y - P_Cx\rangle \leq 0, \tag{2.3}$$

$$\|x - y\|^2 \leq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \ \forall x \in H, y \in C, \tag{2.4}$$

$$\|(x - y) - (P_Cx - P_Cy)\|^2 \geq \|x - y\|^2 - \|P_Cx - P_Cy\|^2, \ x, y \in H. \tag{2.5}$$

A mapping $B : C \to H$ is said to be monotone if

$$\langle Bu - Bv, u - v\rangle \geq 0, \ \forall u, v \in C. \tag{2.6}$$

Moreover, if B satisfies

$$\langle Bu - Bv, u - v\rangle \geq \alpha\|Bu - Bv\|^2, \ \forall u, v \in C, \tag{2.7}$$

for some positive real number α. Then, B is called an α-inverse-strongly monotone mapping. It is clear that every inverse-strongly monotone mapping is monotone.

Lemma 2.1. [35, 37] *Let H be a real Hilbert space, $\lambda \in \mathbb{R}$, then $\forall x, y \in H$, we have*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(ii) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$;
(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(iv) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

Lemma 2.2. [30] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Then*

$$\|y - P_Cx\|^2 + \|x - P_Cx\|^2 \leq \|x - y\|^2, \ \forall x \in H, y \in C.$$

Lemma 2.3. [?] *Let $x_i \in H, (1 \leq i \leq m), \sum_{i=1}^m \alpha_i = 1$, where $\{\alpha_i\} \subseteq (0, 1)$. Then*

$$\left\|\sum_{i=1}^m \alpha_i x_i\right\|^2 = \sum_{i=1}^m \alpha_i\|x_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j\|x_i - x_j\|^2.$$

Lemma 2.4. [24] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and $a \in (R)$, the set $D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Assumption 2.5. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions that satisfy the following conditions:*

(A1) $F_1(x, x) = 0$ *for all* $x \in C$,
(A2) F *is monotone, that is,* $F_1(x, y) + F_1(y, x) \leq 0$ *for all* $x, y \in C$,

(A3) *F is upper hemicontinuous, that is, for all $x, y, z \in C$,* $\lim_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x, y)$,
(A4) *for each $x \in C, y \mapsto F_1(x, y)$ is convex and lower semicontinuous,*
(A5) *$\phi_1(x, x) \geq 0$, for all $x \in C$,*
(A6) *for each $y \in C, x \mapsto \phi_1(x, y)$ is upper semicontinuous,*
(A7) *for each $x \in C, y \mapsto \phi_1(x, y)$ is convex and lower semicontinuous,*

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F_1(y, x) + \phi_1(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \ \forall y \in C \setminus K.$$

Lemma 2.6. [36] *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions that satisfy Assumption 2.5. Assume that ϕ is monotone. For $r > 0$ and and $x \in H$. Define mapping $T_r^{(F,\phi)} : H \to C$ as follows:*

$$T_r^{(F,\phi)}(x) = \big\{z \in C : F(z, y) + \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\big\}$$

for all $x \in H_1$. Then

(1) *for each $x \in H_1$, $T_r^{(F,\phi)} \neq \emptyset$,*
(2) *$T_r^{(F,\phi)}$ is single-valued,*
(3) *$T_r^{(F,\phi)}$ is firmly nonexpansive, that is, for any $x, y \in H_1$,*

$$\|T_r^{(F,\phi)}x - T_r^{(F,\phi)}y\|^2 \leq \langle T_r^{(F,\phi)}x - T_r^{(F,\phi)}y, x - y\rangle,$$

(4) *$F(T_r^{(F,\phi)}) = GEP(F, \phi)$,*
(5) *$GEP(F, \phi)$ is closed and convex.*

Lemma 2.7. [44] *Let X be a Banach space space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that*

$$\lim_{n\to\infty} \|x_n - u\| \text{ and } \lim_{n\to\infty} \|x_n - v\| \text{ exist.}$$

If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Lemma 2.8. [10] *Let $B : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous and monotone mapping. Then, the mapping $A + B$ is a maximal monotone mapping.*

Lemma 2.9. [20] *Let $B : H \to 2^H$ be a maximal monotone operator and $A : H \to H$ be a mapping on H. Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$. Then, we have the following*

$$Fix(T_\lambda) = (A + B)^{-1}(0), \ \forall \lambda > 0. \tag{2.8}$$

Let D be a nonempty subset of H. D is said to be proximal if there exists $y \in D$ such that

$$\|x - y\| = d(x, D), \ x \in H.$$

Let $CC(C), CB(C)$ and $P(C)$ be the family of nonempty closed convex subset of H, nonempty closed bounded subsets of H and nonempty proximal bounded subsets of H respectively. The Hausdorff metric on $CB(C)$ is defined as follows:

$$H(A,B) := \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A),\ \forall A,B \in CB(C)\right\}.$$

Let $S : C \to 2^C$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of S if $x \in Sx$. We say that S satisfies the endpoint condition if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : H \to 2^H$ $(i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say S_i satisfies the common endpoint condition if $S_i(p) = \{p\}$ for all $i \in \mathbb{N},\ p \in \cap_{i=1}^{\infty} F(S_i)$.

Definition 2.10. *Let $A : H \to H$ be a nonlinear operator. Then A is called*

(i) *Lipschitz continuous if for all $L > 0$*

$$\|Ax - Ay\| \leq L\|x - y\|,\ \forall x,y \in H;$$

if $0 \leq L < 1$, then A is a contraction mapping,

(ii) *$\beta-$strongly monotone if for all $\beta > 0$*

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2,\ \forall x,y \in H.$$

Definition 2.11. *Let $S : C \to CB(C)$ be a multivalued mapping. S is said to be*

(i) *nonexpansive if*

$$H(Sx,Sy) \leq \|x - y\|,\ \forall x,y \in C,$$

(ii) *quasi-nonexpansive if $F(S) \neq \emptyset$ such that*

$$H(Sx,Sp) \leq \|x - p\|,\ \forall x \in C, p \in F(S),$$

(iii) *k- strictly pseudo-contractive if there exists a constant $k \in [0,1)$ such that*

$$(H(Sx,Sy))^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2,\ \forall u \in Sx, v \in Sy \quad (2.9)$$

If $k = 1$ in (2.9), then the mapping S is said to be pseudo-contractive.

Clearly, the class of k-strict pseudo-contractive mappings properly contains the class of nonexpansive mappings. That is, S is nonexpansive if and only if S is 0-strict pseudo-contractive. It is known that if S is a k-strict pseudo-contraction and $F(S) \neq \emptyset$, then $F(S)$ is a closed convex subset of H (see [51]). Strict pseudo-contractions have many applications, due to their ties with inverse strongly monotone operators. It is known that, if B is a strongly monotone operator, then $S = I - B$ is a strict pseudo-contraction, and so we can recast a problem of zeros for B as a fixed point problem for S, and vice versa (see e.g. [13, 41]).

Let $S : H \to CB(H)$ be a multivalued mapping. The multivalued mapping $I - S$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset H$ which converges weakly to p and the sequence $\{\|x_n - u_n\|\}$ converges strongly to 0, where $u_n \in Sx_n$, then $p \in F(S)$.

3. Proposed Method

In this section, we present our proposed algorithm.
Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}_{i=1}^m$ be a countable family of k_i-strictly pseudo-contractive multivalued mappings of C into $CB(C)$ such that $I - S_i$ is demiclosed at zero for each $i = 1, 2, \dots, m$, $S_i p = \{p\}$ for each $p \in \cap_{i=1}^m F(S_i)$ and $k = \max\{k_i\}$. Let $F_1, \phi_1 : C \times C \to \mathbb{R}, F_2, \phi_2 : Q \times Q \to$ be bifunctions satisfying Assumptions 2.5. Let ϕ_1, ϕ_2 be monotone, ϕ_1 be upper hemicontinuous, and F_2 and ϕ_2 be upper semicontinuous in the first argument. Let $B : H_1 \to H_1$ be L-Lipschitz continuous and monotone and $D : H_1 \to 2^{H_1}$ be a maximal monotone operator such that $\Gamma = SGEP(F_1, \phi_1, F_2, \phi_2) \cap \cap_{i=1}^m F(S_i) \cap (B+D)^{-1}(0) \neq \emptyset$. We establish the convergence of our algorithm under the following conditions on the control parameters:

(C1) $0 < a \leq \tau_n \leq b < 2, \{r_n\} \subset (0, \infty), \liminf_{n\to\infty} r_n > 0$,
(C2) $\liminf_n \alpha_{n,i}(\alpha_{n,0} - k) > 0$ and $\lim_{n\to\infty} \alpha_{n,i} \in (0, 1)$ exists for all $i \geq 0$.

Now, we present our proposed algorithm as follows:

Algorithm 3.1.

Initialization: *Select* $x_0, x_1 \in H_1$, $s_1 > 0, \mu \in (0, 1), \theta_n \in [-\theta, \theta]$ *for some* $\theta > 0$ *and* $C_1 = C$.

Iterative Step: *Given the current iterate* x_n, *calculate the next iterate as follows:*

Step 1 : *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2 : *Compute*

$$z_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)w_n.$$

Step 3 : *Compute*

$$y_n = \alpha_{n,0} z_n + \sum_{i=1}^{m} \alpha_{n,i} u_{n,i}, \quad u_{n,i} \in S_i z_n.$$

Step 4 : *Compute*

$$\begin{cases} v_n = (I + s_n D)^{-1}(I - s_n B)y_n = J_{s_n}^D (I - s_n B)y_n \\ t_n = v_n - s_n(Bv_n - By_n) \\ C_{n+1} = \{p \in C_n : \|t_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2\} \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases}$$

Step 5 : *Compute*

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}, s_n\right\} & \text{if } By_n - Bv_n \neq 0. \\ s_n & \text{otherwise}, \end{cases} \tag{3.1}$$

Set $n := n + 1$ *and return to* **Step 1**, *where*

$$\gamma_n = \begin{cases} \tau_n \frac{||(I-T_{r_n}^{(F_2,\phi_2)})Aw_n||^2}{||A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n||^2} & \text{If } Aw_n \neq T_{r_n}^{(F_2,\phi_2)} Aw_n \\ \gamma & \text{otherwise } (\gamma \text{ being any non-negative real number}). \end{cases}$$

Remark 3.2. We observe that

(i) The implementation of our proposed algorithm does not require prior knowledge of the operator norm. Hence, this makes our method easily implementable.

(ii) We employ the inertial technique to accelerate the rate of convergence.

(iii) The underlying single-valued operator $B : H_1 \to H_1$ for most of the results on monotone inclusion problem in the literature are either strongly monotone or inverse strongly monotone while the single-valued operator in our proposed algorithm is only required to be monotone and Lipschitz continuous. Moreover, knowledge of the Lipschitz constant of the operator is not required to implement our proposed algorithm. Thus, our method is more applicable than several of the existing methods in the literature.

(iv) Our result extends and improves on the results of Deepho et al. [16], Sitthithakerngkiet et al. [42], Phuengrattana and Lerkchaiyaphum [39], Olona et al. [35] and several other results in the current literature in this direction.

4. Convergence Analysis

In this section, we analyze the convergence of our proposed algorithm.

Lemma 4.1. *Let* $\{s_n\}$ *be a sequence generated by* (3.1). *Then,* $\{s_n\}$ *is a non-increasing sequence and*

$$\lim_{n\to\infty} s_n = s \geq \min\left\{s_1, \frac{\mu}{L}\right\}. \tag{4.1}$$

Proof.
From (3.1), it is clear that $\{s_n\}$ is a nonincreasing sequence. Moreover, observe that if $By_n - Bv_n \neq 0$, then

$$\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|} \geq \frac{\mu}{L}. \tag{4.2}$$

Hence, the sequence $\{s_n\}$ has the lower bound $\min\left\{s_1, \frac{\mu}{L}\right\}$. □

Lemma 4.2. [20] *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.1. *Then the following inequality holds for all* $p \in \Gamma$:

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2,\ p \in \Gamma, \tag{4.3}$$

and

$$\|t_n - v_n\| \le \mu \frac{s_n}{s_{n+1}} \|y_n - v_n\|. \tag{4.4}$$

Proof. By the definition of s_n, we have

$$\|By_n - Bv_n\| \le \frac{\mu}{s_{n+1}} \|y_n - v_n\| \quad \forall\, n \in \mathbb{N}. \tag{4.5}$$

Clearly, if $By_n = Bv_n$, then (4.5) holds. Otherwise , we have

$$s_{n+1} = \min\left\{ \frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}, s_n \right\} \le \frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}.$$

This implies that

$$\|By_n - Bv_n\| \le \frac{\mu}{s_{n+1}} \|y_n - v_n\|.$$

Thus, (4.5) holds when $By_n = Bv_n$ and $By_n \ne Bv_n$. Let $p \in \Gamma$, then by Lemma 2.1, we have

$$\begin{aligned}
\|t_n - p\|^2 &= \|v_n - s_n(Bv_n - By_n) - p\|^2 \\
&= \|v_n - p\|^2 + s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|y_n - v_n\|^2 + 2\langle v_n - y_n, y_n - p\rangle \\
&+ s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|y_n - v_n\|^2 - 2\langle v_n - y_n, v_n - y_n\rangle + 2\langle v_n - y_n, v_n - p\rangle \\
&+ s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - v_n\|^2 + 2\langle v_n - y_n, y_n - p\rangle \\
&+ s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - v_n\|^2 - 2\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle \\
&+ s_n^2\|Bv_n - By_n\|^2.
\end{aligned} \tag{4.6}$$

By applying (4.5) in (4.6), we obtain

$$\|t_n - p\|^2 \le \|y_n - p\|^2 - \left(1 - \mu^2\frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2 - 2\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle. \tag{4.7}$$

We now prove that $\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle \ge 0$. Since $v_n = (I + s_nD)^{-1}(I - s_nB)y_n$, then we have $(I - s_nB)y_n \in (I + s_nD)v_n$. Recall that D is maximal monotone. Then there exists $u_n \in Dy_n$ such that

$$(I - s_nB)y_n = v_n + s_nu_n,$$

from which we obtain

$$u_n = \frac{1}{s_n}(y_n - v_n - s_nBy_n). \tag{4.8}$$

Moreover, we have $0 \in (B + D)p$ and $Bv_n + u_n \in (B + D)v_n$. Since $B + D$ is maximal monotone, we get

$$\langle Bv_n + u_n, v_n - p\rangle \ge 0. \tag{4.9}$$

By substituting (4.8) into (4.9), we obtain

$$\frac{1}{s_n}\langle y_n - v_n - s_n B y_n + s_n B v_n, v_n - p\rangle \geq 0.$$

This implies that

$$\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle \geq 0. \tag{4.10}$$

By applying (4.10) in (4.7), we have

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2. \tag{4.11}$$

On the other hand, one can see that (4.4) follows from (4.5).

□

Remark 4.3. By Lemma 4.1 and $\mu \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mu^2 \frac{s_n^2}{s_{n+1}^2} > \epsilon > 0$ for all $n \geq n_0$. Consequently, it follows from (4.3) that for all $p \in \Gamma$ and $n \geq n_0$

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \epsilon\|y_n - v_n\|^2.$$

Theorem 4.4. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of k_i-strictly pseudo-contractive multi-valued mappings of C into $CB(C)$. Let $F_1, \phi_1 : C\times C \to \mathbb{R}, F_2, \phi_2 : Q\times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions* 2.5. *Suppose ϕ_1, ϕ_2 are monotone, ϕ_1 is upper hemicontinuous, and F_2 and ϕ_2 are upper semicontinuous in the first argument. Let $B : H_1 \to H_1$ be an $L-$Lipschitz continuous monotone mapping and $D : H_1 \to 2^{H_1}$ be a maximal monotone operator such that $\Gamma = SGEP(F_1, \phi_1, F_2, \phi_2) \cap \bigcap_{i=1}^m F(S_i) \cap \Omega \neq \emptyset$, where $\Omega = (B + D)^{-1}(0)$ and $S_i p = \{p\}$ for each $p \in \cap_{i=1}^m F(S_i)$. Let $\{x_n\}$ be a sequence generated by Algorithm* 3.1 *such that conditions (C1) and (C2) hold. Then, the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$.*

Proof. We divide the proof of the strong convergence Theorem 4.4 into various steps as follows:

Step 1: We show that sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded and well defined.

Let $p \in \Gamma$, then we have $p = T_{r_n}^{(F_1,\phi_1)} p$ and $Ap = T_{r_n}^{(F_1,\phi_1)} Ap, S_i p = p$, for all $i = 1, 2, ...m$.

Since $T_{r_n}^{(F_1,\phi_1)}$ is nonexpansive, then by Lemma 2.1 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|T_{r_n}^{(F_1,\phi_1)}(w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n) - p\|^2 \\
&\leq \|w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n - p\|^2 \\
&= \|w_n - p\|^2 + \gamma_n^2 \|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\
&\quad - 2\gamma_n \langle w_n - p, A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle.
\end{aligned} \tag{4.12}$$

By the firmly nonexpansivity of $I - T_{r_n}^{(F_2,\phi_2)}$, we get

$$\begin{aligned}\langle w_n - p, A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle &= \langle Aw_n - Ap, (I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle \\ &= \langle Aw_n - Ap, (I - T_{r_n}^{(F_2,\phi_2)})Aw_n \\ &- (I - T_{r_n}^{(F_2,\phi_2)})Ap\rangle \\ &\geq \|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2. \qquad (4.13)\end{aligned}$$

By substituting (4.13) in (4.12) and applying the condition on τ_n, we have

$$\begin{aligned}\|z_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\ &- 2\gamma_n\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\ &= \|w_n - p\|^2 - \gamma_n\big[2\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\ &- \gamma_n\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2\big] \\ &= \|w_n - p\|^2 - \gamma_n(2 - \tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \qquad (4.14) \\ &\leq \|w_n - p\|^2. \qquad (4.15)\end{aligned}$$

By Lemma 2.3 and applying the fact that $S_i,\ i = 1, 2, \ldots, m$ is strictly pseudo-contractive together with condition (C2), we get

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_{n,0}z_n + \sum_{i=1}^m \alpha_{n,i}u_{n,i} - p\|^2 \\ &= \alpha_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \alpha_{n,i}\|u_{n,i} - p\|^2 \\ &- \sum_{i=1}^m \alpha_{n,0}\alpha_{n,i}\|u_{n,i} - z_n\|^2 - \sum_{1\leq i<j\leq m} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\ &\leq \alpha_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \alpha_{n,i}\big(H(S_iz_n, S_ip)\big)^2 \\ &- \sum_{i=1}^m \alpha_{n,0}\alpha_{n,i}\|u_{n,i} - z_n\|^2 - \sum_{1\leq i<j\leq m} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\ &\leq \alpha_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \alpha_{n,i}\big(\|z_n - p\|^2 + k_i\|u_{n,i} - z_n\|^2\big) \\ &- \sum_{i=1}^m \alpha_{n,0}\alpha_{n,i}\|u_{n,i} - z_n\|^2 \\ &- \sum_{1\leq i<j\leq m} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\ &\leq \|z_n - p\|^2 - \sum_{i=1}^m \alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2 \qquad (4.16) \\ &\leq \|z_n - p\|^2, \qquad (4.17)\end{aligned}$$

which implies that

$$\|y_n - p\| \leq \|z_n - p\|. \tag{4.18}$$

By applying (4.17) and (4.15) into (4.11), we get

$$\|t_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right) \|y_n - v_n\|^2, \forall p \in \Gamma. \tag{4.19}$$

By Lemma 2.4, we have that C_{n+1} is closed and convex. Furthermore, from (4.19) it follows that $p \in C_{n+1}$. Hence, we have $\Gamma \subset C_{n+1} \subset C_n$ for all n and thus $x_{n+1} = P_{C_{n+1}} x_0$ is well defined. Therefore, $\{x_n\}$ is well defined.
We now show that $\{x_n\}$ is bounded. It is known that Γ is a nonempty closed convex subset of H_1, then there exists a unique $q \in \Gamma$ such that $q = P_\Gamma x_0$. From $x_n = P_{C_n} x_0$ and $x_{n+1} \in C_{n+1}$ for all $n \in \mathbb{N}$, we obtain

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \ \forall \ n \in \mathbb{N}.$$

On the other hand, since $\Gamma \subset C_n$, we get

$$\|x_n - x_0\| \leq \|q - x_0\|, \ \forall \ n \in \mathbb{N}.$$

This implies that $\{\|x_n - x_0\|\}$ is bounded. Hence, $\{x_n\}$ is bounded. Consequently $\{w_n\}, \{t_n\}, \{z_n\}$ and $\{y_n\}$ are bounded. Thus, $\lim_{n\to\infty} \|x_n - x_0\|$ exists.
Step 2: We claim that $\lim_{n\to\infty} x_n = q$, for some $q \in C$.
It is clear from the definition of C_n that $x_m = P_{C_m} x_0 \in C_m \subset C_n$, $m > n \geq 1$. By Lemma 2.2, we obtain

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{4.20}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, then it follows from (4.20) that $\|x_m - x_n\| \to 0$ as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since H_1 is complete and C is closed, there exists $q \in C$ such that $x_n \to q$ as $n \to \infty$.
Step 3: We now show that $q \in \Gamma$.
From (4.20), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.21}$$

From the definition of w_n and by applying (4.21), we get

$$\|w_n - x_n\| = |\theta_n| \|x_n - x_{n-1}\| \leq |\theta| \|x_n - x_{n-1}\| \to 0, \quad n \to \infty. \tag{4.22}$$

From (4.21) and (4.22), we obtain

$$\|w_n - x_{n+1}\| \to 0, \quad n \to \infty. \tag{4.23}$$

We known that $x_{n+1} \in C_{n+1}$. Then, from the definition of C_{n+1} we obtain

$$\|t_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2.$$

Combining this with (4.23) gives

$$\lim_{n\to\infty} \|t_n - x_{n+1}\| = 0. \tag{4.24}$$

From (4.21) and (4.24), we obtain

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{4.25}$$

From (4.22) and (4.25), we obtain

$$\lim_{n\to\infty} \|t_n - w_n\| = 0. \tag{4.26}$$

By applying (4.17) and (4.15) into Remark 4.3, we have

$$\|t_n - p\|^2 \leq \|w_n - p\|^2 - \epsilon\|y_n - v_n\|^2.$$

From which we get

$$\begin{aligned}\epsilon\|y_n - v_n\|^2 &\leq \|w_n - p\|^2 - \|t_n - p\|^2\\ &\leq \|w_n - t_n\|(\|w_n - p\| + \|t_n - p\|),\end{aligned}$$

which together with (4.26) implies that

$$\|y_n - v_n\| \to 0, \quad n \to \infty. \tag{4.27}$$

Applying Lemma 4.1 together with (4.27) to (4.4), we have

$$\|t_n - v_n\| \to 0, \quad n \to \infty. \tag{4.28}$$

From (4.26)-(4.28), we obtain

$$\|y_n - w_n\| \to 0, \quad n \to \infty. \tag{4.29}$$

From (4.15) and (4.16), we obtain

$$\|y_n - p\|^2 \leq \|w_n - p\|^2 - \sum_{i=1}^{m} \alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2.$$

From this we have

$$\begin{aligned}\alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2 &\leq \sum_{i=1}^{m} \alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2\\ &\leq \|w_n - p\|^2 - \|y_n - p\|^2\\ &\leq (\|w_n - y_n\|)(\|w_n - p\| + \|y_n - p\|).\end{aligned}$$

By applying Condition (C2) and (4.29), we get

$$\|u_{n,i} - z_n\| \to 0, \quad n \to \infty. \tag{4.30}$$

From the definition of y_n and by applying (4.30), we get

$$\|y_n - z_n\| \leq \alpha_{n,0}\|z_n - z_n\| + \sum_{i=1}^{m} \alpha_{n,i}\|u_{n,i} - z_n\| \to 0, \quad n \to \infty. \tag{4.31}$$

Also, by applying (4.22), (4.29) and (4.31), we obtain

$$\lim_{n\to\infty}\|w_n - z_n\| = 0; \quad \lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{4.32}$$

From (4.14), we have

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \gamma_n(2-\gamma_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2,$$

which implies that

$$\begin{aligned}\gamma_n(2-\gamma_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 &\leq \|w_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|w_n - z_n\|(\|w_n - p\| + \|z_n - p\|).\end{aligned}$$

Using the definition of γ_n, the condition on τ_n and applying (4.32), it follows that

$$\frac{\tau_n(2-\tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^4}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)Aw_n})Aw_n\|^2} \to 0 \quad \text{as } n \to \infty.$$

From which we get

$$\frac{\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|} \to 0 \quad \text{as } n \to \infty.$$

Since $\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|$ is bounded, then it follows that

$$\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\| \to 0, \ n \to \infty. \tag{4.33}$$

Consequently, we have

$$\begin{aligned}\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n)\| &\leq \|A^*\|\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n)\| \\ &= \|A\|\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n)\| \to 0 \text{ as } n \to \infty.\end{aligned} \tag{4.34}$$

Since $t_n = v_n - s_n(Bv_n - By_n)$ and B is Lipschitz continuous, then by applying (4.27) we have

$$\|t_n - v_n\| = \|v_n - s_n(Bv_n - By_n) - v_n\| = s_n\|By_n - Bv_n\| \to 0, \quad n \to \infty.$$

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $q \in w_\omega(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ as $k \to \infty$. Let $z \in w_\omega(x_n)$ and $\{x_{n_j}\} \subset \{x_n\}$ be such that $x_{n_j} \rightharpoonup z$ as $j \to \infty$. From (4.32), we get $z_{n_k} \rightharpoonup q$ and $z_{n_j} \rightharpoonup z$. Since $I - S_i$ is demiclosed at zero for each $i = 1, 2, \ldots, m$, then it follows from (4.30) that $q, z \in F(S_i)$ for all $i = 1, 2, \ldots, m$, which implies that $q, z \in \cap_{i=1}^m F(S_i)$.

Next, let $(g, h) \in \text{Graph}(B + D)$, that is $h - Bg \in Dg$. Since $v_{n_k} = (I + s_{n_k}D)^{-1}(I - s_{n_k}B)y_{n_k}$, we have

$$(I - s_{n_k}B)y_{n_k} \in (I + s_{n_k}D)v_{n_k},$$

which implies that

$$\frac{1}{s_{n_k}}(y_{n_k} - v_{n_k} - s_{n_k}By_{n_k}) \in Dv_{n_k}.$$

Since D is maximal monotone, we get

$$\left\langle g - v_{n_k}, h - Bg - \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k} - s_{n_k}By_{n_k})\right\rangle \geq 0.$$

From this we obtain

$$\begin{aligned}\langle g - v_{n_k}, h\rangle &\geq \left\langle g - v_{n_k}, Bg + \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k} - s_{n_k}By_{n_k})\right\rangle \\ &= \langle g - v_{n_k}, Bg - By_{n_k}\rangle + \left\langle g - v_{n_k}, \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k})\right\rangle \\ &= \langle g - v_{n_k}, Bg - Bv_{n_k}\rangle + \langle g - v_{n_k}, Bv_{n_k} - By_{n_k}\rangle \\ &+ \left\langle g - v_{n_k}, \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k})\right\rangle \\ &\geq \langle g - v_{n_k}, Bv_{n_k} - By_{n_k}\rangle + \left\langle g - v_{n_k}, \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k})\right\rangle.\end{aligned}$$

Since B is Lipschitz continuous and $\lim_{n\to\infty}\|v_n - y_n\| = 0$, we have $\lim_{n\to\infty}\|Bv_{n_k} - By_{n_k}\| = 0$. Applying this together with $\lim_{n\to\infty} s_n = s \geq \min\left\{s_1, \frac{\mu}{L}\right\}$, we get

$$\langle g - q, h\rangle = \lim_{k\to\infty}\langle g - v_{n_k}, h\rangle \geq 0. \tag{4.35}$$

Following similar argument, we obtain

$$\langle g - z, h\rangle = \lim_{j\to\infty}\langle g - v_{n_j}, h\rangle \geq 0. \tag{4.36}$$

By the maximal monotonicity of $(B + D)$, it follows from (4.35) and (4.36) that $q, z \in (B + D)^{-1}(0)$.
Next, since $z_{n_k} = T_{r_{n_k}}^{(F_1,\phi_1)}(I - \gamma_{n_k}A^*(I - T_{r_{n_k}}^{F_2,\phi_2})A)w_{n_k}$, then by applying Lemma 2.6, we get

$$\begin{aligned}&F_1(z_{n_k}, y) + \phi_1(z_{n_k}, y) \\ &\qquad + \frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k} - \gamma_{n_k}A^*(I - T_{r_{n_k}}^{(F_2,\phi_2)})Aw_{n_k}\rangle \\ &\qquad \geq 0,\ \forall y \in C,\end{aligned}$$

which implies that

$$\begin{aligned}&F_1(z_{n_k}, y) + \phi_1(z_{n_k}, y) \\ &\qquad + \frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k}\rangle \\ &\qquad - \frac{1}{r_{n_k}}\langle y - z_{n_k}, \gamma_{n_k}A^*(I - T_{r_{n_k}}^{(F_2,\phi_2)})Aw_{n_k}\rangle \\ &\qquad \geq 0,\ \forall y \in C.\end{aligned}$$

From the monotonicity of F_1 and ϕ_1, it follows that

$$\begin{aligned}\frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k}\rangle \\ &- \frac{1}{r_{n_k}}\langle y - z_{n_k}, \gamma_{n_k} A^*(I - T_{r_{n_k}}^{(F_2,\phi_2)})Aw_{n_k}\rangle \\ &\geq F_1(y, z_{n_k}) + \phi_1(y, z_{n_k}),\ \forall y \in C.\end{aligned}$$

By (4.32) and $x_{n_k} \rightharpoonup q$, we obtain $z_{n_k} \rightharpoonup q$. Applying condition $(C1)$, (4.32), (4.34) and Assumption 2.5 (A1)-(A7), we obtain

$$0 \geq F_1(y, q) + \phi_1(y, q),\ \forall y \in C.$$

Suppose $y_t = ty + (1-t)q, \forall t \in (0, 1]$ and $y \in C$. Then, $y_t \in C$ and $F_1(y_t, q) + \phi_1(y_t, q) \leq 0$. Therefore, by Assumption 2.5 (A1)-(A7), we get

$$\begin{aligned}0 &\leq F_1(y_t, y_t) + \phi_1(y_t, y_t) \\ &\leq t\big(F_1(y_t, y) + \phi_1(y_t, y)\big) + (1-t)\big(F_1(y_t, q) + \phi_1(y_t, q)\big) \\ &\leq t\big(F_1(y_t, y) + \phi_1(y_t, y)\big).\end{aligned}$$

Thus, we have

$$F_1(y_t, y) + \phi_1(y_t, y) \geq 0,\ \forall y \in C.$$

Letting $t \to 0$, and applying condition (A3) together with the upper hemicontinuity of ϕ_1, we have

$$F_1(q, y) + \phi_1(q, y) \geq 0,\ \forall y \in C. \tag{4.37}$$

By similar argument, we have

$$F_1(z, y) + \phi_1(z, y) \geq 0,\ \forall y \in C. \tag{4.38}$$

It follows from (4.37) and (4.38) that $q, z \in GEP(F_1, \phi_1)$.
Next, we show that $Aq, Az \in GEP(F_2, \phi_2)$. Since A is a bounded linear operator, then by (4.22) we have $Aw_{n_k} \rightharpoonup Aq$. Hence, from (4.33), we obtain

$$T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k} \rightharpoonup Aq, \quad k \to \infty. \tag{4.39}$$

By the definition of $T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}$, we have

$$\begin{aligned}F_2(T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}, y) \\ &+ \phi_2(T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}, y) \\ &+ \frac{1}{r_{n_k}}\langle y - T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}, T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k} - Aw_{n_k}\rangle \\ &\geq 0,\ \forall y \in Q.\end{aligned}$$

Since F_2 and ϕ_2 are upper semicontinuous in the first argument, then by (4.33), (4.39) and $\liminf_{k\to\infty} r_{n_k} > 0$, we have

$$F_2(Aq, y) + \phi_2(Aq, y) \geq 0,\ \forall y \in Q. \tag{4.40}$$

Following similar argument, we have

$$F_2(Az, y) + \phi_2(Az, y) \geq 0, \ \forall y \in Q. \tag{4.41}$$

From (4.40) and (4.41), it follows that $Aq, Az \in GEP(F_2, \phi_2)$. Therefore $q, z \in SGEP(F_1, \phi_1, F_2, \phi_2)$. By Invoking Lemma 2.7, we get $q = z$. Hence, we have that $q \in \Gamma$.

Step 4. Lastly, we show that $q = P_\Gamma x_0$.

Since $x_n = P_{C_n} x_0$ and $\Gamma \subset C_n$, we have $\langle x_0 - x_n, x_n - p\rangle \geq 0$ for all $p \in \Gamma$. By taking limit as $n \to \infty$, we have $\langle x_0 - q, q - p\rangle \geq 0$ for all $p \in \Gamma$. This shows that $q = P_\Gamma x_0$.

Therefore, we can conclude by the steps above that $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$. This completes the proof. □

If $\phi_1 = \phi_2 = 0$ in (1.3)-(1.4), then the split generalized equilibrium problem reduces to split equilibrium problem. Hence from Theorem 3.1 , we obtain the following consequent result.

Corollary 4.5. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}_{i=1}^m$ be a countable family of k_i-strictly pseudo-contractive multivalued mappings of C into $CB(C)$ such that $I - S_i$ is demiclosed at zero for each $i = 1, 2, \dots, m$, $S_i p = \{p\}$ for each $p \in \cap_{i=1}^m F(S_i)$ and $k = \max\{k_i\}$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.5 such that F_2 is upper semicontinuous in the first argument. Let $B : H_1 \to H_1$ be L-Lipschitz continuous and monotone and $D : H_1 \to 2^{H_1}$ be a maximal monotone operator such that $\Gamma = SGEP(F_1, F_2) \cap_{i=1}^m F(S_i) \bigcap (B+D)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 4.6.

Initialization: *Select $x_0, x_1 \in H_1$, $\mu \in (0,1), \theta_n \in [-\theta, \theta]$ for some $\theta > 0$ and $C_1 = C$.*

Iterative Step: *Given the current iterate x_n, calculate the next iterate as follows:*

Step 1 : *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2 : *Compute*

$$z_n = T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A)w_n.$$

Step 3 : *Compute*

$$y_n = \alpha_{n,0} z_n + \sum_{i=1}^{m} \alpha_{n,i} u_{n,i}, \quad u_{n,i} \in S_i z_n.$$

Step 4 *: Compute*

$$\begin{cases} v_n = (I + s_n D)^{-1}(I - s_n B)y_n \\ t_n = v_n - s_n(Bv_n - By_n) \\ C_{n+1} = \{p \in C_n : \|t_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n \langle x_n - p, x_{n-1} - x_n \rangle \\ +\theta_n^2 \|x_{n-1} - x_n\|^2\} \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases}$$

Step 5 *: Compute*

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}, s_n\right\} & \textit{if } By_n - Bv_n \neq 0. \\ s_n & \textit{otherwise}, \end{cases} \tag{4.42}$$

Set $n := n + 1$ *and return to* **Step 1**. *where*

$$\gamma_n = \begin{cases} \tau_n \frac{||(I - T_{r_n}^{(F_2)})Aw_n||^2}{||A^*(I - T_{r_n}^{(F_2)})Aw_n||^2} & \textit{If } Aw_n \neq T_{r_n}^{(F_2}Aw_n \\ \gamma & \textit{otherwise } (\gamma \textit{ being any non-negative real number.}) \end{cases}$$

Suppose other conditions of Theorem 3.1 *hold. Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Gamma x_0$.

5. Applications

5.1. Split Minimization Problem

Let H_1, H_2 be two real Hilbert spaces, and let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed, and convex subsets. Let $f : C \to \mathbb{R}$, $g : Q \to \mathbb{R}$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator. The split minimization problem (SMP) is formulated as finding

$$x^* \in C \quad \text{such that} \quad f(x^*) \leq f(x),\ \forall x \in C, \tag{5.1}$$

and

$$y^* = Ax^* \quad \text{such that} \quad g(y^*) \leq g(y),\ y \in Q. \tag{5.2}$$

Let Ω denote the set of solution of SMP (5.1)-(5.2), and we assume $\Omega \neq \emptyset$. Let $\phi_1 = \phi_2 = 0$, and

$$F_1(x, y) := f(y) - f(x) \quad \text{for all } x, y \in C;$$

and

$$F_2(u, v) := g(v) - g(u) \quad \text{for all } u, v \in Q.$$

Suppose f and g are convex and lower semi-continuous on C and Q, respectively. Then, F_1, F_2, ϕ_1 and ϕ_2 satisfy all the conditions of Assumption 2.5. Consequently, from Theorem 3.1 we obtain a strong convergence theorem for approximating a common solution of split minimization problem, monotone variational inclusion problem and fixed point problem for a countable family of strict pseudo-contractive multivalued mappings in Hilbert spaces.

5.2. Split Variational Inequality Problem

Let C be a nonempty closed convex subset of a real Hilbert space H, and $f : H \to H$ be a single-valued mapping. The variational inequality problem (VIP) introduced independently by Fichera [19] and Stampacchia [43] is formulated as follows:

$$\text{find } x^* \in C \text{ such that } \langle y - x^*, fx^* \rangle \geq 0, \qquad \forall\, y \in C. \tag{5.3}$$

The VIP can be modelled to solve several optimization problems and has vast applications in different fields, such as in physics, engineering, economics, etc, (see [4, 8, 12, 16, 31, 34, 42]).
The split variational inequality problem (SVIP), which was first introduced by Censor et al. [12] is defined as finding a point:

$$x^* \in C \text{ such that } \langle x - x^*, f(x^*) \rangle \geq 0 \qquad \forall\, x \in C, \tag{5.4}$$

and

$$y^* = Ax^* \in Q \text{ solves } \langle y - y^*, g(y^*) \rangle \geq 0 \qquad \forall\, y \in Q, \tag{5.5}$$

where C and Q are nonempty, closed, convex subsets of real Hilbert spaces H_1 and H_2, respectively, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are monotone mappings, and $A : H_1 \to H_2$ is a bounded linear operator, see [25]. Let $\Omega \neq \emptyset$ denote the set of solution of SVIP (5.4)-(5.5). By setting $\phi_1 = \phi_2 = 0$, and

$$F_1(x, y) := \langle y - x, f(x) \rangle \quad \text{for all } x, y \in C;$$

and

$$F_2(u, v) := \langle v - u, g(u) \rangle \quad \text{for all } u, v \in Q.$$

Then, F_1, F_2, ϕ_1 and ϕ_2 satisfy all the conditions of Assumption 2.5. Hence, from Theorem 3.1, we obtain a strong convergence theorem for approximating a common solution of split variational inequality problem, monotone variational inclusion problem and fixed point problem for a countable family of strict pseudo-contractive multivalued mappings in Hilbert spaces.

6. Numerical Examples

In this section, we present a numerical experiments to illustrate the performance of our Algorithm 3.1 as well as comparing it with Algorithm (1.14), Algorithm (1.15), Algorithm (1.16) and Algorithm (1.17) in the literature.
In our computation, we choose $\alpha_{n,0} = \frac{n}{2n+1}, \alpha_{n,i} = \frac{n+1}{5(2n+1)}, \ i = 1, 2, \ldots, 5, \tau_n = 1.5, \theta_n = 1.9, r_n = 2.0, s_0 = 0.1$ and $\mu = 0.7$ in our Algorithm 3.1. $Gx = \frac{1}{3}x, fx = \frac{2}{3}x, Kx = \frac{2}{5}x, \lambda_n = \frac{2n}{5n+1}, \alpha_n = \frac{2}{2n+3}, \beta_n = \frac{n+1}{2n+3}, \eta = \frac{2}{5}, \gamma = 0.2, T_j x = \frac{2}{(3+j)}x$ in Algorithm (1.14),$\beta_n = \xi_n = \frac{1}{2}(1 - \alpha_n), \sigma_n = \frac{2}{2n+1}$ in Algorithm (1.15) while in Algorithm (1.16) and Algorithm (1.17). Let the

sequences $\{\delta_{n,j}\}$ be defined as follows for each $j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$:

$$\delta_{n,j} = \begin{cases} \frac{1}{b^{j+1}}(\frac{n}{n+1}), & n > j, \\ 1 - \frac{n}{n+1}(\sum_{k=1}^{n} \frac{1}{b^k}), & n = j, \\ 0, & n < j, \end{cases} \tag{6.1}$$

where $b > 1$.

Example 6.1. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 10]$. Let $A : H_1 \to H_2$ be defined by $Ax = \frac{x}{5}$ for all $x \in H_1$. Then, we have that $A^*y = \frac{y}{5}$ for all $y \in H_2$. For $x \in C, j \in \mathbb{N}$ and $i = 1, 2, \ldots, 5$, let $P_j, S_i : C \to CB(C)$ be multivalued mappings defined as follows:

$$P_j(x) = \left[0, \frac{x}{10j}\right], \quad S_i(x) = \left[0, \frac{x}{10i}\right]. \tag{6.2}$$

One can easily verify that P_j and S_i are nonexpansive and strictly pseudo-contractive, respectively. Define mappings $B : H_1 \to H_1$ by $Bx = 2x$, $D : H_1 \to H_1$ by $Dx = 3x$, and let the bifunctions $F_1, \phi_1 : C \times C \to \mathbb{R}$ be defined by $F_1(x, y) = y^2 + 3xy - 4x^2$ and $\phi_1(x, y) = y^2 - x^2$ for $x, y \in C$, and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ by $F_2(w, v) = 2v^2 + wv - 3w^2$ and $\phi_2(w, v) = w - v$ for $w, v \in Q$. It is easy to verify that all the conditions of Theorem 4.4 are satisfied. Next, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $u \in C$ such that for all $z \in C$

$$\begin{aligned} 0 &\leq F_1(u, z) + \phi_1(u, z) + \frac{1}{r}\langle z - u, u - x\rangle \\ &= 2z^2 + 3uz - 5u^2 + \frac{1}{r}\langle z - u, u - x\rangle \\ &\Leftrightarrow \\ 0 &\leq 2rz^2 + 3ruz - 5ru^2 + (z - u)(u - x) \\ &= 2rz^2 + 3ruz - 5ru^2 + uz - xz - u^2 + ux \\ &= 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux). \end{aligned}$$

Suppose $h(z) = 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux)$. Then, $h(z)$ is a quadratic function of z with coefficients $a = 2r, b = 3ru + u - x$, and $c = -5ru^2 - u^2 + ux$. We determine the discriminant $\triangle$ of $h(z)$ as follows:

$$\begin{aligned} \triangle &= (3ru + u - x)^2 - 4(2r)(-5ru^2 - u^2 + ux) \\ &= 49r^2u^2 + 14ru^2 - 14rux + u^2 - 2ux + x^2 \\ &= ((7r + 1)u - x)^2. \end{aligned} \tag{6.3}$$

By Lemma 2.6, $T_r^{(F_1,\phi_1)}$ is single-valued. Thus, it follows that $h(z)$ has at most one solution in $\mathbb{R}$. Hence, from (6.3), we have that $u = \frac{y}{7r+1}$. This implies that $T_r^{(F_1,\phi_1)}(y) = \frac{y}{7r+1}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(y)$. Find $w \in Q$ such that for all $d \in Q$

$$T_r^{(F_2,\phi_2)}(y) = \left\{ w \in Q : F_2(w, d) + \phi_2(w, d) + \frac{1}{r}\langle d - w, w - y\rangle \geq 0, \quad \forall\, d \in Q \right\}.$$

By following similar procedure as above, we obtain $w = \frac{y+r}{5r+1}$. This implies that $T_r^{(F_2,\phi_2)}(y) = \frac{y+r}{5r+1}$.
In this example, we set the parameter b on $\{\delta_{n,i}\}$ in (6.1) to be $b = 40, v = 3.5$ and we choose different initial values as follows:
Case I: $x_0 = 7, x_1 = 3$;
Case II: $x_0 = 6, x_1 = 2$;
Case III: $x_0 = 8, x_1 = 4$;
Case IV: $x_0 = 9, x_1 = 5$.

We compare the performance of our Algorithm 3.1 with Algorithms (1.14), (1.15), (1.16) and (1.17). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

TABLE 1. **Numerical results for Example** 6.1

		Alg. (1.14)	Alg. (1.15)	Alg. (1.16)	Alg. (1.17)	Alg. 3.1
Case I	No. of Iter.	9	20	4	9	2
	CPU time (sec)	0.0057	0.0078	1.6693	0.3383	0.0032
Case II	No. of Iter.	8	20	4	8	2
	CPU time (sec)	0.0051	0.0059	1.6884	0.3124	0.0039
Case III	No. of Iter.	9	20	4	9	2
	CPU time (sec)	0.0053	0.0057	1.6625	0.3566	0.0041
Case IV	No. of Iter.	9	20	4	9	2
	CPU time (sec)	0.0054	0.0067	1.6623	0.3449	0.0039

Example 6.2. Let $H_1 = H_2 = L_2([0,1])$ with the inner product defined as

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x, y \in L_2([0,1]).$$

Let

$$C := \{x \in H_1 : \langle a, x\rangle \geq d\},$$

where $a = 2t^2$ and $d = 0$. Here, we have

$$P_C(x) = x + \frac{d - \langle a, x\rangle}{||a||^2}a.$$

Also, let

$$Q := \{x \in H_2 : \langle c, x\rangle \leq e\},$$

where $c = \frac{t}{3}$, $e = 1$ and we have

$$P_Q(x) = x + \max\left\{0, \frac{e - \langle c, x\rangle}{||c||^2}c\right\}.$$

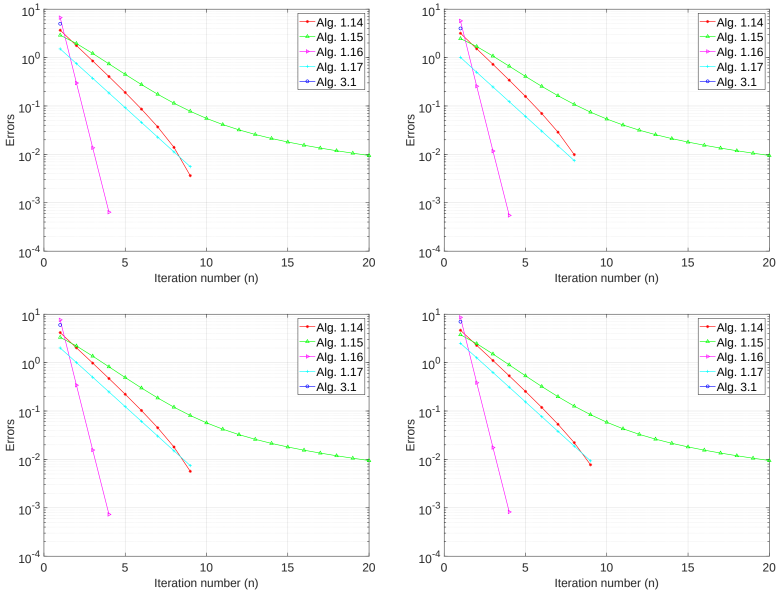

FIGURE 1. Top left: Case I ; Top right: Case II; Bottom left: Case III ; Bottom right: Case IV.

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be defined as $F_1(x, y) = \langle L_1 x, y - x\rangle$ and $F_2(x, y) = \langle L_2 x, y - x\rangle$, where $L_1 x(t) = \frac{x(t)}{3}$ and $L_2 x(t) = \frac{x(t)}{4}$. It can easily be verified that F_1 and F_2 satisfy conditions (A1)-(A4). Also, let $\phi_1 = \phi_2 = 0$. Furthermore, define $B : H_1 \to H_1$ by $Bx = 3x$, $D : H_1 \to H_1$ by $Dx = 7x$, and let $A : L_2([0, 1]) \to L_2([0, 1])$ be defined by $Ax(t) = \frac{x(t)}{3}$ and $A^* y(t) = \frac{y(t)}{3}$. Then, A is a bounded linear operator. We consider the case for which the multivalued mappings $\{S_j\}$ and $\{S_i\}$ are single-valued. Let $S_j, S_i : L^2([0, 1]) \to L^2([0, 1])$ be defined by

$$(S_j x)(t) = \int_0^1 t^j x(s)ds \qquad \text{and} \qquad (S_i x)(t) = \int_0^1 t^i x(s)ds \quad \text{for all } t \in [0, 1].$$

Note that S_i and S_j are nonexpansive for each i, j. Select $r_n = \frac{2n}{2n+1}, \theta_n = 0.8, \tau_n = 0.7$. It can easily be checked that all the conditions of Theorem 4.4 are satisfied. Now, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $z \in C$ such that for all

$y \in C$

$$\begin{aligned} &F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \\ &\Leftrightarrow \langle \frac{z}{2}, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \\ &\Leftrightarrow \frac{z}{3}(y - z) + \frac{1}{r}(y - z)(z - x) \geq 0 \\ &\Leftrightarrow (y - z)[rz + 3(z - x)] \geq 0 \\ &\Leftrightarrow (y - z)[(r + 3)z - 3x] \geq 0. \end{aligned} \tag{6.4}$$

By Lemma 2.6, we obtain

$$T_r^{(F_1,\phi_1)}(x) = \left\{ z \in C : F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C \right\},$$

$(\forall\, x \in H_1)$, is single-valued. Thus, from (6.4) we obtain $z = \frac{3x}{r+3}$. This implies that $T_r^{(F_1,\phi_1)}(x) = \frac{3x}{r+3}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(v)$. We find $w \in Q$ such that for all $d \in Q$

$$T_s^{(F_2,\phi_2)}(v) = \left\{ w \in Q : F_2(w,d) + \phi_2(w,d) + \frac{1}{s}\langle d - w, w - v\rangle \geq 0, \quad \forall\, d \in Q \right\}.$$

By using similar approach as above, we obtain $w = \frac{4v}{s+4}$. This implies that $T_s^{(F_2,\phi_2)}(v) = \frac{4v}{s+4}$.

Here, we set the parameter b on $\{\delta_{n,i}\}$ in (6.1) to be $b = 3, v = t^2$ and we choose different initial values as follows:

Case I: $x_0 = t^4, x_1 = t^2 + t^4 + t^6 + 3$;
Case II: $x_0 = t^5, x_1 = t^2 + t^5 + 2$;
Case III: $x_0 = t^4, x_1 = t^3 + t^5 + t^7 + 2$;
Case IV: $x_0 = t^5, x_1 = t + t^2 + 1$.

We compare the performance of our Algorithm 3.1 with Algorithms (1.14), (1.15), (1.16) and (1.17). The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2 and Table 2.

7. Conclusion

In this article, we proposed a new modified inertial shrinking projection algorithm for finding common solution of split generalized equilibrium problem, monotone inclusion problem and fixed point problems for a countable family of strictly pseudo-contractive multivalued mappings. We established strong convergence result for the proposed method. We applied our results to study related optimization problems and presented some numerical examples to demonstrate the efficiency of our proposed method in comparison with other

TABLE 2. **Numerical results for Example** 6.2

		Alg. (1.14)	Alg. (1.15)	Alg. (1.16)	App. (1.17)	Alg. 3.1
Case I	No. of Iter.	10	14	10	6	6
	CPU time (sec)	0.7297	0.7237	1.2541	0.2548	0.3256
Case II	No. of Iter.	9	14	9	6	6
	CPU time (sec)	0.6743	0.7004	1.1791	0.2628	0.3091
Case III	No. of Iter.	9	14	9	6	6
	CPU time (sec)	0.6507	0.6825	1.1474	0.2599	0.3087
Case IV	No. of Iter.	9	13	8	6	6
	CPU time (sec)	0.6353	0.6458	1.1130	0.2631	0.3166

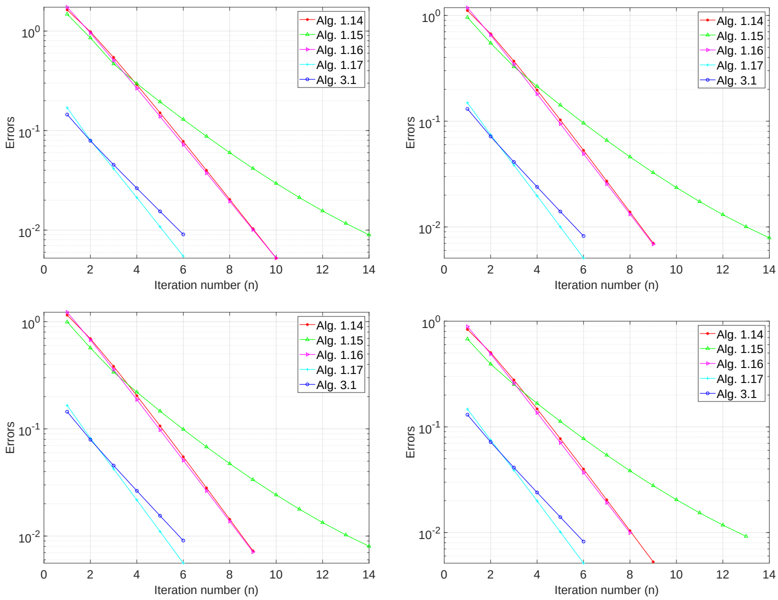

FIGURE 2. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

existing methods. Our results extend and improve several existing results in this direction in the current literature.

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