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Global well-posedness for the generalized Keller-Segel system in critical Besov-Morrey spaces with variable exponent

Ahmed El Idrissi, Fatima Ouidirne, Brahim El Boukari and Jalila El Ghordaf

Abstract. This article is devoted to studying the generalized Keller-Segel system (GKS) in homogeneous variable exponent Besov-Morrey spaces. By making use of the Littlewood-Paley theory and the Chemin mono-norm methods, we obtain, when $\frac{1}{2} < \beta \leq 1$, a global well-posedness result for GKS system with small initial data in the critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^n)$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$, $1 \leq h \leq \infty$. In the limit case $\beta = \frac{1}{2}$, we show the global well-posedness for small initial data in $\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}(\mathbb{R}^n)$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$.

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1. Introduction

We are concerned with the generalized Keller-Segel system given by the following fractional diffusion:

$$\begin{cases} u_t + (-\Delta)^\beta u = -\nabla \cdot (u \nabla \psi) & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ -\Delta \psi = u & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.1}$$

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where $n \geq 2,\ u = u(x,t)$ denotes the unknown density of cells, $\psi = \psi(x,t)$ represents the unknown concentration of the chemo-attractant, u_0 is the initial data, ∇ is the gradient operator, $(-\Delta)^\beta$ is the Laplacian operator, which is the Fourier multiplier with symbol $|\xi|^{2\beta}$, and $\frac{1}{2} \leq \beta \leq 1$, that is, the abnormal (normal) diffusion is modeled by a fractional power of the Laplacian.

Note that the function ψ, which is determined by the Poisson equation, is given by the second equation of (1.1) as the volume potential of v:

$$\psi(x,t) = (-\Delta)^{-1} u(x,t).$$

We can therefore eliminate ψ from the system (1.1) and get the following equivalent problem:

$$\begin{cases} u_t + (-\Delta)^\beta u = -\nabla \cdot (u \nabla (-\Delta)^{-1} u) & \text{in } \ \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \ \mathbb{R}^n. \end{cases} \tag{1.2}$$

For $\beta = 1$, (1.1) corresponds to the classical Keller-Segel equation which is a simplified system of

$$\begin{cases} u_t - \Delta u = -\nabla \cdot (u \nabla \psi) & \text{in } \ \mathbb{R}^n \times (0,\infty), \\ \psi_t - \Delta \psi = u - \psi & \text{in } \ \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad \psi(x,0) = \psi_0(x) & \text{in } \ \mathbb{R}^n. \end{cases} \tag{1.3}$$

The system (1.3) was introduced by Keller and Segel [17] in 1970. It describes a chemotaxis mathematical model, and it is also linked to astrophysical models of gravitational auto-interaction of huge particles in a cloud or nebula, the reader may refer to [6]. The well-posedness of classical Keller-Segel models has been studied by several researchers in various spaces. Recently, making use of the smoothing effect of the heat semigroup, Iwabuchi [16] proved the global well-posedness of the system (1.3) in $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ where $n \geq 1$ and $\max\{1, n/2\} < p < \infty$, under the condition of smallness of the initial data. Later, by the same method, Nogayama and Sawano [18] extended this well-posedness result, where they established global well-posedness in the Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}(\mathbb{R}^n)$ with $\max\left\{1, \frac{n}{2}\right\} < p < \infty$ and $1 \leq h \leq p$.

For the general case $\frac{1}{2} < \beta < 1$, (1.1) was initially considered by Escudero [11], in which it was utilized to characterize the spatio-temporal distribution of a population density of random walkers subjected to Lévy flights. Furthermore, in that paper, it has been established that (1.1) in this case, has global in time solutions. There are many studies on (1.1) by several researchers in various spaces. Recently, Zhao [21] obtained well-posedness results of (1.1) in the classical Besov spaces $\dot{B}^{-2\beta+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ with $\frac{1}{2} \leq \beta \leq 1$ and $1 \leq p, r \leq \infty$. We mention that certain aspects of these results were also extended to the fractional power bipolar type drift-diffusion system. Further information on this topic can be found in [14, 12] and the relevant references cited therein.

Inspired by this work, we aim to investigate, by making use of the Chemin mono-norm methods, global well-posedness of the generalized Keller-Segel system (1.1) with initial data in the critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^n)$ with $\frac{1}{2} \leq \beta \leq 1,\ 1 \leq r(\cdot) \leq q(\cdot) < \infty$ and $1 \leq h \leq \infty$.

In general, variable exponent function spaces have garnered significant attention from researchers in recent times. This interest extends beyond theoretical aspects, encompassing their pivotal role in various applications, such as fluid dynamics [20] and resolving specific equations [10, 4]. Notably, variable exponent Besov-Morrey space, based on variable exponent Morrey spaces, is a new large framework compared to Besov space, i.e. variable exponent Besov-Morrey spaces are strictly broader than classical Besov spaces (also refer to Remark 2). However, there are many challenges in addressing the well-posedness of equations in these spaces. Replacing the L^p-norm by the $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-norm is not sufficient to ensure a direct transition from Besov spaces to variable exponent Besov-Morrey spaces. One of the main difficulties comes from the collapse of certain essential embedding features and the inapplicability of certain classical theories, like the multiplier theorem and Young's inequality, within Besov-Morrey spaces with variable exponents, unlike classical Besov spaces. To overcome these challenges, the present paper primarily relies on the properties described in Section 2 to look at the global well-posedness result. For an in-depth exploration of these variable exponent function spaces, we direct the reader to [1, 8, 9, 10, 19, 13, 20, 15, 2, 3] and the associated references therein.

To address the system (1.1), passing via (1.2), we think about the following equivalent integral equations:

$$u(t) = e^{-t(-\Delta)^\beta} u_0 - \int_0^t e^{-(t-t')(-\Delta)^\beta} \nabla \cdot \big(u \nabla (-\Delta)^{-1} u\big) \, \mathrm{d}t', \tag{1.4}$$

where $e^{-t(-\Delta)^\beta} := \mathcal{F}^{-1}(e^{-t|\xi|^{2\beta}} \mathcal{F})$ is the fractional heat semigroup operator.

Organization of the paper: In Section 2, we present some basic background information on the Littlewood-Paley theory and some different laws on products in variable exponent Besov-Morrey spaces, and then, in Section 3, we state and prove our main theorem.

2. Preliminaries

We introduce some background knowledge on Littlewood-Paley theory and variable exponent Besov-Morrey spaces, and present some propositions relevant to our objectives. Firstly, we start by introducing some of the notations used in the present paper, $E \lesssim H$ designates having a constant $C > 0$, which can be different at different places, such that $E \leq CH$ and $E \sim H$ designates having two constants $C_1, C_2 > 0$ such that $C_1 H \leq E \leq C_2 H$. We define, for two Banach spaces X and Y, and $u \in X \cap Y$, the norm $\|\cdot\|_{X \cap Y}$ as

$$\|u\|_{X \cap Y} := \|u\|_X + \|u\|_Y \, .$$

Definition 2.1. [3] *For the measurable function $r(\cdot)$, let*

$$\mathcal{P}_0(\mathbb{R}^n) := \left\{ r(\cdot) : \mathbb{R}^n \to (0, \infty]; 0 < r_- = \operatorname*{essinf}_{x \in \mathbb{R}^n} r(x), \operatorname*{esssup}_{x \in \mathbb{R}^n} r(x) = r_+ < \infty \right\}$$

The Lebesgue space with variable exponent is defined by

$$L^{r(\cdot)}(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable, } \int_{\mathbb{R}^n} |u(x)|^{r(x)} dx < \infty \right\},$$

with norm

$$\begin{aligned} \|u\|_{L^{r(\cdot)}} &:= \inf \left\{ \lambda > 0 : \varrho_{r(\cdot)}(u/\lambda) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|u(x)|}{\lambda} \right)^{r(x)} dx \leq 1 \right\}. \end{aligned}$$

We use the following notation to separate variable exponents from constant exponents: $r(\cdot)$ for variable exponents, r for constant exponents. Also $(L^{r(\cdot)}(\mathbb{R}^n), \|u\|_{L^{r(\cdot)}})$ is a Banach space.

$L^{r(\cdot)}$ doesn't have the same features as L^r. Therefore, to assure the boundedness of the maximal Hardy-Littlewood operator M on $L^{r(\cdot)}(\mathbb{R}^n)$, the following standard conditions are assumed:

1. (Locally log-Hölder's continuous)[3] There exists a constant $C_{\log}(r)$ such that

$$|r(x) - r(y)| \leq \frac{C_{\log}(r)}{\log(e + |x - y|^{-1})}, \text{ for all } x, y \in \mathbb{R}^n,\ x \neq y.$$

2. (Globally log-Hölder's continuous)[3] There exist two constants $C_{\log}(r)$ and r_∞ such that

$$|r(x) - r_\infty| \leq \frac{C_{\log}(r)}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

$C_{\log}(\mathbb{R}^n)$ denotes the set of all functions $r(\cdot) : \mathbb{R}^n \to \mathbb{R}$ that satisfy 1 and 2.

Definition 2.2. [2] *Let $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < r_- \leq r(\cdot) \leq q(\cdot) \leq \infty$, the variable exponent Morrey space $\mathcal{M}^{q(\cdot)}_{r(\cdot)} := \mathcal{M}^{q(\cdot)}_{r(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions on $\mathbb{R}^n$ such that*

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, R > 0} \left\| R^{\frac{n}{q(x)} - \frac{n}{r(x)}} u \right\|_{L^{r(\cdot)}(B(x_0,R))} < \infty.$$

Here we give an important lemma.

Lemma 2.3. [2] *Let $r(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Then for any measurable function u*

$$\sup_{x_0 \in \mathbb{R}^n, R > 0} \varrho_{r(\cdot)} \left(u \chi_{B(x_0,R)} \right) = \varrho_{r(\cdot)}(u),$$

and $\|u\|_{\mathcal{M}^{r(\cdot)}_{r(\cdot)}} = \|u\|_{L^{r(\cdot)}}$.

We now recall the Littlewood-Paley decomposition (refer to [5] for further information). Consider $\varphi \in \mathcal{S}(\mathbb{R}^n)$ a smooth radial function such that

$$0 \leq \varphi \leq 1,$$
$$\operatorname{supp} \varphi \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\},$$
$$\sum_{j\in Z} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0,$$

and we denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Then for every $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the frequency localization operators for all $j \in \mathbb{Z}$, as follows

$$\Delta_j u = \mathcal{F}^{-1}\varphi_j * u \qquad \text{and} \qquad S_j u = \sum_{k\leq j-1} \Delta_k u. \tag{2.1}$$

One observes here that $\dot{\Delta}_j$ has frequency $\{|\xi| \sim 2^j\}$ and that $\dot{S}_j$ has frequency $\{|\xi| \lesssim 2^j\}$, and one also notes that the quasi-orthogonality property holds for the Littlewood-Paley decomposition, that is, for every $u, v \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$,

$$\dot{\Delta}_i\dot{\Delta}_j u = 0 \quad \text{if} \quad |i-j| \geq 2, \quad \dot{\Delta}_i\left(\dot{S}_{j-1}u\dot{\Delta}_j v\right) = 0 \qquad \text{if} \quad |i-j| \geq 5, \tag{2.2}$$

with $\mathcal{P}(\mathbb{R}^n)$ denoting the collection of all polynomials over $\mathbb{R}^n$.

All through this document, we will use the following Bony paraproduct decomposition:

$$uv = T_u v + T_v u + R(u,v), \tag{2.3}$$

with

$$T_u v = \sum_j S_{j-1}u\Delta_j v, \quad R(u,v) = \sum_j \sum_{|j-l|\leq 1} \Delta_j u \Delta_l v.$$

Definition 2.4. [2] *Let $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $r(\cdot) \leq q(\cdot)$, the mixed Morrey-sequence space $\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})$ is the set of all sequences $\{a_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that*

$$\|\{a_j\}_{j\in\mathbb{Z}}\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} := \inf\left\{\lambda > 0 : \varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\{a_j/\lambda\}_{j\in\mathbb{Z}}) \leq 1\right\},$$

where

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\{a_j\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \inf\left\{\nu > 0 : \int_{\mathbb{R}^n}\left(\frac{|R^{\frac{n}{q(x)}-\frac{n}{r(x)}}a_j\chi_{B(x_0,R)}|}{\nu^{\frac{1}{h(x)}}}\right)^{r(x)} dx \leq 1\right\}$$

Notice that if $h_+ < \infty$ and $r(\cdot) \leq h(\cdot)$, then

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\{a_j\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \sup_{x_0\in\mathbb{R}^n, R>0}\left\|\left(R^{\frac{n}{q(x)}-\frac{n}{r(x)}}u\right)^{h(x)}\right\|_{L^{\frac{r(\cdot)}{h(\cdot)}}(B(x_0,R))}.$$

Definition 2.5. [2] *Let* $s(\cdot) \in C_{\log}(\mathbb{R}^n)$ *and* $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $0 < r_- \leq r(\cdot) \leq q(\cdot) \leq \infty$. *The variable exponent homogeneous Besov-Morrey space* $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$ *is defined by*

$$\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} := \left\{ u \in \mathcal{D}'(\mathbb{R}^n) : \|u\|_{\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} < \infty \right\},$$

with norm

$$\|u\|_{\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} := \left\| \left\{ 2^{js(\cdot)} \Delta_j u \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},$$

and $\mathcal{D}'(\mathbb{R}^n)$ *represents the dual space of*

$$\mathcal{D}(\mathbb{R}^n) = \{ u \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha u)(0) = 0, \ \textit{for all multi-index } \alpha \}.$$

For $T > 0$ and $1 \leq h, \rho \leq \infty$. The mixed space-time space $\mathcal{L}^\rho\left(0, T; \mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)$ is the set of all tempered distribution u satisfying

$$\|u\|_{\mathcal{L}^\rho\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} := \left(\sum_{j\in\mathbb{Z}} \left\| 2^{js(\cdot)} \Delta_j u \right\|^h_{L^\rho_T\left(\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} \right)^{\frac{1}{h}} < \infty,$$

where

$$\|u\|_{L^\rho_T\left(\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} := \left(\int_0^T \|u(\cdot,t)\|^\rho_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \, \mathrm{nt} \right)^{\frac{1}{\rho}}.$$

With the standard modification if $h = \infty$ or $\rho = \infty$.

Proposition 2.6. *The following inclusions hold for variable exponent Morrey spaces.*

1. *(Hölder's inequality)*[1] *Let* $r(\cdot), r_1(\cdot), r_2(\cdot), q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ *satisfying* $r(\cdot) \leq q(\cdot)$, $r_i(\cdot) \leq q_i(\cdot) (i = 1, 2)$, $\frac{1}{r(\cdot)} = \frac{1}{r_1(\cdot)} + \frac{1}{r_2(\cdot)}$ *and* $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$. *Then for all* $u \in \mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}$ *and* $v \in \mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}$, *there is a constant* C *depending only on* r_- *and* r_+ *such that*
$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C \|u\|_{\mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}} \|v\|_{\mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}}. \tag{2.4}$$
And for all $u \in L^\infty(\mathbb{R}^n)$ *and* $v \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, *there is a constant* C *such that*
$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C \|u\|_{L^\infty} \|v\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}. \tag{2.5}$$
2. *(Sobolev-type embedding)* [1] *Let* $r_1(\cdot), r_2(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $0 < h < \infty$ *and* $s_1(\cdot), s_2(\cdot) \in L^\infty \cap C_{\log}(\mathbb{R}^n)$ *with* $s_1(\cdot) > s_2(\cdot)$. *If* $\frac{1}{h}$ *and*
$$s_1(\cdot) - \frac{n}{r_1(\cdot)} = s_2(\cdot) - \frac{n}{r_2(\cdot)}$$
are locally log-Hölder continuous, then
$$\mathcal{N}^{s_1(\cdot)}_{r_1(\cdot),q_1(\cdot),h} \hookrightarrow \mathcal{N}^{s_2(\cdot)}_{r_2(\cdot),q_2(\cdot),h}. \tag{2.6}$$

3. *(Mollification inequality)* [2] *Let* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *and* $\phi \in L^1(\mathbb{R}^n)$, *suppose* $\Phi(y) = \sup_{x \notin B(0,|y|)} |\phi(x)|$ *is integrable. Then for all* $u \in \mathcal{M}_{r(\cdot)}^{q(\cdot)}$, *there is a constant* C *depending only on* d *such that*

$$\|u * \phi_\varepsilon\|_{\mathcal{M}_{r(\cdot)}^{q(\cdot)}} \leq C \|u\|_{\mathcal{M}_{r(\cdot)}^{q(\cdot)}} \|\Phi\|_{L^1}, \tag{2.7}$$

where $\phi_\varepsilon = \frac{1}{\varepsilon^d}\phi(\varepsilon)$.

Lemma 2.7. [2] *Let* $\mathcal{C}$ *be a ring, and* $\mathcal{B}$ *a ball in* $\mathbb{R}^n$, *and let* $k \in \mathbb{N}$, $j \in \mathbb{Z}$, $\lambda > 0$, *and* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $r(\cdot) \leq q(\cdot) < \infty$.

1. *Assume that* $u \in \mathcal{M}_{r(\cdot)}^{q(\cdot)}$ *satisfying supp*$\mathcal{F}(u) \subset \lambda\mathcal{B}$, *then*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{\mathcal{M}_{r(\cdot)}^{q(\cdot)}} \leq C^{k+1}\lambda^k \|u\|_{\mathcal{M}_{r(\cdot)}^{q(\cdot)}}.$$

2. *Assume that* $u \in \mathcal{M}_{r(\cdot)}^{q(\cdot)}$ *satisfying supp*$\mathcal{F}(u) \subset \lambda^j\mathcal{B}$, *then*

$$\|u\|_{L^\infty} \leq C\lambda^{j\frac{n}{q(\cdot)}} \|u\|_{\mathcal{M}_{r(\cdot)}^{q(\cdot)}}.$$

where C *is a constant independent of* λ.

Lemma 2.8. *Let* $m \in \mathbb{R}$, $s(\cdot) \in C_{\log}(\mathbb{R}^n)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $r(\cdot) \leq q(\cdot)$, *and let* $0 < h < \infty$. *Then*

$$\partial_\xi^m : \mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)+m} \to \mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)}$$

is bounded.

Proof. For the proof, we can use the same idea as in [14, Lemma 2] □

Lemma 2.9. *(*[5, Lemma 5.5]*) Let* X *be a Banach space with norm* $\|\cdot\|_X$ *and* B *be a bounded bilinear operator from* $X \times X$ *to* X *satisfying*

$$\|B(x_1, x_2)\|_X \leq C_0 \|x_1\|_X \|x_2\|_X,$$

for all $x_1, x_2 \in X$ *and a constant* $C_0 > 0$. *Then for any* $a \in X$ *such that* $\|a\|_X < \frac{1}{4C_0}$, *the equation* $x = a + B(x, x)$ *has a solution* x *in* X. *Moreover, the solution is such that* $\|x\|_X \leq 2\|a\|_X$, *and it is the only one such that* $\|x\|_X < \frac{1}{2C_0}$.

3. Well-posedness

In this section, we state our main theorem, and then prove it for the case $\frac{1}{2} < \beta \leq 1$ and the case $\beta = \frac{1}{2}$ in Subsection 3.1 and in Subsection 3.2, respectively.

Theorem 3.1. *Let* $n \geq 2$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$ *and* $1 \leq h \leq \infty$.

1. *Let* $\frac{1}{2} < \beta \leq 1$. *Then there exists a constant* $\varepsilon > 0$ *such that for any* $u_0 \in \mathcal{N}_{r(\cdot),q(\cdot),h}^{-2\beta+\frac{n}{q(\cdot)}}$ *satisfying* $\|u_0\|_{\dot{\mathcal{N}}_{r(\cdot),q(\cdot),h}^{-2\beta+\frac{n}{q(\cdot)}}} \leq \varepsilon$, *the system* (1.1) *admits a unique time-global solution* $u \in \mathcal{X}_\varepsilon$, *where*

$$\mathcal{X}_\varepsilon := \left\{u \in \mathcal{X}^0 \cap \mathcal{X}^1 : \ \|u\|_{\mathcal{X}^0} < \infty, \ \|u\|_{\mathcal{X}^1} \lesssim \varepsilon\right\},$$

with

$$\mathcal{X}^0 := \mathcal{L}^\infty\left(\mathbb{R}_+; \mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right),$$

$$\mathcal{X}^1 := \mathcal{L}^{\gamma_1}\left(\mathbb{R}_+; \mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_2}\left(\mathbb{R}_+; \mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right),$$

and

$$\gamma_1 = \frac{2\beta}{2\beta - 1 + \varepsilon}, \;\; \gamma_2 = \frac{2\beta}{2\beta - 1 - \varepsilon}, \;\; 0 < \varepsilon < 2\beta - 1,$$
$$s_1(\cdot) = -1 + \frac{n}{q(\cdot)} + \varepsilon, \;\; s_2(\cdot) = -1 + \frac{n}{q(\cdot)} - \varepsilon.$$

2. *Let* $\beta = \frac{1}{2}$. *Assume that* $u_0 \in \mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}$ *is small enough. Then the system* (1.1) *admits a unique global solution* v *satisfying*
$$u \in \mathcal{L}^\infty\left(\mathbb{R}_+; \mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right).$$

The above result requires some further comments.

Remark 3.2.

1. The results of this work remain valid if we take variable exponent Besov space $\mathcal{B}^{s(\cdot)}_{r(\cdot),h(\cdot)}$ instead of variable exponent Besov-Morrey space $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$. Indeed, if we have $r(\cdot) = q(\cdot)$, then $\mathcal{N}^{s(\cdot)}_{r(\cdot),r(\cdot),h(\cdot)} = \mathcal{B}^{s(\cdot)}_{r(\cdot),h(\cdot)}$.
2. Theorem 3.1 extends the corresponding well-posedness results of [21], where the author considered the system (1.1) in Besov spaces, which is a particular case of our framework which is variable exponent Besov-Morrey spaces $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$. Moreover, we have $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} \not\subset \dot{B}^{s'}_{p',r'}$ for any $s' \in \mathbb{R}$, $1 \leq p' < \infty$ and $1 \leq r' < \infty$.

In order to prove Theorem 3.1, we consider the following linear equation:

$$\begin{cases} u_t + (-\Delta)^\beta u = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{3.1}$$

for which we get the following linear estimate:

Proposition 3.3. *(Linear estimate) Let* $0 < T \leq \infty$, $\frac{1}{2} \leq \beta \leq 1$, $s(\cdot) \in C_{\log}(\mathbb{R}^n)$, $1 \leq h, \gamma \leq \infty$, *and let* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $1 \leq r(\cdot) \leq q(\cdot) < \infty$. *Assume that* $u_0 \in \mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}$ *and* $f \in \mathcal{L}^\gamma(0,T; \mathcal{N}^{s(\cdot)+\frac{2\beta}{\gamma}-2\beta}_{r(\cdot),q(\cdot),h})$. *Then* (3.1) *has a unique solution* v *satisfying, for any* $\rho \in [\gamma, \infty]$,

$$\|u\|_{\mathcal{L}^\rho\left(0,T;\mathcal{N}^{s(\cdot)+\frac{2\beta}{\rho}}_{r(\cdot),q(\cdot),h}\right)} \leq C\left(\|u_0\|_{\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}} + \|f\|_{\mathcal{L}^\gamma\left(0,T;\mathcal{N}^{s(\cdot)+\frac{2\beta}{\gamma}-2\beta}_{r(\cdot),q(\cdot),h}\right)}\right), \tag{3.2}$$

where $C > 0$ *is a constant depending only on* β *and* d.

Before proving this proposition, we need to get estimates for the localisations of the fractional heat semigroup $\{e^{-t(-\Delta)^\beta}\}_{t\geq 0}$ in our framework.

Proposition 3.4. *Let $t > 0$, $j \in \mathbb{Z}$ and $1 \le r(\cdot) \le q(\cdot) < \infty$. Then for all $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\Delta_j u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, we have*

$$\left\| \Delta_j (e^{-t(-\Delta)^\beta} v) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le \mathcal{K} e^{-\kappa t 2^{2\beta j}} \left\| \Delta_j v \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}},$$

where $\mathcal{K}$ and κ are tow constants independent of j and t.

Proof. Recalling that $supp(\mathcal{F}(\Delta_j v)) \subset 2^j \mathcal{C}$ (Δ_j is a frequency to $\{|\xi| \sim 2^j\}$), and considering a function $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with $\phi \equiv 1$ in a neighborhood of the ring $\mathcal{C}$, then one has

$$\begin{aligned} \Delta_j (e^{-t(-\Delta)^\beta} v) &= e^{-t(-\Delta)^\beta} \Delta_j v \\ &= \phi(2^j \cdot) e^{-t(-\Delta)^\beta} \Delta_j v \\ &= \mathcal{F}^{-1} \left(\phi(2^j \xi) e^{-t|\xi|^{2\beta}} \right) * \Delta_j v. \end{aligned}$$

Hence, Proposition 2.6, gives

$$\begin{aligned} \left\| \Delta_j (e^{-t(-\Delta)^\beta} v) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} &\le \left\| \mathcal{F}^{-1} \left(\phi(2^j \xi) e^{-t|\xi|^{2\beta}} \right) \right\|_{L^1} \left\| \Delta_j v \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \\ &\le \mathcal{K} e^{-\kappa t 2^{2\beta j}} \left\| \Delta_j v \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}, \end{aligned}$$

as desired. □

Proof of Proposition 3.3. Since $u_0 \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ and $f \in \mathcal{S}'([0,T) \times \mathbb{R}^n)/\mathcal{P}$, we can obtain $u \in \mathcal{S}'([0,T) \times \mathbb{R}^n)/\mathcal{P}$. And then, applying Δ_j to (3.1) and taking the $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-norm, we get

$$\left\| \Delta_j u(t) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le \left\| e^{-t(-\Delta)^\beta} \Delta_j u_0 \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + \int_0^t \left\| e^{-(t-t')(-\Delta)^\beta} \Delta_j f(t') \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \, dt'.$$

According to Proposition 3.4, we obtain for some $\kappa > 0$,

$$\left\| \Delta_j u(t) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \lesssim e^{-\kappa t 2^{2\beta j}} \left\| \Delta_j u_0 \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + \int_0^t e^{-\kappa 2^{2\beta j}(t-t')} \left\| \Delta_j f(t') \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \, dt'.$$

Set $\frac{1}{\theta} = 1 + \frac{1}{\rho} - \frac{1}{\gamma}$. Young's inequality in L^ρ gives us,

$$\begin{aligned} &\left\| \Delta_j u(t) \right\|_{L^\rho_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\ &\lesssim \left(\frac{1 - e^{-\kappa T 2^{2\beta j} \rho}}{\kappa 2^{2\beta j} \rho} \right)^{\frac{1}{\rho}} \left\| \Delta_j u_0 \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + \left(\frac{1 - e^{-\kappa T 2^{2\beta j} \theta}}{\kappa 2^{2\beta j} \theta} \right)^{\frac{1}{\theta}} \left\| \Delta_j f(t') \right\|_{L^\gamma_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\ &\lesssim 2^{-\frac{2\beta}{\rho} j} \left\| \Delta_j u_0 \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + 2^{-2\beta(1 + \frac{1}{\rho} - \frac{1}{\gamma}) j} \left\| \Delta_j f(t') \right\|_{L^\gamma_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}. \end{aligned}$$

Finally, multiplying by $2^{(s(\cdot) + \frac{2\beta}{\rho}) j}$, and taking l^h-norm of both sides in the above inequality, we obtain the desired estimate. And this completes the proof of Proposition 3.3. □

3.1. Proof of Theorem 3.1 (1) **(The case** $\frac{1}{2} < \beta \le 1$**)**

In this part, we aim at proving global well-posedness for small initial data of the system (1.1) in critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}$ with $\frac{1}{2} < \beta \le 1$, $1 \le r(\cdot) \le q(\cdot) < \infty$ and $1 \le h \le \infty$. Firstly, we get the following key bilinear estimate.

Lemma 3.5. *Let* $0 < T \le \infty$, $s(\cdot) \in C_{\log}(\mathbb{R}^n)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $s(\cdot) > -1$, $1 \le r(\cdot) \le q(\cdot) < \infty$, *and let* $1 \le h, \gamma, \gamma_1, \gamma_2 \le \infty$ *satisfying* $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. *Then for any* $\varepsilon > 0$, *one has*

$$\begin{aligned}
&\left\| f\nabla(-\Delta)^{-1}g + g\nabla(-\Delta)^{-1}f \right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\quad + \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}. \qquad (3.3)
\end{aligned}$$

Proof. Using the following paraproduct decomposition due to J. M. Bony [7],

$$f\nabla(-\Delta)^{-1}g + g\nabla(-\Delta)^{-1}f := J_1 + J_2 + J_3, \qquad (3.4)$$

where,

$$\begin{aligned}
J_1 &:= \sum_{l\in\mathbb{Z}} \Delta_l f\nabla(-\Delta)^{-1}S_{l-1}g + \Delta_l g\nabla(-\Delta)^{-1}S_{l-1}f, \\
J_2 &:= \sum_{l\in\mathbb{Z}} S_{l-1}f\nabla(-\Delta)^{-1}\Delta_l g + S_{l-1}g\nabla(-\Delta)^{-1}\Delta_l f, \\
J_3 &:= \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\le 1} \Delta_l f\nabla(-\Delta)^{-1}\Delta_{l'} g + \Delta_l g\nabla(-\Delta)^{-1}\Delta_{l'} f.
\end{aligned}$$

Below, we estimate J_1, J_2 and J_3 separately. For J_1, we consider the estimate of its first term only, while the second one can be treated similarly. So, by the facts (2.1) and (2.2), Proposition 2.6, Hölder's inequality in L^p-space, and Lemmas 2.7 and 2.8, when $\varepsilon > 0$, one has

$$\begin{aligned}
&\left\| \Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f\nabla(-\Delta)^{-1}S_{l-1}g \right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \left\| \left\|\mathcal{F}^{-1}\varphi_j\right\|_{L^1} \left\|\Delta_l f\nabla(-\Delta)^{-1}S_{l-1}g\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right\|_{L^{\gamma}_T} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-1}S_{l-1}g\right\|_{L^{\gamma_2}_T(L^\infty)} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \sum_{k\le l-2} 2^{k\left(-1+\frac{n}{q(\cdot)}\right)} \|\Delta_k g\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},
\end{aligned}$$

then,

$$\begin{aligned}
&\left\|\Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left(\sum_{k\le l-2} 2^{\varepsilon k h'}\right)^{1/h'} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim 2^{-s(\cdot)j} \sum_{|l-j|\le 4} 2^{-s(\cdot)(l-j)} 2^{(s(\cdot)+\varepsilon)l} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.
\end{aligned}$$

Multiplying by $2^{s(\cdot)j}$, and taking l^h-norm of both sides in the above estimate, we obtain

$$\begin{aligned}
\left\|\sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\quad \times \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|J_1\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\quad + \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}. \qquad (3.5)
\end{aligned}$$

Similarly for J_2: By applying Hölder's inequality and Lemma 2.7, we get

$$\begin{aligned}
&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \left\|S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \sum_{k\le l-2} 2^{k\frac{n}{q(\cdot)}} \|\Delta_k f\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},
\end{aligned}$$

Lemma 2.8 gives us again,

$$\begin{aligned}
&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \sum_{k\le l-2} 2^{(-1+\frac{n}{q(\cdot)}-\varepsilon)k} 2^{(1+\varepsilon)k} \|\Delta_k f\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{-l} \|\Delta_l g\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} 2^{-l} \|\Delta_l g\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left(\sum_{k\le l-2} 2^{(1+\varepsilon)kh'}\right)^{1/h'} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.
\end{aligned}$$

Since $\varepsilon > 0$, then

$$\begin{aligned}&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1}\Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim \sum_{|l-j|\le 4} 2^{-sl}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\left\|f\right\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}\left\|\sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1}\Delta_l g\right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\\&\quad\times\|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|J_2\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\\&\quad+\|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}\tag{3.6}$$

We are now moving on to the last term J_3. We use the following formula, based on an analysis of the algebraic structure of Equation (1.1) [21]:

$$(J_3)_i = \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\le 1} \Delta_l f \partial_i(-\Delta)^{-1}\Delta_{l'} g + \Delta_l g \partial_i(-\Delta)^{-1}\Delta_{l'} f = K_i^1 + K_i^2 + K_i^3,$$

for $i = 1, 2, ..., n$. Where $(J_3)_i$ is the i-th exponent of (J_3) and

$$\begin{aligned}K_i^1 &:= \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\le 1} (-\Delta)\left[\left((-\Delta)^{-1}\Delta_l f\right)\left(\partial_i(-\Delta)^{-1}\Delta_{l'} g\right)\right],\\K_i^2 &:= \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\le 1} 2\nabla\cdot\left[\left((-\Delta)^{-1}\Delta_l f\right)\left(\partial_i\nabla(-\Delta)^{-1}\Delta_{l'} g\right)\right],\\K_i^3 &:= \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\le 1} \partial_i\left[\left((-\Delta)^{-1}\Delta_l f\right)\Delta_{l'} g\right].\end{aligned}$$

In order to estimate the above three terms, we use Hölder's inequality in L^p-space and $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-space (2.5), and Lemma 2.8 as follows: From (2.2), there is $d_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\left\|\Delta_j K_i^1\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim 2^{2j} \sum_{l\ge j-d_0} \sum_{|l-l'|\le 1} \left\|\left((-\Delta)^{-1}\Delta_l f\right)\left(\partial_i(-\Delta)^{-1}\Delta_{l'} g\right)\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim 2^{2j} \sum_{l\ge j-d_0} \sum_{|l-l'|\le 1} 2^{-2l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} 2^{(-1+\frac{n}{q(\cdot)})l'}\left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim \sum_{l\ge j-d_0} 2^{-s(\cdot)j}2^{-(2+s(\cdot))(l-j)}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\left\|\Delta_j K_i^2\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim 2^{j} \sum_{l\ge j-d_0} \sum_{|l-l'|\le 1} \left\|\left((-\Delta)^{-1}\Delta_l f\right)\left(\partial_i\nabla(-\Delta)^{-1}\Delta_{l'} g\right)\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim 2^{j} \sum_{l\ge j-d_0} \sum_{|l-l'|\le 1} 2^{-2l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} 2^{\frac{n}{q(\cdot)}l'}\left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim \sum_{l\ge j-d_0} 2^{-s(\cdot)j}2^{-(1+s(\cdot))(l-j)}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)},
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\left\|\Delta_j K_i^3\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim 2^{j} \sum_{l\ge j-d_0} \sum_{|l-l'|\le 1} \left\|\left((-\Delta)^{-1}\Delta_l f\right)\Delta_{l'} g\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim 2^{j} \sum_{l\ge j-d_0} \sum_{|l-l'|\le 1} 2^{-2l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} 2^{\frac{n}{q(\cdot)}l'}\left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\\
&\lesssim \sum_{l\ge j-d_0} 2^{-s(\cdot)j}2^{-(1+s(\cdot))(l-j)}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})}\|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)}.
\end{aligned} \tag{3.9}$$

Thus, (3.7), (3.8) and (3.9) give us, when $s(\cdot)+1>0$,

$$\begin{aligned}
\|J_3\|_{\mathcal{L}^\gamma\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)}\right)} &\le \sum_{i=1}^{n}\sum_{k=1}^{3}\left\|K_i^k\right\|_{\mathcal{L}^\gamma\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)}\right)}\\
&\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)+\varepsilon}\right)}\|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)}.
\end{aligned} \tag{3.10}$$

Finally, by combining (3.5), (3.6) and (3.10) with (3.4), we get (3.3). This completes the proof of Lemma 3.5. □

Now, by using Lemma 2.9, we can start to prove the existence of local and global solutions of the system (1.1) in the case $\frac{1}{2} < \beta \leq 1$. We define

$$\mathcal{X}^1 := \mathcal{L}^{\gamma_1}\left(\mathbb{R}_+; \mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_2}\left(\mathbb{R}_+; \mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right),$$

with

$$s_1(\cdot) = -1 + \frac{n}{q(\cdot)} + \varepsilon,\ \ s_2(\cdot) = -1 + \frac{n}{q(\cdot)} - \varepsilon,\ \ \gamma_1 = \frac{2\beta}{2\beta - 1 + \varepsilon},$$
$$\gamma_2 = \frac{2\beta}{2\beta - 1 - \varepsilon},\ \ 0 < \varepsilon < 2\beta - 1.$$

Due to Duhamel's principle, the solution of the system (1.1) can be written as

$$u(t) = e^{-t(-\Delta)^\beta} u_0 - \int_0^t e^{-(t-t')(-\Delta)^\beta} \nabla \cdot \left(u \nabla (-\Delta)^{-1} u\right) \mathrm{d}t'. \tag{3.11}$$

Set

$$\mathcal{A}(u,w) := \int_0^t e^{-(t-t')(-\Delta)^\beta} \nabla \cdot \left(u \nabla (-\Delta)^{-1} w\right) \mathrm{d}t'.$$

We note that $\mathcal{A}(u,w)$ can be considered as the solution of the dissipative equation (3.1) with $u_0 = 0$ and $f = \nabla \cdot \left(u\nabla(-\Delta)^{-1}w\right)$. Then by applying Proposition 3.3 and Lemma 3.5, with $\gamma = \frac{\beta}{2\beta-1}$, we see that

$$\begin{aligned}
\|\mathcal{A}(u,w)\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \left\|\nabla \cdot \left(u\nabla(-\Delta)^{-1}w\right)\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-1}}\left(\mathbb{R}_+;\mathcal{N}^{-2+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \left\|u\nabla(-\Delta)^{-1}w\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-1}}\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \|u\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \|w\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \\
&\quad + \|w\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \|u\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \|u\|_{\mathcal{X}^1} \|w\|_{\mathcal{X}^1},
\end{aligned}$$

and similarly,

$$\|\mathcal{A}(u,w)\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|u\|_{\mathcal{X}^1} \|w\|_{\mathcal{X}^1}.$$

Thus,

$$\|\mathcal{A}(u,w)\|_{\mathcal{X}^1} \leq C \|u\|_{\mathcal{X}^1} \|w\|_{\mathcal{X}^1}. \tag{3.12}$$

On the other hand, $e^{-t(-\Delta)^\beta} u_0$ can also be considered as the solution of the dissipative equation (3.1) with $u_0 = u_0$ and $f = 0$. Then we can directly deduce from Proposition 3.3 that,

$$\left\|e^{-t(-\Delta)^\beta} u_0\right\|_{\mathcal{X}^1} \leq C \|u_0\|_{\dot{\mathcal{N}}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}}.$$

So, if $\|u_0\|_{\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} \leq \varepsilon$ with $\varepsilon = \frac{1}{4C^2}$, then by Lemma 2.9, the integral equation (3.11) admits a unique solution u such that $\|u\|_{\mathcal{X}^1} \leq 2C\varepsilon$, which is the unique solution of the system (1.1). Furthermore, Proposition 3.3 and Lemma 3.5 once again give us

$$\|u\|_{\mathcal{L}^\infty\left(0,T;\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|u_0\|_{\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|u\|^2_{\mathcal{X}^1} < \infty.$$

Finally, $u \in \mathcal{X}_\varepsilon$. This completes the proof of the first assertion of Theorem 3.1.

3.2. Proof of Theorem 3.1 (2) (The case $\beta = \frac{1}{2}$)

In this part, we establish the global well-posedness for the system (1.1) in the limit case $\beta = \frac{1}{2}$, with initial data in critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}$ with $1 \le r(\cdot) \le q(\cdot) < \infty$. Firstly, by making a slight modification to the proof of Lemma 3.5, we obtain the following estimate:

Lemma 3.6. *For any* $f, g \in \mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)$, *one has*

$$\left\|f\nabla(-\Delta)^{-1}g + g\nabla(-\Delta)^{-1}f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \times \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{3.13}$$

Proof. We estimate the first term of J_1 as follows:

$$\begin{aligned}
&\left\|\Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-1} S_{l-1} g\right\|_{L^\infty_\infty(L^\infty)} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \sum_{k\le l-2} 2^{k\left(-1+\frac{n}{q(\cdot)}\right)} \|\Delta_k g\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned}$$

Multiplying by $2^{(-1+\frac{n}{q(\cdot)})j}$, and taking l^1-norm of both sides in the above estimate, we obtain

$$\begin{aligned}
&\left\|\sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \\
&\qquad \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned}$$

And then,

$$\|J_1\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{3.14}$$

Similarly, for J_2,

$$\begin{aligned}&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1}\Delta_l g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim \sum_{|l-j|\le 4}\sum_{k\le l-2} 2^{(-1+\frac{n}{q(\cdot)})k} 2^k \left\|\Delta_k f\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{-l}\left\|\Delta_l g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim \sum_{|l-j|\le 4}\left\|\Delta_l g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\left\|f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},\end{aligned}$$

which gives us that

$$\begin{aligned}&\left\|\sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1}\Delta_l g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\&\qquad\lesssim \left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\left\|f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned}$$

Thus, we get

$$\left\|J_2\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\lesssim \left\|f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{3.15}$$

Moreover for the final term $J_3 = K^1 + K^2 + K^3$, and since K^3 is similar to K^2, we only estimate $K^1$and K^2 as follows:

$$\begin{aligned}&\left\|\Delta_j K^1_i\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{2j}\sum_{l\ge j-d_0}\sum_{|l-l'|\le 1} 2^{-2l}\left\|\Delta_l f\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(-1+\frac{n}{q(\cdot)})l'}\left\|\Delta_{l'} g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{(1-\frac{n}{q(\cdot)})j}\sum_{l\ge j-d_0} 2^{-(1+\frac{n}{q(\cdot)})(l-j)} 2^{(-1+\frac{n}{q(\cdot)})l}\left\|\Delta_l f\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\qquad\times\left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\left\|\Delta_j K^2_i\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{j}\sum_{l\ge j-d_0}\sum_{|l-l'|\le 1} 2^{-2l}\left\|\Delta_l f\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{\frac{n}{q(\cdot)}l'}\left\|\Delta_{l'} g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{(1-\frac{n}{q(\cdot)})j}\sum_{l\ge j-d_0} 2^{-\frac{n}{q(\cdot)}(l-j)} 2^{(-1+\frac{n}{q(\cdot)})l}\left\|\Delta_l f\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\end{aligned}\tag{3.17}$$

Hence, from (3.16) and (3.17), we arrive at

$$\begin{aligned}\left\|J_3\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}&\le\sum_{i=1}^{n}\sum_{k=1}^{3}\left\|K^k_i\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\&\lesssim \left\|f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned}\tag{3.18}$$

Finally, putting the estimates (3.14), (3.15) and (3.18) together, we get (3.13). The proof of Lemma 3.6 is complete. □

We are now in a position to demonstrate the second assertion of Theorem 3.1. By considering the resolution space $\mathcal{L}^\infty(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1})$ and returning to the integral equation (3.11), Proposition 3.3 with $\beta=\frac{1}{2}$ and $\gamma=\infty$, and Lemma 3.6, give us

$$\begin{aligned}\|\mathcal{A}(u,w)\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} &\le C\left\|\nabla\cdot\left(u\nabla(-\Delta)^{-1}w\right)\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-2+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\ &\le C\|u\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\|w\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned}$$

Applying Proposition 3.3 for $\beta=\frac{1}{2}$ again, we can obtain

$$\left\|e^{-t(-\Delta)^{\frac{1}{2}}}u_0\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\le C\|u_0\|_{\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}}.$$

If $\|u_0\|_{\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}}$ is sufficiently small, by using the fixed point argument as in Subsection 3.1, we get the global solution of the system (1.1) in $\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)$. The proof of Theorem 3.1 is complete, as desired.

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Ahmed El Idrissi
LMACS Laboratory, FST, Sultan Moulay Slimane University,
Beni Mellal, 23000, Morocco
e-mail: `ahmed.elidrissi@usms.ma`

Fatima Ouidirne
LMACS Laboratory, FST, Sultan Moulay Slimane University,
Beni Mellal, 23000, Morocco
e-mail: `fati.ouidirne@gmail.com`

Brahim El Boukari
LMACS Laboratory, FST, Sultan Moulay Slimane University,
Beni Mellal, 23000, Morocco
e-mail: `elboukaribrahim@yahoo.fr`

Jalila El Ghordaf
LMACS Laboratory, FST, Sultan Moulay Slimane University,
Beni Mellal, 23000, Morocco
e-mail: elg_jalila@yahoo.fr