Possibly infinite generalized iterated function systems comprising φ-max contractions

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Abstract. One way to generalize the concept of iterated function system was proposed by R. Miculescu and A. Mihail under the name of generalized iterated function system (for short GIFS). More precisely, given $m \in \mathbb{N}^*$ and a metric space (X, d), a generalized iterated function system of order m is a finite family of functions $f_1, ..., f_n : X^m \to X$ satisfying certain contractive conditions. Another generalization of the notion of iterated function system, due to F. Georgescu, R. Miculescu and A. Mihail, is given by those systems consisting of φ-max contractions. Combining these two lines of research, we prove that the fractal operator associated to a possibly infinite generalized iterated function system comprising φ-max contractions is a Picard operator (whose fixed point is called the attractor of the system). We associate to each possibly infinite generalized iterated function system comprising φ-max contractions $\mathcal{F}$ (of order m) an operator $H_{\mathcal{F}} : \mathcal{C}^m \to \mathcal{C}$, where $\mathcal{C}$ stands for the space of continuous and bounded functions from the shift space on the metric space corresponding to the system. We prove that $H_{\mathcal{F}}$ is a Picard operator whose fixed point is the canonical projection associated to $\mathcal{F}$.

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INTRODUCTION

One way to generalize the concept of iterated function system was proposed by R. Miculescu and A. Mihail (see [6] and [8]) under the name of generalized iterated function system. More precisely, given $m \in \mathbb{N}^*$ and a metric space (X, d), a generalized iterated function system (for short a GIFS) of order m is a finite family of functions $f_1, ..., f_n : X^m \to X$ satisfying certain contractive conditions.

They proved that there exists a unique attractor of a GIFS, studied some of its properties and provided examples showing that GIFSs are real generalizations of iterated function systems. In addition, F. Strobin (see [13]) proved that, for any $m \in \mathbb{N}$, $m \geq 2$, there exists a Cantor subset of the plane which is the attractor of some GIFS of order m, but is not the attractor of any GIFS of order $m-1$. This kind of iterated function system was generalized in several ways (see [1], [2], [10], [12], [14] and [15]). In addition, the Hutchinson measure associated with a GIFS was studied in [7] (for GIFS with probabilities), in [4] (for generalized iterated function systems with place dependent probabilities) and in [11]

Another generalization of the notion of iterated function system in given by those systems consisting of φ-max-contractions (see [3]).

Combining these lines of research, we prove that the fractal operator associated to a possibly infinite generalized iterated function system comprising φ-max contractions is a Picard operator (whose fixed point is called the attractor of the system).

The main tool in the study of topological properties of the attractor of an iterated function system is the canonical projection. Paper [9] inspired us to associate to each possibly infinite generalized iterated function system comprising φ-max contractions $\mathcal{F}$ (of order m) an operator $H_{\mathcal{F}} : \mathcal{C}^m \to \mathcal{C}$, where $\mathcal{C}$ stands for the space of continuous and bounded functions from the shift space on the metric space corresponding to the system. We prove that $H_{\mathcal{F}}$ is a Picard operator whose fixed point is the canonical projection associated to $\mathcal{F}$.

PRELIMINARIES

For a metric space (X, d) and $m \in \mathbb{N}^*$, we consider:

- $P_{b,cl}(X)$ the set of all non-empty, bounded and closed subsets of X;

- the Hausdorff-Pompeiu metric $h : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, \infty)$ given by

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

for every $A, B \in P_{b,cl}(X)$, where $d(A, B) = \sup\limits_{x \in A} \inf\limits_{y \in B} d(x, y)$;

- the Cartesian product X^m endowed with the maximum metric $d_{\max}$ defined by

$$d_{\max}((x_1, ..., x_m), (y_1, ..., y_m)) = \max\{d(x_1, y_1), ..., d(x_m, y_m)\},$$

for all $(x_1, ..., x_m), (y_1, ..., y_m) \in X^m$;

• the spaces X_1, X_2, ..., X_k, ..., defined inductively in the following way:

$$X_1 = \underbrace{X \times X \times ... \times X}_{m \text{ times}} = X^m$$

and

$$X_{k+1} = \underbrace{X_k \times X_k \times ... \times X_k}_{m \text{ times}}$$

for every $k \in \mathbb{N}^*$. We endow X_k with the maximum metric for every $k \in \mathbb{N}^*$. Note that X_k is isometric to X^{m^k} with the maximum metric for every $k \in \mathbb{N}^*$ and that we will identify X_k and X^{m^k};

• $\mathcal{F}_i^p = \{\sigma : \{1, 2, ..., m^i\} \to \{1, 2, ..., m^p\}\}$, where $p \in \mathbb{N}^*$ and $i \in \{0, 1, ..., p-1\}$

• $x_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(m^i)})$ and $y_\sigma = (y_{\sigma(1)}, ..., y_{\sigma(m^i)})$, where $x = (x_1, x_2, ..., x_{m^p})$, $y = (y_1, y_2, ..., y_{m^p}) \in X^{m^p}$, $p \in \mathbb{N}^*$, $i \in \{0, 1, ..., p-1\}$ and $\sigma \in \mathcal{F}_i^p$.

Definition 1.1. *A possibly infinite generalized iterated function system of order $m \in \mathbb{N}^*$ is a pair $\mathcal{F} = ((X, d), (f_i)_{i \in I})$, where (X, d) is a metric space, $f_i : X^m \to X$ is continuous for every $i \in I$ and the family of functions $(f_i)_{i \in I}$ is bounded (i.e. $\underset{i \in I}{\cup} f_i(B)$ is bounded for each bounded subset B of X^m).*

The function $F_{\mathcal{F}} : (P_{b,cl}(X))^m \to P_{b,cl}(X)$, described by

$$F_{\mathcal{F}}(B_1, ..., B_m) = \overline{\underset{i \in I}{\cup} f_i(B_1 \times ... \times B_m)},$$

for all $(B_1, ..., B_m) \in (P_{b,cl}(X))^m$, is called the fractal operator associated to $\mathcal{F}$.

If there exists a unique $A \in P_{b,cl}(X)$ such that $F_{\mathcal{F}}(A, ..., A) = A$, then we say that $\mathcal{F}$ has attractor and A, which is denoted by $A_{\mathcal{F}}$, is called the attractor of $\mathcal{F}$.

Now we recall the concept of code space associated to a possibly infinite generalized iterated function system which was considered by A. Mihail and F. Strobin & J. Swaczyna.

Let us consider $m \in \mathbb{N}^*$ and a set I. One can define inductively the sets Ω_1, Ω_2, ..., Ω_k, ... in the following way:

$$\Omega_1 = I \text{ and } \Omega_{k+1} = \underbrace{\Omega_k \times \Omega_k \times ... \times \Omega_k}_{m \text{ times}},$$

for every $k \in \mathbb{N}^*$.

We are also dealing in the sequel with the following sets:

$$\Omega = \Omega_1 \times \Omega_2 \times ... \times \Omega_k \times ...$$

and

$${}_k\Omega = \Omega_1 \times \Omega_2 \times ... \times \Omega_k,$$

where $k \in \mathbb{N}^*$.

For $i \in \{1, 2, ..., m\}$, $k \in \mathbb{N}$, $k \geq 2$ and $\alpha = \alpha^1\alpha^2...\alpha^k \in {}_k\Omega$, where $\alpha^2 = \alpha_1^2\alpha_2^2...\alpha_m^2 \in \Omega_2$, ..., $\alpha^k = \alpha_1^k\alpha_2^k...\alpha_m^k \in \Omega_k$, we consider

$$\alpha(i) = \alpha_i^2\alpha_i^3...\alpha_i^k \in {}_{k-1}\Omega.$$

For $\alpha \in \Omega$ and $i \in \{1, 2, ..., m\}$, $\alpha(i) \in \Omega$ could be similarly defined in a similar manner.

Definition 1.2. *Ω is called the Mihail-Strobin&Swaczyna generalized code space.*

Ω becomes a complete metric space if it is furnished with the metric d given by

$$d(\alpha, \beta) = \sum_{k \in \mathbb{N}} C^k d(\alpha^k, \beta^k),$$

for every $\alpha = \alpha^1\alpha^2...\alpha^i\alpha^{i+1}...$, $\beta = \beta^1\beta^2...\beta^i\beta^{i+1}... \in \Omega$, where $d(\alpha^k, \beta^k) = \{\begin{array}{ll} 1, & \alpha^k \neq \beta^k \\ 0, & \alpha^k = \beta^k \end{array}$ and $C \in (0, 1)$. Moreover, the metric space (Ω, d) is compact provided that I is finite.

To a possibly infinite generalized iterated function system $\mathcal{F} = ((X, d), (f_i)_{i \in I})$ of order m, one can associate, for every $k \in \mathbb{N}^*$, a family of functions $\{f_\alpha : X_k \to X \mid \alpha \in {}_k\Omega\}$ defined inductively in the following way:

i) For $k = 1$, the family is $(f_i)_{i \in I}$.

ii) If the functions f_α, where $\alpha \in {}_k\Omega$, have been defined, then, we set

$$f_\alpha(x_1, x_2, ..., x_m) = f_{\alpha^1}(f_{\alpha(1)}(x_1), ..., f_{\alpha(m)}(x_m))$$

for every $\alpha = \alpha^1\alpha^2...\alpha^k\alpha^{k+1} \in {}_{k+1}\Omega$, $(x_1, x_2, ..., x_m) \in X_{k+1} = X_k \times \underbrace{X_k \times ... \times X_k}_{m \text{ times}}$.

Note that if $m = 1$, then ${}_k\Omega = I^k$ and if $\omega = \omega^1\omega^2...\omega^k \in {}_k\Omega$, then $f_\omega = f_{\omega^1} \circ ... \circ f_{\omega^k}$. Hence the above introduced families of functions are natural generalizations of compositions of functions.

Given a set X, $m \in \mathbb{N}^*$ and a function $f : X^m \to X$, we define inductively a family of functions $f^{[k]} : X^{m^k} \to X$, $k \in \mathbb{N}^*$, in the following way:

i)

$$f^{[1]} = f;$$

ii) assuming that we have defined $f^{[k]}$, then

$$f^{[k+1]}(x_1, ..., x_m) = f(f^{[k]}(x_1), ..., f^{[k]}(x_m)),$$

for every $(x_1, ..., x_m) \in \underbrace{X^{m^k} \times ... \times X^{m^k}}_{m \text{ times}} = X^{m^{k+1}} = X_{k+1}$.

Note that for $m = 1$, we have $f^{[k]} = \underbrace{f \circ ... \circ f}_{k \text{ times}}$. We remark that maps $f^{[k]}$ are special cases of f_α defined earlier.

Definition 1.3. *Given a set X, $m \in \mathbb{N}^*$ and a function $f : X^m \to X$, an element x of X such that $f(x, ..., x) = x$ is called a fixed point of f.*

Definition 1.4. *Given a metric space (X, d) and $m \in \mathbb{N}^*$, a function $f : X^m \to X$ is called contraction if there exists $C \in [0, 1)$ such that $d(f(x), f(y)) \le Cd_{\max}(x, y)$ for all $x, y \in X^m$.*

Definition 1.5. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called comparison function provided that it satisfies the following properties:*

i) it is nondecreasing;

ii) it is right-continuous;

iii) $\varphi(t) < t$ for every $t > 0$.

Definition 1.6.

a) *Given a metric space (X, d), $m \in \mathbb{N}^*$ and a comparison function φ, a function $f : X^m \to X$ is called φ-contraction if $d(f(x), f(y)) \le \varphi(d_{\max}(x, y))$ for all $x, y \in X^m$.*

b) *Given a metric space (X, d), a comparison function φ and $m \in \mathbb{N}^*$, a function $f : X^m \to X$ is called φ-max generalized contraction if there exists $p \in \mathbb{N}^*$ such that*

$$d(f^{[p]}(x), f^{[p]}(y)) \le \varphi(\max\{\max_{\sigma \in \mathcal{F}_i^p} d(f^{[i]}(x_\sigma), f^{[i]}(y_\sigma)) \mid i \in \{0, 1, 2, ..., p-1\}\}),$$

for all $x, y \in X^{m^p}$.

Now let us introduce an important tool that will be used in this paper, namely the operator $H_{\mathcal{F}}$ associated to a generalized possibly infinite generalized iterated function system $\mathcal{F}$.

To a possibly infinite generalized iterated function system $\mathcal{F} = ((X, d), (f_i)_{i\in I})$ of order m, we associate the operator $H_{\mathcal{F}} : \mathcal{C}^m \to \mathcal{C}$ described by

$$H_{\mathcal{F}}(g_1, ..., g_m)(\alpha) = f_{\alpha^1}(g_1(\alpha(1)), ..., g_m(\alpha(m))),$$

for every $g_1, ..., g_m \in \mathcal{C}$ and every $\alpha = \alpha^1\alpha^2...\alpha^k... \in \Omega$, where the metric space $(\mathcal{C}, d_u)$ is described by $\mathcal{C} = \{f : \Omega \to X \mid f \text{ is continuous and bounded}\}$ and $d_u(f, g) = \sup\limits_{\alpha\in\Omega} d(f(\alpha), g(\alpha))$ for every $f, g \in \mathcal{C}$.

Remark 1.7.

i) $H_{\mathcal{F}}(g_1, ..., g_m)$ is continuous for all $g_1, ..., g_m \in \mathcal{C}$. This results from the following facts: the maps $\alpha \to \alpha(i)$ are continuous, $\Omega = \underset{i\in I}{\cup}\Omega^i$, where $\Omega^i = \{\alpha = \alpha^1\alpha^2...\alpha^i\alpha^{i+1}... \in \Omega \mid \alpha^1 = i\}$, and the restriction of $H_{\mathcal{F}}(g_1, ..., g_m)$ to the open set Ω^i is continuous for every $i \in I$.

ii) $H_{\mathcal{F}}(g_1, ..., g_m)$ is bounded for all $g_1, ..., g_m \in \mathcal{C}$. This results from the boundedness of the family of functions $(f_i)_{i\in I}$, the boundedness of the functions $g_1, ..., g_m$ and from the fact that

$$H_{\mathcal{F}}(g_1, ..., g_m)(\Omega) = H_{\mathcal{F}}(g_1, ..., g_m)(\underset{i\in I}{\cup}\Omega^i) =$$

$$= \underset{i\in I}{\cup} H_{\mathcal{F}}(g_1, ..., g_m)(\Omega^i) = \underset{i\in I}{\cup} f_i(g_1(\Omega) \times ... \times g_m(\Omega)).$$

iii) $H_{\mathcal{F}}$ *is well defined*. This results from i) and ii).

Remark 1.8. $(\mathcal{C}, d_u)$ *is complete provided that* (X, d) *is complete.*

Finally we introduce the canonical projection associated to a possibly infinite generalized iterated function system $\mathcal{F}$.

Definition 1.9. *A possibly infinite generalized iterated function system* $\mathcal{F} = ((X, d), (f_i)_{i\in I})$ *of order* $m \in \mathbb{N}^*$ *admits canonical projection if has attractor (denoted by* $A_{\mathcal{F}}$*) and for every* $\alpha = \alpha^1...\alpha^n... \in \Omega$ *the set* $\underset{n\in\mathbb{N}}{\cap} \overline{f_{\alpha^1...\alpha^n}(A_{\mathcal{F}}, ..., A_{\mathcal{F}})}$ *consists of a single element denoted by* $\pi(\alpha)$. *In this case the function* $\pi : \Omega \to X$ *is called the canonical projection associated to* $\mathcal{F}$.

THE MAIN RESULTS

Theorem 2.1. *Let (X,d) be a complete metric space, $\varphi : [0,\infty) \to [0,\infty)$ a comparison function, $m, p \in \mathbb{N}^*$ and a continuous function $f : X^m \to X$ such that*

$$d(f^{[p]}(x), f^{[p]}(y)) \le \varphi(\max\{\max_{\sigma \in \mathcal{F}_i^p} d(f^{[i]}(x_\sigma), f^{[i]}(y_\sigma)) \mid i \in \{0,1,2,...,p-1\}\}),$$

for all $x, y \in X^{m^p}$.

Then:

a) There exists a unique $\alpha \in X$ such that $f(\alpha,...,\alpha) = \alpha$.

b) If f is bounded on bounded subsets of X^m, then, for every $B \in P_{b,cl}(X)$ and every $x_k \in B^{m^k}$, $\lim_{k\to\infty} f^{[k]}(x_k) = \alpha$, the convergence being uniform with respect to x_k.

Proof.

a) Note that the continuous function $g : X \to X$ given by $g(x) = f(x,...,x)$ satisfies the inequality

$$d(g^{[p]}(x), g^{[p]}(y)) \le \varphi(\max\{d(g^{[i]}(x), g^{[i]}(y)) \mid i \in \{0,1,...,p-1\}\}), \qquad (1)$$

for all $x, y \in X$. Then, based on (1), using Theorem 3.1 from [5], we infer that there exists a unique $\alpha \in X$ such that $g(\alpha) = \alpha$ and $\lim_{n\to\infty} g^{[n]}(x) = \alpha$ for every $x \in X$. Hence there exists a unique $\alpha \in X$ such that $f(\alpha,...,\alpha) = \alpha$.

b) In the sequel, for $B \in P_{b,cl}(X)$ and $k \in \mathbb{N}$, we shall use the following notations:

$$M_k(B) \overset{not}{=} \sup_{x \in B^{m^k}} d(\alpha, f^{[k]}(x))$$

and

$$N_k(B) \overset{not}{=} \max\{M_{k+i}(B) \mid i \in \{0,1,...,p-1\}\}.$$

As $M_n(f(B)) = \sup_{y \in (f(B))^{m^n}} d(\alpha, f^{[k]}(y)) = \sup_{x \in B^{m^{n+1}}} d(\alpha, f^{[n+1]}(x)) = M_{n+1}(B)$ for all $B \in P_{b,cl}(X)$ and all $n \in \mathbb{N}$, the mathematical induction method leads us to the following conclusion:

$$M_m(f^{[n]}(B)) = M_{m+n}(B), \qquad (2)$$

for every $B \in P_{b,cl}(X)$, $m, n \in \mathbb{N}$.

Moreover, we have

$$M_{n+p}(B) \le \varphi(\max\{M_{n+i}(B) \mid i \in \{0,1,...,p-1\}\}), \tag{3}$$

for every $B \in P_{b,cl}(X)$ and $n \in \mathbb{N}$.

Indeed,

$$M_{n+p}(B) \overset{(2)}{=} M_n(f^{[p]}(B)) = \sup_{x \in B^{m^{n+p}}} d(\alpha, f^{[m+p]}(x)) \le$$

$$\le \sup_{x \in B^{m^{n+p}}} \varphi(\max\{d(\alpha, f^{[n+i]}(x)) \mid i \in \{0,1,...,p-1\}\}) \le$$

$$\le \varphi(\max\{\sup_{x \in B^{m^{n+i}}} d(\alpha, f^{[n+i]}(x)) \mid i \in \{0,1,...,p-1\}\}) =$$

$$= \varphi(\max\{M_{n+i}(B) \mid i \in \{0,1,...,p-1\}\}).$$

In addition, from (3), we have $N_{n+1}(B) \le N_n(B) \le ... \le N_0(B) < \infty$ and $N_{n+p}(B) \le \varphi(N_n(B))$ for every $n \in \mathbb{N}$.

Hence $N_n(B) \le \varphi^{[\frac{n}{p}]}(\max\{M_i(B) \mid i \in \{0,1,...,p-1\}\})$ and consequently $\lim\limits_{n\to\infty} N_n(B) = \lim\limits_{n\to\infty} M_n(B) = 0$ for every $B \in P_{b,cl}(X)$. $\square$

Theorem 2.2. *Let $\mathcal{F} = ((X,d),(f_i)_{i\in I})$ be a possibly infinite generalized iterated function system of order $m \in \mathbb{N}^*$ and $p \in \mathbb{N}$ such that*

$$d(f_\alpha(x), f_\alpha(y)) \le \varphi(\max\{\max_{\sigma \in \mathcal{F}_q^p} d(f_\beta(x_\sigma), f_\beta(y_\sigma)) \mid \beta \in_q \Omega, q \in \{0,1,...,p-1\}),$$

for all $x, y \in X^{m^p}$. Then:

a) There exists a unique $A_\mathcal{F} \in P_{b,cl}(X)$ such that $F_\mathcal{F}(A_\mathcal{F},...,A_\mathcal{F}) = A_\mathcal{F}$, i.e. $\mathcal{F}$ has attractor.

b) $\lim\limits_{n\to\infty} F_\mathcal{F}^{[n]}(B_n) = A_\mathcal{F}$ for all $B \in P_{b,cl}(X)$ and $B_n = (B_1^n,...,B_{m^n}^n) \subseteq B^{m^n}$ with $B_i^n \in P_{b,cl}(X)$ for all $i \in \{1,2,...,m^n\}$.

c) For all $\alpha = \alpha^1...\alpha^n... \in \Omega$, the set $\bigcap\limits_{n\in\mathbb{N}} \overline{f_{\alpha^1\alpha^2...\alpha^n}(A_\mathcal{F},...,A_\mathcal{F})}$ has only one element denoted by a_α, so $\mathcal{F}$ admits canonical projection.

d) For all $\alpha = \alpha^1...\alpha^n... \in \Omega$, $B \in P_{b,cl}(X)$ and $B_n = (B_1^n,...,B_{m^n}^n) \subseteq B^{m^n}$ with $B_i^n \in P_{b,cl}(X)$ for each $i \in \{1,2,...,m^n\}$, we have $\lim\limits_{n\to\infty} f_{\alpha^1\alpha^2...\alpha^n}(B_n) = \{a_\alpha\}$ and the convergence is uniform with respect to α and the sets B_n.

Proof.

a) The function $F : P_{b,cl}(X) \to P_{b,cl}(X)$ given by $F(B) = F_{\mathcal{F}}(B, ..., B)$ for every $B \in P_{b,cl}(X)$ has the property that

$$h(F^{[p]}(B_1), F^{[p]}(B_2)) \leq \varphi(\max\{h(F^{[i]}(B_1), F^{[i]}(B_2)) \mid i \in \{0, 1, ..., p-1\}\}),$$

for all $B_1, B_2 \in P_{b,cl}(X)$. Theorem 2.1 assures the existence and the uniqueness of a set $A_{\mathcal{F}} \in P_{b,cl}(X)$ such that $F(A_{\mathcal{F}}) = A_{\mathcal{F}}$ (i.e. $F_{\mathcal{F}}(A_{\mathcal{F}}, ..., A_{\mathcal{F}}) = A_{\mathcal{F}}$) and $\lim\limits_{n\to\infty} F^{[n]}(B) = A_{\mathcal{F}}$ for every $B \in P_{b,cl}(X)$.

b) For $B_1, B_2 \in P_{b,cl}(X)$, $p, n \in \mathbb{N}$ and $\alpha \in {}_p\Omega$, in the sequel, we shall use the following notations:

$$M_\alpha(B_1, B_2) = \sup_{x\in B_1^{m^p}, y\in B_2^{m^p}} d(f_\alpha(x), f_\alpha(y)),$$

$$M_p(B_1, B_2) = \sup_{\alpha\in {}_p\Omega} M_\alpha(B_1, B_2)$$

and

$$N_n(B_1, B_2) = \max\{M_n(B_1, B_2), ..., M_{n+p-1}(B_1, B_2)\}.$$

Then, we have

$$d(f_\alpha(x), f_\alpha(y)) \leq \varphi(\max\{\max_{\sigma\in\mathcal{F}_q^p} d(f_\beta(x_\sigma), f_\beta(y_\sigma)) \mid \beta \in {}_q\Omega,\ q \in \{0, 1, ..., p-1\}) \leq$$

$$\leq \varphi(\max\{\max_{\omega\in {}_i\Omega} M_\omega(B_1, B_2)) \mid i \in \{0, 1, ..., p-1\}) \leq$$

$$\leq \varphi(\max\{M_i(B_1, B_2)) \mid i \in \{0, 1, ..., p-1\}),$$

for all $B_1, B_2 \in P_{b,cl}(X)$ and $x \in B_1^{m^p}, y \in B_2^{m^p}$, so

$$M_\alpha(B_1, B_2) \leq \varphi(\max\{M_0(B_1, B_2), ..., M_{p-1}(B_1, B_2)\})$$

and

$$M_p(B_1, B_2) \leq \varphi(\max\{M_0(B_1, B_2), ..., M_{p-1}(B_1, B_2)\}), \tag{1}$$

for all $B_1, B_2 \in P_{b,cl}(X)$ and $\alpha \in {}_p\Omega$. Moreover

$$M_{i+j}(B_1, B_2) = M_j(F_{\mathcal{F}}^{[i]}(B_1, ..., B_1), F_{\mathcal{F}}^{[i]}(B_2, ..., B_2)), \tag{2}$$

for all $B_1, B_2 \in P_{b,cl}(X)$, $i, j \in \mathbb{N}$. By replacing, in (1), the set B_1 by $F_{\mathcal{F}}^{[n]}(B_1, ..., B_1)$ and the set B_2 by $F_{\mathcal{F}}^{[n]}(B_2, ..., B_2)$, we get

$$M_{n+p}(B_1, B_2) \leq \varphi(\max\{M_n(B_1, B_2), ..., M_{n+p-1}(B_1, B_2)), \tag{3}$$

for all $B_1, B_2 \in P_{b,cl}(X)$, $n \in \mathbb{N}$. From (3) we infer that

$$N_{n+1}(B_1, B_2) \leq N_n(B_1, B_2) \text{ and } N_{n+p}(B_1, B_2) \leq \varphi(N_n(B_1, B_2)),$$

for all $B_1, B_2 \in P_{b,cl}(X)$ and $n \in \mathbb{N}$. Therefore

$$N_n(B_1, B_2) \leq \varphi^{[\frac{n}{p}]}(\max\{M_0(B_1, B_2), ..., M_{p-1}(B_1, B_2)\}),$$

for all $B_1, B_2 \in P_{b,cl}(X)$ and $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} N_n(B_1, B_2) = \lim_{n\to\infty} M_n(B_1, B_2) = \lim_{n\to\infty} h(F_{\mathcal{F}}^{[n]}(B_1, ..., B_1), F_{\mathcal{F}}^{[n]}(B_2, ..., B_2)) = 0, \tag{4}$$

for all $B_1, B_2 \in P_{b,cl}(X)$. In particular, for $B_2 = A_{\mathcal{F}}$, we obtain that

$$\lim_{n\to\infty} h(F_{\mathcal{F}}^{[n]}(B_1, ..., B_1), A_{\mathcal{F}}) = 0, \text{ i.e. } \lim_{n\to\infty} F_{\mathcal{F}}^{[n]}(B_1, ..., B_1) = A_{\mathcal{F}},$$

for each $B_1 \in P_{b,cl}(X)$. Moreover, we have

$$M_\alpha(B_1, B_2) \leq M_\alpha(C_1, C_2) \text{ and } M_n(B_1, B_2) \leq M_n(C_1, C_2),$$

for all $B_1, B_2, C_1, C_2 \in P_{b,cl}(X)$, $B_1 \subseteq C_1$, $B_2 \subseteq C_2$, $n \in \mathbb{N}$ and $\alpha \in {}_n\Omega$.

If for $B, C \in P_{b,cl}(X)$ and $n \in \mathbb{N}$, $B_n = (B_1^n, ..., B_{m^n}^n)$, $C_n = (C_1^n, ..., C_{m^n}^n) \subseteq B^{m^n}$, with $B_i^n, C_i^n \in P_{b,cl}(X)$ and $B_i^n \subseteq B, C_i^n \subseteq C$ for all $i \in \{1, ..., m^n\}$, then $\lim_{n\to\infty} F_{\mathcal{F}}^{[n]}(B_n) = A_{\mathcal{F}}$. Indeed, we have only to take into account (4) and the inequality

$$h(F_{\mathcal{F}}^{[n]}(B_n), F_{\mathcal{F}}^{[n]}(C_n)) \leq M_n(B, C),$$

which is valid for all $n \in \mathbb{N}$, for $C = A_{\mathcal{F}}$.

c) Let us note that, as $h(f_\alpha(B_n), f_\alpha(C_n)) \leq M_n(B, C)$ for all $\alpha \in {}_n\Omega$, taking into account (4), we infer that $\lim_{n\to\infty} h(f_\alpha(B_n), f_\alpha(C_n)) = 0$ for all $B, C \in P_{b,cl}(X)$ and $\alpha \in {}_n\Omega$.

In the sequel, for $\alpha = \alpha^1...\alpha^n... \in \Omega$, we shall use the following notation:

$$f_{\alpha^1...\alpha^n}(A_{\mathcal{F}}, ..., A_{\mathcal{F}}) \overset{not}{=} A_{\alpha^1...\alpha^n}.$$

Note that $diam(A_{\alpha^1...\alpha^n}) = M_{\alpha^1...\alpha^n}(A_{\mathcal{F}}, A_{\mathcal{F}})$ for all $\alpha = \alpha^1...\alpha^n... \in \Omega$ and $n \in \mathbb{N}$. As $A_{\alpha^1...\alpha^n\alpha^{n+1}} \subseteq A_{\alpha^1...\alpha^n}$, we obtain that $diam(A_{\alpha^1...\alpha^n\alpha^{n+1}}) \leq diam(A_{\alpha^1...\alpha^n}) \leq M_n(A_{\mathcal{F}}, A_{\mathcal{F}})$ for all $\alpha = \alpha^1...\alpha^n... \in \Omega$ and $n \in \mathbb{N}$ and, based on (4), we conclude that the set $\bigcap_{n\in\mathbb{N}} \overline{f_{\alpha^1\alpha^2...\alpha^n}(A_{\mathcal{F}}, ..., A_{\mathcal{F}})}$ has only one element denoted by a_α.

Let us note that $h(f_{\alpha^1...\alpha^n}(A_{\mathcal{F}},...,A_{\mathcal{F}}),\{a_\alpha\}) \leq diam(f_{\alpha^1...\alpha^n}(A_{\mathcal{F}},...,A_{\mathcal{F}})) \leq M_n(A_{\mathcal{F}},A_{\mathcal{F}})$ for all $\alpha = \alpha^1...\alpha^n... \in \Omega$ and $n \in \mathbb{N}$. Therefore, using (4), we get $\lim_{n\to\infty} f_{\alpha^1\alpha^2...\alpha^n}(A_{\mathcal{F}},...,A_{\mathcal{F}}) = \{a_\alpha\}$.

d) Because $\lim_{n\to\infty} f_{\alpha^1...\alpha^n}(A_{\mathcal{F}},...,A_{\mathcal{F}}) = \{a_\alpha\}$ and $\lim_{n\to\infty} f_{\alpha^1...\alpha^n}(A_{\mathcal{F}},...,A_{\mathcal{F}}) = \lim_{n\to\infty} f_{\alpha^1...\alpha^n}(B_1^n,...,B_{m^n}^n)$, we conclude that $\lim_{n\to\infty} f_{\alpha^1\alpha^2...\alpha^n}(B_1^n,...,B_{m^n}^n) = 0$ for all $\alpha = \alpha^1...\alpha^n... \in \Omega$, $B \in P_{b,cl}(X)$ and $B_n = (B_1^n,...,B_{m^n}^n) \subseteq B^{m^n}$ with $B_i^n \in P_{b,cl}(X)$ for each $i \in \{1,2,...,m^n\}$.

Concerning the rate of the convergence we have the following estimation:

$$h(f_{\alpha^1\alpha^2...\alpha^n}(B_n),\{a_\alpha\}) \leq h(f_{\alpha^1\alpha^2...\alpha^n}(B_n),A_{\alpha^1...\alpha^n}) + h(A_{\alpha^1...\alpha^n},\{a_\alpha\}) \leq$$

$$\leq M_n(A_{\mathcal{F}},B)+M_n(A_{\mathcal{F}},A_{\mathcal{F}}) \leq 2\varphi^{[\frac{n}{p}]}(\max\{M_i(A_{\mathcal{F}},B) \mid i \in \{0,1,...,p-1\}\}),$$

for all $\alpha = \alpha^1...\alpha^n... \in \Omega$ and $n \in \mathbb{N}$. $\square$

Theorem 2.3. *Let $\mathcal{F} = ((X,d),(f_i)_{i\in I})$ be a possibly infinite generalized iterated function system of order $m \in \mathbb{N}^*$ and $p \in \mathbb{N}$ such that*

$$d(f_\alpha(x),f_\alpha(y)) \leq \varphi(\max\{\max_{\sigma\in\mathcal{F}_q^p} d(f_\beta(x_\sigma),f_\beta(y_\sigma)) \mid \beta \in_q \Omega, q \in \{0,1,...,p-1\}),$$

for all $x,y \in X^{m^p}$. Then there exists a unique $\pi \in \mathcal{C}$ such that:

a) $H_{\mathcal{F}}(\pi,...,\pi) = \pi$ and $\overline{\pi(\Omega)} = A_{\mathcal{F}}$.

b) $\lim_{n\to\infty} H_{\mathcal{F}}^{[n]}(f_n) = \pi$ for all $B \in P_{b,cl}(X)$ and $f_n = (f_1^n,...,f_{m^n}^n) \in \mathcal{C}_B^{m^n}$, where $\mathcal{C}_B = \{f : \Omega \to X \mid f$ is continuous$\}$ is endowed with the uniform metric, the convergence being uniform with respect to B.

c) π is the canonical projection associated to $\mathcal{F}$.

Proof.

a) Using the mathematical induction method, one can easily prove that

$$H_{\mathcal{F}}^{[n]}(g_1,...,g_{m^n})(\alpha) =$$

$$= f_{\alpha^1\alpha^2...\alpha^n}(g_1(\alpha(11...1)),...,g_m(\alpha(11...m)),...,g_{m^n}(\alpha(mm...m))), \quad (1)$$

for all $\alpha = \alpha^1...\alpha^n... \in \Omega$ and all $n \in \mathbb{N}$, where we adopted the following notation:

$$\alpha(i_1)(i_2)...(i_k) \overset{not}{=} \alpha(i_1i_2...i_k).$$

For a fixed $n \in \mathbb{N}$, for each $l \in \{1,...,m^n\}$ there exists a unique ordered subset $\{l_1,...,l_n\}$ of $\{1,2...,m\}$ such that $l-1 = l_1m^{n-1} + l_2m^{n-2} + ... + l_n$, so we can consider the function $u : \{1, 2, ..., m^n\} \to \{1, 2, ..., m\}^n$ given by $u(l) = (l_1 + 1, l_2 + 1, ..., l_n + 1)$ for all $l \in \{1, 2, ..., m^n\}$ and rewrite (1) in the following form:

$$H_{\mathcal{F}}^{[n]}(g_1, ..., g_{m^n})(\alpha) = f_{\alpha^1\alpha^2...\alpha^n}(g_1(\alpha(u(1))), ..., g_{m^n}(\alpha(u(m^n)))),$$

for all $\alpha = \alpha^1...\alpha^n... \in \Omega$ and all $n \in \mathbb{N}$.

Claim. $H_{\mathcal{F}}$ is a φ-max generalized contraction.

Justification of the claim. Indeed, we have

$$d_u(H_{\mathcal{F}}^{[p]}(g_1, ..., g_{m^p}), H_{\mathcal{F}}^{[p]}(h_1, ..., h_{m^p})) =$$

$$\sup_{\alpha\in\Omega} d(H_{\mathcal{F}}^{[p]}(g_1, ..., g_{m^p})(\alpha), H_{\mathcal{F}}^{[p]}(h_1, ..., h_{m^p})(\alpha)) \leq$$

$$\leq \sup_{\alpha\in\ _m\Omega} \sup_{\alpha(1),...,\alpha(m^p)\in\Omega} \varphi(\max_{i\in\{0,1,...,p-1\}} \max_{\beta\in\ _i\Omega} \max_{\sigma\in\mathcal{F}_i^p} d(f_\beta(g_{\sigma(i)}(\alpha(\sigma(u(i))))), f_\beta(h_{\sigma(i)}(\alpha(\sigma(u(i))))))) \leq$$

$$\leq \varphi(\sup_{\alpha\in\ _m\Omega} \max_{i\in\{0,1,...,p-1\}} \max_{\sigma\in\mathcal{F}_i^p} \max_{\beta\in\ _i\Omega} \sup_{\alpha(1),...,\alpha(m^p)\in\Omega} d(f_\beta(g_{\sigma(i)}(\alpha(\sigma(u(i))))), f_\beta(h_{\sigma(i)}(\alpha(\sigma(u(i)))))) \leq$$

$$\leq \varphi(\max\{\max_{\sigma\in\mathcal{F}_i^p} d_u(H_{\mathcal{F}}^{[i]}(g_\sigma), H_{\mathcal{F}}^{[i]}(h_\sigma)) \mid i \in \{0, 1, ..., p-1\}\}),$$

for all $g_1, ..., g_{m^p}, h_1, ..., h_{m^p} \in \mathcal{C}$.

The Claim and Theorem 2.1 assure us that there exists a unique $\pi \in \mathcal{C}$ such that $H_{\mathcal{F}}(\pi, ..., \pi) = \pi$.

Moreover, we have $\overline{\pi(\Omega)} = A_{\mathcal{F}}$.

Indeed,

$$\overline{\pi(\Omega)} = \overline{H_{\mathcal{F}}(\pi, ..., \pi)(\Omega)} =$$

$$= \overline{\bigcup_{i\in I} \bigcup_{\alpha_1,...,\alpha_m\in\Omega} f_i(\pi(\alpha_1), ..., \pi(\alpha_m))} = \overline{\bigcup_{i\in I} f_i(\pi(\Omega) \times ... \times \pi(\Omega))} =$$

$$\overset{f_i \text{ continuous}}{=} \overline{\bigcup_{i\in I} f_i(\overline{\pi(\Omega)} \times ... \times \overline{\pi(\Omega)})} = F_{\mathcal{F}}(\overline{\pi(\Omega)} \times ... \times \overline{\pi(\Omega)})$$

and $\overline{\pi(\Omega)} \in P_{b,cl}(X)$ (since $\pi \in \mathcal{C}$). In view of Theorem 2.2, a), we conclude that $\overline{\pi(\Omega)} = A_{\mathcal{F}}$.

b) Let us consider $B \in P_{b,cl}(X)$ and $f_n = (f_1^n, ..., f_{m^n}^n) \in \mathcal{C}_B^{m^n}$, $n \in \mathbb{N}$. Note that the family of function $(f_i^n)_{i\in\{1,2,...,m^n\}}$ is bounded (as $\bigcup_{i\in\{1,2,...,m^n\}} f_i^n(\Omega) \subseteq B$) for all $n \in \mathbb{N}$.

Claim 1. $H_{\mathcal{F}}(\mathcal{C}_1 \times ... \times \mathcal{C}_1)$ is bounded for every bounded subset $\mathcal{C}_1$ of $\mathcal{C}$.

Justification of Claim 1. Let us consider $\mathcal{C}_1$ a bounded (with respect to d_u) subset of $\mathcal{C}$. Then there exists $g \in \mathcal{C}$ and $r > 0$ such that $\mathcal{C}_1 \subseteq$

$B(g,r)$. It follows that $\overline{\underset{f\in\mathcal{C}_1}{\cup} f(\Omega)} \subseteq \overline{B(g(\Omega),r)}$ and we shall use the following notation: $B \overset{not}{=} \overline{\underset{f\in\mathcal{C}_1}{\cup} f(\Omega)} \in P_{b,cl}(X)$. The inclusion $H_{\mathcal{F}}(\mathcal{C}_1,...,\mathcal{C}_1) \subseteq C(\Omega, F_{\mathcal{F}}(B,...,B)) = \{f : \Omega \to F_{\mathcal{F}}(B,...,B) \mid f \text{ is bounded}\}$ is valid as

$$H_{\mathcal{F}}(f_1,...,f_m)(\Omega) = \underset{i\in I}{\cup} \underset{\alpha(1),...,\alpha(m)\in\Omega}{\cup} f_i(f_1(\alpha(1)),...,f_m(\alpha(m))) \subseteq$$

$$\subseteq \underset{i\in I}{\cup} f_i(f_1(\Omega),...,f_m(\Omega)) \subseteq \underset{i\in I}{\cup} f_i(B,...,B) \subseteq F_{\mathcal{F}}(B,...,B),$$

for all $f_1,...,f_m \in \mathcal{C}_1$. Hence $d_u(H_{\mathcal{F}}(f_1,...,f_m), H_{\mathcal{F}}(g_1,...,g_m)) \leq diam(F_{\mathcal{F}}(B,...,B))$ for all $f_1,...,f_m, g_1,...,g_m \in \mathcal{C}_1$, so $H_{\mathcal{F}}(\mathcal{C}_1 \times ... \times \mathcal{C}_1)$ is bounded for every bounded subset $\mathcal{C}_1$ of $\mathcal{C}$. The justification of the claim is done.

Let $\mathcal{C}_1$ be a bounded subset of $\mathcal{C}$. Since $d_u(H_{\mathcal{F}}^{[n]}(g_1^n,...,g_{m^n}^n), H_{\mathcal{F}}^{[n]}(h_1^n,...,h_{m^n}^n)) \leq diam(F_{\mathcal{F}}^{[n]}(B,...B))$ for all $n \in \mathbb{N}$ and $g_1^n,...,g_{m^n}^n, h_1^n,...,h_{m^n}^n \in \mathcal{C}_1 \cup \{\pi\}$, using Theorem 2.1, b), we conclude that $\lim\limits_{n\to\infty} H_{\mathcal{F}}^{[n]}(f_n) = \pi$.

c) Note that

$$\pi(\alpha) = H_{\mathcal{F}}(\pi,...,\pi)(\alpha) = f_{\alpha^1}(\pi(\alpha(1)),...,\pi(\alpha(m))), \tag{2}$$

for all $\alpha \in \Omega$.

Claim 2.

$$\pi(F_{\alpha^1\alpha^2...\alpha^n}(\Lambda_1,...,\Lambda_{m^n})) = f_{\alpha^1\alpha^2...\alpha^n}(\pi(\Lambda_1) \times ... \times \pi(\Lambda_{m^n})), \tag{3}$$

for all $n \in \mathbb{N}^*$, $\alpha^1 \in I, \alpha^2 \in \Omega_2, ..., \alpha^n \in \Omega_n$ and $\Lambda_1,...,\Lambda_{m^n} \subseteq \Omega$.

Justification of Claim 2.

We are going to use the mathematical induction method.

Using (2), we get Claim 2 for $n = 1$.

Let us suppose that (3) is valid for n. We shall prove that it is also true for $n+1$. We have

$$\pi(F_{\alpha^1\alpha^2...\alpha^n\alpha^{n+1}}(\Lambda_1,...,\Lambda_{m^{n+1}})) =$$

$$= \pi((F_{\alpha(1)}(\Lambda_1,...,\Lambda_{m^n}),...,F_{\alpha(m)}(\Lambda_{m^{n+1}-m^n+1},...,\Lambda_{m^{n+1}}))) \overset{(2)}{=}$$

$$= f_{\alpha^1}(\pi(F_{\alpha(1)}(\Lambda_1,...,\Lambda_{m^n})),...,\pi(F_{\alpha(m)}(\Lambda_{m^{n+1}-m^n+1},...,\Lambda_{m^{n+1}})))) =$$

$$= f_{\alpha^1}(f_{\alpha(1)}(\pi(\Lambda_1),...,\pi(\Lambda_{m^n})),...,f_{\alpha(m)}(\pi(\Lambda_{m^{n+1}-m^n+1}),...,\pi(\Lambda_{m^{n+1}}))) \overset{\text{Claim 2 for } n}{=}$$

$$= f_{\alpha^1\alpha^2...\alpha^n\alpha^{n+1}}(\pi(\Lambda_1) \times ... \times \pi(\Lambda_{m^{n+1}})),$$

for all $\Lambda_1, ..., \Lambda_{m^{n+1}} \subseteq \Omega$, where $\alpha = \alpha^1\alpha^2...\alpha^n....$

Finally, we have

$$\pi(\alpha) \in \pi(\bigcap_{n\in\mathbb{N}^*} F_{\alpha^1\alpha^2...\alpha^n}(\Omega, ..., \Omega)) \subseteq \bigcap_{n\in\mathbb{N}^*} \pi(F_{\alpha^1\alpha^2...\alpha^n}(\Omega, ..., \Omega)) \overset{\text{Claim 2}}{=}$$

$$= \bigcap_{n\in\mathbb{N}^*} f_{\alpha^1\alpha^2...\alpha^n}(\pi(\Omega), ..., \pi(\Omega)) \subseteq \bigcap_{n\in\mathbb{N}^*} \overline{f_{\alpha^1\alpha^2...\alpha^n}(A_{\mathcal{F}}, ..., A_{\mathcal{F}})},$$

for all $\alpha = \alpha^1\alpha^2...\alpha^n... \in \Omega$, so, based on Theorem 2.2, b), π is the canonical projection associated to $\mathcal{F}$. $\square$

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