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On a Riemann-Liouville fractional anti-periodic boundary value problem in a weighted space

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Abstract. This paper is concerned with the existence of solutions for a fractional anti-periodic boundary value problem of order $\alpha \in (2, 3]$ involving Riemann–Liouville fractional derivative and integral operators in a weighted space. The existence of solutions for the given problem is shown by means of the Leray-Schauder's alternative, while the uniqueness of its solutions is established with the aid of the Banach's fixed point theorem. We also discuss the Ulam–Hyers stability for the problem at hand. Examples are presented for illustration of the main results.

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1. Introduction

Riemann–Liouville fractional differential equations have been extensively studied in the literature. It has been mainly due to the ability of fractional operators to describe memory and hereditary characteristics of several materials and processes. One can find applications of Riemann–Liouville fractional derivative operators in the study of complex systems like polymers, biological tissue and self-similar protein dynamics [27], real materials [31], projectile motion [5], electrical circuits [9], backward diffusion problems [32], viscoelasticity [25], bioengineering [26], fractional dynamics and control [11], modeling framework [35], etc.

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Influenced by the occurrence of Riemann–Liouville fractional operators in the mathematical modeling of several real-world phenomena, many researchers have shown a keen interest in developing theoretical aspects of Riemann–Liouville type fractional boundary value problems, for instance, see [23, 2, 3, 33, 7, 12, 28]. Anti-periodic boundary conditions form a special case of non-separated boundary conditions and have been extensively studied in the literature, for instance, see the papers [1, 19, 15, 8]. In [4], the authors solved a fractional integro-differential equation equipped with a new class of dual anti-periodic boundary conditions. In [6], the authors studied a Riemann-Liouville fractional differential equation of order $\alpha \in (1, 2]$ with fractional anti-periodic boundary conditions.

In 1940, Ulam [34] introduced the notion of stability for a functional equation to find a set of conditions ensuring an approximate solution of this equation to be close to its exact solution. Hyers [16] discussed the Ulam's idea of stability more rigorously in the context of Banach spaces in 1941. Later, it was known as the Ulam–Hyers stability. Then, Rassias [29] applied the idea of Ulam–Hyers stability to a wide class of functional equations, which is now referred to as Ulam–Hyers-Rassias stability [17]. The Ulam-Hyers stability for Black-Scholes equation was studied in [24]. For some recent results on Ulam–Hyers stability for fractional differential equations, for instance, see [37, 20, 22, 13, 10, 14, 36].

In this paper, motivated by the foregoing discussion, we discuss the existence, uniqueness and Ulam-Hyers stability for solutions of a nonlinear Riemann–Liouville fractional differential equation equipped with fractional anti-periodic boundary conditions in a weighted space. Precisely, we consider a nonlinear Riemann–Liouville fractional differential equation

$$D^{\alpha}x(t) = f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)),\ 2 < \alpha \leq 3,\ t \in \mathcal{J} = [0, T],\ T > 0, \tag{1.1}$$

complemented with fractional anti-periodic boundary conditions

$$\begin{cases} D^{\alpha-1}x(0^+) + D^{\alpha-1}x(T^-) = 0, \\ D^{\alpha-2}x(0^+) + D^{\alpha-2}x(T^-) = 0, \\ D^{\alpha-3}x(0^+) + D^{\alpha-3}x(T^-) = 0, \end{cases} \tag{1.2}$$

where D^{α} denote the Riemann–Liouville fractional derivative operator of order α, $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an appropriate continuous function and

$$(\lambda_1 x)(t) = \int_0^t \Psi_1(t, s)\, x(s)\, ds, \quad (\lambda_2 x)(t) = \int_0^t \Psi_2(t, s)\, x(s)\, ds, \tag{1.3}$$

with Ψ_1 and Ψ_2 being continuous functions on $\mathcal{J} \times \mathcal{J}$. The relationship between the Green's functions of lower- and higher-order anti-periodic fractional boundary value problems is also described.

We derive the existence and uniqueness results for the problem (1.1)-(1.2) with the aid of the Leray-Schauder's alternative and Banach's contraction mapping principle.

We arrange the remaining content of the paper as follows. Section 2 contains some preliminary concepts and a subsidiary result related to the linear version of the given nonlinear problem. An interesting discussion concerning the Green's function is given in Section 3. We prove the existence and uniqueness results for the problem (1.1)-(1.2) in Section 4, which are well-illustrated with examples in Section 5. We discuss the Ulam–Hyers stability for the problem at hand in Section 6.

2. A subsidiary result

Let us first recall some basic concepts of fractional calculus from the text [21].

Definition 2.1. *For $\varphi \in L_1[a,b]$, the (left) Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$, denoted by $I_{a+}^{\alpha}\varphi$, is defined as*

$$I_{a+}^{\alpha}\varphi(t) = (\varphi * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds,$$

where $K_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ denotes the Euler gamma function.

Definition 2.2. *Let $\varphi, \varphi^{(m)} \in L_1[a,b], a, b \in \mathbb{R}$ and $\alpha \in (m-1, m]$, $m \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order α, denoted by $D_{a+}^{\alpha}\varphi$, is defined as*

$$D_{a+}^{\alpha}\varphi(t) = \frac{d^m}{dt^m}I_{a+}^{1-\alpha}\varphi(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-1-\alpha}\varphi(s)ds.$$

In the current work, we write the Riemann–Liouville fractional integral and derivative operators I_{a+}^{q} and D_{a+}^{q} as I^q and D^q when $a=0$, respectively.

Lemma 2.3. *Let p and q be positive reals. If φ is a continuous function, then*
(*i*) $I^pI^q\varphi(t) = I^{p+q}\varphi(t)$,
(*ii*) $D^pI^q\varphi(t) = I^{q-p}\varphi(t)$ *for* $q > p > 0$.

Note that $D^p t^{p-i} = 0,\ i = 1, 2, \ldots, [p]+1$, where $[p]$ is the largest integer less than p and

$$D^p t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-p+1)}t^{\lambda-p},\ \lambda > -1,\ \lambda \neq p-1, p-2, \ldots, p-n.$$

In the following lemma, we solve the linear version of the equation (1.1) complemented with the boundary data (1.2).

Lemma 2.4. *For $g \in C(\mathcal{J}, \mathbb{R})$, the unique solution of the linear equation*

$$D^\alpha x(t) = g(t),\ 2 < \alpha \leq 3,\ t \in \mathcal{J}, \tag{2.1}$$

subject to the boundary conditions (1.2) *is given by*

$$\begin{aligned} x(t) &= \int_0^T \left[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\right]g(s)ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\mu_1(t) &= \frac{-t^{\alpha-1}}{2\Gamma(\alpha)}+\frac{Tt^{\alpha-2}}{4\Gamma(\alpha-1)},\ \mu_2(t)=\frac{-t^{\alpha-2}}{2\Gamma(\alpha-1)}+\frac{Tt^{\alpha-3}}{4\Gamma(\alpha-2)},\\ \mu_3(t) &= \frac{-t^{\alpha-3}}{2\Gamma(\alpha-2)}.\end{aligned} \tag{2.3}$$

Proof. Operating the integral operator I^α to (2.1), we get

$$x(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+I^\alpha g(t), \tag{2.4}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ are unknown arbitrary constants. From (2.4), we have

$$\begin{cases} D^{\alpha-1}x(t)=c_1\Gamma(\alpha)+I^1g(t),\\ D^{\alpha-2}x(t)=c_1\Gamma(\alpha)t+c_2\Gamma(\alpha-1)+I^2g(t),\\ D^{\alpha-3}x(t)=c_1\dfrac{\Gamma(\alpha)}{2}t^2+c_2\Gamma(\alpha-1)t+c_3\Gamma(\alpha-2)+I^3g(t).\end{cases} \tag{2.5}$$

Using (2.5) in the boundary condition (1.2), we obtain

$$\begin{cases} 2c_1\Gamma(\alpha)+I^1g(T)=0,\\ 2c_2\Gamma(\alpha-1)+I^2g(T)+c_1\Gamma(\alpha)T=0,\\ 2c_3\Gamma(\alpha-2)+I^3g(T)+\dfrac{c_1\Gamma(\alpha)T^2}{2}+c_2\Gamma(\alpha-1)T=0.\end{cases} \tag{2.6}$$

From (2.6), we get

$$\begin{aligned} c_1 &= \frac{-1}{2\Gamma(\alpha)}I^1g(T),\ c_2=\frac{1}{2\Gamma(\alpha-1)}\left(\frac{T}{2}I^1g(T)-I^2g(T)\right),\\ c_3 &= \frac{1}{2\Gamma(\alpha-2)}\left(\frac{T}{2}I^2g(T)-I^3g(T)\right).\end{aligned} \tag{2.7}$$

Inserting the above values of c_1, c_2 and c_3 in (2.4) together with the notation (2.3), we obtain the solution (2.2). The converse of the lemma follows by direct computation. The proof is completed. □

Remark 2.5. The solution (2.2) of the equation (2.1) subject to the boundary conditions (1.2) can be expressed in terms of the Green's function as

$$x(t)=\int_0^T G(t,s,\alpha)x(s)ds,$$

where

$$G(t,s,\alpha)=\begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\dfrac{t^{\alpha-1}}{2\Gamma(\alpha)}+\dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T)\\ \quad+\underbrace{\dfrac{st^{\alpha-3}(T-s)}{4\Gamma(\alpha-2)}}, & s\le t,\\[2ex] \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)}+\dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T)+\underbrace{\dfrac{st^{\alpha-3}(T-s)}{4\Gamma(\alpha-2)}}, & t\le s.\end{cases} \tag{2.8}$$

3. An interesting analogy

It is interesting to note that the Green's function (2.8) contains the expressions for the Green's function for the problem

$$\begin{cases} D^{\alpha}x(t) = g(t),\ 1 < \alpha \le 2,\ t \in \mathcal{J}, \\ D^{\alpha-1}x(0^+) = -D^{\alpha-1}x(T^-),\ D^{\alpha-2}x(0^+) = -D^{\alpha-2}x(T^-), \end{cases} \tag{3.1}$$

given by

$$G(t,s,\alpha) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}}{2\Gamma(\alpha)} + \underbrace{\dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T)}, & s \le t, \\ \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)} + \underbrace{\dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T)}, & t \le s. \end{cases} \tag{3.2}$$

Likewise, the Green's function (3.2) contains the one for the problem

$$\begin{cases} D^{\alpha}x(t) = g(t),\ 0 < \alpha \le 1,\ t \in \mathcal{J}, \\ D^{\alpha-1}x(0^+) = -D^{\alpha-1}x(T^-), \end{cases} \tag{3.3}$$

given by

$$G(t,s,\alpha) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}}{2\Gamma(\alpha)}, & s \le t, \\ \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)}, & t \le s. \end{cases} \tag{3.4}$$

Thus, we deduce that the Green's function for a higher order fractional boundary value problem involving a Riemann-Liouville fractional differential equation contains the expressions for the Green's functions associated with lower order Riemann-Liouville fractional anti-periodic boundary value problems. The additional terms in (2.8) and (3.2) with reference to the Green's functions for the lower order problems (3.1) and (3.3) respectively are indicated with the under-braces.

4. Main Results

Let $C(\mathcal{J},\mathbb{R})$ be the Banach space of all continuous real-valued functions from $\mathcal{J} \to \mathbb{R}$ endowed with the supremum norm $\|x\| = \sup_{t\in\mathcal{J}} |x(t)|$. For $t \in \mathcal{J}$, we define $x_{\tilde r}(t) = t^{\tilde r}x(t)$, $\tilde r > 0$, and let $C_{\tilde r}(\mathcal{J},\mathbb{R})$ be the space of all functions $x_{\tilde r}$ such that $x \in C(\mathcal{J},\mathbb{R})$ which turns out to be a Banach space when endowed with the norm $\|x\|_{\tilde r} = \sup_{t\in\mathcal{J}}\{t^{\tilde r}|x(t)|\}$.

By Lemma 2.4, we transform the problem (1.1)-(1.2) into a fixed point problem as

$$x = \mathcal{T}x,$$

where $\mathcal{T} : C_{3-\alpha}(\mathcal{J},\mathbb{R}) \to C_{3-\alpha}(\mathcal{J},\mathbb{R})$ is an operator defined by

$$\begin{aligned}(\mathcal{T}x)(t) &= \int_0^T \Big[\mu_1(t)+\mu_2(t)(T-s)+\mu_3(t)\frac{(T-s)^2}{2}\Big]\times\\ &\quad \times f(s,x(s),(\lambda_1 x)(s),(\lambda_2 x)(s))ds\\ &\quad +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),(\lambda_1 x)(s),(\lambda_2 x)(s))ds. \qquad (4.1)\end{aligned}$$

Observe that the fixed points of the operator $\mathcal{T}$ are solution to the problem (1.1)-(1.2).

Lemma 4.1. *Assume that* $f : \mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is continuous. Then, the operator* $\mathcal{T} : C_{3-\alpha}(\mathcal{J},\mathbb{R}) \to C_{3-\alpha}(\mathcal{J},\mathbb{R})$ *is compact.*

Proof. Let us first note that continuity of $\mathcal{T}$ follows from that of f. Let $\mathcal{G}$ be a bounded set in $C_{3-\alpha}(\mathcal{J},\mathbb{R})$. Then, there exists a positive constant N_f such that $|f(t,x(t),(\lambda_1 x)(t),(\lambda_2 x)(t))| \le N_f$, $\forall x\in\mathcal{G}$, $t\in\mathcal{J}$. In consequence, we have

$$\begin{aligned}&\|\mathcal{T}x\|_{3-\alpha}\\ =\ & \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg|\int_0^T \Big[\mu_1(t)+\mu_2(t)(T-s)+\mu_3(t)\frac{(T-s)^2}{2}\Big]\times\\ & \times f(s,x(s),(\lambda_1 x)(s),(\lambda_2 x)(s))ds\\ & +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),(\lambda_1 x)(s),(\lambda_2 x)(s))ds\Bigg|\Bigg\}\\ \le\ & \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[N_f\int_0^T\Big[|\mu_1(t)|+|\mu_2(t)|(T-s)+|\mu_3(t)|\frac{(T-s)^2}{2}\Big]ds\\ & +N_f\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg]\Bigg\}\\ \le\ & N_f\Big[\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_1(t)|T+\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_2(t)|\frac{T^2}{2}+\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_3(t)|\frac{T^3}{6}\Big]\\ & +N_f\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\}\\ =\ & \Big[\delta_1 T+\delta_2\frac{T^2}{2}+\delta_3\frac{T^3}{6}+\frac{T^3}{\Gamma(\alpha+1)}\Big]N_f,\end{aligned}$$

where

$$\delta_i = \sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_i(t)|,\ i=1,2,3, \qquad (4.2)$$

and $\mu_i, i = 1,2,3$, are given in (2.3). Thus, we have that $\mathcal{T}(\mathcal{G}) < \infty$. Hence $\mathcal{T}(\mathcal{G})$ is uniformly bounded. For verifying that $\mathcal{T}(\mathcal{G})$ is equicontinuous, we take $\tau_1,\tau_2\in\mathcal{J}$ with $\tau_1<\tau_2$. Then, we obtain

$$|\tau_2^{3-\alpha}(\mathcal{T}x)(\tau_2)-\tau_1^{3-\alpha}(\mathcal{T}x)(\tau_1)|$$

$$
\begin{aligned}
= \;& \Bigg| \tau_2^{3-\alpha} \Bigg[\int_0^T \Big[\mu_1(\tau_2) + \mu_2(\tau_2)(T-s) + \mu_3(\tau_2) \frac{(T-s)^2}{2} \Big] \times \\
& \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \\
& + \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \Bigg] \\
& - \tau_1^{3-\alpha} \Bigg[\int_0^T \Big[\mu_1(\tau_1) + \mu_2(\tau_1)(T-s) + \mu_3(\tau_1) \frac{(T-s)^2}{2} \Big] \times \\
& \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \\
& + \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \Bigg] \Bigg| \\
\leq \;& \Big[|\tau_2^{3-\alpha} \mu_1(\tau_2) - \tau_1^{3-\alpha} \mu_1(\tau_1)| T + |\tau_2^{3-\alpha} \mu_2(\tau_2) - \tau_1^{3-\alpha} \mu_2(\tau_1)| \frac{T^2}{2} \\
& + |\tau_2^{3-\alpha} \mu_3(\tau_2) - \tau_1^{3-\alpha} \mu_3(\tau_1)| \frac{T^3}{6} \Big] N_f + \frac{2N_f}{\Gamma(\alpha+1)} \tau_2^{3-\alpha} (\tau_2 - \tau_1)^\alpha \\
& + \frac{N_f}{\Gamma(\alpha+1)} |\tau_2^3 - \tau_1^3| \\
\leq \;& \frac{TN_f}{2\Gamma(\alpha)} |\tau_2^2 - \tau_1^2| + \frac{TN_f}{4\Gamma(\alpha-1)} |\tau_2 - \tau_1| \big(T+1\big) + \frac{2N_f}{\Gamma(\alpha+1)} \tau_2^{3-\alpha} (\tau_2 - \tau_1)^\alpha \\
& + \frac{N_f}{\Gamma(\alpha+1)} |\tau_2^3 - \tau_1^3|, \qquad (4.3)
\end{aligned}
$$

which tends to zero as $\tau_2 \to \tau_1$ independent of $x \in \mathcal{G}$. Thus, $\mathcal{T}(\mathcal{G})$ is equicontinuous. From the preceding steps, it follows that $\mathcal{T}$ is compact. □

Now, we prove our first existence result for the problem (1.1)-(1.2) by applying the Leray-Schauder alternative [18, Theorem 2.4, p.4].

Theorem 4.2. *Assume that $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. In addition, we suppose that there exist a constant $N_f > 0$ such that $|f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t))| \leq N_f$, $\forall x \in \mathbb{R}$, $t \in \mathcal{J}$. Then, the problem* (1.1)-(1.2) *has at least one solution on $\mathcal{J}$.*

Proof. From Lemma 4.1, we know that $\mathcal{T}$ is completely continuous. So, the conclusion of the Leray-Schauder alternative will be applicable when it is shown that the set $V = \{t^{3-\alpha} x \in \mathbb{R} : t^{3-\alpha} x = \xi t^{3-\alpha} \mathcal{T} x, 0 < \xi < 1\}$ is bounded. For $x \in V$, we have $|t^{3-\alpha} x(t)| = |\xi t^{3-\alpha} \mathcal{T} x(t)| < t^{3-\alpha} |\mathcal{T} x(t)|$. Following the method of proof of Lemma 4.1, we obtain

$$
\|x\|_{3-\alpha} < \Big[\delta_1 T + \delta_2 \frac{T^2}{2} + \delta_3 \frac{T^3}{6} + \frac{T^3}{\Gamma(\alpha+1)} \Big] N_f < \infty, \qquad (4.4)
$$

which implies that the set V is bounded. Thus, by the Leray-Schauder alternative, we deduce that the operator $\mathcal{T}$ has at least one fixed point, which is indeed a solution of the problem (1.1)-(1.2). □

Next, we establish the existence of a unique solution to the problem (1.1)–(1.2) with the aid of Banach's fixed point theorem.

Theorem 4.3. *Let $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition:*

(H_1) *There exist positive functions $\mathbb{L}_1(t), \mathbb{L}_2(t), \mathbb{L}_3(t)$ such that*

$$\begin{aligned} & |f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) - f(t, y(t), (\lambda_1 y(t), (\lambda_2 y)(t))| \\ \leq \; & \mathbb{L}_1(t)|x - y| + \mathbb{L}_2(t)|\lambda_1 x - \lambda_1 y| + \mathbb{L}_3(t)|\lambda_2 x - \lambda_2 y|, \ \forall t \in \mathcal{J}, \ x, y \in \mathbb{R}. \end{aligned}$$

Then, the problem (1.1)–(1.2) has a unique solution in $\mathcal{J}$, provided that

$$\Lambda = \left(1 + \frac{\psi_1 + \psi_2}{(\alpha - 2)}\right)\left(\delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 + T^{3-\alpha} e_4\right) < 1, \tag{4.5}$$

where $\delta_m, \ m = 1, 2, 3,$ are given by (4.2),

$\psi_1 = \sup_{t,s \in \mathcal{J}} |\Psi_1(t, s)|, \ \psi_2 = \sup_{t,s \in \mathcal{J}} |\Psi_2(t, s)|,$
$e_1 = \max\{|I\mathbb{L}_1(T)T^{\alpha-3}|, |I\mathbb{L}_2(T)T^{\alpha-2}|, |I\mathbb{L}_3(T)T^{\alpha-2}|\},$
$e_2 = \max\{|I^2\mathbb{L}_1(T)T^{\alpha-3}|, |I^2\mathbb{L}_2(T)T^{\alpha-2}|, |I^2\mathbb{L}_3(T)T^{\alpha-2}|\},$
$e_3 = \max\{|I^3\mathbb{L}_1(T)T^{\alpha-3}|, |I^3\mathbb{L}_2(T)T^{\alpha-2}|, |I^3\mathbb{L}_3(T)T^{\alpha-2}|\},$
$e_4 = \sup_{t \in \mathcal{J}}\{|I^\alpha\mathbb{L}_1(t)t^{\alpha-3}|, |I^\alpha\mathbb{L}_2(t)t^{\alpha-2}|, |I^\alpha\mathbb{L}_3(t)t^{\alpha-2}|\},$

I^α denotes the Riemann-Liouville integral operator of order α and

$$I^n \phi(t) = \frac{1}{\Gamma(n)} \int_0^t (t - s)^{n-1} \phi(s) ds, \ n = 1, 2, 3.$$

Proof. For verifying the hypotheses of Banach's fixed point theorem, we consider a closed ball $\mathcal{E}_\rho = \{x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R}) : \|x\|_{3-\alpha} \leq \rho\}$ with

$$\rho \geq (\overline{f}\ \overline{\delta})(1 - \Lambda)^{-1}, \tag{4.6}$$

where $\sup_{t \in \mathcal{J}} |f(t, 0, 0, 0)| = \overline{f}$,

$$\overline{\delta} \;=\; \delta_1 T + \delta_2 \frac{T^2}{2} + \delta_3 \frac{T^3}{6} + \frac{T^3}{\Gamma(\alpha + 1)}, \tag{4.7}$$

$\delta_m, \ m = 1, 2, 3,$ are given by (4.2). Now, we establish that $\mathcal{T}\mathcal{E}_\rho \subset \mathcal{E}_\rho$, where $\mathcal{T} : \mathcal{E}_\rho \longrightarrow C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is given by (4.1). By (H_1), we have

$$\begin{aligned} & |f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t))| \\ \leq \; & |f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\ \leq \; & \mathbb{L}_1(t)|x| + \mathbb{L}_2(t)|\lambda_1 x| + \mathbb{L}_3(t)|\lambda_2 x| + \overline{f}. \end{aligned} \tag{4.8}$$

For $x \in \mathcal{E}_\rho$, it follows by using (4.8) that

$$\begin{aligned} & \|\mathcal{T}x\|_{3-\alpha} \\ = \; & \sup_{t \in \mathcal{J}} \left\{ t^{3-\alpha} \left| \int_0^T \left[\mu_1(t) + \mu_2(t)(T - s) + \mu_3(t)\frac{(T - s)^2}{2}\right] \times \right.\right. \\ & \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),(\lambda_1 x)(s),(\lambda_2 x)(s))ds\Bigg|\Bigg\}\\
\le\;& \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[\int_0^T\Big[|\mu_1(t)|+|\mu_2(t)|(T-s)+|\mu_3(t)|\frac{(T-s)^2}{2}\Big]\\
&\times\Big(\mathbb{L}_1(s)|x|+\mathbb{L}_2(s)|\lambda_1 x|+\mathbb{L}_3(s)|\lambda_2 x|+\overline{f}\Big)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\mathbb{L}_1(s)|x|+\mathbb{L}_2(s)|\lambda_1 x|+\mathbb{L}_3(s)|\lambda_2 x|+\overline{f}\Big)ds\Bigg]\Bigg\}\\
\le\;& \int_0^T\Big[\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_1(t)|+\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_2(t)|(T-s)+\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_3(t)|\frac{(T-s)^2}{2}\Big]\\
&\times\Big(\mathbb{L}_1(s)|x|+\mathbb{L}_2(s)|\lambda_1 x|+\mathbb{L}_3(s)|\lambda_2 x|+\overline{f}\Big)ds\\
&+\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\mathbb{L}_1(s)|x|+\mathbb{L}_2(s)|\lambda_1 x|+\mathbb{L}_3(s)|\lambda_2 x|+\overline{f}\Big)ds\Bigg\}\\
\le\;& \int_0^T\Big[\delta_1+\delta_2(T-s)+\delta_3\frac{(T-s)^2}{2}\Big]\\
&\times\Bigg(\Big(\mathbb{L}_1(s)s^{\alpha-3}+\mathbb{L}_2(s)\frac{\psi_1 s^{\alpha-2}}{(\alpha-2)}+\mathbb{L}_3(s)\frac{\psi_2 s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{f}\Bigg)ds\\
&+\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Bigg(\Big(\mathbb{L}_1(s)s^{\alpha-3}+\mathbb{L}_2(s)\frac{\psi_1 s^{\alpha-2}}{(\alpha-2)}\\
&+\mathbb{L}_3(s)\frac{\psi_2 s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{f}\Bigg)ds\Bigg\}\\
\le\;& \Bigg[\Big(1+\frac{\psi_1}{(\alpha-2)}+\frac{\psi_2}{(\alpha-2)}\Big)\Big(\delta_1e_1+\delta_2e_2+\delta_3e_3+T^{3-\alpha}e_4\Big)\Bigg]\rho\\
&+\overline{f}\Big[\delta_1T+\delta_2\frac{T^2}{2}+\delta_3\frac{T^3}{6}+\frac{T^3}{\Gamma(\alpha+1)}\Big]\\
\le\;& \Lambda\rho+\overline{f}\,\overline{\delta}. \qquad (4.9)
\end{aligned}
$$

Combining (4.9) with (4.6), we obtain

$$\|\mathcal{T}x\|_{3-\alpha}\le\Lambda\rho+\overline{f}\,\overline{\delta}\le\rho,$$

which shows that $\mathcal{T}x\in\mathcal{E}_\rho$. Hence, $\mathcal{T}\mathcal{E}_\rho\subset\mathcal{E}_\rho$ since $x\in\mathcal{E}_\rho$ is an arbitrary element.

Next, we show that the operator $\mathcal{T}$ is a contraction. For that, let $x,y\in C_{3-\alpha}(\mathcal{J},\mathbb{R})$. Then, for any $t\in\mathcal{J}$ and using (H_1) together with the relation

$$
\begin{aligned}
&\mathbb{L}_1(t)|x-y|+\mathbb{L}_2(t)|\lambda_1x-\lambda_1y|+\mathbb{L}_3(t)|\lambda_2x-\lambda_2y|\\
\le\;&\Big(\mathbb{L}_1(t)t^{\alpha-3}+\mathbb{L}_2(t)\frac{\psi_1t^{\alpha-2}}{(\alpha-2)}+\mathbb{L}_3(s)\frac{\psi_2t^{\alpha-2}}{(\alpha-2)}\Big)\|x-y\|_{3-\alpha}, \qquad (4.10)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
& \|\mathcal{T}x - \mathcal{T}y\|_{3-\alpha} \\
\leq\ & \sup_{t\in\mathcal{J}} \Bigg\{ t^{3-\alpha} \Bigg[\int_0^T \Big[|\mu_1(t)| + |\mu_2(t)|(T-s) + |\mu_3(t)| \frac{(T-s)^2}{2} \Big] \\
& \times |f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) - f(s, y(s), (\lambda_1 y)(s), (\lambda_2 y)(s))| ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) \\
& - f(s, y(s), (\lambda_1 y)(s), (\lambda_2 y)(s))| ds \Bigg]\Bigg\} \\
\leq\ & \sup_{t\in\mathcal{J}} \Bigg\{ t^{3-\alpha} \Bigg[\int_0^T \Big[|\mu_1(t)| + |\mu_2(t)|(T-s) + |\mu_3(t)| \frac{(T-s)^2}{2} \Big] \\
& \times \Big(\mathbb{L}_1(s)|x-y| + \mathbb{L}_2(s)|\lambda_1 x - \lambda_1 y| + \mathbb{L}_3(s)|\lambda_2 x - \lambda_2 y| \Big) ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\mathbb{L}_1(s)|x-y| + \mathbb{L}_2(s)|\lambda_1 x - \lambda_1 y| \\
& + \mathbb{L}_3(s)|\lambda_2 x - \lambda_2 y| \Big) ds \Bigg]\Bigg\} \\
\leq\ & \int_0^T \Big[\delta_1 + \delta_2(T-s) + \delta_3 \frac{(T-s)^2}{2} \Big] \\
& \times \left(\mathbb{L}_1(s)s^{\alpha-3} + \mathbb{L}_2(s)\frac{\psi_1 s^{\alpha-2}}{(\alpha-2)} + \mathbb{L}_3(s)\frac{\psi_2 s^{\alpha-2}}{(\alpha-2)} \right) \|x-y\|_{3-\alpha} ds \\
& + \sup_{t\in\mathcal{J}} \Bigg\{ t^{3-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\mathbb{L}_1(s)s^{\alpha-3} + \mathbb{L}_2(s)\frac{\psi_1 s^{\alpha-2}}{(\alpha-2)} \right. \\
& \left. + \mathbb{L}_3(s)\frac{\psi_2 s^{\alpha-2}}{(\alpha-2)} \right) \|x-y\|_{3-\alpha} ds \Bigg\} \\
& \leq \Lambda \|x-y\|_{3-\alpha},
\end{aligned}
$$

which shows that the operator $\mathcal{T}$ is a contraction as $\Lambda < 1$ by the assumption (4.5). Thus, the hypotheses of Banach's fixed point theorem is verified and hence its conclusion implies that the operator $\mathcal{T}$ has a unique fixed point. Therefore, there exists a unique solution to the problem (1.1)-(1.2) on $\mathcal{J}$. This finish the proof. □

As a special case of Theorem 4.3, by taking $\Psi_1(t,s) = \dfrac{(t-s)^{p-1}}{\Gamma(p)}$, $\Psi_2(t,s) = \dfrac{(t-s)^{q-1}}{\Gamma(q)}$, $0 < p, q$, in (1.1), we get a nonlinear fractional differential equation involving both Riemann-Liouville derivative and integral operators given by

$$D^\alpha x(t) = f(t, x(t), I^p x(t), I^q x(t)). \tag{4.11}$$

Now we present a uniqueness result for fractional differential equation (4.1) subject to the boundary conditions (1.2).

Theorem 4.4. *Assume that* $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and the following condition holds:*

(H_2) *There exist positive functions* $\overline{\nu}_1(t)$, $\overline{\nu}_2(t)$ *and* $\overline{\nu}_3(t)$, *such that:*

$$\begin{aligned} & |f(t, x(t), (I^p x)(t), (I^q x)(t)) - f(t, y(t), (I^p y)(t), (I^q y)(t))| \\ \leq \;& \overline{\nu}_1(t)|x - y| + \overline{\nu}_2(t) I^p|x - y| + \overline{\nu}_3(t) I^q|x - y|, \ \forall\ t \in \mathcal{J}, x, y \in \mathbb{R}. \end{aligned} \tag{4.12}$$

Then, there exists a unique solution to the Riemann-Liuoville fractional differential equation (4.11) *subject to the boundary conditions* (1.2) *on* $\mathcal{J}$, *provided that*

$$\begin{aligned} \Lambda_1 \;=\; & \left(1 + \frac{\Gamma(\alpha-2)}{\Gamma(p+\alpha-2)} + \frac{\Gamma(\alpha-2)}{\Gamma(q+\alpha-2)}\right)\Big(\delta_1 k_1 + \delta_2 k_2 + \delta_3 k_3 \\ & + k_4 T^{3-\alpha}\Big) < 1, \end{aligned} \tag{4.13}$$

where $\delta_m, m = 1, 2, 3,$ *are given in* (4.2),

$k_1 = \max\{|I\overline{\nu}_1(T)T^{\alpha-3}|, |I\overline{\nu}_2(T)T^{p+\alpha-3}|, |I\overline{\nu}_3(T)T^{q+\alpha-3}|\}$,
$k_2 = \max\{|I^2\overline{\nu}_1(T)T^{\alpha-3}|, |I^2\overline{\nu}_2(T)T^{p+\alpha-3}|, |I^2\overline{\nu}_3(T)T^{q+\alpha-3}|\}$,
$k_3 = \max\{|I^3\overline{\nu}_1(T)T^{\alpha-3}|, |I^3\overline{\nu}_2(T)T^{p+\alpha-3}|, |I^3\overline{\nu}_3(T)T^{q+\alpha-3}\}$,
$k_4 = \sup_{t\in\mathcal{J}}\{|I^\alpha\overline{\nu}_1(t)t^{\alpha-3}|, |I^\alpha\overline{\nu}_2(t)t^{p+\alpha-3}|, |I^\alpha\overline{\nu}_3(t)t^{q+\alpha-3}|\}$.

Proof. We omit the proof as it is similar to that of Theorem 4.3. □

5. Examples

In this section, we present examples illustrating the results obtained in the last section.

Example 5.1. Let us consider the nonlinear fractional differential equation

$$D^\alpha x(t) = f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)), \ t \in [0, 1], \tag{5.1}$$

subject to the Riemann–Liouville boundary conditions

$$\begin{cases} D^{\alpha-1}x(0^+) + D^{\alpha-1}x(T^-) = 0, \\ D^{\alpha-2}x(0^+) + D^{\alpha-2}x(T^-) = 0, \\ D^{\alpha-3}x(0^+) + D^{\alpha-3}x(T^-) = 0. \end{cases} \tag{5.2}$$

Here, $\alpha = 8/3, \mathcal{J} = [0, 1], T = 1$, and

$$\begin{aligned} & f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) \\ =\; & \frac{\sin x(t)}{\sqrt{t^2 + 144}} + \int_0^t \frac{e^{-(s^2-t)}}{600} x(s)ds + \int_0^t \frac{\sin(s-t)}{(t+225)^2} x(s)ds. \end{aligned}$$

Using the given data, it is found that $\delta_1 \approx 0.05538661, \delta_2 \approx 0.36924406, \delta_3 \approx 0.36924406, \psi_1 \approx 0.00045350, \psi_2 \approx 0.00001662, e_1 \approx 0.06136364, e_2 \approx 0.22500000, e_3 \approx 0.60000000, e_4 \approx 0.09748313$. Clearly, (H_1) is satisfied with $L_1(t) = \dfrac{1}{\sqrt{t^2 + 144}}$, $L_2(t) = L_3(t) = 1$ and $\Lambda \approx 0.23661991 < 1$, that is, the condition (4.5) is verified. As the hypothesis of Theorem 4.4 holds true, so its conclusion implies that the boundary value problem (5.1)-(5.2) has a unique solution on $[0, 1]$.

Example 5.2. Consider the nonlinear equation

$$D^{\alpha}x(t) = f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)),\ t \in [0, 1], \tag{5.3}$$

subject to the Riemann–Liouville boundary conditions in (5.2), where $\alpha = 8/3$, and

$$\begin{aligned} & f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) \\ = \ & \frac{\tan^{-1} x(t)}{\sqrt{t^2 + 144}} + \frac{1}{6}\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} x(s)ds + \frac{1}{25}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x(s)ds, \end{aligned}$$

with $p = 0.6,\ q = 0.8$.

Using the given values, we find that $\delta_1 \approx 0.05538661,\ \delta_2 \approx 0.36924406,\ \delta_3 \approx 0.36924406,\ k_1 \approx 0.13157895,\ k_2 \approx 0.07497047,\ k_3 \approx 0.02811867,\ k_4 \approx 0.04060714$, and the assumption (H_2) holds true with $\overline{\nu}_1(t) = \dfrac{1}{\sqrt{t^2 + 144}}, \overline{\nu}_2(t) = 1/6,\ \overline{\nu}_3(t) = 1/25$, and the condition (4.13) is satisfied as $\Lambda_1 \approx 0.34628001 < 1$. Thus, by the conclusion of Theorem 4.4, the equation (5.3) with the boundary conditions (5.2) has a unique solution on $[0, T]$.

6. Ulam–Hyers Stability Analysis

Let us first develop the arguments for the Ulam–Hyers stability [30] of the problem (1.1)-(1.2).

For $\epsilon > 0$ and $t \in \mathcal{J}$, let us consider the inequality

$$\left| D^{\alpha}\widehat{x}(t) - f(t, \widehat{x}(t), (\lambda_1\widehat{x})(t)(\lambda_2\widehat{x})(t)) \right| \le \epsilon, \tag{6.1}$$

with boundary conditions (1.2).

If $\widehat{x} \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is a solution to the inequality (6.1) with boundary conditions (1.2), then there exists a function $\kappa \in C(\mathcal{J}, \mathbb{R})$ such that $|\kappa(t)| \le \epsilon$, $t \in \mathcal{J}$, and the function $\widehat{x}$ satisfies the Riemann-Liouville fractional differential equation

$$D^{\alpha}\widehat{x}(t) = f(t, \widehat{x}(t), (\lambda_1\widehat{x})(t)(\lambda_2\widehat{x})(t)) + \kappa(t),$$

with boundary conditions (1.2). Thus, we consider the boundary value problem

$$\begin{cases} D^{\alpha}\widehat{x}(t) = f(t, \widehat{x}(t), (\lambda_1\widehat{x})(t)(\lambda_2\widehat{x})(t)) + \kappa(t),\ t \in \mathcal{J}, \\ D^{\alpha-1}\widehat{x}(0^+) + D^{\alpha-1}\widehat{x}(T^-) = 0, \\ D^{\alpha-2}\widehat{x}(0^+) + D^{\alpha-2}\widehat{x}(T^-) = 0, \\ D^{\alpha-3}\widehat{x}(0^+) + D^{\alpha-3}\widehat{x}(T^-) = 0. \end{cases} \tag{6.2}$$

Definition 6.1. *The problem (1.1)-(1.2) is called Ulam–Hyers stable if we can find $c_f > 0$, such that, for each solution $\widehat{x} \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of (6.2), there exists a unique solution $x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of the problem (1.1)-(1.2) satisfying*

$$\|\widehat{x} - x\|_{3-\alpha} \le c\epsilon,\ t \in \mathcal{J}.$$

Definition 6.2. *If there exists* $\Phi \in C_{3-\alpha}(\mathbb{R}^+, \mathbb{R}^+)$, *with* $\Phi(0) = 0$, *such that, for each solution* $u \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ *of* (6.2), *there exists a unique solution* $x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ *of the problem* (1.1)–(1.2) *satisfying*

$$\|\widehat{x} - x\|_{3-\alpha} \leq \Phi(\epsilon), \ t \in \mathcal{J}.$$

Then, the problem (1.1)–(1.2) *is generalized Ulam–Hyers stable.*

Theorem 6.3. *If the assumption* (H_1) *and the condition (4.5) are satisfied, then the problem* (1.1)–(1.2) *is Ulam–Hyers stable and hence generalized Ulam–Hyers stable in* $C_{3-\alpha}(\mathcal{J}, \mathbb{R})$.

Proof. By Lemma 2.4, the solution of (6.2) can be written as

$$\begin{aligned}
\widehat{x}(t) \ &= \ \int_0^T \Big[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\Big] \times \\
&\quad \times\Big(f(s, \widehat{x}(s), (\lambda_1\widehat{x})(s), (\lambda_2\widehat{x})(s)) + \kappa(s)\Big)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(s, \widehat{x}(s), (\lambda_1\widehat{x})(s), (\lambda_2\widehat{x})(s)) + \kappa(s)\Big)ds.
\end{aligned}$$

Using $|\kappa| < \epsilon$ and (4.7), we get

$$\begin{aligned}
&\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg|\widehat{x}(t) - \int_0^T \Big[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\Big] \times \\
&\times f(s, \widehat{x}(s), (\lambda_1\widehat{x})(s), (\lambda_2\widehat{x})(s))ds \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \widehat{x}(s), (\lambda_1\widehat{x})(s), (\lambda_2\widehat{x})(s))ds\Bigg|\Bigg\} \leq \overline{\delta}\epsilon.
\end{aligned}$$

It follows by the assumption (H_1) together with (4.10) that

$$\begin{aligned}
& \|\widehat{x} - x\|_{3-\alpha} \\
= \ & \sup_{t\in\mathcal{J}}\{t^{3-\alpha}|\widehat{x}(t) - x(t)|\} \\
\leq \ & \overline{\delta}\epsilon + \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[\int_0^T \Big[|\mu_1(t)| + |\mu_2(t)|(T-s) + |\mu_3(t)|\frac{(T-s)^2}{2}\Big] \\
& \times|f(s, \widehat{x}(s), (\lambda_1\widehat{x})(s), (\lambda_2\widehat{x})(s)) - f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))|ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, \widehat{x}(s), (\lambda_1\widehat{x})(s), (\lambda_2\widehat{x})(s)) \\
& -f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))|ds\Bigg]\Bigg\} \\
\leq \ & \overline{\delta}\epsilon + \Lambda\|\widehat{x} - x\|_{3-\alpha},
\end{aligned}$$

which can alternatively be written as

$$\|\widehat{x} - x\|_{3-\alpha} \leq \frac{\overline{\delta}\epsilon}{1-\Lambda}.$$

Letting $c = c_f = \dfrac{\overline{\delta}}{1-\Lambda}$, we get $\|\widehat{x} - x\|_{3-\alpha} \leq c\epsilon$. Hence, the problem (1.1)–(1.2) is Ulam–Hyers stable. Moreover, it is generalized Ulam–Hyers stable as $\|\widehat{x} - x\|_{3-\alpha} \leq \Phi(\epsilon)$, whit $\Phi(\epsilon) = c\epsilon,\ \Phi(0) = 0$. □

Example 6.4. The problems (5.1)-(5.2) and (5.3)-(5.2) are Ulam–Hyers stable, and generalized Ulam–Hyers stable as $\Lambda \approx 0.23661991 < 1$ and $\Lambda_1 \approx 0.34628001 < 1$, respectively.

7. Conclusions

We explored the criteria ensuring the existence and uniqueness of solutions for nonlinear Riemann–Liouville fractional integro-differential equations of order $\alpha \in (2, 3]$ subject to fractional anti-periodic boundary conditions in a weighted space. We applied the Leray-Schauder's alternative and Banach's fixed point theorem to accomplish the desired results. A special case for the nonlinearity of (1.1) depending upon the Riemann–Liouville fractional integrals is also discussed. We also studied the Ulam–Hyers stability for the problem at hand. It is imperative to point out that the solution of the problem (1.1)–(1.2) contains the solution of the Riemann-Liouville fractional differential equation of order $\alpha \in (1, 2]$ complemented with fractional anti-periodic boundary conditions investigated in [6] (for details, see Section 3). Thus, our results are novel in the given configuration and contribute to the known literature on fractional anti-periodic boundary value problems of nonlinear Riemann–Liouville fractional differential equations in the weighted space.

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