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A new computational method based on the Picard iteration method for solving boundary optimal control problems governed by PDEs with two-point boundary conditions

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Abstract. This paper presents a new computational method based on the Picard iteration method for solving boundary optimal control problems governed by parabolic partial differential equations with two-point boundary conditions. The proposed approach adapts the Picard iteration method to solve the necessary optimality conditions derived from Pontryagin's minimum principle, yielding a solution expressed as a truncated power series. To evaluate the effectiveness of the proposed method, a numerical example is provided, and the obtained results are compared with those derived from an alternative approach, demonstrating the accuracy and reliability of the method.

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1. Introduction and problem statement

Optimal control problems, particularly those governed by partial differential equations (PDEs), constitute an important area of research with numerous practical applications, such as chemical reactions, diffusion–reaction phenomena, gas dynamics, blood cancer diagnosis, and many others, see, e.g., [3, 4, 5, 6, 8, 9, 10, 15, 16, 18, 21]. The objective of such problems is to determine the control signal that optimizes a

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cost functional subject to the governing PDE constraints. Due to the importance in applications, several numerical and semi-analytical methods have been proposed in the literature to approximate the solution, including the shooting method [7], the variational iteration method (VIM) [9], and the Picard iteration method (PIM) [19]. Comparative studies in the literature have demonstrated their efficiency (see, e.g. [6, 9, 20]).

The Picard iteration method (PIM) [19, 20] is a well-established approach for solving a wide range of differential equations, including homogeneous, nonhomogeneous, linear, and nonlinear cases. It has proven to be a powerful tool in addressing various problems. For example, PIM has been applied to time–space fractional differential equations in [12], to the numerical solution of two-point nonlinear boundary value problems in [2], and to fractional differential–algebraic systems in [14]. The PIM is based on representing the solution of an ordinary differential equation (ODE) as an infinite series, which is then approximated by truncating the series after a finite number of terms. A key advantage of PIM is that it provides solutions without imposing restrictive assumptions, and, unlike the variational iteration method (VIM), it does not require the computation of Lagrange multipliers. In fact, several studies documented in the literature demonstrate favourable convergence results for the PIM, as seen in the reference [2, 19]. However, this method has yet to be implemented in optimal control problems governed by PDEs due to its complexity, which serves as the impetus for this work.

In this paper, we focus on using and evaluating the numerical performance of the Picard iteration method to solve an optimal control problem governed by a one-dimensional parabolic-type partial differential equation (PDE) with two-point boundary conditions, formulated as follows

$$\min_{u_0,u_\ell} J = \int_0^{t_f}\int_0^{\ell} y^2(x,t)\,dx\,dt + \int_0^{t_f}\left[q_0\,u_0^2(t) + q_\ell\,u_\ell^2(t)\right]dt, \tag{1.1}$$

subject to the one-dimensional parabolic differential equation

$$\frac{\partial y}{\partial t}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) \qquad \text{in Q}, \tag{1.2}$$

with initial and final conditions

$$y(x,0) = y_0(x),\ \ y(x,t_f) = y_{t_f}(x) \qquad \text{in } \ \Omega, \tag{1.3}$$

and with two-points boundary conditions

$$y(0,t) = u_0(t),\ \ y(\ell,t) = u_\ell(t) \qquad \text{on } \ \Sigma, \tag{1.4}$$

where $Q = \Omega \times (0,t_f)$, $\Sigma = \partial\Omega \times (0,t_f)$ with $\Omega = (0,\ell)$ is a segment of the real axis, q_0 and q_ℓ are positive weighting factors. The function $u_0(t)$ and $u_\ell(t)$ are the control variables on the two-point boundary conditions.

In this paper, we propose a new computational procedure based on the PIM to determine the control variable $u(x,t)$ that minimizes the objective function (1.1) subject to the PDE (1.2) and the constraints (1.3)-(1.4). First, by using finite-difference approximation [1], we transform the problem into a control problem governed by a system of ordinary differential equations (ODEs). Then the Pontryagin minimum principle [17] is applied to derive the necessary optimality conditions, which are expressed by the Hamilton–Pontryagin (HP) equations. Next, the PIM is adapted to iteratively obtain an approximate analytical solution of these equations in the form of a truncated power series.

The rest of the paper is organized as follows. In Section 2, we present a numerical method based on the PIM method for solving the control problem (1.1)-(1.4). This section also discusses Pontryagin's minimum principle and examines the convergence of the PIM when applied to the HP equations. In Section 3, we summarize the proposed algorithm for solving the problem using the PIM. A numerical example is presented in Section 4 to demonstrate the effectiveness of the proposed approach. Finally, Section 5 concludes the paper.

2. Numerical analysis of the proposed method

2.1. Finite difference approximation

In this section, the finite difference method is employed to transform problem (1.1)-(1.4) into an optimal control problem governed by ODEs. First, the parabolic equation (1.2) is discretized in the spatial variable x, resulting in a system of ODEs. Then, the trapezoidal rule is applied to the objective function (1.1) to derive a weighted quadratic objective function expressed in terms of the state and control variables.

By dividing the interval $[0, \ell]$ into $n \in \mathbb{N}$ intervals, and define the step size $h = \dfrac{\ell}{n}$, the problem (1.1)-(1.4) becomes

$$\min_{u_0, u_\ell} J = (\frac{h}{2} + q_0) \int_0^{t_f} u_0^2(t)\, dt + h \sum_{k=1}^{n-1} \int_0^{t_f} y_k^2(t)\, dt \quad +(\frac{h}{2} + q_l) \int_0^{t_f} u_\ell^2(t)\, dt, \tag{2.1}$$

subject to the system of ODEs

$$\begin{cases} \dot{y}_1(t) = \dfrac{1}{h^2}(y_2(t) - 2y_1(t) + u_0(t)), \\ \dot{y}_k(t) = \dfrac{1}{h^2}(y_{k+1}(t) - 2y_k(t) + y_{k-1}(t)), \ k = 2, \dots, n-2 \\ \dot{y}_{n-1}(t) = \dfrac{1}{h^2}(u_\ell(t) - 2y_{n-1}(t) + y_{n-2}(t)), \end{cases} \tag{2.2}$$

with boundary conditions

$$y(x_k, 0) = y_k(0), \quad y(x_k, t_f) = y_k(t_f), \quad k = 1, \dots, n-1. \tag{2.3}$$

2.2. Pontryagin's Minimum Principle

The minimum principle of Pontryagin developed by the Russian mathematician Lev Pontryagin and his colleagues [17], is a fundamental concept in optimal control theory. It provides the necessary conditions for optimality in solving extremum problems in optimal control. According to this principle, the solution of (2.1)-(2.3) is obtained by minimizing the Hamiltonian function $\mathcal{H}$, which is defined as follows

$$\mathcal{H} = (\frac{h}{2} + q_0) u_0^2(t) + h \sum_{k=1}^{n-1} y_k^2(t) + (\frac{h}{2} + q_l) u_\ell^2(t) + p(t)^T f(y(t), u(t)), \tag{2.4}$$

where $p(t) \in \mathbb{R}^{n-1}$ is the adjoint vector, $f : \mathbb{R}^{n-1} \times \mathbb{R}^2 \to \mathbb{R}^{n-1}$ is a vector function, and the system (2.2) can be written as

$$\dot{y}(t) = f(y(t), u(t)), \quad y(t) \in \mathbb{R}^{n-1}, \; u(t) \in \mathbb{R}^2.$$

The optimal control law is then obtained by minimizing the Hamiltonian (2.4) as follows

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial u_0}(y(t), u(t), p(t)) = 0, \\ \dfrac{\partial \mathcal{H}}{\partial u_\ell}(y(t), u(t), p(t)) = 0, \end{cases} \tag{2.5}$$

which yields

$$\begin{cases} u_0(t) = -\dfrac{p_1(t)}{h^3 + 2\,h^2 q_0}, \\ u_\ell(t) = -\dfrac{p_{n-1}(t)}{h^3 + 2\,h^2 q_\ell}. \end{cases} \tag{2.6}$$

By substituting (2.6) into the Hamiltonian (2.4) and applying the minimum principle of Pontryagin, the Hamilton–Pontryagin (HP) equations are obtained as follows

$$\begin{cases} \dot{y}(t) = \dfrac{\partial \mathcal{H}}{\partial p(t)}(y(t), p(t)), \\ \dot{p}(t) = -\dfrac{\partial \mathcal{H}}{\partial y(t)}(y(t), p(t)), \end{cases} \tag{2.7}$$

which yields

$$\begin{cases} \dot{y}_1(t) = \dfrac{1}{h^2}\left(y_2(t) - 2y_1(t) - \dfrac{p_1(t)}{h^3 + 2\,h^2 q_0}\right), \\ \dot{y}_k(t) = \dfrac{1}{h^2}\Big(y_{k+1}(t) - 2y_k(t) + y_{k-1}(t)\Big), \; k = 2, \dots, n-2, \\ \dot{y}_{n-1}(t) = \dfrac{1}{h^2}\left(y_{n-2}(t) - 2y_{n-1}(t) - \dfrac{p_{n-1}(t)}{h^3 + 2\,h^2 q_\ell}\right), \end{cases} \tag{2.8}$$

$$\begin{cases} \dot{p}_1(t) = -\left(2\,h\,y_1(t) - 2\dfrac{p_1(t)}{h^2} + \dfrac{p_2(t)}{h^2}\right), \\ \dot{p}_k(t) = -\left(2\,h\,y_k(t) + \dfrac{p_{k-1}(t)}{h^2} - 2\dfrac{p_k(t)}{h^2} + \dfrac{p_{k+1}(t)}{h^2}\right), \; k = 2, \dots, n-2, \\ \dot{p}_{n-1}(t) = -\left(2\,h\,y_{n-1}(t) + \dfrac{p_{n-2}(t)}{h^2} - 2\dfrac{p_{n-1}(t)}{h^2}\right), \end{cases} \tag{2.9}$$

subject to the following boundary conditions:

$$y(0) \in E_0, \tag{2.10}$$
$$y(t_f) \in E_f, \tag{2.11}$$
$$p(0) = \sum_{i=1}^{d} \mu_i \nabla E_{0i}(y(0)), \tag{2.12}$$
$$p(t_f) = -\sum_{j=1}^{b} \nu_j \nabla E_{fj}(y(t_f)), \tag{2.13}$$

where E_0 and E_f are manifolds in $\mathbb{R}^{n-1}$ given as follows:

$$E_0 = \left\{ y(t) \in \mathbb{R}^{n-1} \,|\, E_{01}(y(t) = E_{02}(y(t) = \cdots = E_{0d}(y(t)) = 0 \right\},$$
$$E_f = \left\{ y(t) \in \mathbb{R}^{n-1} \,|\, E_{f1}(y(t) = E_{f2}(y(t) = \cdots = E_{fb}(y(t)) = 0 \right\},$$

where the functions $E_{0i} : \mathbb{R}^{n-1} \mapsto \mathbb{R}$, $i = 1, \cdots, d$, (with $d \leq n-1$) and $E_{fj} : \mathbb{R}^{n-1} \mapsto \mathbb{R}$, $j = 1, \cdots, b$, (with $b \leq n-1$) are assumed to be continuously differentiable. $\mu = [\mu_1, \mu_2, \cdots, \mu_d]^T$ and $\nu = [\nu_1, \nu_2, \cdots, \nu_b]^T$ are the vectors of additional Lagrange multipliers associated with $E_{0i} = (E_{01}, E_{02}, \cdots, E_{0d})$ and $E_{fj} = (E_{f1}, E_{f2}, \cdots, E_{fb})$ respectively.

Remark 2.1. From [13, 22], we can observe that the following property holds:
1. If E_0 is reduced to a single point, that is $E_0 = \{y(0) = y_0\}$, then condition (2.12) becomes vacuous.
2. If $E_0 = \mathbb{R}^{n-1}$ meaning that the initial point is not specified, we obtain $p(0) = 0$.
3. If E_f is reduced to a single point, that is $E_f = \left\{y(t_f) = y_{t_f}\right\}$, then condition (2.13) becomes vacuous.
4. If $E_f = \mathbb{R}^{n-1}$ meaning that the final point is free, we obtain $p(t_f) = 0$.

2.3. Application of the Picard iteration method

2.3.1. A brief description. Before applying the Picard iteration method to solve the HP equations (2.8)–(2.9), it is important to first illustrate its fundamental principle. Consider the following differential equation

$$\dot{y}(t) = f(t, y(t)), \; y(t_0) = y_0, \; t > t_0, \tag{2.14}$$

where f is a continuous function. By integrating both sides of (2.14) with respect to t, we get the following integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau. \tag{2.15}$$

To solve (2.14) using the PIM method, we generate a sequence of functions $y_k(t)$ iteratively as

$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(\tau,\, y_k(\tau))\, d\tau. \tag{2.16}$$

where $y_0(t)$ is the initial guess, which can be chosen based on the initial condition of problem (2.14), and each approximation $y_{k+1}(t)$, $k \geq 0$ is defined in terms of the previous approximation $y_k(t)$ using the iterative formula (2.16). Therefore, the solution of problem (2.10) is given as the limit of the sequence of functions $\{y_k\}$ generated by the formula (2.16), that is

$$y(t) = \lim_{k\to\infty} y_k(t). \tag{2.17}$$

2.3.2. Convergence analysis. This section presents the convergence analysis of the PIM method for solving the HP equations (2.8)–(2.9). We set

$$\begin{cases} v(t) = \begin{pmatrix} y(t) \\ p(t) \end{pmatrix}, \\ \\ \varphi(t,\, v(t)) = \begin{pmatrix} \dot{y}(t) \\ \dot{p}(t) \end{pmatrix}, \end{cases} \tag{2.18}$$

where $\varphi(t,\, v(t)) : [0,\, t_f] \times \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ is assumed to be continuous in its arguments for all $t \in [0,\, t_f]$. Thus, the HP equations (2.8)–(2.9) can be conveniently rewritten as follows

$$\dot{v}(t) = \varphi(t,\, v(t)),\ v(0) = v_0 = (y_0,\, p_0). \tag{2.19}$$

Applying the PIM method to the IVP (2.19), we immediately obtain the following integral equation

$$v(t) = v_0 + \int_0^t \varphi(\tau,\, v(\tau))\, d\tau, \tag{2.20}$$

and the sequence of Picard iterates $\{v_k\}$ of the IVP (2.19) is given as

$$v_{k+1}(t) = v_0 + \int_0^t \varphi(\tau,\, v_k(\tau))\, d\tau, \qquad k \geq 0. \tag{2.21}$$

The following lemma establishes the equivalence between the IVP (2.15) and the integral equation (2.20).

Lemma 2.2. *Assume that $\mathcal{D}$ is an open subset of $[0,\, t_f] \times \mathbb{R}^{2n}$, $\varphi(t,\, v(t)) : \mathcal{D} \mapsto \mathbb{R}^{2n}$ is continuous, and $(0,\, v_0) \in \mathcal{D}$; then $v(t)$ is a solution of the IPV (2.19) on $[0,\, t_f]$ if and only if $v(t)$ is a solution of (2.20).*

Proof. We refer to the proof of [11, Lemma 8.6] for details. □

Definition 2.3. *A vector function $\varphi(t,\, v(t)) : \mathcal{D} \mapsto \mathbb{R}^{2n}$ is said to satisfy a uniform Lipschitz condition with respect to $v(t)$ on $\mathcal{D} \subset [0,\, t_f] \times \mathbb{R}^{2n}$ if there exists a positive constant C such that*

$$\|\varphi(t,\, v(t)) - \varphi(t,\, u(t))\| \leqslant C\, \|(v(t) - u(t))\|, \tag{2.22}$$

for all $(t, v(t)), (t,\, u(t)) \in \mathcal{D}$. The constant C is called a Lipschitz constant for $\varphi(t,\, v(t))$ with respect to $v(t)$ on the open set $\mathcal{D}$.

According to the Picard-Lidelof theorem [11, Theorem 8.13], we obtain the following result regarding the sufficient conditions for the convergence of the PIM method applied to the HP equations (2.18).

Theorem 2.4. *Assume that* $\varphi(t,\, v(t)) : \mathcal{D} \mapsto \mathbb{R}^{2n}$ *is a continuous 2n-dimensional vector function on the parallelepiped*

$$\Gamma = \{(t,\, v(t)) \,:\, 0 \leq t \leq s,\ \|v(t) - v_0(t)\| \leq l\},\, l > 0, \tag{2.23}$$

and assume that $\varphi(t,\, v(t))$ *satisfies a uniform Lipschitz condition with respect to* $v(t)$ *on* Γ*, and let*

$$\lambda = \max\{\|\varphi(t,\, v(t))\| : (t,\, v(t)) \in \Gamma\},\, \gamma = \min\left\{s,\, \frac{l}{\lambda}\right\}. \tag{2.24}$$

Then the sequence of Picard iterates $\{v_k(t)\}$ *given by (2.21) converges to the exact solution* v *of the IVP (2.19) on* $[0,\, \gamma]$*, such that*

$$\|v(t) - v_0(t)\| \leq l,\ \forall\, t \in [0,\, \gamma]. \tag{2.25}$$

Proof. The proof is inspired by [11]. We first prove by induction that each Picard iterate v_k from the sequence of Picard iterates $\{v_k(t)\},\ k \geq 0$, defined by (2.21), is continuous on the interval$[0,\, \gamma]$ with respect to $v(t)$ on Γ. It is clear that $v_0(t)$ satisfies these conditions. Suppose now that $v_k(t)$ is well defined and continuous on $[0,\, \gamma]$, and satisfies the required assumptions

$$\|v_k(t) - v_0\| \leq l, \quad t \in [0,\, \gamma]. \tag{2.26}$$

Using Lemma 2.2, it follows that

$$v_{k+1}(t) = v_0 + \int_0^t \psi(\tau,\, v_k(\tau))\, d\tau, \quad t \in [0,\, \gamma], \tag{2.27}$$

is well defined and continuous on $[0,\, \gamma]$.
Moreover, for all $t \in [0,\, \gamma]$ we have

$$\begin{aligned}\|v_{k+1}(t) - v_0\| &\leq \int_0^t |\, \psi(\tau,\, v_k(\tau))\,|\, d\tau \\ &\leq \lambda(t) \leq \lambda\gamma \leq l,\end{aligned}$$

and the induction is complete.

Let C be a Lipschitz constant for $\varphi(t,\, v(t))$ with respect to $v(t)$ on Γ, and we consider the following hypothesis:

$$\|v_{k+1}(t) - v_k(t)\| \leq \frac{\lambda\, C^k\, (t)^{k+1}}{(k+1)!}, \quad k \geq 0,\ t \in [0,\, \gamma]. \tag{2.28}$$

Let us verify this hypothesis by induction. The result is trivially true for $k = 0$. Now, assume that it holds for $k - 1$. Then, using the Lipschitz condition (2.22) together with the induction assumption, we obtain

$$
\begin{aligned}
\|v_{k+1}(t) - v_k(t)\| &= \| \int_0^t [\varphi(\tau, v_k(\tau)) - \varphi(\tau, v_{k-1}(\tau))] \, d\tau \| \\
&\leq \int_0^t \| \varphi(\tau, v_k(\tau)) - \varphi(\tau, v_{k-1}(\tau)) \| \, d\tau \\
&\leq C \int_0^t \| v_k(\tau) - v_{k-1}(\tau) \| \, d\tau \\
&\leq \lambda \, C^k \int_0^t \frac{(\tau)^k}{k!} \, d\tau \\
&= \frac{\lambda \, C^k \, (t)^{k+1}}{(k+1)!}, \quad holds \ for \ all \ \ t \in [0, \gamma].
\end{aligned}
$$

Hence the induction is complete.

We now proceed to prove the convergence of the sequence of Picard iterates $\{v_k(t)\}$. For this purpose, we rewrite $\{v_k(t)\}$ as the sequence of partial sums of an infinite series, given by:

$$
\{v_k(t)\}_{k=1}^{\infty} = \{\varphi_0(t) + \sum_{m=0}^{k-1} (v_{m+1}(t) - v_m(t)) \}. \tag{2.29}
$$

Obviously, we have

$$
\|v_{m+1}(t) - v_m(t)\| \leq \frac{\lambda \, (C \, \gamma)^{m+1}}{C \, (m+1)!}, \quad holds \ for \ all \ \ t \in [0, \gamma], \tag{2.30}
$$

and

$$
\sum_{m=0}^{\infty} \frac{\lambda \, (C \, \gamma)^{m+1}}{C \, (m+1)!} \ \text{converge.}
$$

Consequently, by applying the Weierstrass M-test [23], it follows that the sequence of Picard iterates $\{v_k(t)\}$ converges uniformly on $[0, \gamma]$. Therefore, we can write:

$$
v(t) = \lim_{k \to \infty} \ v_k(t), \ \ \forall \, t \in [0, \gamma]. \tag{2.31}
$$

It follows that (2.25) holds for all $t \in [0, \gamma]$. Moreover, since the Lipschitz condition (2.22) is satisfied,

$$
\|\varphi(t, v_k(t)) - \varphi(t, v(t))\| \leqslant C \, \|(v_k(t) - v(t))\|, \ \ t \in \ [0, \gamma], \tag{2.32}
$$

we deduce that

$$
\lim_{k \to \infty} \ \varphi(t, v_k(t)) = \psi(t, v(t)), \tag{2.33}
$$

uniformly on $[0, \gamma]$.

Now, applying the limits (2.31) and (2.33) to both sides of Eq. (2.27), we obtain

$$
v(t) = v_0 + \int_0^t \varphi(\tau, v(\tau)) \, d\tau, \quad t \in [0, \gamma], \tag{2.34}
$$

which means that (2.34) is a solution of the IVP (2.19).

To prove the uniqueness, suppose that g is another solution of the IVP (2.15) on $[0,\, \mu]$, where $0 \le \mu \le \gamma$, and we prove that $g = v$. From Lemma 2.2, it follows that g satisfies the following integral equation:

$$g(t) = v_0 + \int_0^t \varphi(\tau,\, g(\tau))\, d\tau, \quad t \in [0,\, \mu]. \tag{2.35}$$

Similarly, we can prove by induction that

$$\|g(t) - v_k(t)\| \le \frac{\lambda\, C^k\, (t)^{k+1}}{(k+1)!}, \quad k \ge 0,\ t \in [0,\, \mu]. \tag{2.36}$$

Consequently, we get

$$g(t) = \lim_{k\to\infty} v_k(t) = v_k(t), \quad t \in [0,\, \mu],$$

and this completes the proof. □

The following result follows immediately from Theorem 2.4.

Corollary 2.5. *Let $\{v_k(t)\}$ be the sequence of the Picard iterates generated by the iterative formula (2.13) of the IVP (2.15), and assume that the hypotheses of Theorem 2.4 hold. If φ is the exact solution of the (2.15), then*

$$\|v(t) - v_k(t)\| \le \frac{\lambda\, C^k\, (t)^{k+1}}{(k+1)!}, \quad k \ge 0,\ t \in [0,\, \gamma], \tag{2.37}$$

where C be a Lipschitz constant for $\varphi(t,\, v(t))$ with respect to $v(t)$ on Γ.

3. Algorithm for the proposed method

In this section a proposed algorithm based on the PIM to approximate the solution of the optimal control problem (1.1)–(1.4) is proposed. The different steps of the proposed approach can be summarized as follows

Algorithm 3.1. Algorithm for the proposed approach

Step 1. Transform the optimal control problem (1.1)–(1.4) into an optimal control problem governed by a system of ODEs using the finite difference approximation.
Step 2. Determine the necessary optimality conditions for the an approximation of $J(u_0(t), u_\ell(t))$ using the PMP.
Step 3. Establish the sequence of Picard iterates $y_{k+1}(t)$ and $p_{k+1}(t)$ of the IVP (2.8)-(2.9) using the PIM method.
Step 4. Choose a desired threshold $\epsilon > 0$, set $k = 0$, and let initial approximations $y_0(t) = y(0),\ t \in [0,\, t_f]$ and $p_0(t) = \Lambda$, where Λ is a vector which can be determined by using boundary conditions.
Step 5. Determine the approximate solution $y_{k+1}(t)$ and $p_{k+1}(t)$ using the iterative formula from step 3.

Step 6. Deduce the optimal control $u_0^k(t)$ and $u_\ell^k(t)$ using expression (2.6), and evaluate the corresponding objective function.
Step 7. Stopping criterion. if

$$|J(u_0^{k+1}(t), u_\ell^{k+1}(t)) - J(u_0^k(t), u_\ell^k(t))| \leq \epsilon,$$

stop the iteration process. Otherwise, set $k = k + 1$, and go to step 6.

4. Numerical example and discussions

In this section, the proposed method is examined through numerical experiments. To demonstrate its efficiency, the results obtained using the proposed method are compared with those obtained using the variational iteration method (VIM) [3]. In the calculations, the following data is used:
$\ell = \pi, t_f = 1, q_0 = q_\ell = 0, y(0) = 0, y(t_f) = \frac{3}{2}, n = 50$ and the step size $h = \dfrac{2500}{\pi^2}$.

Therefore, according to the PIM we construct the following iterative formulas:

$$\begin{cases} y_1^{(k+1)}(t) &= y_1(0) + \displaystyle\int_0^t \frac{1}{h^2}\left(y_2^{(k)}(\tau) - 2\,y_1^{(k)} - \frac{p_1^{(k)}(\tau)}{h^3}\right) d\tau, \\ y_m^{(k+1)}(t) &= y_m(0) + \displaystyle\int_0^t \frac{1}{h^2}\left(y_{m+1}^{(k)}(\tau) - 2\,y_m^{(k)} - y_{m-1}^{(k)}\right) d\tau,\ m = 2, \dots, 48, \\ y_{49}^{(k+1)}(t) &= y_{49}(0) + \displaystyle\int_0^t \frac{1}{h^2}\left(y_{48}^{(k)}(\tau) - 2\,y_{49}^{(k)} - \frac{p_{49}^{(k)}(\tau)}{h^3}\right) d\tau. \end{cases} \tag{4.1}$$

$$\begin{cases} p_1^{(k+1)}(t) = p_1(0) - \displaystyle\int_0^t \left(2h\,y_1^{(k)}(\tau) - 2\frac{p_1^{(k)}(\tau)}{h^2} + \frac{p_2^{(k)}(\tau)}{h^2}\right) d\tau, \\ p_m^{(k+1)}(t) = p_m(0) - \displaystyle\int_0^t \left(2h\,y_m^{(k)}(\tau) + \frac{p_{m-1}^{(k)}}{h^2} - 2\frac{p_m^{(k)}}{h^2} + \frac{p_{m+1}^{(k)}}{h^2}\right) d\tau, \\ \text{for, } m = 2, \dots, 48, \\ p_{49}^{(k+1)}(t) = p_{49}(0) - \displaystyle\int_0^t \left(2h\,y_{49}^{(k)}(\tau) + \frac{p_{48}^{(k)}(\tau)}{h^2} - 2\frac{p_{49}^{(k)}(\tau)}{h^2}\right) d\tau. \end{cases} \tag{4.2}$$

From (2.5) and (2.6), the optimal control law is given as:

$$\begin{cases} u_0^*(t) = y_0(t) = -\dfrac{p_1(t)}{h^3}, \\ u_\ell^*(t) = y_{50}(t) = -\dfrac{p_{49}(t)}{h^3}, \end{cases} \tag{4.3}$$

where, the zeroth approximation can be chosen as

$$y_m(0) = 0 \text{ and } p_m(0) = \lambda_m,\ \forall\, m = 1, ..., 49,$$

where λ_m are unknown parameters to be determined by imposing the final condition $y(t_f)$. The results of the iterative process are presented in Table 1.

TABLE 1. Iterations result

k	$\lvert J^{k+1} - J^k \rvert$	k	$\lvert J^{k+1} - J^k \rvert$
0	/	6	0.02877604
1	1.67698260	7	0.01246203
2	1.04756755	8	0.00409854
3	0.10723897	9	0.00083735
4	0.10156208	10	0.00009973
5	0.08039717	11	0.00000073

Based on the obtained results, the optimal control is determined using a convergence threshold of $\epsilon = 10^{-6}$. Therefore, we conclude that the proposed method converges after 11 iterations, and yields the following approximate optimal control law:

$$\begin{aligned} u_0^*(t) = \left(-\tfrac{50}{\pi}\right)^3 \Big[& \left(\tfrac{1.65611e{+}12\, t^{11}}{\pi^{22}}\right) + \left(\tfrac{1.94656e{+}11\, t^{10}}{\pi^{20}}\right) + \left(\tfrac{5.18143e{+}10\, t^{9}}{\pi^{18}}\right) \\ & + \left(\tfrac{7.8365e{+}9\, t^{8}}{\pi^{16}}\right) + \left(\tfrac{8.30182e{+}8\, t^{7}}{\pi^{14}}\right) + \left(\tfrac{1.0815e{+}8\, t^{6}}{\pi^{12}}\right) + \left(\tfrac{6.7437e{+}6\, t^{5}}{\pi^{10}}\right) \\ & + \left(\tfrac{6.3835e{+}5\, t^{4}}{\pi^{8}}\right) + \left(\tfrac{2.1615e{+}4\, t^{3}}{\pi^{6}}\right) + \left(\tfrac{1.2509e{+}3\, t^{2}}{\pi^{4}}\right) + \left(\tfrac{15.15\, t}{\pi^{2}}\right) + 0.987 \Big] \end{aligned}$$

$$\begin{aligned} u_l^*(t) = \left(-\tfrac{50}{\pi}\right)^3 \Big[& \left(\tfrac{6.1665e{+}12\, t^{11}}{\pi^{22}}\right) + \left(\tfrac{1.6388e{+}12\, t^{10}}{\pi^{20}}\right) + \left(\tfrac{1.1178e{+}11\, t^{9}}{\pi^{18}}\right) \\ & + \left(\tfrac{2.3576e{+}10\, t^{8}}{\pi^{16}}\right) + \left(\tfrac{1.4122e{+}9\, t^{7}}{\pi^{14}}\right) + \left(\tfrac{2.2684e{+}8\, t^{6}}{\pi^{12}}\right) + \left(\tfrac{1.1203e{+}7\, t^{5}}{\pi^{10}}\right) \\ & + \left(\tfrac{1.2864e{+}6\, t^{4}}{\pi^{8}}\right) + \left(\tfrac{4.7711e{+}4\, t^{3}}{\pi^{6}}\right) + \left(\tfrac{3.5206e{+}3\, t^{2}}{\pi^{4}}\right) + \left(\tfrac{79.75\, t}{\pi^{2}}\right) + 2.846 \Big] \end{aligned}$$

In Figure 1, the approximate optimal trajectories of the control variables obtained using the proposed approach are plotted alongside those obtained with the Variational Iteration Method (VIM)[3]. The plots show a high level of agreement between the two methods.

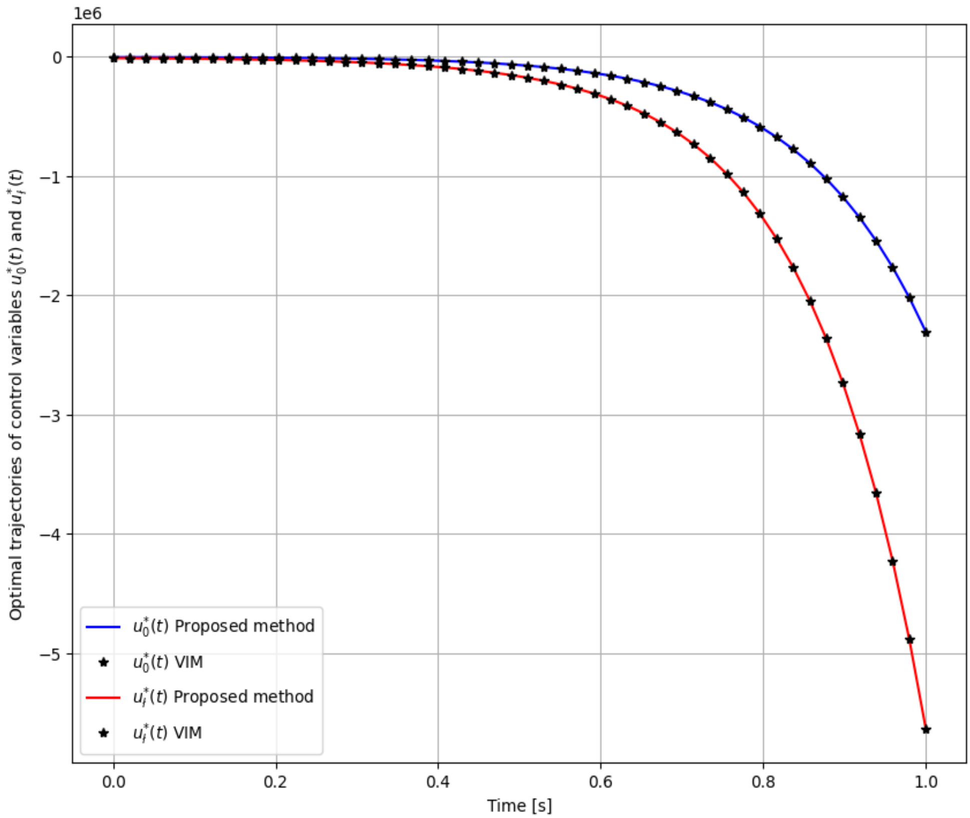

FIGURE 1. Optimal trajectories of control variables

5. Conclusion

In this work, a new approach based on the Picard iteration method is successfully employed to obtain an approximate solution to an optimal control problem governed by a one-dimensional parabolic partial differential equation with two-point boundary conditions. In fact, the Picard iteration method is adapted to solve the necessary optimality conditions derived from Pontryagin's minimum principle. The resulting solution is expressed as a truncated power series.

The proposed approach is illustrated by a numerical example. To demonstrate its efficiency, a comparison is made between the obtained results and those obtained using the variational iteration method (VIM), showing that the results are very close. The main advantage of the proposed method over the variational iteration method is that it provides a solution to the problem without requiring any restrictive assumptions. Furthermore, unlike the variational iteration method, it does not require the computation of a Lagrange multiplier.

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