

# Fuzzy Differential Subordinations Connected with Convolution

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**Abstract.** The object of the present paper is to obtain several fuzzy differential subordinations associated with Linear operator  $\mathcal{D}_{n,\delta,g}^m f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)c^n(\delta)]^m a_j b_j z^j$ . Using the operator  $\mathcal{D}_{n,\delta,g}^m$ , we also introduce a class  $\mathcal{H}_{n,m,\delta}^F(\eta, g)$  of univalent analytic functions for which we give some properties.

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## 1. Introduction

Let  $\Omega \subset \mathbb{C}$ ,  $H(\Omega)$  the class of holomorphic functions on  $\Omega$  and denote by  $H_d(\Omega)$  the class of holomorphic and univalent functions on  $\Omega$ . In this paper, we denote by  $H(\Delta)$  the class of holomorphic functions in the open unit disk  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  with  $B_\Delta = \{z \in \mathbb{C} : |z| = 1\}$  the boundary of the unit disk. For  $\beta \in \mathbb{C}$  and  $d \in \mathbb{N}$ , we denote

$$H[\beta, d] = \left\{ f \in H(\Delta) : f(z) = \beta + \sum_{j=d+1}^{\infty} a_j z^j, \quad z \in \Delta \right\},$$

$$\mathbb{A}_d = \left\{ f \in H(\Delta) : f(z) = z + \sum_{j=d+1}^{\infty} a_j z^j, \quad z \in \Delta \right\} \quad \text{with } \mathbb{A}_1 = \mathbb{A},$$

and,

$$\mathcal{S} = \{f \in \mathbb{A} : f \text{ is a univalent function in } \Delta\}.$$

We denote by

$$\mathcal{C} = \left\{ f \in \mathbb{A} : \Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) > 0, \quad z \in \Delta \right\},$$

is convex functions in  $\Delta$ .

**Definition 1.1.** [2, 11] *Let  $f_1$  and  $f_2$  are analytic function in  $\Delta$ , then  $f_1$  is subordinate to  $f_2$ , written  $f_1 \prec f_2$  if there exists a Schwarz function  $w$ , which is analytic in  $\Delta$  with  $w(0) = 0$  and  $|w(z)| < 1$  for all  $z \in \Delta$ , such that  $f_1(z) = f_2(w(z))$ . Furthermore, if the function  $f_2$  is univalent in  $\Delta$ , then we have the following equivalence:*

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\Delta) \subset f_2(\Delta).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

**Definition 1.2.** [8] *Assume that the set  $\mathcal{Y} \neq \emptyset$ . Application  $\mathcal{F} : \mathcal{Y} \rightarrow [0, 1]$  is fuzzy subset. A pair  $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ , where  $\mathcal{F}_{\mathcal{B}} : \mathcal{Y} \rightarrow [0, 1]$  and*

$$A = \{x \in \mathcal{Y} : 0 < \mathcal{F}_{\mathcal{B}}(x) \leq 1\} = \sup(\mathcal{B}, \mathcal{F}_{\mathcal{B}}), \quad (1.1)$$

*is said fuzzy subset. A function  $\mathcal{F}_{\mathcal{B}}$  is said to be the fuzzy set  $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ .*

**Proposition 1.3.** [12] *(i) If  $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) = (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$ , then we have  $\mathcal{B} = \mathcal{U}$ , where  $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$  and  $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$ ;*

*(ii) If  $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) \subseteq (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$ , then we have  $\mathcal{B} \subseteq \mathcal{U}$ , where  $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$  and  $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$ .*

Let  $f, g \in H(\Omega)$ , we denote by

$$f(\Omega) = \{f(z) : 0 < \mathcal{F}_{f(\Omega)}f(z) \leq 1, z \in \Omega\} = \sup(f(\Omega), \mathcal{F}_{f(\Omega)}), \quad (1.2)$$

and,

$$g(\Omega) = \{g(z) : 0 < \mathcal{F}_{g(\Omega)}g(z) \leq 1, z \in \Omega\} = \sup(g(\Omega), \mathcal{F}_{g(\Omega)}). \quad (1.3)$$

**Definition 1.4.** [12] *Let  $z_0 \in \Omega$  be a fixed point and let the functions  $f, g \in H(\Omega)$ . The function  $f$  is said to be fuzzy subordinate to  $g$  and write  $f \prec_{\mathcal{F}} g$  or  $f(z) \prec_{\mathcal{F}} g(z)$ , which is satisfied the following conditions:*

$$(i) f(z_0) = g(z_0)$$

$$(ii) \mathcal{F}_{f(\Omega)}f(z) \leq \mathcal{F}_{g(\Omega)}g(z), \quad z \in \Omega.$$

**Proposition 1.5.** [12] *Assume that  $z_0 \in \Omega$  is a fixed point and the functions  $f, g \in H(\Omega)$ . If  $f(z) \prec_{\mathcal{F}} g(z)$ ,  $z \in \Omega$ , then*

$$(i) f(z_0) = g(z_0)$$

$$(ii) f(\Omega) \subseteq g(\Omega), \quad \mathcal{F}_{f(\Omega)}f(z) \leq \mathcal{F}_{g(\Omega)}g(z), \quad z \in \Omega,$$

*where  $f(\Omega)$  and  $g(\Omega)$  are defined by (1.2) and (1.3), respectively.*

**Definition 1.6.** [13] *Assume that  $\Phi : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}$  and  $h \in \mathcal{S}$ , with  $\Phi(\alpha, 0, 0; 0) = h(0) = \alpha$ . If  $p$  is analytic in  $\Delta$ , with  $p(0) = \alpha$  and satisfies the second order fuzzy differential subordination*

$$\mathcal{F}_{\Phi(\mathbb{C}^3 \times \Delta)}\Phi\left(p(z), zp'(z), z^2p''(z); z\right) \leq \mathcal{F}_{h(\Delta)}h(z),$$

$$\text{i.e. } \Phi\left(p(z), zp'(z), z^2p''(z); z\right) \prec_{\mathcal{F}} h(z), \quad z \in \Delta. \quad (1.4)$$

Then  $p$  is said to be a fuzzy solution of the fuzzy differential subordination, the univalent function  $q$  is called a fuzzy dominant of the fuzzy solutions for the fuzzy differential subordination if

$$\mathcal{F}_{p(\Delta)}p(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad \text{i.e. } p(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all  $p$  satisfying (1.4).

A fuzzy dominant  $\tilde{q}$  that satisfies

$$\mathcal{F}_{\tilde{q}(\Delta)}\tilde{q}(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad \text{i.e. } \tilde{q}(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all fuzzy dominants  $q$  of (1.4) is called the fuzzy best dominant of (1.4).

Making use the binomial series

$$(1 - \delta)^n = \sum_{i=0}^n \binom{n}{i} (-1)^i \delta^i \quad (n \in \mathbb{N} = \{1, 2, \dots\}),$$

for  $f \in \mathbb{A}$ , we introduced the linear differential operator as follows:

$$\mathcal{D}_{n,\delta,g}^0 f(z) = (f * g)(z),$$

$$\mathcal{D}_{n,\delta,g}^1 f(z) = \mathcal{D}_{n,\delta,g} f(z) = (1 - \delta)^n (f * g)(z) + [1 - (1 - \delta)^n] z (f * g)'(z)$$

$$= z + \sum_{j=2}^{\infty} [1 + (j - 1) c^n(\delta)] a_j b_j z^j$$

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$$\mathcal{D}_{n,\delta,g}^m f(z) = \mathcal{D}_{n,\delta,g} \left( \mathcal{D}_{n,\delta,g}^{m-1} f(z) \right)$$

$$= (1 - \delta)^n \mathcal{D}_{n,\delta,g}^{m-1} f(z) + [1 - (1 - \delta)^n] z \left( \mathcal{D}_{n,\delta,g}^{m-1} f(z) \right)'$$

$$= z + \sum_{j=2}^{\infty} [1 + (j - 1) c^n(\delta)]^m a_j b_j z^j \tag{1.5}$$

$$(\delta > 0, n \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where

$$c^n(\delta) = \sum_{i=1}^n \binom{n}{i} (-1)^{i+1} \delta^i \quad (n \in \mathbb{N}).$$

From (1.5), we obtain that

$$c^n(\delta) z \left( \mathcal{D}_{n,\delta,g}^m f(z) \right)' = \mathcal{D}_{n,\delta,g}^{m+1} f(z) - [1 - c^n(\delta)] \mathcal{D}_{n,\delta,g}^m f(z).$$

By specializing the parameters  $n$ ,  $\delta$  and  $b_j$ , we note that

(i) Putting  $b_j = 1$  (or  $g(z) = \frac{z}{1-z}$ ), then  $\mathcal{D}_{n,\delta,\frac{z}{1-z}}^m = \mathcal{D}_{n,\delta}^m$  defined by Yousef et al.[17];

(ii) Putting  $b_j = 1$  (or  $g(z) = \frac{z}{1-z}$ ) and  $n = 1$ , then  $\mathcal{D}_{1,\delta,\frac{z}{1-z}}^m = \mathcal{D}_{\delta}^m$  defined by Al-Oboudi [1];

(iii) Putting  $b_j = 1$  (or  $g(z) = \frac{z}{1-z}$ ) and  $n = \delta = 1$ , then  $\mathcal{D}_{1,1,\frac{z}{1-z}}^m = \mathcal{D}^m$  defined by Salagean.[15];

(iv) Putting  $b_j = \left(\frac{\ell+1}{\ell+j}\right)^\alpha$  ( $\alpha > 0$ ,  $\ell > -1$ ) and  $n = 1$ , then  $\mathcal{D}_{1,\delta,g}^m = \mathcal{I}_{\ell,\delta}^{m,\alpha} f(z)$  defined by El-Deeb and Lupas [6];

(v) Putting  $b_j = \left(\frac{\alpha+1}{\alpha+j}\right)^n \frac{m^{j-1}}{(j-1)!} e^{-m}$  ( $m, \alpha \geq 0$ ,  $n \in \mathbb{N}_0$ ) and  $m = 0$ , then  $\mathcal{D}_{n,\delta,g}^0 = \mathcal{H}_{\alpha,m}^n f(z)$  defined by El-Deeb and Oros [7];

(vi) Putting  $b_j = \frac{(-1)^{k-1} \Gamma(v+1)}{4^{k-1} (k-1)! \Gamma(k+v)} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$ , ( $v > 0$ ,  $\lambda > -1$ ,  $0 < q < 1$ ) studied by El-Deeb and Bulboacă [4] and El-Deeb [3], we obtain the operator  $\mathcal{N}_{v,n,\delta}^{m,\lambda,q}$ , defined as follows:

$$\begin{aligned} \mathcal{N}_{v,n,\delta}^{m,\lambda,q} f(z) &= z + \sum_{j=2}^{\infty} [1 + (j-1) c^n(\delta)]^m \frac{(-1)^{j-1} \Gamma(v+1)}{4^{j-1} (j-1)! \Gamma(j+v)} a_j z^j \\ &(\lambda > -1; 0 < q < 1; \delta, v > 0; n \in \mathbb{N}; m \in \mathbb{N}_0); \end{aligned}$$

(vi) Putting  $b_j = \left(\frac{\ell+1}{\ell+j}\right)^\alpha \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$ , ( $\alpha > 0$ ,  $n \geq 0$ ,  $\lambda > -1$ ,  $0 < q < 1$ ) studied by El-Deeb and Bulboacă [5] and Srivastava and El-Deeb [16], we obtain the operator  $\mathcal{M}_{\ell,n,\delta,\alpha}^{m,\lambda,q}$ , defined as follows:

$$\begin{aligned} \mathcal{M}_{\ell,n,\delta,\alpha}^{m,\lambda,q} f(z) &= z + \sum_{j=2}^{\infty} [1 + (j-1) c^n(\delta)]^m \left(\frac{n+1}{n+k}\right)^\alpha \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_j z^j \\ &(\alpha > 0; \lambda > -1; \ell \geq 0; 0 < q < 1; \delta > 0; n \in \mathbb{N}; m \in \mathbb{N}_0). \end{aligned}$$

## 2. Preliminary

To prove our results, we need the following lemmas.

**Lemma 2.1.** [11] *Let  $\psi \in \mathbb{A}$  and  $\mathcal{G}(z) = \frac{1}{z} \int_0^z \psi(t) dt$ ,  $z \in \Delta$ . If*

$$\Re \left\{ 1 + \frac{z\psi''(z)}{\psi'(z)} \right\} > \frac{-1}{2}, \quad z \in \Delta, \text{ then } \mathcal{G} \in \mathcal{K}.$$

**Lemma 2.2.** [14, Theorem 2.6] *Let  $\psi$  be a convex function with  $\psi(0) = \beta$  and  $\nu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$  with  $\Re(\nu) \geq 0$ . If  $p \in H[\beta, d]$  with  $p(0) = \beta$ ,  $\Phi : \mathbb{C}^2 \times \Delta \rightarrow \mathbb{C}$ ,  $\Phi(p(z), zp'(z); z) = p(z) + \frac{1}{\nu} zp'(z)$  is analytic function in  $\Delta$  and*

$$\mathcal{F}_{\Phi(\mathbb{C}^2 \times \Delta)} \left( p(z) + \frac{1}{\nu} zp'(z) \right) \leq \mathcal{F}_{h(\Delta)} h(z) \rightarrow p(z) + \frac{1}{\nu} zp'(z) \prec_{\mathcal{F}} h(z), \quad z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)} p(z) \leq \mathcal{F}_{q(\Delta)} q(z) \leq \mathcal{F}_{h(\Delta)} h(z) \rightarrow p(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta,$$

where

$$q(z) = \frac{\nu}{dz^{\frac{\nu}{d}}} \int_0^z \psi(t) t^{\frac{\nu}{d}-1} dt, \quad z \in \Delta.$$

The function  $q$  is convex and it is the fuzzy best dominant.

**Lemma 2.3.** [14, Theorem 2.7] *Let  $g$  be a convex function in  $\Delta$  and  $\psi(z) = g(z) + d\gamma z g'(z)$ , where  $z \in \Delta$ ,  $d \in \mathbb{N}$  and  $\gamma > 0$ . If*

$$p(z) = g(0) + p_d z^d + p_{d+1} z^{d+1} + \dots$$

belongs to  $H(\Delta)$ , and

$$\mathcal{F}_{p(\Delta)} \left( p(z) + \gamma z p'(z) \right) \leq \mathcal{F}_{\psi(\Delta)} \psi(z) \quad \rightarrow \quad p(z) + \gamma z p'(z) \prec_{\mathcal{F}} \psi(z), \quad z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)} (p(z)) \leq \mathcal{F}_{g(\Delta)} g(z) \quad \rightarrow \quad p(z) \prec_{\mathcal{F}} g(z), \quad z \in \Delta.$$

This result is sharp.

For the general theory of fuzzy differential subordination and its applications, we refer the reader to [9, 10].

In the next section, we obtain several fuzzy differential subordinations associated with the differential operator  $\mathcal{D}_{n,\delta,g}^m f(z)$  by using the method of fuzzy differential subordination.

### 3. Main results

Assume that  $\eta \in [0, 1)$ ,  $\delta > 0$ ,  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$ ,  $\lambda > 0$  and  $z \in \Delta$  are mentioned through this paper:

By using the integral operator  $\mathcal{D}_{n,\delta,g}^m$ , we define a class of analytic functions and we derive several fuzzy differential subordinations for this class.

**Definition 3.1.** *Let the function  $f \in \mathbb{A}$  belongs to the class  $\mathcal{H}_{n,m,\delta}^F(\eta, g)$  for all  $\eta \in [0, 1)$ ,  $n \in \mathbb{N}_0$ ,  $m > 0$  and  $\alpha \geq 0$  if it satisfies the inequality:*

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} \left( (\mathcal{D}_{n,\delta,g}^m f(z))' \right) > \eta, \quad (z \in \Delta).$$

**Theorem 3.2.** *Let  $k$  belongs to  $\mathcal{C}$  in  $\Delta$  and suppose that  $h(z) = k(z) + \frac{1}{\lambda+2} z k'(z)$ . If  $f \in \mathcal{H}_{n,m,\delta}^F(\eta, g)$  and*

$$G(z) = I^\lambda f(z) = \frac{\lambda+2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt, \tag{3.1}$$

then

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} \left( (\mathcal{D}_{n,\delta,g}^m f(z))' \right) \leq F_{h(\Delta)} h(z) \quad \rightarrow \quad (\mathcal{D}_{n,\delta,g}^m f(z))' \prec_{\mathcal{F}} h(z), \tag{3.2}$$

implies

$$F_{(\mathcal{D}_{n,\delta,g}^m G)'(\Delta)} \left( (\mathcal{D}_{n,\delta,g}^m G(z))' \right) \leq F_{k(\Delta)} k(z) \quad \rightarrow \quad (\mathcal{D}_{n,\delta,g}^m G(z))' \prec_{\mathcal{F}} k(z),$$

and this result is sharp.

*Proof.* Since

$$z^{\lambda+1}G(z) = (\lambda + 2) \int_0^z t^\lambda f(t) dt,$$

by differentiating, we obtain

$$(\lambda + 1)G(z) + zG'(z) = (\lambda + 2)f(z),$$

and,

$$(\lambda + 1)\mathcal{D}_{n,\delta,g}^m G(z) + z(\mathcal{D}_{n,\delta,g}^m G(z))' = (\lambda + 2)\mathcal{D}_{n,\delta,g}^m f(z), \quad (3.3)$$

and also, by differentiating (3.3) we obtain

$$(\mathcal{D}_{n,\delta,g}^m G(z))' + \frac{1}{(\lambda + 2)}z(\mathcal{D}_{n,\delta,g}^m G(z))'' = (\mathcal{D}_{n,\delta,g}^m f(z))' \quad (3.4)$$

By using (3.4), the fuzzy differential subordination (3.2) is

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} \left( (\mathcal{D}_{n,\delta,g}^m G(z))' + \frac{1}{(\lambda + 2)}z(\mathcal{D}_{n,\delta,g}^m G(z))'' \right) \leq F_{h(\Delta)} \left( k(z) + \frac{1}{(\lambda + 2)}zk'(z) \right). \quad (3.5)$$

We denote

$$q(z) = (\mathcal{D}_{n,\delta,g}^m G(z))', \quad \text{so } q \in \mathcal{H}[1, n]. \quad (3.6)$$

Putting (3.6) in (3.5), we have

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} \left( q(z) + \frac{1}{(\lambda + 2)}zq'(z) \right) \leq F_{h(\Delta)} \left( k(z) + \frac{1}{(\lambda + 2)}zk'(z) \right), \quad (3.7)$$

and applying Lemma (2.3), we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z), \quad \text{i.e. } F_{(\mathcal{D}_{n,\delta,g}^m G(z))'(\Delta)}(\mathcal{D}_{n,\delta,g}^m G(z))' \leq F_{k(\Delta)}k(z),$$

therefore  $(\mathcal{D}_{n,\delta,g}^m G(z))' \prec_{\mathcal{F}} k(z)$ , and  $k$  is the fuzzy best dominant.  $\square$

**Theorem 3.3.** Assume that  $h(z) = \frac{1+(2\eta-1)z}{1+z}$ ,  $\eta \in [0, 1)$ ,  $\lambda > 0$  and  $\mathcal{I}^\lambda$  is given by (3.1), then

$$\mathcal{I}^\lambda [\mathcal{H}_{n,m,\delta}^F(\eta, g)] \subset \mathcal{H}_{n,m,\delta}^F(\eta^*, g), \quad (3.8)$$

where

$$\eta^* = 2\eta - 1 + (\lambda + 2)(2 - 2\eta) \int_0^1 \frac{t^{\lambda+2}}{t+1} dt. \quad (3.9)$$

*Proof.* A function  $h$  belongs to  $\mathcal{C}$  and using the same technique in the proof of Theorem 3.2, we obtain from the hypothesis of Theorem 3.3 that

$$F_{q(\Delta)} \left( q(z) + \frac{1}{(\lambda + 2)}zq'(z) \right) \leq F_{h(\Delta)}h(z),$$

where  $q(z)$  is defined in (3.6). By using Lemma 2.2, we obtain

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z) \leq F_{h(\Delta)}h(z),$$

which implies

$$F_{(\mathcal{D}_{n,\delta,g}^m G)'(\Delta)} \left( \mathcal{D}_{n,\delta,g}^m G(z) \right)' \leq F_{k(\Delta)}k(z) \leq F_{h(\Delta)}h(z),$$

where

$$\begin{aligned} k(z) &= \frac{\lambda + 2}{z^{\lambda+2}} \int_0^z t^{\lambda+1} \frac{1 + (2\eta - 1)t}{1 + t} dt \\ &= (2\eta - 1) + \frac{(\lambda + 2)(2 - 2\eta)}{z^{\lambda+2}} \int_0^z \frac{t^{\lambda+1}}{1 + t} dt. \end{aligned}$$

$k$  belongs to  $\mathcal{C}$  and  $k(\Delta)$  is symmetric with respect to the real axis, so we conclude

$$F_{(\mathcal{D}_{n,\delta,g}^m G)'(\Delta)} \left( \mathcal{D}_{n,\delta,g}^m G(z) \right)' \geq \min_{|z|=1} F_{k(\Delta)}k(z) = F_{k(\Delta)}k(1), \quad (3.10)$$

and  $\eta^* = k(1) = 2\eta - 1 + (\lambda + 2) \int_0^1 \frac{t^{\lambda+2}}{t+1} dt$ . □

**Theorem 3.4.** *Let  $k$  belongs to  $\mathcal{C}$  in  $\Delta$ ,  $k(0) = 1$ , and  $h(z) = k(z) + zk'(z)$ . If  $f \in \mathbb{A}$  and satisfies the fuzzy differential subordination*

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} \left( \mathcal{D}_{n,\delta,g}^m f(z) \right)' \leq F_{h(\Delta)}h(z) \quad \rightarrow \quad \left( \mathcal{D}_{n,\delta,g}^m f(z) \right)' \prec_{\mathcal{F}} h(z), \quad (3.11)$$

holds, then

$$F_{\mathcal{D}_{n,\delta,g}^m f(\Delta)} \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \leq F_{k(\Delta)}k(z) \quad \rightarrow \quad \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \prec_{\mathcal{F}} k(z). \quad (3.12)$$

The result is sharp.

*Proof.* For

$$\begin{aligned} q(z) &= \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} [1 + (j - 1) c^n(\delta)]^m a_j b_j z^j}{z} \\ &= 1 + \sum_{j=2}^{\infty} [1 + (j - 1) c^n(\delta)]^m a_j b_j z^{j-1}, \end{aligned}$$

we obtain that  $q(z) + zq'(z) = \left( \mathcal{D}_{n,\delta,g}^m f(z) \right)'$ , so

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} \left( \mathcal{D}_{n,\delta,g}^m f(z) \right)' \leq F_{h(\Delta)}h(z)$$

implies

$$F_{q(\Delta)} \left( q(z) + zq'(z) \right) \leq F_{h(\Delta)}h(z) = F_{k(\Delta)} \left( k(z) + zk'(z) \right).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z) \quad \rightarrow \quad F_{\mathcal{D}_{n,\delta,g}^m f(\Delta)} \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \leq F_{k(\Delta)}k(z),$$

and we get

$$\frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \prec_{\mathcal{F}} k(z).$$

The result is sharp.  $\square$

**Theorem 3.5.** Consider  $h \in \mathcal{H}(\Delta)$  with  $h(0) = 1$ , which satisfies  $\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > \frac{-1}{2}$ . If  $f \in \mathbb{A}$  and the fuzzy differential subordination

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} (\mathcal{D}_{n,\delta,g}^m f(z))' \leq F_{h(\Delta)}h(z) \quad \rightarrow \quad (\mathcal{D}_{n,\delta,g}^m f(z))' \prec_{\mathcal{F}} h(z), \quad (3.13)$$

then

$$F_{\mathcal{D}_{n,\delta,g}^m f(\Delta)} \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \leq F_{k(\Delta)}k(z) \quad \text{i.e.} \quad \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \prec_{\mathcal{F}} k(z), \quad (3.14)$$

where

$$k(z) = \frac{1}{z} \int_0^z h(t) dt,$$

the function  $k$  is convex and it is the fuzzy best dominant.

*Proof.* Let

$$q(z) = \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} = 1 + \sum_{j=2}^{\infty} [1 + (j-1)c^n(\delta)]^m a_j b_j z^{j-1}, \quad q \in \mathcal{H}[1, 1],$$

where  $\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > \frac{-1}{2}$ . From Lemma 2.1, we have

$$k(z) = \frac{1}{z} \int_0^z h(t) dt$$

belongs to the class  $\mathcal{C}$ , which satisfies the fuzzy differential subordination (3.13). Since

$$k(z) + zk'(z) = h(z),$$

it is the fuzzy best dominant.

We have  $q(z) + zq'(z) = (\mathcal{D}_{n,\delta,g}^m f(z))'$ , then (3.13) becomes

$$F_{q(\Delta)} \left( q(z) + zq'(z) \right) \leq F_{h(\Delta)}h(z).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z), \quad \text{i.e.} \quad F_{\mathcal{D}_{n,\delta,g}^m f(\Delta)} \frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \leq F_{k(\Delta)}k(z),$$

then

$$\frac{\mathcal{D}_{n,\delta,g}^m f(z)}{z} \prec_{\mathcal{F}} k(z).$$

□

Putting  $h(z) = \frac{1+(2\beta-1)z}{1+z}$  in Theorem 3.5, we obtain the following corollary:

**Corollary 3.6.** *Let  $h = \frac{1+(2\beta-1)z}{1+z}$  a convex function in  $\Delta$ , with  $h(0) = 1$ ,  $0 \leq \beta < 1$ . If  $f \in \mathbb{A}$  and verifies the fuzzy differential subordination*

$$F_{(\mathcal{D}_{n,\delta,g}^m f)'(\Delta)} (\mathcal{D}_{n,\delta,g}^m f(z))' \leq F_{h(\Delta)} h(z), \quad \text{i.e.} \quad (\mathcal{D}_{n,\delta,g}^m f(z))' \prec_{\mathcal{F}} h(z),$$

then

$$k(z) = 2\beta - 1 + \frac{2(1 - \beta)}{z} \ln(1 + z),$$

the function  $k$  is convex and it is the fuzzy best dominant.

**Concluding**, all the above results give us information about fuzzy differential subordinations for the operator  $\mathcal{D}_{n,\delta,g}^m$ , we give some properties for the class  $\mathcal{H}_{\alpha,m}^F(n, \eta)$  of univalent analytic functions.

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