Fuzzy Differential Subordinations Connected with Convolution

[1]S. M. El-Deeb and [2]Alina Alb Lupas
[1]Department of Mathematics, Faculty of Science,
Damietta University, New Damietta 34517, Egypt
Current address: Department of Mathematics,
College of Science and Arts,
Al-Qassim University, Saudi Arabia
[2]Department of Mathematics and Computer Science,
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
[1] shezaeldeeb@yahoo.com [2]dalb@uoradea.ro

Abstract

The object of the present paper is to obtain several fuzzy differential subordinations associated with Linear operator $\mathcal{D}_{n,\delta,g}^{m}f(z) = z + \sum\limits_{j=2}^{\infty}\left[1+(j-1)\,c^{n}(\delta)\right]^{m} a_j b_j z^j$. Using the operator $\mathcal{D}_{n,\delta,g}^{m}$, we also introduce a class $\mathcal{H}_{n,m,\delta}^{F}(\eta, g)$ of univalent analytic functions for which we give some properties.

Keywords and phrases: Fuzzy differential subordination, fuzzy best dominant, linear differential operator, convolution.
2010 Mathematics Subject Classification: 30C45, 30A20.

1 Introduction

Let $\Omega\subset\mathbb{C}$, $H(\Omega)$ the class of holomorphic functions on Ω and denote by $H_d(\Omega)$ the class of holomorphic and univalent functions on Ω. In this paper, we denote by $H(\Delta)$ the class of holomorphic functions in the open unit disk $\Delta = \{z\in\mathbb{C} : |z| < 1\}$ with $B_\Delta = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disk. For $\beta\in\mathbb{C}$ and $d\in\mathbb{N}$, we denote

$$H[\beta, d] = \left\{ f\in H(\Delta) : \ f(z) = \beta + \sum_{j=d+1}^{\infty} a_j z^j, \quad z\in\Delta \right\},$$

$$\mathbb{A}_d = \left\{ f\in H(\Delta) : \ f(z) = z + \sum_{j=d+1}^{\infty} a_j z^j, \quad z\in\Delta \right\} \quad \text{with} \quad \mathbb{A}_1 = \mathbb{A},$$

and,

$$\mathcal{S} = \{ f \in \mathbb{A} : \ f \text{ is a univalent function in } \Delta \}.$$

We denote by

$$\mathcal{C} = \left\{ f \in \mathbb{A} : \ \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0, \ z \in \Delta \right\},$$

is convex functions in Δ.

Definition 1.1 *[2, 9] Let f_1 and f_2 are analytic function in Δ, then f_1 is subordinate to f_2, written $f_1 \prec f_2$ if there exists a Schwarz function* w, *which is analytic in Δ with $\mathrm{w}(0) = 0$ and $|\mathrm{w}(z)| < 1$ for all $z \in \Delta$, such that $f_1(z) = f_2(\mathrm{w}(z))$. Furthermore, if the function f_2 is univalent in Δ, then we have the following equivalence:*

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \ \textit{and} \ f_1(\Delta) \subset f_2(\Delta).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

Definition 1.2 *[6] Assume that the set $\mathcal{Y} \neq \varnothing$. Application $\mathcal{F} : \mathcal{Y} \to [0,1]$ is fuzzy subset. A pair $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$, where $\mathcal{F}_{\mathcal{B}} : \mathcal{Y} \to [0,1]$ and*

$$A = \{ x \in \mathcal{Y} : 0 < \mathcal{F}_{\mathcal{B}}(x) \leq 1 \} = \sup (\mathcal{B}, \mathcal{F}_{\mathcal{B}}), \tag{1.1}$$

is said fuzzy subset. A function $\mathcal{F}_{\mathcal{B}}$ is said to be the fuzzy set $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$.

Proposition 1.1 *[10] (i) If $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) = (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, then we have $\mathcal{B} = \mathcal{U}$, where $\mathcal{B} = \sup (\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ and $\mathcal{U} = \sup (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$;*

(ii) If $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) \subseteq (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, then we have $\mathcal{B} \subseteq \mathcal{U}$, where $\mathcal{B} = \sup (\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ and $\mathcal{U} = \sup (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$.

Let $f, g \in H(\Omega)$, we denote by

$$f(\Omega) = \left\{ f(z) : \ 0 < \mathcal{F}_{f(\Omega)} f(z) \leq 1, \ z \in \Omega \right\} = \sup \left(f(\Omega), \mathcal{F}_{f(\Omega)} \right), \tag{1.2}$$

and,

$$g(\Omega) = \left\{ g(z) : \ 0 < \mathcal{F}_{g(\Omega)} g(z) \leq 1, \ z \in \Omega \right\} = \sup \left(g(\Omega), \mathcal{F}_{g(\Omega)} \right). \tag{1.3}$$

Definition 1.3 *[10] Let $z_0 \in \Omega$ be a fixed point and let the functions $f, g \in H(\Omega)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, which is satisfied the following conditions:*

(i) $f(z_0) = g(z_0)$

(ii) $\mathcal{F}_{f(\Omega)} f(z) \leq \mathcal{F}_{\mathrm{g}(\Omega)} g(z), \quad z \in \Omega.$

Proposition 1.2 *[10] Assume that $z_0 \in \Omega$ is a fixed point and the functions $f, \mathrm{g} \in H(\Omega)$. If $f(z) \prec_{\mathcal{F}} g(z)$, $z \in \Omega$, then*

(i) $f(z_0) = g(z_0)$

(ii) $f(\Omega) \subseteq g(\Omega), \quad \mathcal{F}_{f(\Omega)} f(z) \leq \mathcal{F}_{\mathrm{g}(\Omega)} g(z), \ z \in \Omega,$

where $f(\Omega)$ and $\mathrm{g}(\Omega)$ are defined by (1.2) and (1.3), respectively.

Definition 1.4 *[11] Assume that $\Phi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ and $h \in \mathcal{S}$, with $\Phi(\alpha, 0, 0; 0) = h(0) = \alpha$. If p is analytic in Δ, with $p(0) = \alpha$ and satisfies the second order fuzzy differential subordination*

$$\mathcal{F}_{\Phi(\mathbb{C}^3\times\Delta)}\Phi\left(p(z), zp^{'}(z), z^2p^{''}(z); z\right) \leq \mathcal{F}_{h(\Delta)}h(z),$$

$$i.e. \quad \Phi\left(p(z), zp^{'}(z), z^2p^{''}(z); z\right) \prec_{\mathcal{F}} h(z), \quad z \in \Delta. \tag{1.4}$$

Then p is said to be a fuzzy solution of the fuzzy differential subordination, the univalent function q is called a fuzzy dominant of the fuzzy solutions for the fuzzy differential subordination if

$$\mathcal{F}_{p(\Delta)}p(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad i.e. \quad p(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all p satisfying (1.4).

A fuzzy dominant $\widetilde{q}$ that satisfies

$$\mathcal{F}_{\widetilde{q}(\Delta)}\widetilde{q}(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad i.e. \quad \widetilde{q}(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all fuzzy dominants q of (1.4) is called the fuzzy best dominant of (1.4).

Making use the binomial series

$$(1-\delta)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \ \delta^i \qquad (n \in \mathbb{N} = \{1, 2, ...\})$$

For $f \in \mathbb{A}$, we introduced the linear differential operator as follows:

$$\mathcal{D}^0_{n,\delta,g}f(z) = (f * g)(z),$$

$$\begin{aligned}
\mathcal{D}^1_{n,\delta,g}f(z) &= \mathcal{D}_{n,\delta,g}f(z) = (1-\delta)^n (f*g)(z) + [1-(1-\delta)^n]\, z\,(f*g)^{'}(z) \\
&= z + \sum_{j=2}^{\infty} [1 + (j-1)\, c^n(\delta)]\, a_j b_j z^j \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
\mathcal{D}^m_{n,\delta,g}f(z) &= \mathcal{D}_{n,\delta,g}\left(\mathcal{D}^{m-1}_{n,\delta,g}f(z)\right) \\
&= (1-\delta)^n\, \mathcal{D}^{m-1}_{n,\delta,g}f(z) + [1-(1-\delta)^n]\, z \left(\mathcal{D}^{m-1}_{n,\delta,g}f(z)\right)^{'} \\
&= z + \sum_{j=2}^{\infty} [1 + (j-1)\, c^n(\delta)]^m\, a_j b_j z^j \qquad (1.5) \\
&\quad\ (\delta > 0,\ n \in \mathbb{N},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),
\end{aligned}$$

where

$$c^n(\delta) = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \ \delta^i \qquad (n \in \mathbb{N}).$$

From (1.5), we obtain that

$$c^n(\delta)\ z\ \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'} = \mathcal{D}^{m+1}_{n,\delta,g}f(z) - \left[1-c^n(\delta)\right]\mathcal{D}^m_{n,\delta,g}f(z).$$

By specializing the parameters $n,\ \delta$ and b_j, we note that
(i) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$), then $\mathcal{D}^m_{n,\delta,\frac{z}{1-z}} = \mathcal{D}^m_{n,\delta}$ defined by Yousef et al.[14];
(ii) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$) and $n = 1$, then $\mathcal{D}^m_{1,\delta,\frac{z}{1-z}} = \mathcal{D}^m_{\delta}$ defined by Al-Oboudi [1];
(iii) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$) and $n = \delta = 1$, then $\mathcal{D}^m_{1,1,\frac{z}{1-z}} = \mathcal{D}^m$ defined by Salagean.[13];
(iv) Putting $b_j = \left(\frac{\ell+1}{\ell+j}\right)^{\alpha}$ $(\alpha > 0,\ \ell > -1)$ and $n = 1$, then $\mathcal{D}^m_{1,\delta,g} = \mathcal{I}^{m,\alpha}_{\ell,\delta}f(z)$ defind by El-Deeb and Oros [5];
(v) Putting $b_j = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$, $(\upsilon > 0,\ \lambda > -1,\ 0 < q < 1)$ studied by El-Deeb and Bulboacă [3], we obtain the operator $\mathcal{N}^{m,\lambda,q}_{\upsilon,n,\delta}$, defined as follows:

$$\begin{aligned}\mathcal{N}^{m,\lambda,q}_{\upsilon,n,\delta}f(z) &= z + \sum_{j=2}^{\infty}\left[1+(j-1)\,c^n(\delta)\right]^m \frac{(-1)^{j-1}\Gamma(\upsilon+1)}{4^{j-1}(j-1)!\Gamma(j+\upsilon)}a_j z^j \\ &\quad (\lambda > -1;\ 0 < q < 1;\ \delta, \upsilon > 0;\ n \in \mathbb{N};\ m \in \mathbb{N}_0)\,;\end{aligned}$$

(vi) Putting $b_j = \left(\frac{\ell+1}{\ell+j}\right)^{\alpha} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$, $(\alpha > 0,\ n \geq 0,\ \lambda > -1,\ 0 < q < 1)$ studied by El-Deeb and Bulboacă [4], we obtain the operator $\mathcal{M}^{m,\lambda,q}_{\ell,n,\delta,\alpha}$, defined as follows:

$$\begin{aligned}\mathcal{M}^{m,\lambda,q}_{\ell,n,\delta,\alpha}f(z) &= z + \sum_{j=2}^{\infty}\left[1+(j-1)\,c^n(\delta)\right]^m \left(\frac{n+1}{n+k}\right)^{\alpha} \frac{[k,q]!}{[\lambda+1,q]_{k-1}}a_j z^j \\ &\quad (\alpha > 0;\ \lambda > -1;\ \ell \geq 0;\ 0 < q < 1;\ \delta > 0;\ n \in \mathbb{N};\ m \in \mathbb{N}_0)\,.\end{aligned}$$

2 Preliminary

To prove our results, we need the following lemmas.

Lemma 2.1 *[9] Let $\psi \in \mathbb{A}$ and $\mathcal{G}(z) = \frac{1}{z}\int\limits_0^z \psi(t)dt,\ z \in \Delta$. If $\Re\left\{1 + \frac{z\psi^{''}(z)}{\psi^{'}(z)}\right\} > \frac{-1}{2},\ z \in \Delta$, then $\mathcal{G} \in \mathcal{K}$.*

Lemma 2.2 *[12, Theorem 2.6] Let ψ be a convex function with $\psi(0) = \beta$ and $\nu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with $\Re(\nu) \geq 0$. If $p \in H[\beta, d]$ with $p(0) = \beta,\ \Phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$, $\Phi\left(p(z), zp^{'}(z); z\right) = p(z) + \frac{1}{\nu}zp^{'}(z)$ is analytic function in Δ and*

$$\mathcal{F}_{\Phi(\mathbb{C}^2\times\Delta)}\left(p(z) + \frac{1}{\nu}zp^{'}(z)\right) \leq \mathcal{F}_{h(\Delta)}h(z) \quad \to p(z) + \frac{1}{\nu}zp^{'}(z) \prec_{\mathcal{F}} h(z), \quad z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)}p(z) \leq \mathcal{F}_{q(\Delta)}q(z) \leq \mathcal{F}_{h(\Delta)}h(z) \ \rightarrow \ p(z) \prec_{\mathcal{F}} q(z), \ z \in \Delta,$$

where

$$q(z) = \frac{\nu}{dz^{\frac{\nu}{d}}} \int_0^z \psi(t) t^{\frac{\nu}{d}-1} dt, \ z \in \Delta.$$

The function q is convex and is the fuzzy best dominant.

Lemma 2.3 *[12, Theorem 2.7] For g be a convex function in Δ and $\psi(z) = g(z) + d\gamma z g^{'}(z)$, where $z \in \Delta$, $d \in \mathbb{N}$ and $\gamma > 0$. If*

$$p(z) = g(0) + p_d \ z^d + p_{d+1} \ z^{d+1} + ...$$

belongs to in $H(\Delta)$, and

$$\mathcal{F}_{p(\Delta)} \left(p(z) + \gamma z p^{'}(z) \right) \leq \mathcal{F}_{\psi(\Delta)} \psi(z) \quad \rightarrow \quad p(z) + \gamma z p^{'}(z) \prec_{\mathcal{F}} \psi(z), \ z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)} \left(p(z) \right) \leq \mathcal{F}_{g(\Delta)} g(z) \quad \rightarrow \quad p(z) \prec_{\mathcal{F}} g(z), \quad z \in \Delta.$$

This result is sharp.

For the general theory of fuzzy differential subordination and its applications, we refer the reader to [7, 8].

In the next section is to obtain several fuzzy differential subordinations associated with the diferential operator $\mathcal{D}^m_{n,\delta,g} f(z)$ by using the method of fuzzy differential subordination.

3 Main results

Assume that $\eta \in [0, 1)$, $\delta > 0$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\lambda > 0$ and $z \in \Delta$ are mentioned through this paper:

By using the integral operator $\mathcal{D}^m_{n,\delta,g}$, we define a class of analytic functions and we derive several fuzzy differential subordinations for this class.

Definition 3.1 *Let the function $f \in \mathbb{A}$ belongs to the class $\mathcal{H}^F_{n,m,\delta}(\eta, g)$ for all $\eta \in [0, 1)$, $n \in \mathbb{N}_0$, $m > 0$ and $\alpha \geq 0$ if it satisfies the inequality:*

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)^{'}(\Delta)} \left(\mathcal{D}^m_{n,\delta,g} f(z) \right)^{'} > \eta, \qquad (z \in \Delta).$$

Theorem 3.1 *Let k belongs to $\mathcal{C}$ in Δ and suppose that $h(z) = k(z) + \frac{1}{\lambda+2} z k^{'}(z)$. If $f \in \mathcal{H}^F_{n,m,\delta}(\eta, g)$ and*

$$G(z) = I^\lambda f(z) = \frac{\lambda + 2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt, \tag{3.1}$$

then

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \leq F_{h(\Delta)}h(z) \quad \rightarrow \quad \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \prec_{\mathcal{F}} h(z), \tag{3.2}$$

implies

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \leq F_{k(\Delta)}k(z) \quad \rightarrow \quad \left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \prec_{\mathcal{F}} k(z),$$

and this result is sharp.

Proof. Since

$$z^{\lambda+1}G(z) = (\lambda+2)\int_0^z t^{\lambda}f(t)dt,$$

by differentiating, we obtain

$$(\lambda+1)\,G(z) + zG'(z) = (\lambda+2)\,f(z),$$

and,

$$(\lambda+1)\,\mathcal{D}^m_{n,\delta,g}G(z) + z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' = (\lambda+2)\,\mathcal{D}^m_{n,\delta,g}f(z), \tag{3.3}$$

and also, by differentiating (3.3) we obtain

$$\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' + \frac{1}{(\lambda+2)}z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)'' = \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \tag{3.4}$$

By using (3.4), the fuzzy differential subordination (3.2) is

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' + \frac{1}{(\lambda+2)}z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)''\right) \leq$$

$$F_{h(\Delta)}\left(k(z) + \frac{1}{(\lambda+2)}zk'(z)\right). \tag{3.5}$$

We denote

$$q(z) = \left(\mathcal{D}^m_{n,\delta,g}G(z)\right)', \quad \text{so} \quad q \in \mathcal{H}\,[1,n]\,. \tag{3.6}$$

Putting (3.6) in (3.5), we have

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(q(z) + \frac{1}{(\lambda+2)}zq'(z)\right) \leq F_{h(\Delta)}\left(k(z) + \frac{1}{(\lambda+2)}zk'(z)\right), \tag{3.7}$$

and applying Lemma (2.3), we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z), \quad \text{i.e} \quad F_{\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \leq F_{k(\Delta)}k(z),$$

therefore $\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \prec_{\mathcal{F}} k(z)$, and k be the fuzzy best dominant. ■

Theorem 3.2 *Assume that* $h(z)=\frac{1+(2\eta-1)z}{1+z}$, $\eta\in[0,1)$, $\lambda>0$ *and* $\mathcal{I}^{\lambda}$ *is given by (3.1), then*

$$\mathcal{I}^{\lambda}\left[\mathcal{H}^{F}_{n,m,\delta}(\eta,g)\right]\subset\mathcal{H}^{F}_{n,m,\delta}(\eta^{*},g), \tag{3.8}$$

where

$$\eta^{*}=2\eta-1+(\lambda+2)(2-2\eta)\int_{0}^{1}\frac{t^{\lambda+2}}{t+1}dt. \tag{3.9}$$

Proof. A function h belongs to $\mathcal{C}$ and using the same technique in the proof of Theorem 3.1, we obtain from the hypothesis of Theorem 3.2 that

$$F_{q(\Delta)}\left(q(z)+\frac{1}{(\lambda+2)}zq^{'}(z)\right)\leq F_{h(\Delta)}h(z),$$

where $q(z)$ is defined in (3.6). By using Lemma 2.2, we obtain

$$F_{q(\Delta)}q(z)\leq F_{k(\Delta)}k(z)\leq F_{h(\Delta)}h(z),$$

which implies

$$F_{\left(\mathcal{D}^{m}_{n,\delta,g}G\right)^{'}(\Delta)}\left(\mathcal{D}^{m}_{n,\delta,g}G(z)\right)^{'}\leq F_{k(\Delta)}k(z)\leq F_{h(\Delta)}h(z),$$

where

$$\begin{aligned} k(z) &= \frac{\lambda+2}{z^{\lambda+2}}\int_{0}^{z}t^{\lambda+1}\frac{1+(2\eta-1)t}{1+t}dt \\ &= (2\eta-1)+\frac{(\lambda+2)(2-2\eta)}{z^{\lambda+2}}\int_{0}^{z}\frac{t^{\lambda+1}}{1+t}dt. \end{aligned}$$

k belongs to $\mathcal{C}$ and $k(\Delta)$ is symmetric with respect to the real axis, so we conclude

$$F_{\left(\mathcal{D}^{m}_{n,\delta,g}G\right)^{'}(\Delta)}\left(\mathcal{D}^{m}_{n,\delta,g}G(z)\right)^{'}\geq\min_{|z|=1}F_{k(\Delta)}k(z)=F_{k(\Delta)}k(1), \tag{3.10}$$

and $\eta^{*}=k(1)=2\eta-1+(\lambda+2)(2-2\eta)\int_{0}^{1}\frac{t^{\lambda+2}}{t+1}dt$. ■

Theorem 3.3 *Let* k *belongs to* $\mathcal{C}$ *in* Δ, $k(0)=1$, *and* $h(z)=k(z)+zk^{'}(z)$. *If* $f\in\mathbb{A}$ *and satisfies the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^{m}_{n,\delta,g}f\right)^{'}(\Delta)}\left(\mathcal{D}^{m}_{n,\delta,g}f(z)\right)^{'}\leq F_{h(\Delta)}h(z)\quad\rightarrow\quad\left(\mathcal{D}^{m}_{n,\delta,g}f(z)\right)^{'}\prec_{\mathcal{F}}h(z), \tag{3.11}$$

then

$$F_{\mathcal{D}^{m}_{n,\delta,g}f(\Delta)}\frac{\mathcal{D}^{m}_{n,\delta,g}f(z)}{z}\leq F_{k(\Delta)}k(z)\quad\rightarrow\quad\frac{\mathcal{D}^{m}_{n,\delta,g}f(z)}{z}\prec_{\mathcal{F}}k(z). \tag{3.12}$$

The result is sharp.

Proof. For

$$q(z) = \frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} [1 + (j-1) c^n(\delta)]^m a_j b_j z^j}{z}$$
$$= 1 + \sum_{j=2}^{\infty} [1 + (j-1) c^n(\delta)]^m a_j b_j z^{j-1},$$

and we obtain that $q(z) + zq^{'}(z) = \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'}$, so

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)^{'}(\Delta)} \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \leq F_{h(\Delta)} h(z)$$

implies

$$F_{q(\Delta)} \left(q(z) + zq^{'}(z)\right) \leq F_{h(\Delta)} h(z) = F_{k(\Delta)} \left(k(z) + zk^{'}(z)\right).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)} q(z) \leq F_{k(\Delta)} k(z) \quad \rightarrow \quad F_{\mathcal{D}^m_{n,\delta,g} f(\Delta)} \frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \leq F_{k(\Delta)} k(z),$$

and we get

$$\frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \prec_{\mathcal{F}} k(z).$$

The result is sharp. ■

Theorem 3.4 *Consider $h \in \mathcal{H}(\Delta)$ with $h(0) = 1$, which satisfies $\Re\left(1 + \frac{zh^{''}(z)}{h^{'}(z)}\right) > \frac{-1}{2}$. If $f \in \mathbb{A}$ and the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)^{'}(\Delta)} \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \leq F_{h(\Delta)} h(z) \quad \rightarrow \quad \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \prec_{\mathcal{F}} h(z), \tag{3.13}$$

then

$$F_{\mathcal{D}^m_{n,\delta,g} f(\Delta)} \frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \leq F_{k(\Delta)} k(z) \quad i.e \quad \frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \prec_{\mathcal{F}} k(z), \tag{3.14}$$

where

$$k(z) = \frac{1}{z} \int_0^z h(t) dt,$$

the function k is convex and it is the fuzzy best dominant.

Proof. Let

$$q(z) = \frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} = 1 + \sum_{j=2}^{\infty} [1 + (j-1) c^n(\delta)]^m a_j b_j z^{j-1}, \quad q \in \mathcal{H}[1,1],$$

where $\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}$. From Lemma 2.1, we have

$$k(z) = \frac{1}{z}\int_0^z h(t)dt$$

belongs to the class $\mathcal{C}$, which satisfies the fuzzy differential subordination (3.13). Since

$$k(z) + zk'(z) = h(z),$$

the fuzzy best dominant.

We have $q(z) + zq'(z) = \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)'$, then (3.13) becomes

$$F_{q(\Delta)}\left(q(z) + zq'(z)\right) \leq F_{h(\Delta)}h(z).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z), \quad \text{i.e.} \quad F_{\mathcal{D}^m_{n,\delta,g}f(\Delta)}\frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \leq F_{k(\Delta)}k(z),$$

then

$$\frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \prec_{\mathcal{F}} k(z).$$

■

Putting $h(z) = \frac{1+(2\beta-1)z}{1+z}$ in Theorem 3.4, we obtain the following corollary:

Corollary 3.5 *Let* $h = \frac{1+(2\beta-1)z}{1+z}$ *a convex function in* Δ, *with* $h(0) = 1$, $0 \leq \beta < 1$. *If* $f \in \mathbb{A}$ *and verifies the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \leq F_{h(\Delta)}h(z), \quad i.e \quad \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \prec_{\mathcal{F}} h(z),$$

then

$$k(z) = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z),$$

the function k *is convex and it is the fuzzy best dominant.*

Concluding, all the above results give us information about fuzzy differential subordinations for the operator $\mathcal{D}^m_{n,\delta,g}$, we give some properties for the class $\mathcal{H}^F_{\alpha,m}(n,\eta)$ of univalent analytic functions.

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Internat. J. Math. Math. Sci., 27 (2004), 1429-1436.

[2] T. Bulboaca, *Differential Subordinations and Superordinations*, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[3] S. M. El-Deeb and T. Bulboacă, Fekete-Szegő inequalities for certain class of analytic functions connected with q-anlogue of Bessel function, J. Egyptian Math. Soc., (2019), 1-11, https://doi.org/10.1186/s42787-019-0049-2.

[4] S. M. El-Deeb and T. Bulboacă, Differential sandwich-type results for symmetric functions connected with a q-analog integral operator, Mathematics, 7(2019), no. 12, 1-17, https://doi.org/10.3390/math7121185.

[5] S. M. El-Deeb and G. Oros, Fuzzy Differential Subordinations Connected with the Linear Operator, Mathematica Bohemica,in press.

[6] S. Gh. Gal and A. I. Ban, *Elemente de matematica fuzzy*, Editura Univ. din Oradea, 1996.

[7] A. Alb Lupas, *On special fuzzy differerential subordinations using convolution product of Salagean operator and Ruscheweyh derivative*, J. Comput. Anal. Appl., 15 (2013), no. 8, 1-6.

[8] A. Alb Lupas and Gh. Oros, *On special fuzzy differerential subordinations using Salagean and Ruscheweyh operators*, Appl. Math. Comput., 261(2015), 119-127.

[9] S. S. Miller and P. T. Mocanu, *Differential Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[10] G. I. Oros and Gh. Oros, *The notation of subordination in fuzzy sets theory*, General Mathematics, 19 (2011), no. 4,97-103.

[11] G. I. Oros and Gh. Oros, *Fuzzy differential subordination*, Acta Univ. Apulensis, 30 (2012), 55-64.

[12] G. I. Oros and Gh. Oros, *Dominant and best dominant for fuzzy differential subordinations*, Stud. Univ. Babes-Bolyai Math., 57(2012), no. 2, 239-248.

[13] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Math., 1013, Springer Verlage, Berlin, (1983), 362-372.

[14] F. Yousef, T. Al-Hawary and G. Murugusundaramoorthy