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Coefficient estimates for a subclass of meromorphic bi-univalent functions defined by subordination

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Abstract. In this work, we use the Faber polynomial expansion by a new method to find upper bounds for $|b_n|$ coefficients for meromorphic bi-univalent functions class Σ' which is defined by subordination. Further, we generalize and improve some of the previously published results.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \quad \text{and} \quad f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

So, if F is the inverse of a function $f \in \mathcal{S}$, then F has the following representation

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \tilde{a}_n w^n \tag{1.2}$$

which is valid in some neighborhood of the origin.

In 1936, Robertson [23] introduced the concept of starlike functions of order α for $0 \leq \alpha < 1$. A function $f \in \mathcal{A}$ is said to be starlike of order α if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{ST}(\alpha)$. Note that $\mathcal{ST}(0) = \mathcal{ST}$.

Definition 1.1. [8] For two functions f and g which are analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Ma and Minda [20] have given a unified treatment of various subclass consisting of starlike functions by replacing the superordinate function $q(z) = \frac{1+z}{1-z}$ by a more general analytic function. For this purpose, they considered an analytic function φ with positive real part on $\mathbb{U}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, symmetric with respect to the real axis. The class $\mathcal{ST}(\varphi)$ of Ma-Minda starlike functions consists of functions $f \in \mathcal{S}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad \text{for } z \in \mathbb{U}.$$

It is clear that for special choices of φ, this class envelop several well-known subclasses of univalent function as special cases. The idea of subordination was used for defining many of classes of functions studied in the Geometric Function Theory, for example see [7, 21].

Let Σ' denote the class of meromorphic univalent functions g defined in $\Delta := \{z \in \mathbb{C} : 1 < |z| < \infty\}$ of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}. \tag{1.3}$$

Since $g \in \Sigma'$ is univalent, it has an inverse $g^{-1} = G$ that satisfy

$$g^{-1}(g(z)) = z \quad (z \in \Delta) \quad \text{and} \quad g(g^{-1}(w)) = w \quad (M < |w| < \infty,\ M > 0).$$

Furthermore, the inverse function $g^{-1} = G$ has a series expansion of the form

$$G(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{\tilde{b}_n}{w^n} \qquad (M < |w| < \infty). \tag{1.4}$$

A simple calculation shows that the inverse function $g^{-1} = G$, is given by

$$G(w) = g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} + \cdots. \tag{1.5}$$

Let $(\mathcal{ST})'(\varphi)$ denote the class of functions $g \in \Sigma'$ which satisfy

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} \prec \varphi(z), \quad \text{for } z \in \mathbb{U}.$$

The mapping $f(z) \mapsto g(z) := 1/f(1/z)$ establishes a one-to-one correspondence between functions in the classes $\mathcal{S}$ and Σ' and also between functions in the classes $\mathcal{ST}(\varphi)$ and $(\mathcal{ST})'(\varphi)$ because (see for more details [5])

$$\frac{zg'(z)}{g(z)} = \frac{z(1/f(1/z))'}{1/f(1/z)} = \frac{1}{z}\frac{f'(1/z)}{f(1/z)}, \quad \text{for } |z| > 1.$$

Noth that if $g \in (\mathcal{ST})'(\varphi)$, then there exists a unique function $f \in \mathcal{ST}(\varphi)$ such that $g(z) = 1/f(1/z)$. Also, it can be easily verified that $G(w) = 1/F(1/w)$, where $F(w)$ is the inverse of $f(z)$.

Analogous to the bi-univalent analytic functions, a function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$. Examples of the meromorphic bi-univalent functions are

$$z + \frac{1}{z}, \qquad z - 1, \qquad -\frac{1}{\log\left(1 - \frac{1}{z}\right)}.$$

Determination of the sharp coefficient estimates of inverse functions in various subclasses of the class of analytic and univalent functions is an interesting problem in geometric function theory. Schiffer [24] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma'$ with $b_0 = 0$ and Duren [8] gave an elementary proof of the inequality $|b_n| \leq \frac{2}{n+1}$ on the coefficient of meromorphic univalent functions $g \in \Sigma'$ with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. But the interest on coefficient estimates of the meromorphic univalent functions keep on by many researchers, see for example, [18, 19, 25, 26]. Several authors by using Faber polynomial expansions obtained coefficient estimates $|a_n|$ for classes meromorphic bi-univalent functions and bi-univalent functions, see for example [10, 12, 13, 14, 15, 16, 17, 28, 27]. First we recall some definitions and lemmas that used in this work.

Faber [9] introduced the Faber polynomials which play an important role in various areas of mathematical sciences, especially in geometric function theory. By using the Faber polynomial expansion of functions $g \in \Sigma'$ of the form (1.3), the coefficients of its inverse map $g^{-1} = G$ defined in (1.5) may be expressed, (see for details [2] and [3]),

$$G(w) = g^{-1}(w) = w - b_0 - \sum_{n \geq 1} \frac{1}{n} K_{n+1}^{n} \frac{1}{w^n}, \tag{1.6}$$

where

$$\begin{aligned} K_{n+1}^{n} &= n b_0^{n-1} b_1 + n(n-1) b_0^{n-2} b_2 + \frac{n(n-1)(n-2)}{2} b_0^{n-3}(b_3 + b_1^2) \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{3!} b_0^{n-3}(b_4 + 3b_1 b_2) + \sum_{j \geq 5} b_0^{n-j} V_j, \end{aligned}$$

such that V_j with $5 \leq j \leq n$ is a homogeneous polynomial in the variables $b_1, b_2, \cdots, b_n$, (see for details [3]).

Definition 1.2. [4] Let φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, symmetric with respect to the real axis. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 > 0). \tag{1.7}$$

Lemma 1.3. [8] *Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathbb{U}$ with*

$$u(0) = v(0) = 0 \qquad \textit{and} \qquad \max\{|u(z)|\,,\ |v(z)|\} < 1.$$

We suppose also that

$$u(z) = \sum_{n=1}^{\infty} p_n z^n \qquad \textit{and} \qquad v(z) = \sum_{n=1}^{\infty} q_n z^n \quad (z \in \mathbb{U}). \tag{1.8}$$

Then

$$|p_1| \leq 1, \quad |p_2| \leq 1 - |p_1|^2, \quad |q_1| \leq 1, \quad |q_2| \leq 1 - |q_1|^2. \tag{1.9}$$

Lemma 1.4. [1, 2] *Let the function $f \in \mathcal{A}$ be given by (1.1). Then for any $p \in \mathbb{Z}$, there are the polynomials K_n^p, such that*

$$(1 + a_2 z + a_3 z^2 + \cdots + a_k z^{k-1} + \cdots)^p = 1 + \sum_{n=1}^{\infty} K_n^p(a_2, a_3, \cdots, a_{n+1}) z^n,$$

where

$$K_n^p(a_2, \cdots, a_{n+1}) = p a_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!(n)!} D_n^n,$$

and

$$D_n^m(a_2, a_3, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1! \cdots \mu_n!}, \text{ for } m \in \mathbb{N} = \{1, 2, \ldots\} \text{ and } m \leq n,$$

the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that $D_n^n(a_2, a_3, \cdots, a_n) = a_2^n$. In particular,

$$\begin{aligned} K_n^1 &= a_{n+1}, \qquad K_1^2 = 2a_2, \qquad K_2^2 = 2a_3 + a_2^2, \\ K_3^2 &= 2a_4 + 2a_2 a_3, \qquad K_4^2 = 2a_5 + 2a_2 a_4 + a_3^2. \end{aligned}$$

Lemma 1.5. [2, 3] and [6, page 52] *Let the function $g \in \Sigma'$ be given by (1.3). Then we have the following expansion*

$$\frac{zg'(z)}{g(z)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, \cdots b_n) \frac{1}{z^{n+1}},$$

where

$$F_{n+1}(b_0, b_1, \cdots b_n) = \sum_{i_1 + 2i_2 + \cdots + (n+1)i_{n+1} = n+1} A(i_1, i_2, \cdots, i_{n+1})(b_0^{i_1} b_1^{i_2} \cdots b_n^{i_{n+1}}),$$

and

$$A(i_1, i_2, \cdots, i_{n+1}) := (-1)^{(n+1)+2i_1+\cdots+(n+2)i_{n+1}} \frac{(i_1 + i_2 + \cdots + i_{n+1} - 1)!(n+1)}{i_1! i_2! \cdots i_{n+1}!}.$$

The first four terms of the Faber polynomials F_n are given by

$$\begin{aligned} F_1 &= -b_0, \qquad F_2 = b_0^2 - 2b_1, \qquad F_3 = -b_0^3 + 3b_1 b_0 - 3b_2, \\ F_4 &= b_0^4 - 4b_0^2 b_1 + 4b_0 b_2 + 2b_1^2 - 4b_3. \end{aligned}$$

In this work, by using the Faber polynomial expansion we find upper bounds for $|b_n|$ coefficients by a new method for meromorphic bi-univalent functions class Σ' which is defined by subordination. Further, we generalize and improve some of the previously published results.

2. Main results

In this section, first we obtain estimates of coefficients $|b_n|$ of meromorphic bi-univalent functions in the class $(\mathcal{ST})'(\varphi)$. Next we obtain an improvement of the bounds $|b_0|$ and $|b_1|$ for special choices of φ.

Theorem 2.1. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'(\varphi)$, where φ is given by Definition* 1.2. *If $b_k = 0$ for $0 \le k \le n-1$, then*

$$|b_n| \le \frac{B_1}{n+1}.$$

Proof. From $g \in (\mathcal{ST})'(\varphi)$, we obtain

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} = \frac{1 - b_1 z^2 - 2b_2 z^3 - \cdots}{1 + b_0 z + b_1 z^2 + \cdots} = 1 - b_0 z + (b_0^2 - 2b_1)z^2 + \cdots. \tag{2.1}$$

Similar to Lemma 1.5, for function $g \in (\mathcal{ST})'(\varphi)$ and for its inverse map $g^{-1} = G$, we have

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, \cdots b_n) z^{n+1}, \tag{2.2}$$

$$\frac{1}{w}\frac{G'(1/w)}{G(1/w)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(\tilde{b}_0, \tilde{b}_1, \cdots \tilde{b}_n) w^{n+1}, \tag{2.3}$$

respectively, where $\tilde{b}_0 = -b_0,\ \tilde{b}_n = \frac{1}{n} K_{n+1}^n$.

On the other hand, since $g,\ G \in (\mathcal{ST})'(\varphi)$, by the Definition 1.1, there exist two Schwarz functions $u, v : \mathbb{U} \to \mathbb{U}$ where $u,\ v$ are given by (1.8), so that

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} = \varphi(u(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(p_1, p_2, \cdots, p_n) z^n, \tag{2.4}$$

and

$$\frac{1}{w}\frac{G'(1/w)}{G(1/w)} = \varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(q_1, q_2, \cdots, q_n) w^n. \tag{2.5}$$

Comparing the corresponding coefficients of (2.2) and (2.4), we get that

$$F_{n+1}(b_0, b_1, \cdots b_n) = \sum_{k=1}^{n+1} B_k D_{n+1}^k(p_1, p_2, \cdots, p_{n+1}). \tag{2.6}$$

Similarly, by comparing the corresponding coefficients of (2.3) and (2.5), we get that

$$F_{n+1}(\tilde{b}_0, \tilde{b}_1, \cdots \tilde{b}_n) = \sum_{k=1}^{n+1} B_k D_{n+1}^k(q_1, q_2, \cdots, q_{n+1}). \tag{2.7}$$

Note that $b_k = 0$ for $0 \leq k \leq n-1$, yields $\tilde{b}_n = -b_n$ and hence from (2.6) and (2.7), respectively, we get

$$-(n+1)b_n = B_1 p_{n+1},$$

and

$$-[-(n+1)]b_n = B_1 q_{n+1}.$$

By solving either of the above two equations for b_n and applying $|p_{n+1}| \leq 1, |q_{n+1}| \leq 1$, we obtain

$$|b_n| \leq \frac{B_1}{n+1},$$

this completes the proof. □

Corollary 2.2. *Let the function g given by (1.3) and its inverse map $g^{-1} = G$ given by (1.4) be in the class $(\mathcal{ST})'\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. If $b_k = 0$ for $0 \leq k \leq n-1$, then*

$$|b_n| \leq \frac{2\alpha}{n+1} \qquad (0 < \alpha \leq 1).$$

Corollary 2.3. [13] *Let the function g given by (1.3) and its inverse map $g^{-1} = G$ given by (1.4) be in the class $(\mathcal{ST})'\left(\frac{1+(1-2\beta)z}{1-z}\right)$. If $b_k = 0$ for $0 \leq k \leq n-1$, then*

$$|b_n| \leq \frac{2(1-\beta)}{n+1} \qquad (0 \leq \beta < 1).$$

Corollary 2.4. *Let the function f given by (1.1) and its inverse map $f^{-1} = F$ given by (1.2) be in the class $\mathcal{ST}(\varphi)$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{B_1}{n-1}.$$

Proof. Setting $f(1/z) := 1/g(z)$ and $F(1/w) = 1/G(w)$ in Theorem 2.1 we obtain the result and this completes the proof. □

Corollary 2.5. ([14, Theorem 2.1]) *Let the function f given by (1.1) and its inverse map $f^{-1} = F$ given by (1.2) be in the class $\mathcal{ST}\left(\frac{1+Az}{1+Bz}\right)$, where A and B are real numbers so that $-1 \leq B < A \leq 1$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{A-B}{n-1}.$$

Theorem 2.6. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'(\varphi)$, where φ is given by Definition* 1.2. *Then*

$$|b_0| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 - B_2| + B_1}} \tag{2.8}$$

and

$$|b_1| \leq \frac{B_1}{2}. \tag{2.9}$$

Proof. The equations (2.6) and (2.7) for $n = 0$ and $n = 1$, respectively, imply

$$-b_0 = B_1 p_1, \tag{2.10}$$

$$b_0^2 - 2b_1 = B_1 p_2 + B_2 p_1^2, \tag{2.11}$$

$$b_0 = B_1 q_1, \tag{2.12}$$

$$b_0^2 + 2b_1 = B_1 q_2 + B_2 q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we have

$$p_1 = -q_1 \tag{2.14}$$

and

$$2b_0^2 = B_1^2\left(p_1^2 + q_1^2\right). \tag{2.15}$$

Also by adding (2.11) and (2.13), and considering (2.15) we have

$$\begin{aligned} 2b_0^2 &= B_1\left(p_2 + q_2\right) + B_2\left(p_1^2 + q_1^2\right) \\ &= B_1\left(p_2 + q_2\right) + \frac{2B_2 b_0^2}{B_1^2}. \end{aligned}$$

So we obtain

$$b_0^2 = \frac{B_1^3\left(p_2 + q_2\right)}{2\left(B_1^2 - B_2\right)}.$$

By (1.9), (2.10), (2.14) and the above equality give

$$\begin{aligned} |b_0|^2 &\leq \frac{B_1^3\left(1 - |p_1|^2\right)}{|B_1^2 - B_2|} \\ &\leq \frac{B_1^3}{|B_1^2 - B_2|}\left(1 - \frac{|b_0|^2}{B_1^2}\right). \end{aligned}$$

Therefore we obtain

$$|b_0|^2 \leq \frac{B_1^3}{|B_1^2 - B_2| + B_1}, \tag{2.16}$$

which is the desired estimate on the coefficient $|b_0|$ as asserted in (2.8).
On the other hand, by subtracting (2.13) from (2.11) and considering (2.14) we get

$$-4b_1 = B_1\left(p_2 - q_2\right).$$

Taking the absolute values and considering (1.9) we obtain the desired estimate on the coefficient $|b_1|$ as asserted in (2.9). This completes the proof. $\square$

Theorem 2.7. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then*

$$|b_0| \leq \frac{2\alpha}{\sqrt{\alpha+1}},$$

and

$$|b_1| \leq \alpha.$$

Remark 2.8. Theorem 2.7 is an refinement of estimate for $|b_0|$ obtained by Panigrahi [22, Corollary 2.3]. Also, for $|b_1|$ if $\frac{1}{\sqrt{5}} < \alpha \leq 1$ and $|b_0|$, Theorem 2.7 is an refinement of estimates obtained by Halim *et al.* [11, Theorem 2].

Theorem 2.9. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'\left(\frac{1+(1-2\beta)z}{1-z}\right)$. Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\beta)} & , \quad 0 \leq \beta \leq \frac{1}{2} \\ \frac{\sqrt{2}(1-\beta)}{\sqrt{\beta}} & , \quad \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|b_1| \leq 1-\beta.$$

Remark 2.10. Theorem 2.9 is an improvement of the estimates obtained by Panigrahi [22, Corollary 3.3] and also obtained by Halim et al. [11, Theorem 1].

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