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On two modified Phillips operators

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Abstract. In this note we introduce two new modified Phillips operators G_n^1 and G_n^2. We obtain direct estimates for approximation of bounded continuous functions, defined on $[0,\infty)$ by G_n^1, as well as for approximation of unbounded continuous functions by G_n^2. We improve some previous results on this topic.

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Dedicated to Professor Heiner Gonska on the occasion of his 70th birthday

1. Introduction

The Phillips operator [20] is defined as

$$S_n(f;x) = n\sum_{k=1}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-nt}\frac{(nt)^{k-1}}{(k-1)!}f(t)\,dt + e^{-nx}f(0).$$

These operators preserve constant as well linear functions. Some approximation results on these operators- commutativity, direct and strong converse inequalities, inverse estimates, linear combinations etc. have been discussed in [8, 21, 22, 11, 14, 23, 24, 12, 15, 13] . Usually we denote the basis functions by

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}. \tag{1}$$

In the last decade a lot of papers appeared devoted to such modifications of classical positive linear operators, which preserve certain exponential type functions. For the new modified Szász–Mirakyan operators we refer the reader to recent papers [1, 2, 3, 5, 4].In the same way , Gupta and Tachev [12] considered Phillips type operators fixing e^{-t} and e^{At}, $A \in \mathbb{R}$, but not both together. Very recently Gupta and Lopez-Moreno in [13] defined a

modification of the Phillips so as to fix both e^{at} and e^{bt} for any two real numbers, different or not , $a, b \in \mathbb{R}$ and studied their approximation behaviour. In most of the results, mentioned above - the modification consists of modelling the basis functions (for example $s_{n,k}(x)$ in case of Szász-Mirakyan operator) such that certain exponential functions are reproduced. The second approach is to modify the argument of the function, to be approximated. The start of this method was given by the work of King [17] on the classical Bernstein operator and further developed for the Szász-Mirakyan operator in [7]. To combine the both methods Aral, Inoan and Raşa generalized Szász-Mirakyan operator in [3] introducing the function $\rho(x)$ in both places - in basis functions, as well as in the argument of the approximated function. Our method is different from all , mentioned above and we modify simultaneously the basis functions $s_{n,k}(x)$ and multiply the approximated function f by $e^{At}, A = -1, 1$. The case of arbitrary $A \in \mathbb{R}$ is similar and we omit the details. In [12] the following modification of Phillips operator was introduced

$$P_n\left(f;x\right) = n\sum_{k=1}^{\infty} e^{-n\alpha_n(x)} \frac{\left(n\alpha_n\left(x\right)\right)^k}{k!} \int_0^{\infty} e^{-nt} \frac{\left(nt\right)^{k-1}}{(k-1)!} f\left(t\right) dt + e^{-n\alpha_n(x)} f(0). \tag{2}$$

By Lemma 1 in [12] it follows for $A \in \mathbb{R}$

$$P_n(e^{At};x) = e^{\frac{An\alpha_n(x)}{n-A}}. \tag{3}$$

It was shown in [12] that if we choose

$$\alpha_n(x) = \frac{x(n+1)}{n},\ x \in [0,\infty), \tag{4}$$

then

$$P_n(e^{-t};x) = e^{-x}.$$

If we choose

$$\alpha_n(x) = x(1-\frac{A}{n}),\ A > 0 \tag{5}$$

then from [12] we have

$$P_n(e^{t};x) = e^{x}. \tag{6}$$

Further we restrict ourselves in (5) for $A = 1$. Now we define our two modified Phillips operators as follows:

$$G_n^1\left(f;x\right) = e^{-x}\left(n\sum_{k=1}^{\infty} s_{n,k}(\alpha_n(x)) \int_0^{\infty} s_{n,k-1}(t) f(t) e^t\, dt + e^{-n\alpha_n(x)} f(0)\right), \tag{7}$$

where $\alpha_n(x) = x(1-\frac{1}{n})$. The second modification is given by

$$G_n^2(f;x) = e^x \left(n \sum_{k=1}^{\infty} s_{n,k}(\alpha_n(x)) \int_0^{\infty} s_{n,k-1}(t) f(t) e^{-t}\, dt + e^{-n\alpha_n(x)} f(0) \right), \tag{8}$$

where $\alpha_n(x) = x(1+\frac{1}{n})$ Further we adopt that 2 in definition of G_n^2 surves as an index and not as a power factor. It is clear that

$$G_n^1(f;x) = e^{-x} P_n\left(f(t)e^t, x\right), \tag{9}$$

with $\alpha_n(x)$ from (7) and

$$G_n^2(f;x) = e^{x} P_n\left(f(t)e^{-t}, x\right), \tag{10}$$

with $\alpha_n(x)$ from (8). We apply the operator G_n^1 to approximate bounded continuous function $f \in C^{\star}[0,\infty)$-the subspace of real-valued continuous functions , which possess finite limit at infinity, endowed with the uniform norm. The operator G_n^2 will be used to approximate unbounded continuous $f \in C[0,\infty)$. In Section 2 we study the approximation order of G_n^1 and compare our direct estimate with previous known results on this topic. In Section 3 using some ideas from [2] we discuss an uniform error estimate for the operator G_n^2 measured in weighted norm.

2. Estimate for G_n^1.

In Boyanov, Veselinov -[6] the uniform convergence of linear positive operators was established. Later Holhoş in [16] established the following quantitative estimate for a sequence of linear positive operators:

Theorem A. [16] If a sequence of linear positive operators $L_n : C^*[0,\infty) \to C^*[0,\infty)$ satisfy the equalities

$$||L_n e_0 - 1||_{[0,\infty)} = \alpha_n$$
$$||L_n(e^{-t}) - e^{-x}||_{[0,\infty)} = \beta_n$$
$$||L_n(e^{-2t}) - e^{-2x}||_{[0,\infty)} = \gamma_n$$

then

$$||L_n f - f||_{[0,\infty)} \leq \|f\|_{\infty} \cdot \alpha_n + (2+\alpha_n) \cdot \omega^*(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}), f \in C^*[0,\infty).$$

The modulus of continuity used in the above theorem is defined as:

$$\omega^*(f,\delta) := \sup_{\substack{|e^{-x}-e^{-t}|\leq\delta \\ x,t\geq 0}} |f(t) - f(x)|.$$

Our first result states the following

Theorem 2.1. *For $f \in C^*[0,\infty)$ we have*

$$\|G_n^1 f - f\|_{C[0,\infty)} \leq 2\omega^*\left(f, \sqrt{\gamma_n}\right), \tag{11}$$

where

$$\gamma_n = \|G_n^1(e^{-2t};x) - e^{-2x}\|_{C[0,\infty)} = \frac{1}{n+1}\left(\frac{n+1}{n}\right)^{-n} < \frac{1}{2(n+1)}. \tag{12}$$

Proof. From definition of the operator G_n^1-(9) and (4) it follows that $G_n^1(1;x) = 1$, $G_n^1(e^{-t};x) = e^{-x}$, i.e. $\alpha_n = \beta_n = 0$. Simple calculations and (3) imply (12). Now the proof follows immediately from Theorem A. □

Our further uniform estimate is based on the suitable transformation which reduces the uniform approximation problem on $C^*[0,\infty)$ to that one on $C[0,1]$. This observation was developed by Gonska in [10] and by Paltanea in [18] and quantitative results were obtained in [19]. In [1] it was shown that the spaces $\left(C^*[0,\infty), \|\cdot\|_{C[0,\infty)}\right)$ and $\left(C[0,1], \|\cdot\|_{C[0,1]}\right)$ are isometrically isomorfic. If we define

$$\psi(t) = e^{-t},\ t \in [0,\infty) \text{ and}$$

$S^* : C^*[0,\infty) \to C^*[0,\infty)$ is a positive linear operator, reproducing the constant functions , then the following statement was proved in [1] (see Theorem 9 there):

Theorem B. [1] If $S^* : C^*[0,\infty) \to C^*[0,\infty)$, then for all $f^* \in C^*[0,\infty)$ and $0 < h \leq \frac{1}{2}$, the following inequality holds

$$\|S^* f^* - f^*\|_{C[0,\infty)} \leq$$

$$\leq \frac{1}{h}\|S^*(\psi) - \psi\|_{C[0,\infty)}\omega_1(f,h) +$$

$$+ \left[1 + \frac{1}{2h^2}\left(\|S^*(\psi^2) - \psi^2\|_{C[0,\infty)} + 2\|S^*(\psi) - \psi\|_{C[0,\infty)}\right)\right]\omega_2(f,h). \quad (13)$$

Here $f = f^* \circ \psi^{-1}$, i.e. $f \in C[0,1]$ and ω_1, ω_2 are the usual first and second order moduli of continuity. Our second result states the following:

Theorem 2.2. *For $f^* \in C^*[0,\infty)$ we have*

$$\|G_n^1 f^* - f^*\|_{C[0,\infty)} \leq \frac{5}{4}\omega_2\left(f, \frac{1}{\sqrt{n+1}}\right), \quad (14)$$

where $f = f^* \circ \psi^{-1}$.

Proof. The proof follows directly from Theorem B, (12) and the fact that G_n^1 reproduces constant and $e^{-t} = \psi(t)$. □

3. Estimates for G_n^2.

In this section we study the approximation of unbounded functions, satisfying certain exponential growth by the operator G_n^2 with $\alpha_n(x) = x(1 + \frac{1}{n})$. Set $\varphi(x) = 1 + e^x$, $x \in \mathbb{R}^+$ and consider the following weighted spaces

$$B_\varphi(\mathbb{R}^+) = \left\{f : \mathbb{R}^+ \to \mathbb{R} : |f(x)| \leq M_f \cdot \varphi(x),\ x \geq 0\right\},$$

$$C_\varphi(\mathbb{R}^+) = C(\mathbb{R}^+) \cap B_\varphi(\mathbb{R}^+),$$

$$C_{\varphi}^{k}(\mathbb{R}^{+})=\left\{f\in C_{\varphi}(\mathbb{R}^{+}):\lim_{x\to\infty}\frac{f(x)}{\varphi(x)}=k_{f}\text{ exists and it is finite, }\right\}$$

where M_f, k_f are constants depending on f. All three spaces are normed with the norm

$$\|f\|_{\varphi}=\sup\ \frac{|f(x)|}{\varphi(x)}.$$

Obviously we have

$$G_n^2(1;x)=1,\text{ and }G_n^2(e^t;x)=e^x. \tag{15}$$

Further from (10) we obtain

$$G_n^2(f;x)=e^xP_n\left(\frac{f(t)}{1+e^t}\cdot(1+e^t)e^{-t};x\right)\leq$$

$$\leq e^x\|f\|_{\varphi}P_n\left(e^{-t}+1;\alpha_n(x)\right)=e^x\|f\|_{\varphi}\left(e^{-x}+1\right)=$$

$$=\|f\|_{\varphi}\left(1+e^x\right).$$

Consequently we have

$$\|G_n^2f\|_{\varphi}\leq\|f\|_{\varphi}$$

and we conclude that G_n^2 maps $C_{\varphi}(\mathbb{R}^+)$ to $C_{\varphi}(\mathbb{R}^+)$. Following the general result obtained by Gadziev -[9] if we choose as a weight function $\varphi(x)=1+e^{2ax},\ a=1$, instead of $\varphi(x)=1+e^x$, to conclude that for each function $f\in C_{\varphi}^{k}(\mathbb{R}^+)$

$$\lim_{n\to\infty}\|G_n^2f-f\|_{\varphi}=0$$

it is enough to verify the three conditions

$$\lim_{n\to\infty}\|G_n^2e^{it}-e^{ix}\|_{\varphi}=0,\ i=0,1,2.$$

For $i=0,1$ this follows from (15). Unfortunately the condition for $i=2$ is not satisfied. Indeed we have

$$G_n^2\left(e^{2t};x\right)=e^xP_n\left(e^{2t}e^{-t};x\right)=$$

$$=e^xP_n\left(e^t;\alpha_n(x)\right)=e^x\cdot e^{\frac{n}{n-1}x\frac{(n+1}{n}}=e^{x\frac{2n}{n-1}}.$$

Therefore

$$\frac{G_n^2\left(e^{2t};x\right)-e^{2x}}{1+e^{2x}}=\frac{e^{2x\frac{n}{n-1}}-e^{2x}}{1+e^{2x}}\to\infty,$$

when $x\to\infty$ for $n\geq 2$. This is the reason to choose $\varphi(x)=1+e^x$ instead of $\varphi(x)=1+e^{2x}$. Now , according to the Korovkin- type theorem, established in [9] we need to verify

$$\lim_{n\to\infty}\|G_n^2\left(e^{\frac{t}{2}};x\right)-e^{\frac{x}{2}}\|_{\varphi}=0. \tag{16}$$

With $\alpha_n(x)=\frac{x(n+1)}{n}$ and using (3) we calculate

$$G_n^2\left(e^{\frac{t}{2}};x\right)=e^xP_n\left(e^{-\frac{t}{2}};\alpha_n(x)\right)=$$

$$e^x\cdot e^{-\frac{1}{2}\frac{n}{(n+\frac{1}{2})}\frac{n+1}{n}x}=e^{x\cdot\frac{n}{2n+1}}.$$

Therefore

$$\frac{G_n^2\left(e^{\frac{t}{2}};x\right)-e^{\frac{x}{2}}}{1+e^x}=\frac{e^{x\cdot\frac{n}{2n+1}}-e^{\frac{x}{2}}}{1+e^x}= \tag{17}$$

$$\frac{y^{\frac{2n}{2n+1}}-y}{1+y^2}:=g(y),$$

where we set $e^{\frac{x}{2}}=y\in[1,\infty)$.

Hence

$$g(y)=\frac{y^2}{(1+y^2)}\left[y^{-\frac{2n+2}{2n+1}}-y^{-1}\right].$$

Consequently

$$\|g\|_{C[1,\infty)}\leq\|\frac{y^2}{1+y^2}\|_{C[1,\infty)}\|y^{-\frac{2n+2}{2n+1}}-y^{-1}\|_{C[1,\infty)}\leq \tag{18}$$

$$\leq\|y^{-\frac{2n+2}{2n+1}}-y^{-1}\|_{C[1,\infty)}.$$

Simple computations imply

$$\|y^{-\frac{2n+2}{2n+1}}-y^{-1}\|_{C[1,\infty)}=\frac{1}{(2n+1)}\left(1-\frac{1}{2n+2}\right)^{2n+2}<\frac{1}{2(2n+1)}, \tag{19}$$

where we used

$$\lim_{n\to\infty}\left(1-\frac{1}{2n+2}\right)^{2n+2}=e^{-1}<\frac{1}{2}.$$

It is clear that (16) follows from (17),(18),(19). Our next statement is:

Theorem 3.1. *For each function $f\in C_\varphi^k(\mathbb{R}^+)$, $\varphi(x)=1+e^x$ we have*

$$\lim_{n\to\infty}\|G_n^2f-f\|_\varphi=0.$$

Proof. The proof is straightforward corollary from (15) and (16). □

Remark 3.2. If in the definition of G_n^2 from (10) instead of $\alpha_n(x)=x(1+\frac{1}{n})$ we use $\alpha_n(x)=x(1-\frac{1}{n})$ then (3) implies in this case

$$G_n^2(e^t;x)=e^x,\ G_n^2(e^{2t};x)=e^{2x}.$$

But for the third test function of Korovkin-type theorem $e^{0t}=1$ simple calculations show

$$\lim_{n\to\infty}\|G_n^2(1;x)-1\|_\varphi=\infty,$$

with $\varphi(x)=1+e^{2x}$. Similar examples for nonconvergence in weighted norm can be found for Phillips operator in [13].

Remark 3.3. Further results for the operators G_n^1, G_n^2, like representation of moments, central moments, images of monomials, quantitative Voronovskaja-type and direct estimates etc. will be subject of another paper.

Remark 3.4. In [13] Gupta and Lopez-Moreno considered for Phillips operators different Korovkin test system $\{1, t, e^{bt}(1+t^r)\}$. But in their settings we need to verify the weighted approximation for two test functions with more complicated calculations, than presented in our note.

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