When (H_3), take

$$L > \frac{N(N-2)\left(b^{N-2} - a^{N-2}\right)}{\left(1 - \left(\frac{a}{b}\right)^{N-2}\right)\lambda_1}$$

there exists $N_1 > 0$ such that $f(t,u) \geq Lu,\ for\ u \geq N_1,\ t \in [a,b]$.
Set $K_2 = \{u : \|u\| < NN_1\}$. Then $\overline{K_1} \subset K_2$. If $u \in \partial K_2$, we have

$$\|u\| = \|T_\lambda u\| = (T_\lambda u)\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2} - a^{N-2}\right)}\left(1 - \left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f(s, u(s))\, ds$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2} - a^{N-2}\right)}\left(1 - \left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, \frac{1}{N}\|u\|\right) ds$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2} - a^{N-2}\right)}\left(1 - \left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, \frac{1}{N}\|u(s)\|\right) ds$$

$$\geq \frac{\lambda L\left(1 - \left(\frac{a}{b}\right)^{N-2}\right)}{N(N-2)\left(b^{N-2} - a^{N-2}\right)}\|u\| > \|u\|.$$

Consequently, Applying Theorem 3.1 that T_λ has a fixed point $u_{\lambda,2} \in \overline{K_2} \setminus K_1$. Equation (3.3) implies that T_λ has no fixed point in ∂K_1. In conclusion, for $\lambda \in (0, \lambda_*]$, T_λ has at least two fixed points $u_{\lambda,1}$ and $u_{\lambda,2}$ in K. The proof is complete. □

We present an example to illustrate the applicability of the results shown before.

Example 3.7. Consider in $\mathbb{R}^3$ the elliptic boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda\left(|x| + u + ln(1+u)\right), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

To the system (3.4) we associate the the second order boundary value problem

$$\begin{cases} -u''(t) = \lambda\left(t + u + ln(1+u)\right), \ t \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$ $-\frac{2}{t}u(t)$

By direct computation, we have

$$F_\infty = 2,\ F_0 = \frac{1}{4},\ F_1 = \frac{1}{2} + \frac{2}{3}(1 + ln(2)) \text{ and } \lambda_* = \frac{48 - 9\pi}{6 + 8(1 + ln(2))}.$$

So, the assumptions (H_1), (H_2) and (H_3) are satisfied, it follows from Theorem 3.4 there exists $\lambda^* = 3 \geq \lambda_*$ such that boundary value problem (3.4) has at least one positive solution for $0 < \lambda \leq 3$ and has no positive solution for $\lambda > \lambda^*$.