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Multiplicative perturbations of local C-cosine functions

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Abstract. We establish some left and right multiplicative perturbations of a local C-cosine function $C(\cdot)$ on a complex Banach space X with non-densely defined generator, which can be applied to obtain some new additive perturbation results concerning $C(\cdot)$.

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1. Introduction

Let X be a Banach space over $\mathbb{F}$ (=$\mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$, and let $L(X)$ denote the set of all bounded linear operators on X. For each $0 < T_0 \leq \infty$ and each injection $C \in L(X)$, a family $C(\cdot)(= \{C(t)\,|\,0 \leq t < T_0\})$ in $L(X)$ is called a local C-cosine function on X if it is strongly continuous, $C(0) = C$ on X and satisfies

(1.1) $2C(t)C(s) = C(t+s)C + C(|t-s|)C$ on X for all $0 \leq t, s, t+s < T_0$

(see [7], [10], [14], [20], [22], [24], [26]). In this case, the generator of $C(\cdot)$ is a linear operator A in X defined by

$$D(A) = \{x \in X \mid \lim_{h\to 0^+} 2(C(h)x - Cx)/h^2 \in R(C)\}$$

and $Ax = C^{-1}\lim_{h\to 0^+} 2(C(h)x - Cx)/h^2$ for $x \in D(A)$. Moreover, we say that $C(\cdot)$ is

(1.2) locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|C(t+h) - C(t)\| \leq K_{t_0}h$ for all $0 \leq t, h, t+h \leq t_0$;

(1.3) exponentially bounded, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|C(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$;

(1.4) exponentially Lipschitz continuous, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|C(t+h) - C(t)\| \leq Khe^{\omega(t+h)}$ for all $t, h \geq 0$.

In general, a local C-cosine function is also called a C-cosine function if $T_0 = \infty$ (see [17], [6], [4], [13]), a C-cosine function may not be exponentially bounded (see [13]), and the generator of a local C-cosine function may not be densely defined (see [17], [6]). Moreover, a local C-cosine function is not necessarily extendable to the half line $[0, \infty)$ (see [22]) except for $C = I$ (identity operator on X). Perturbations of local C-cosine functions with or without the exponential boundedness have been extensively studied by many authors appearing in [2,6,8-17,19,23,25]. Some interesting applications of this topic are also illustrated there. In particular, Li has obtained some right-multiplicative perturbation theorems for local C-cosine functions in which the operator C may not commute with the bounded perturbation operator B on X, which satisfies an estimation that is similar to the condition (2.6) below. In this case, $C^{-1}A(I+B)C$ generates a local C-cosine function on X when $CA(I+B) \subset A(I+B)C$ (see [18]). Along this line, Li and Liu also establish some left-multiplicative perturbation theorems for local C-cosine functions on X with densely defined generators. In this case, $(I+B)A$ generates a local C-cosine function on X when $C^{-1}(I+B)AC = (I+B)A$ (see [20]). Just as continuous work of this topic, Kuo shows that $A + B$ generates a local C-cosine function on X when either B is a bounded linear operator from $[D(A)]$ into $R(C)$ such that $R(C^{-1}B) \subset D(A)$ (see [14]) or B is a bounded linear operator on X which commutes with $C(\cdot)$ on X (see [15] or Theorem 2.13 below). The purpose of this paper is to establish some left and right multiplicative perturbation theorems for local C-cosine functions just as results in [18,20] when the generator A of a perturbed local C-cosine function $C(\cdot)$ may not be densely defined, the perturbation operator B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$, and the assumption of $C^{-1}(I+B)AC = (I+B)A$ is not necessary, which together with Theorem 2.13 can be applied to obtain some new Miyadera type additive perturbation theorems just as results in [15] for local C-cosine functions (see Theorems 2.14 and 2.16 below). An illustrative example concerning these results is also presented in the final part of this paper.

2. Perturbation theorems

In this section, we first note some basic properties of a local C-cosine function and known results about connections between the generator of a local C-cosine function and strong solutions of the following abstract Cauchy problem:

$$\mathrm{ACP}(A, f, x, y) \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0) \\ u(0) = x, u'(0) = y, \end{cases}$$

where $x, y \in X$ and f is an X-valued function defined on a subset of $[0, T_0)$.

Proposition 2.1. *(see* [4], [11], [13], [22]*). Let A be the generator of a local C-cosine function $C(\cdot)$ on X. Then*

(2.1) *A is closed and $C^{-1}AC = A$;*
(2.2) *$C(t)x \in D(A)$ and $C(t)Ax = AC(t)x$ for all $x \in D(A)$ and $0 \le t < T_0$;*

(2.3) $\int_0^t\int_0^s C(r)xdrds \in D(A)$ *and* $A\int_0^t\int_0^s C(r)xdrds = C(t)x - Cx$ *for all* $x \in D(A)$ *and* $0 \le t < T_0$;

(2.4) $D(A) = \{x \in X | C(t)x - Cx = \int_0^t\int_0^s C(r)y_x drds$ *for all* $0 \le t < T_0$ *and for some* $y_x \in X\}$ *and* $Ax = y_x$ *for each* $x \in D(A)$;

(2.5) $R(C(t)) \subset \overline{D(A)}$ *for* $0 \le t < T_0$.

Definition 2.2. Let $A : D(A) \subset X \to X$ be a closed linear operator in a Banach space X with domain $D(A)$ and range $R(A)$. A function $u : [0, T_0) \to X$ is called a (strong) solution of ACP(A, f, x, y), if $u \in C^2((0, T_0), X) \cap C^1([0, T_0), X) \cap C((0, T_0), [D(A)])$ and satisfies ACP(A, f, x, y). Here $[D(A)]$ denotes the Banach space $D(A)$ with norm $|\cdot|$ defined by $|x| = \|x\| + \|Ax\|$ for $x \in D(A)$.

Theorem 2.3. (see [11], [13]) *A generates a local C-cosine function $C(\cdot)$ on X if and only if $C^{-1}AC = A$ and for each $x \in X$, ACP$(A, Cx, 0, 0)$ has a unique (strong) solution $u(\cdot, x)$ in $C^2([0, T_0), X)$. In this case, we have*

$$u(t, x) = j_1 * C(t)x \left(= \int_0^t j_1(t-s)C(s)xds \right)$$

for all $x \in X$ and $0 \le t < T_0$. Here $j_k(t) = t^k/k!$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$.

Proposition 2.4. *(see [11], [13]) Let A be the generator of a local C-cosine function $C(\cdot)$ on X, $x, y \in X$ and $f \in L^1_{loc}([0, T_0), X) \cap C((0, T_0), X)$. Then ACP$(A, Cf, Cx, Cy)$ has a (strong) solution u in $C^2([0, T_0), X)$ if and only if*

$$v(\cdot) = C(\cdot)x + S(\cdot)y + S * f(\cdot) \in C^2([0, T_0), X).$$

In this case, $u = v$ on $[0, T_0)$. Here $S(\cdot) = j_0 * C(\cdot)$ and $S * f(\cdot) = \int_0^{\cdot} S(\cdot - s)f(s)ds$.

We next establish a new right-multiplicative perturbation theorem for locally Lipschitz continuous and exponentially Lipschitz continuous local C-cosine functions in which B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$.

Theorem 2.5. *Let $C(\cdot)$ be a locally Lipschitz continuous local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that $(S * C^{-1}Bf)(t) \in D(A)$ and*

$$\|A(S * C^{-1}B)[f(t) - f(s)]\| \le M_{t_0} \int_s^t \|f(r)\|dr \tag{2.6}$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le s < t \le t_0$. Then $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on X satisfying

$$T(\cdot)x = C(\cdot)x + A(S * C^{-1}BT)(\cdot)x \quad on\ [0, T_0) \tag{2.7}$$

for all $x \in X$.

Proof. Let $x \in X$ and $0 < t_0 < T_0$ be fixed.
We define $U : C([0,t_0],\overline{D(A)}) \to C([0,t_0],\overline{D(A)})$ by

$$U(f)(\cdot) = C(\cdot)x + A(S * C^{-1}Bf)(\cdot)$$

on $[0,t_0]$ for all $f \in C([0,t_0],\overline{D(A)})$. Then U is well-defined. By induction, we obtain from (2.6) that

$$\begin{aligned} \|U^n f(t) - U^n g(t)\| &= \|U(U^{n-1}f)(t) - U(U^{n-1}g)(t)\| \\ &= \|AS * C^{-1}B(U^{n-1}f - U^{n-1}g)(t)\| \\ &\le M_{t_0}^n \int_0^t j_{n-1}(t-s)\|f(s)-g(s)\|ds \\ &\le M_{t_0}^n j_n(t_0)\|f-g\| \end{aligned}$$

for all $f,g \in C([0,t_0],\overline{D(A)})$, $0 \le t \le t_0$ and $n \in \mathbb{N}$. Here

$$\|f-g\| = \max_{0\le s\le t_0} \|f(s)-g(s)\|.$$

It follows from the contraction mapping theorem that there exists a unique function w_{x,t_0} in $C([0,t_0],\overline{D(A)})$ such that

$$w_{x,t_0}(\cdot) = C(\cdot)x + AS * C^{-1}Bw_{x,t_0}(\cdot)$$

on $[0,t_0]$. In this case, we set $w_x(t) = w_{x,t_0}(t)$ for all $0 \le t \le t_0 < T_0$, then $w_x(\cdot)$ is a unique function in $C([0,T_0),\overline{D(A)})$ such that

$$w_x(\cdot) = C(\cdot)x + AS * C^{-1}Bw_x(\cdot)$$

on $[0,T_0)$. Since

$$\begin{aligned} j_1 * w_x(\cdot) &= j_1 * C(\cdot)x + Aj_1 * S * C^{-1}Bw_x(\cdot) \\ &= j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) - Bj_1 * w_x(\cdot) \end{aligned}$$

on $[0,T_0)$, we have

$$(I+B)j_1 * w_x(t) = j_0 * S(t)x + S * C^{-1}Bw_x(t) \in D(A)$$

for all $0 \le t < T_0$. Clearly, $j_1 * w_x$ is the unique function u_x in $C^2([0,T_0),X)$ such that

$$u_x(\cdot) = j_0 * S(\cdot)x + AS * C^{-1}Bu_x(\cdot)$$

on $[0,T_0)$. Since $j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) \in C^2([0,T_0),X)$, we obtain from Proposition 2.4 that

$$j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) = (I+B)j_1 * w_x(\cdot)$$

is the unique solution of ACP$(A, Cx + Bw_x, 0, 0)$ in $C^2([0,T_0),X)$. This implies that

$$A(I+B)j_1 * w_x + Cx + Bw_x = (I+B)w_x$$

on $[0,T_0)$, and so $A(I+B)j_1 * w_x + Cx = w_x$ on $[0,T_0)$. Hence, $j_1 * w_x$ is a solution of ACP$(A(I+B), Cx, 0, 0)$ in $C^2([0,T_0),X)$. To prove the uniqueness of solutions of

ACP$(A(I+B), Cx, 0, 0)$.
Suppose that $u \in C([0,T_0), X)$ and satisfies $A(I+B)j_1 * u + Cx = u$ on $[0,T_0)$. Then

$$\begin{aligned} j_1 * (S * u - S * j_0 Cx) =& j_1 * S * A(I+B)j_1 * u \\ =& Aj_1 * S * (I+B)j_1 * u \\ =& S * (I+B)j_1 * u - Cj_1 * (I+B)j_1 * u \\ =& S * j_1 * u + S * Bj_1 * u - Cj_1 * (I+B)j_1 * u \end{aligned}$$

on $[0,T_0)$, and so $-S * j_2(\cdot)Cx = S * Bj_1 * u(\cdot) - Cj_1 * (I+B)j_1 * u(\cdot)$ on $[0,T_0)$. Hence,

$$\begin{aligned} -S * j_0(\cdot)x =& (S * C^{-1}Bj_1 * u)''(\cdot) - (I+B)j_1 * u(\cdot) \\ =& AS * C^{-1}Bj_1 * u(\cdot) + Bj_1 * u(\cdot) - (I+B)j_1 * u(\cdot) \\ =& AS * C^{-1}Bj_1 * u(\cdot) - j_1 * u(\cdot) \end{aligned}$$

on $[0,T_0)$, which implies that $j_1 * u(\cdot) = S * j_0(\cdot)x + AS * C^{-1}Bj_1 * u(\cdot)$ on $[0,T_0)$. Consequently, $j_1 * u = j_1 * w_x$ on $[0,T_0)$ or equivalently, $u = w_x$ on $[0,T_0)$. Clearly, $A(I+B)$ is closed and $A(I+B)C = CA(I+B)$ on $D(A(I+B))$. It follows from Proposition 2.4 that $C^{-1}A(I+B)C$ generates a local C-cosine function $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Just as in the proof of [27, Theorem 2.5], we have $C^{-1}A(I+B)C = A(I+C^{-1}BC)$. By (2.6), $T(\cdot)$ is also locally Lipschitz continuous. □

Since the condition (2.6) in the proof of Theorem 2.5 is only used to show that $T(\cdot)$ is locally Lipschitz continuous. By slightly modifying the proof of Theorem 2.5, we can obtain the next right-multiplicative perturbation theorem for local C-cosine functions without the local Lipschitz continuity.

Theorem 2.6. *Let $C(\cdot)$ be a local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that $(S * C^{-1}Bf)(t) \in D(A)$ and*

$$\|A(S * C^{-1}Bf)(t)\| \leq M_{t_0} \int_0^t \|f(s)\| ds \tag{2.8}$$

for all $f \in C([0,t_0], \overline{D(A)})$ and $0 \leq t \leq t_0$. Then $A(I+C^{-1}BC)$ generates a local C-cosine function $T(\cdot)$ on X satisfying (2.7)

Corollary 2.7. *Let $C(\cdot)$ be a locally Lipschitz continuous local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$ and $C^{-1}Bx \in \overline{D(A)}$ for all $x \in \overline{D(A)}$. Then $A(I+C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Moreover, $T(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is.*

Proof. Clearly, it suffices to show that for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that (2.6) holds for all $f \in C([0,t_0], \overline{D(A)})$ and $0 \leq s < t \leq t_0$. Suppose that

$C_1(t)$ denotes the restriction of $C(t)$ to $\overline{D(A)}$, $C_1^{'}(t)$ the strong derivative of $C_1(t)$ on $\overline{D(A)}$ for all $0 \le t < T_0$, and D^2 the second order derivative of a function. Then $C_1(t)x = Cx + Aj_0 * S(t)x$ and $C_1^{'}(t)x = AS(t)x$ for all $x \in \overline{D(A)}$ and $0 \le t < T_0$. In particular, $AS(\cdot)$ is a strongly continuous family of bounded linear operators on $\overline{D(A)}$, which is also exponentially bounded if $C(\cdot)$ is exponentially Lipschitz continuous. Let $0 < t_0 < T_0$ be given, then $S * C^{-1}Bf(\cdot)$ is twice continuously differentiable on $[0, t_0]$,

$$D^2(S * C^{-1}Bf)(\cdot) = AS * C^{-1}Bf(\cdot) + Bf(\cdot) = C_1^{'} * C^{-1}Bf(\cdot) + Bf(\cdot)$$

on $[0, t_0]$ and

$$\begin{aligned}\|A(S * C^{-1}B[f(t) - f(s)])\| =& \|C_1^{'} * C^{-1}B[f(t) - f(s)]\| \\ &\le \sup_{0 \le r \le t_0} \|AS(r)\| \|C^{-1}B\| \int_s^t \|f(r)\| dr\end{aligned}$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le s < t \le t_0$. It follows from Theorem 2.3 that $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Combining the local Lipschitz continuity of $C^{-1}BT(\cdot)$ with the exponential boundedness of $AS(\cdot)$, we get that $AS * C^{-1}BT)(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is. Consequently, $T(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is. □

Corollary 2.8. *Let $C(\cdot)$ be a local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$ and $C^{-1}Bx \in D(A)$ for all $x \in \overline{D(A)}$. Then $A(I + C^{-1}BC)$ generates a local C-cosine function $T(\cdot)$ on X satisfying*

$$T(\cdot)x = C(\cdot)x + S * AC^{-1}BT(\cdot)x \quad \text{on } [0, T_0) \tag{2.9}$$

for all $x \in X$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous) if $C(\cdot)$ is.

Proof. By the assumption of $C^{-1}Bx \in D(A)$ for all $x \in \overline{D(A)}$, we can apply the following estimation to replace the condition (2.8):

$$\|(S * AC^{-1}Bf(t))\| \le \sup_{0 \le r \le t_0} \|S(r)\| \|AC^{-1}B\| \int_0^t \|f(r)\| dr$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le t \le t_0$. Clearly, $S(\cdot)AC^{-1}B$ is also exponentially bounded (resp., norm continuous) if $C(\cdot)$ is. By (2.9) and the boundedness of $AC^{-1}B$, we have

$$T(\cdot)x = C(\cdot)x + SAC^{-1}B * T(\cdot)x \quad \text{on } [0, T_0) \tag{2.10}$$

for all $x \in X$, which together with Gronwall's inequality implies that $T(\cdot)$ is exponentially bounded (resp., norm continuous) if $C(\cdot)$ is. □

When $\rho((I + C^{-1}BC)A)$ (resolvent set of $(I + C^{-1}BC)A$) is nonempty, we can apply Theorem 2.5 to obtain the next left-multiplicative perturbation theorem concerning locally Lipschitz continuous local C-cosine functions on X in which the generator A of a perturbed local C-cosine function may not be densely defined, B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$, and $C^{-1}(I+B)AC$ and $(I+B)A$ both may not be equal.

Theorem 2.9. *Under the assumptions of Theorem 2.5. Assume that $\rho((I+C^{-1}BC)A)$ is nonempty. Then $(I+C^{-1}BC)A$ generates a locally Lipschitz continuous local C-cosine function $U(\cdot)$ on X satisfying*

$$\begin{aligned} &U(\cdot)x \\ =&Cx+[\lambda-(I+C^{-1}BC)A](I+C^{-1}BC)j_1*T(\cdot)A[\lambda-(I+C^{-1}BC)A]^{-1}x \end{aligned} \tag{2.11}$$

on $[0,T_0)$ for all $x\in X$. Here $\lambda\in\rho((I+C^{-1}BC)A)$ is fixed and $T(\cdot)$ is given as in (2.7).

Proof. Just as in the proof of [27, Theorem 2.9], we have

$$(I+C^{-1}BC)ACx=C(I+C^{-1}BC)Ax$$

for all $x\in D((I+C^{-1}BC)A)$. We set $P=I+C^{-1}BC$ and

$$u_x(\cdot)=Cx+(\lambda-PA)Pj_1*T(\cdot)A(\lambda-PA)^{-1}x$$

on $[0,T_0)$ for all $x\in X$, then $u_x\in C([0,T_0),X)$ and

$$\begin{aligned} &A(\lambda-PA)^{-1}u_x(\cdot) \\ =&A(\lambda-PA)^{-1}Cx+A(Pj_1*T(\cdot))A(\lambda-PA)^{-1}x \\ =&A(\lambda-PA)^{-1}Cx+T(\cdot)A(\lambda-PA)^{-1}x-CA(\lambda-PA)^{-1}x \\ =&A(\lambda-PA)^{-1}Cx+T(\cdot)A(\lambda-PA)^{-1}x-A(\lambda-PA)^{-1}Cx \\ =&T(\cdot)A(\lambda-PA)^{-1}x \end{aligned}$$

on $[0,T_0)$, and so

$$PA(\lambda-PA)^{-1}j_1*u_x(\cdot)=Pj_1*T(\cdot)A(\lambda-PA)^{-1}x$$

on $[0,T_0)$. Hence,

$$\begin{aligned} -j_1*u_x(\cdot)+\lambda(\lambda-PA)^{-1}j_1*u_x(\cdot)=&PA(\lambda-PA)^{-1}j_1*u_x(\cdot) \\ =&Pj_1*T(\cdot)A(\lambda-PA)^{-1}x \\ =&(\lambda-PA)^{-1}u_x(\cdot)-(\lambda-PA)^{-1}Cx \end{aligned}$$

on $[0,T_0)$, which implies that $j_1*u_x(t)\in D(PA)$ for all $0\le t<T_0$. Consequently,

$$PA(\lambda-PA)^{-1}j_1*u_x(t)\in D(PA)$$

for all $0\le t<T_0$ and $PAj_1*u_x=u_x-Cx$ on $[0,T_0)$. This shows that j_1*u_x is a solution of ACP$(PA,Cx,0,0)$ in $C^2([0,T_0),X)$. In order to show the uniqueness. Suppose that $v\in C([0,T_0),X)$ and $v=PAj_1*v$ on $[0,T_0)$. We set $u=A(\lambda-PA)^{-1}v$ on $[0,T_0)$, then

$$\begin{aligned} Pj_1*u=&PA(\lambda-PA)^{-1}j_1*v \\ =&(\lambda-PA)^{-1}PAj_1*v \\ =&(\lambda-PA)^{-1}v \end{aligned}$$

on $[0, T_0)$, and so $APj_1 * u = A(\lambda - PA)^{-1}v = u$ on $[0, T_0)$. Hence, $u = 0$ on $[0, T_0)$, which implies that $(\lambda - PA)^{-1}v = 0$ on $[0, T_0)$ or equivalently, $v = 0$ on $[0, T_0)$. We conclude from Theorem 2.3 that $(I + C^{-1}BC)A$ generates a local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Clearly, for each $y \in X$,

$$(PA)Pj_1 * T(\cdot)y = P(AP)j_1 * T(\cdot)y = PT(\cdot)y - PCy$$

on $[0, T_0)$. It follows from the right-hand side of (2.11) that $U(\cdot)$ is also locally Lipschitz continuous. □

By slightly modifying the proof of Theorem 2.9, we can obtain the next left-multiplicative perturbation theorem for local C-cosine functions in which the generator A of a perturbed local C-cosine function may not be densely defined, B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$, and $C^{-1}(I + B)AC$ and $(I + B)A$ both may not be equal.

Theorem 2.10. *Under the assumptions of Theorem 2.6. Assume that $\rho((I+C^{-1}BC)A)$ is nonempty. Then $(I + C^{-1}BC)A$ generates a local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Moreover, $U(\cdot)$ is exponentially bounded (resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous) if $T(\cdot)$ is. Here $T(\cdot)$ is given as in (2.7).*

Corollary 2.11. *Under the assumptions of Corollary 2.7.*
Assume that $\rho((I+C^{-1}BC)A)$ is nonempty. Then $(I+C^{-1}BC)A$ generates a locally Lipschitz continuous local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Moreover, $U(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is.

Corollary 2.12. *Under the assumptions of Corollary 2.8.*
Assume that $\rho((I + C^{-1}BC)A)$ is nonempty. Then $(I + C^{-1}BC)A$ generates a local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Moreover, $U(\cdot)$ is also exponentially bounded (resp., norm continuous) if $C(\cdot)$ is.

Theorem 2.13. *(see* [15]*) Let A be the generator of a local C-cosine function $C(\cdot)$ on X. Assume that B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Then $A + B$ is the generator of a local C-cosine function $T_B(\cdot)$ on X satisfying*

$$T_B(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)B^n x ds$$

for all $x \in X$ and $0 \le t < T_0$.

Combining Theorem 2.10 with Theorem 2.13, the next new result concerning the additive perturbations of a local C-cosine function on X is also attained in which the generator of a perturbed local C-cosine function may not be densely defined.

Theorem 2.14. *Let $C(\cdot)$ be a local C-cosine function on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB = BC$ on $D(A)$. Assume that $\rho_C(A)$ and $\rho(A+B)$ both are nonempty, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that*

$$|S * C^{-2}Bf(t)| \le M_{t_0} \int_0^t |f(s)|ds \tag{2.12}$$

for all $f \in C([0,t_0],[D(A)])$ *and* $0 \le t \le t_0$. *Then* $A+B$ *generates a local* C*-cosine function* $V(\cdot)$ *on* X.

Proof. Let $\lambda \in \rho_C(A)$ be fixed. We set $\widetilde{B} = C^{-1}B(A-\lambda)^{-1}C$ and $C(-t) = C(t)$ for all $0 \le t < T_0$. Then $\widetilde{B}$ is a bounded linear operator from X into $R(C)$ such that $C\widetilde{B} = \widetilde{B}C$, $A-\lambda$ is the generator of the local C-cosine function $T_{-\lambda}(\cdot)$ on X satisfying

$$j_0 * T_{-\lambda}(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s) j_n(t-s) S(t-2s)(-\lambda)^n x ds$$

for all $x \in X$ and $0 \le t < T_0$, and $(A-\lambda)^{-1}C^2 = C(A-\lambda)^{-1}C$. Here

$$\int_0^t j_{-1}(s) j_0(t-s) S(t-2s) x ds = S(t)x.$$

Since the norm $|\cdot|_{A-\lambda}$ on $D(A)$ defined by $|x|_{A-\lambda} = \|x\| + \|(A-\lambda)x\|$ for all $x \in D(A)$, is equivalent to $|\cdot|$, we may assume that (2.12) holds under $|\cdot|_{A-\lambda}$. Since

$$(I + C^{-1}\widetilde{B}C)(A-\lambda) = A - \lambda + B$$

and $\rho(A+B)$ is nonempty we have $\rho((I + C^{-1}\widetilde{B}C)(A-\lambda))$ is also nonempty. It is not difficult to see that

$$\begin{aligned} &\int_0^t j_{n-1}(s) j_n(t-s) S(t-2s) x ds \\ =& \sum_{k=0}^{n} \frac{(n-1+k)!}{(n-1)!k!} (-1)^k \frac{1}{2^{n+k}} [j_{n-k}(j_{n-1+k} * S)](t)x \\ &+ \sum_{k=0}^{n-1} \frac{(n+k)!}{n!k!} (-1)^k \frac{1}{2^{n+k+1}} [j_{n-1-k}(j_{n+k} * S)](t)x \end{aligned} \tag{2.13}$$

for each $n \in \mathbb{N}$, $x \in X$ and $0 \le t < T_0$. Let $0 < t_0 < T_0$ and $f \in C([0,t_0],X)$ be fixed. Then

$$\begin{aligned} &[j_{n-k}(j_{n-1+k} * S)] * C^{-1}\widetilde{B}f(t) \\ =& \int_0^t j_{n-k}(t-s)(j_{n-1+k} * S)(t-s) C^{-1}\widetilde{B}f(s) ds \\ =& \frac{1}{(n-k)!} \sum_{m=0}^{n-k} \binom{n-k}{m} (-1)^m t^{n-k-m} \int_0^t j_{n-1+k} * S(t-s) C^{-1}\widetilde{B} s^m f(s) ds \\ =& \sum_{m=0}^{n-k} (-1)^m j_{n-k-m}(t) j_{n-1+k} * S * C^{-1}\widetilde{B} j_m f)(t) \\ =& \sum_{m=0}^{n-k} (-1)^m j_{n-k-m}(t) S * C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t) \end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
&[j_{n-1-k}(j_{n+k} * S)] * C^{-1}\widetilde{B}f(t) \\
=&\int_0^t j_{n-1-k}(t-s)(j_{n+k} * S)(t-s)C^{-1}\widetilde{B}f(s)ds \\
=&\frac{1}{(n-1-k)!}\sum_{m=0}^{n-1-k} \binom{n-1-k}{m}(-1)^m t^{n-1-k-m}\int_0^t j_{n+k} * S(t-s)C^{-1}\widetilde{B}s^m f(s)ds \\
=&\sum_{m=0}^{n-1-k}(-1)^m j_{n-1-k-m}(t)j_{n+k} * S * (C^{-1}\widetilde{B}j_m f)(t) \\
=&\sum_{m=0}^{n-1-k}(-1)^m j_{n-1-k-m}(t)S * C^{-1}\widetilde{B}[j_{n+k} * (j_m f)](t) \qquad (2.15)
\end{aligned}
$$

for all $0 \le t \le t_0$. By (2.12), we have

$$
\begin{aligned}
&\|(A-\lambda)j_{n-k-m}(t)S * C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\| \\
\le& j_{n-k-m}(t_0)\|(A-\lambda)S * C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\| \\
=& j_{n-k-m}(t_0)\|(A-\lambda)S * C^{-2}B(A-\lambda)^{-1}C[j_{n-1+k} * (j_m f)](t)\| \\
\le& j_{n-k-m}(t_0)M_{t_0}\int_0^t |(A-\lambda)^{-1}C[j_{n-1+k} * (j_m f)](s)|_{A-\lambda}ds \\
\le& j_{n-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t \|[j_{n-1+k} * (j_m f)](s)\|ds \qquad (2.16)
\end{aligned}
$$

for all $0 \le t \le t_0$. Since

$$
\begin{aligned}
&\int_0^t \|[j_{n-1+k} * (j_m f)](s)\|ds \\
\le& \int_0^t j_{n-1+k}(s)j_m(s)\int_0^s \|f(s)\|ds \\
=&\frac{(n+k-1+m)!}{(n-1+k)!m!}[j_{n+k+m}(t)\int_0^t \|f(r)\|dr - \int_0^t \|f(s)\|ds] \\
\le&\frac{(n+k-1+m)!}{(n-1+k)!m!}j_{n+k+m}(t)\int_0^t \|f(r)\|dr \qquad (2.17)
\end{aligned}
$$

for all $0 \le t \le t_0$, we have

$$
\begin{aligned}
&\|(A-\lambda)j_{n-k-m}(t)S * (C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\| \qquad (2.18) \\
\le& j_{n-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\frac{(n+k-1+m)!}{(n-1+k)!m!}j_{n+k+m}(t)\int_0^t \|f(r)\|dr
\end{aligned}
$$

for all $0 \leq t \leq t_0$. Similarly, we can apply (2.12) and (2.15) to obtain

$$\begin{aligned}
&\|(A-\lambda)j_{n-1-k-m}(t)S * (C^{-1}\widetilde{B}[j_{n+k} * (j_m f)](t)\| \qquad (2.19)\\
\leq & j_{n-1-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t \|[j_{n+k} * (j_m f)](s)\|ds\\
\leq & j_{n-1-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\frac{(n+k+m)!}{(n+k)!m!}j_{n+k+m-1}(t)\int_0^t \|f(r)\|dr
\end{aligned}$$

for all $0 \leq t \leq t_0$. By (2.13), we have

$$\begin{aligned}
&j_0 * T_{-\lambda} * C^{-1}\widetilde{B}f(t) = S * C^{-1}\widetilde{B}f(t) +\\
&\sum_{n=1}^{\infty}(-\lambda)^n \sum_{k=0}^{n} \frac{(n-1+k)!}{(n-1)!k!}(-1)^k \frac{1}{2^{n+k}}[j_{n-k}(j_{n-1+k} * S)] * C^{-1}\widetilde{B}f(t)\\
&+\sum_{n=1}^{\infty}(-\lambda)^n \sum_{k=0}^{n-1} \frac{(n+k)!}{n!k!}(-1)^k \frac{1}{2^{n+k+1}}[j_{n-1-k}(j_{n+k} * S)] * C^{-1}\widetilde{B}f(t) \qquad (2.20)
\end{aligned}$$

for all $0 \leq t \leq t_0$. By (2.14) and (2.18), we have

$$\begin{aligned}
&\|(A-\lambda)\sum_{k=0}^{n} \frac{(n-1+k)!}{(n-1)!k!}(-1)^k \frac{1}{2^{n+k}}[j_{n-k}(j_{n-1+k} * S)] * C^{-1}\widetilde{B}f(t)\|\\
=&\|(A-\lambda)\sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{(n-1+k)!}{(n-1)!k!}(-1)^{k+m} \frac{1}{2^{n+k}}j_{n-k-m}(t)S\\
&* C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\|\\
\leq&\sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{(n-1+k)!}{(n-1)!k!}\frac{1}{2^{n+k}}\frac{(n-1+k+m)!}{(n-1+k)!m!}j_{n-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\|\\
&+\|C\|)j_{n+k+m}(t)\int_0^t \|f(r)\|dr\\
\leq&\sum_{k=0}^{n}\frac{t_0^{2n}}{n!k!2^{n+k}}\sum_{m=0}^{n-k}\frac{1}{m!}M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t \|f(r)\|dr\\
\leq&\frac{t_0^{2n}}{n!2^n}e^{1/2}eM_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t \|f(r)\|dr. \qquad (2.21)
\end{aligned}$$

Similarly, we can apply (2.15) and (2.19) to show that

$$\begin{aligned}
&\|(A-\lambda)\sum_{k=0}^{n-1}\frac{(n-1+k)!}{n!k!}(-1)^k\frac{1}{2^{n+k+1}}[j_{n-1-k}(j_{n+k}*S)]*C^{-1}\widetilde{B}f(t)\| \\
=&\|(A-\lambda)\sum_{k=0}^{n-1}\sum_{m=0}^{n-1-k}\frac{(n+k)!}{n!k!}(-1)^k\frac{1}{2^{n+k+1}}(-1)^m j_{n-1-k-m}(t)S \\
&*C^{-1}\widetilde{B}[j_{n+k}*(j_m f)](t)\| \\
\le&\sum_{k=0}^{n-1}\sum_{m=0}^{n-1-k}\frac{(n+k)!}{n!k!}\frac{1}{2^{n+k+1}}\frac{(n+k+m)!}{(n+k)!m!}j_{n-1-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| \\
&+\|C\|)j_{n+k+m-1}(t)\int_0^t\|f(r)\|dr \\
\le&\sum_{k=0}^{n-1}\frac{t_0^{2n}}{(n-1)!k!2^{n+k}}\sum_{m=0}^{n-1-k}\frac{1}{m!}M_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t\|f(r)\|dr \\
\le&\frac{t_0^{2n}}{(n-1)!2^n}e^{1/2}eM_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t\|f(r)\|dr. \qquad (2.22)
\end{aligned}$$

Combining (2.20)-(2.22), we get that there exists an $\widetilde{M_{t_0}}>0$ such that

$$\|(A-\lambda)j_0*T_{-\lambda}*C^{-1}\widetilde{B}f(t)\|\le\widetilde{M_{t_0}}\int_0^t\|f(s)\|ds$$

for all $f\in C([0,t_0],X)$ and $0\le t\le t_0$. It follows from Theorem 2.5 that $A+B-\lambda$ generates a local C-cosine function $U(\cdot)$ on X, which implies that $A+B$ generates a local C-cosine function $V(\cdot)$ on X. □

Just as in the proof of Corollary 2.8, we can apply Theorems 2.13 and 2.14 to obtain the next corollary.

Corollary 2.15. *Let $C(\cdot)$ be a local C-cosine function on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB=BC$ on $D(A)$ and $C^{-2}Bx\in D(A)$ for all $x\in D(A)$. Assume that $\rho_C(A)$ and $\rho(A+B)$ both are nonempty. Then $A+B$ generates a local C-cosine function $V(\cdot)$ on X given as in the proof of Theorem 2.14. Moreover, $V(\cdot)$ is exponentially bounded (resp., norm continuous) if $C(\cdot)$ is.*

By slightly modifying the proof of Theorem 2.14, the following additive perturbation results are also attained when $\widetilde{B}$ denotes the restriction of $B(A-\lambda)^{-1}$ to $\overline{D(A)}$, and the assumptions that B is a bounded linear operator from $[D(A)]$ into $R(C^2)$ and $\rho_C(A)$ is nonempty are replaced by assuming that B is a bounded linear operator from $[D(A)]$ into $R(C)$ and $\rho(A)$ is nonempty.

Theorem 2.16. *Let $C(\cdot)$ be a local C-cosine function on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB=BC$ on $D(A)$.*

Assume that $\rho(A)$ *and* $\rho(A+B)$ *both are nonempty, and for each* $0 < t_0 < T_0$ *there exists an* $M_{t_0} > 0$ *such that*

$$|S * C^{-1}Bf(t)| \le M_{t_0} \int_0^t |f(s)|ds \tag{2.23}$$

for all $f \in C([0,t_0],[D(A)])$ *and* $0 \le t \le t_0$. *Then* $A+B$ *generates a local* C*-cosine function on* X.

Corollary 2.17. *Let* $C(\cdot)$ *be a local* C*-cosine function on* X *with generator* A, *and let* B *be a bounded linear operator from* $[D(A)]$ *into* $R(C)$ *such that* $CB = BC$ *on* $D(A)$ *and* $C^{-1}Bx \in D(A)$ *for all* $x \in D(A)$. *Assume that* $\rho(A)$ *and* $\rho(A+B)$ *both are nonempty. Then* $A+B$ *generates a local* C*-cosine function on* X, *which is also exponentially bounded (resp., norm continuous) if* $C(\cdot)$ *is.*

Remark 2.18. The conclusions of Corollaries 2.7 and 2.11 are still true when the assumption that $R(C^{-1}B) \subset \overline{D(A)}$ is replaced by assuming that

$$R(C^{-1}B) \subset \{x \in X \,|\, C(\cdot)x \in C^1([0,T_0),X)\}.$$

We end this paper with a simple illustrative example.

Example 2.19. Let $X = L^\infty(\mathbb{R})$, and $A_0 : D(A_0) \subset X \to X$ be defined by

$$D(A_0) = W^{1,\infty}(\mathbb{R})$$

and $A_0 f = -f'$ for all $f \in D(A_0)$, then $A = A_0^2$ generates a locally Lipschitz continuous local C-cosine function $C(\cdot)(= \{C(t)|0 \le t < T_0\})$ on X and

$$\overline{D(A)} = \overline{W^{2,\infty}(\mathbb{R})} = C_0(\mathbb{R})$$

(see [1, Example 3.15.5] and [17, Theorem 18.3]). Here $C = (\lambda - A_0)^{-1}$ with $\lambda \in \rho(A_0)$ and $0 < T_0 \le \infty$ are fixed. Applying Corollary 2.7, we get that $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on $L^\infty(\mathbb{R})$ satisfying (2.7) when B is a bounded linear operator from $C_0(\mathbb{R})$ into $W^{1,\infty}(\mathbb{R})$ such that $(\lambda - A_0)^{-1}B = B(\lambda - A_0)^{-1}$ on $C_0(\mathbb{R})$ and $R((\lambda - A_0)B) \subset C_0(\mathbb{R})$.

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