

# On the coefficient estimates for a subclass of $m$ -fold symmetric bi-univalent functions

Seyed Hadi Hosseini , Ahmad Motamednezhad , Safa Salehian  and Ágnes Orsolya Páll-Szabó 

**Abstract.** In this work, we introduce and investigate a subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\lambda, \gamma)$  of analytic and bi-univalent functions when both  $f$  and  $f^{-1}$  are  $m$ -fold symmetric in the open unit disk  $\mathbb{U}$ . Moreover, we find upper bounds for the initial coefficients  $|a_{m+1}|$  and  $|a_{2m+1}|$  for functions belonging to this subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ . The results presented in this paper would generalize and improve those that were given in several recent works.

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## 1. Introduction

Let  $\mathcal{A}$  denote the class of functions of the following normalized form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (1.1)$$

which are analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ .


Also, we denote by  $\mathcal{S}$  the class of all functions in the normalized analytic function class  $f \in \mathcal{A}$  which are univalent in  $\mathbb{U}$ .

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk  $\mathbb{U}$ . The *Koebe One-Quarter Theorem* [4] ensures that the image of  $\mathbb{U}$  under every univalent function  $f \in \mathcal{S}$  contains

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a disk of radius  $\frac{1}{4}$ . Hence, every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{1.2}$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $\mathbb{U}$ , if both  $f$  and  $f^{-1}$  are univalent in  $\mathbb{U}$ . The class consisting of bi-univalent functions are denoted by  $\Sigma$ .

Determination of the bounds for the coefficients  $a_n$  is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient  $a_2$  of functions  $f \in \mathcal{S}$  gives the growth and distortion bounds as well as covering theorems.

Lewin [8] investigated the class  $\Sigma$  of bi-univalent functions and showed that  $|a_2| < 1.51$  for the functions belonging to  $\Sigma$ . Subsequently, Brannan and Clunie [2] conjectured that  $|a_2| \leq \sqrt{2}$ . Kedzierawski [7] proved this conjecture for a special case when the function  $f$  and  $f^{-1}$  are starlike functions. Tan [14] obtained the bound for  $|a_2|$  namely  $|a_2| \leq 1.485$  which is the best known estimate for functions in the class  $\Sigma$ . Recently there are interest to study the bi-univalent functions class  $\Sigma$  (see [5, 6, 16, 17]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients  $|a_2|$  and  $|a_3|$ . The coefficient estimate problem i.e. bound of  $|a_n|$  ( $n \in \mathbb{N} - \{1, 2\}$ ) for each  $f \in \Sigma$  given by (1.1) is still an open problem. For each function  $f \in \mathcal{S}$  the function  $h(z)$  given by

$$h(z) = \sqrt[m]{f(z^m)} \quad (z \in \mathbb{U}, m \in \mathbb{N})$$

is univalent and maps the unit disk  $\mathbb{U}$  into a region with  $m$ -fold symmetry. A function is called  $m$ -fold symmetric (see[11, 12, 13]) if the function  $f(z)$  has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}) \tag{1.3}$$

We denote by  $\mathcal{S}_m$  the class of  $m$ -fold symmetric univalent functions in  $\mathbb{U}$ , which are normalized by the series expansion (1.3). In fact, the functions in the class  $\mathcal{S}$  are one-fold symmetric, that is

$$\mathcal{S}_1 = \mathcal{S}$$

Analogous to the concept of  $m$ -fold symmetric univalent functions, we now introduce the concept of  $m$ -fold symmetric bi-univalent functions. Each function  $f \in \Sigma$  generates an  $m$ -fold symmetric bi-univalent function for each integer  $m \in \mathbb{N}$ . The normalized form of  $f$  is given as in (1.3). Furthermore, the series expansion for  $f^{-1}$ , which was recently proven by Srivastava et al. [13], is given as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m + 1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \left[\frac{1}{2}(m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \dots,$$

where  $g = f^{-1}$ .

We denote by  $\Sigma_m$  the class of m-fold symmetric bi-univalent functions in  $\mathbb{U}$ . In the special case when  $m = 1$ , the formula (1.4) for the class  $\Sigma_m$  coincides with the formula (1.2) for the class  $\Sigma$ . Some examples of m-fold symmetric bi-univalent functions are given below:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}} \quad \text{and} \quad [-\log(1 - z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions given by

$$\left(\frac{w^m}{1 - w^m}\right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{w^m} - 1}{e^{w^m}}\right)^{\frac{1}{m}}$$

respectively.

Quite recently, Wanas and Páll-Szabó [15] introduced two new general subclasses  $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$  and  $\mathcal{AS}_{\Sigma_m}^*(\gamma, \lambda; \beta)$  of the m-fold symmetric bi-univalent function class  $\Sigma_m$  consisting of analytic and m-fold symmetric bi-univalent functions in  $\mathbb{U}$  and derived the coefficient bounds for  $|a_{m+1}|$  and  $|a_{2m+1}|$  for functions in each of these new subclasses.

**Definition 1.1.** [15] *A function  $f \in \Sigma_m$  given by (1.3) is said to be in the class  $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$  if it satisfies the following conditions:*

$$\left| \arg \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] \right| < \frac{\alpha\pi}{2},$$

where  $z, w \in \mathbb{U}$ ,  $0 < \gamma \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 < \alpha \leq 1$ ,  $m \in \mathbb{N}$  and  $g = f^{-1}$ .

**Theorem 1.2** ([15]). *Let  $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$  be given by (1.3). Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{2\alpha\gamma(1 + \lambda m) + \gamma(\gamma - \alpha)(1 + \lambda m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2(m + 1)}{m^2\gamma^2(1 + \lambda m)^2} + \frac{\alpha}{m\gamma(1 + 2\lambda m)}.$$

**Definition 1.3.** [15] *A function  $f \in \Sigma_m$  given by (1.3) is said to be in the class  $\mathcal{AS}_{\Sigma_m}^*(\gamma, \lambda; \beta)$ , if it satisfies the following conditions:*

$$\Re \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma > \beta$$

and

$$\Re \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma > \beta,$$

where  $z, w \in \mathbb{U}$ ,  $0 < \gamma \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 \leq \beta < 1$ ,  $m \in \mathbb{N}$  and  $g = f^{-1}$ .

**Theorem 1.4** ([15]). *Let  $f \in \mathcal{AS}^*_{\Sigma_m}(\gamma, \lambda; \beta)$  be given by (1.3). Then*

$$|a_{m+1}| \leq \frac{2}{m} \sqrt{\frac{1 - \beta}{2\gamma(1 + \lambda m) + \gamma(\gamma - 1)(1 + \lambda m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2(m + 1)(1 - \beta)^2}{m^2\gamma^2(1 + \lambda m)^2} + \frac{1 - \beta}{m\gamma(1 + 2\lambda m)}.$$

The main objective of this paper is to present an elegant formula for computing the coefficients of the inverse functions for the class  $\Sigma_m$  of  $m$ -fold symmetric functions by means of the residue calculus. As an application, we introduce a new subclass of bi-univalent functions in which both  $f$  and  $f^{-1}$  are  $m$ -fold symmetric analytic functions and obtain upper bounds for the coefficients  $|a_{m+1}|$  and  $|a_{2m+1}|$  for functions in this subclass. Our results for the bi-univalent function class  $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ , which we shall introduce in section 2, would generalize and improve some recent works by Wanas and Páll-Szabó [15] and some of other researchers[1, 9, 10]

## 2. Coefficient Estimates

In this section, we introduce and investigate the general subclass  $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ .

**Definition 2.1.** *Let  $h, p : \mathbb{U} \rightarrow \mathbb{C}$  be analytic functions and*

$$\min\{\Re(h(z)), \Re(p(z))\} > 0 \quad (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$

*A function  $f$  given by (1.3) is said to be in the class  $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ , if the following conditions are satisfied:*

$$\left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma \in h(\mathbb{U}) \tag{2.1}$$

and

$$\left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma \in p(\mathbb{U}) \tag{2.2}$$

where  $z, w \in \mathbb{U}$ ,  $0 < \gamma \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $m \in \mathbb{N}$  and  $g = f^{-1}$ .

**Remark 2.2.** There are many choices of the functions  $h, p$  which would provide interesting subclasses of the general class  $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ . For example, if we set  $\gamma = 1$ , the subclass  $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$  reduces to the subclass  $f \in \mathcal{M}^{h,p}_{\Sigma_m}(\lambda, 1)$  which was introduced by Motamednezhad et al. [10].

If we let

$$h(z) = p(z) = \left( \frac{1+z^m}{1-z^m} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \dots \quad (0 < \alpha \leq 1),$$

it can easily be verified that the functions  $h(z)$  and  $p(z)$  satisfy the hypotheses of Definition 2.1. Thus, if we have  $f \in \mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ , then

$$\left| \arg \left[ (1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left[ (1-\lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma \right| < \frac{\alpha\pi}{2}.$$

In this case we say that  $f$  belongs to the subclass  $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ . If we put  $h(z) = p(z) = \left( \frac{1+z^m}{1-z^m} \right)^\alpha$  and  $\gamma = 1$ , the subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$  reduces to the subclass  $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$  which was considered by Motamednezad et al. [10].

Also, for  $h(z) = p(z) = \left( \frac{1+z^m}{1-z^m} \right)^\alpha$ ,  $\gamma = 1$  and  $\lambda = 0$ , the subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$  reduces to the subclass  $\mathcal{S}_{\Sigma_m}^\alpha$  which was considered by Altinkaya and Yalcin [1].

On the other hand, if we take

$$h(z) = p(z) = \frac{1 + (1 - 2\beta)z^m}{1 - z^m} = 1 + 2(1 - \beta)z^m + 2(1 - \beta)z^{2m} + \dots \quad (0 \leq \beta < 1).$$

then the conditions of Definition 2.1 are satisfied for both functions  $h(z)$  and  $p(z)$ . Thus, if  $f \in \mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ ; then

$$\Re \left[ (1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma > \beta$$

and

$$\Re \left[ (1-\lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma > \beta.$$

In this case we say that  $f$  belongs to the subclass  $f \in \mathcal{AS}_{\Sigma_m}^*(\gamma, \lambda; \beta)$ . If we put  $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$  and  $\gamma = 1$ , the subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$  reduces to the subclass  $\mathcal{M}_{\Sigma_m}(\beta, \lambda, 1)$  which was considered by Motamednezad et al. [10].

Also, for  $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ ,  $\gamma = 1$  and  $\lambda = 0$ , the subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$  reduces to the subclass  $\mathcal{S}_{\Sigma_m}^\beta$  which was considered by Altinkaya and Yalcin [1].

**Remark 2.3.** For one-Fold symmetric bi-univalent functions, we denote the subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda) = \mathcal{G}_{\Sigma}^{h,p}(\gamma, \lambda)$ . Special cases of this subclass illustrated below:

- (A) By putting  $h(z) = p(z) = \left( \frac{1+z}{1-z} \right)^\alpha$  and  $\gamma = 1$ , then the subclass  $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$  reduces to the subclass  $M_{\Sigma}(\alpha, \lambda)$  studied by Li and Wang [9].
- (B) By putting  $h(z) = p(z) = \left( \frac{1+z}{1-z} \right)^\alpha$ ,  $\gamma = 1$  and  $\lambda = 0$ , then the subclass  $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$  reduces to the subclass  $S_{\Sigma}^*(\alpha)$  studied by Brannan and Taha[3].

- (C) By putting  $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ , and  $\lambda = \gamma = 1$ , then the subclass  $\mathcal{G}_\Sigma^{h,p}(\lambda, \gamma)$  reduces to the subclass  $M_\Sigma(\alpha, 1)$  studied by Li and Wang [9].
- (D) By putting  $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$  and  $\gamma = 1$ , then the subclass  $\mathcal{G}_\Sigma^{h,p}(\lambda, \gamma)$  reduces to the subclass  $B_\Sigma(\beta, \lambda)$  studied by Li and Wang [9].
- (E) By putting  $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $\gamma = 1$  and  $\lambda = 0$ , then the subclass  $\mathcal{G}_\Sigma^{h,p}(\lambda, \gamma)$  reduces to the subclass  $S_\Sigma^*(\beta)$  of bi-starlike functions of order  $\beta (0 \leq \beta < 1)$  studied by Brannan and Taha[3].
- (F) By putting  $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$  and  $\lambda = \gamma = 1$ , then the subclass  $\mathcal{G}_\Sigma^{h,p}(\lambda, \gamma)$  reduces to the subclass  $B_\Sigma(\beta, 1)$  of bi-convex functions of order  $\beta (0 \leq \beta < 1)$  studied by Li and Wang [9].

We are now ready to express the bounds for the coefficients  $|a_{m+1}|$  and  $|a_{2m+1}|$  for the subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$  of the normalized bi-univalent function class  $\Sigma_m$ .

**Theorem 2.4.** *Let the function  $f$  given by (1.3) be in the class  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ . Then*

$$|a_{m+1}| \leq \min \left\{ \sqrt{\frac{|h_{2m}| + |p_{2m}|}{m^2\gamma[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}}, \sqrt{\frac{|h_m|^2 + |p_m|^2}{2[m\gamma(1 + \lambda m)]^2}} \right\} \tag{2.3}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{|h_{2m}| + |p_{2m}|}{4\gamma m(1 + 2\lambda m)} + \frac{(m + 1)(|h_m|^2 + |p_m|^2)}{4\gamma^2 m^2(1 + \lambda m)^2}, \frac{|2m(1 + \lambda m) + 2(m + 1)(1 + 2\lambda m) + m(\gamma - 1)(1 + \lambda m)^2|}{4m^2\gamma(1 + 2\lambda m)[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}|h_{2m}| + \frac{|2(m + 1)(1 + 2\lambda m) - 2m(1 + \lambda m) - m(\gamma - 1)(1 + \lambda m)^2|}{4m^2\gamma(1 + 2\lambda m)[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}|p_{2m}| \right\}. \tag{2.4}$$

*Proof.* The main idea in the proof of Theorem 2.4 is to get the desired bounds for the coefficient  $|a_{m+1}|$  and  $|a_{2m+1}|$ . Indeed, by considering the relations (2.1) and (2.2), we have

$$\left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma = h(z) \tag{2.5}$$

and

$$\left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma = p(z), \tag{2.6}$$

where each of the functions  $h$  and  $p$  satisfies the conditions of Definition 1.3. In light of the following Taylor-Maclaurin series expansions for the functions  $h$  and  $p$ , we get

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \dots \tag{2.7}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \dots \tag{2.8}$$

By substituting the relations (2.7) and (2.8) into (2.5) and (2.6), respectively, we get

$$m\gamma(1 + \lambda m)a_{m+1} = h_m, \tag{2.9}$$

$$\begin{aligned} \gamma m \left[ \frac{(\gamma - 1)}{2} m(1 + \lambda m)^2 - (\lambda m^2 + 2\lambda m + 1) \right] a_{m+1}^2 \\ + 2m\gamma(1 + 2\lambda m)a_{2m+1} = h_{2m}, \end{aligned} \tag{2.10}$$

$$-m\gamma(1 + \lambda m)a_{m+1} = p_m \tag{2.11}$$

and

$$\begin{aligned} \gamma m \left[ (3\lambda m^2 + 2(\lambda + 1)m + 1) + \frac{(\gamma - 1)}{2} m(1 + \lambda m)^2 \right] a_{m+1}^2 \\ - 2m\gamma(1 + 2\lambda m)a_{2m+1} = p_{2m}. \end{aligned} \tag{2.12}$$

Comparing the coefficients (2.9) and (2.11), we obtain

$$h_m = -p_m \tag{2.13}$$

and

$$2m^2\gamma^2(1 + \lambda m)^2 a_{m+1}^2 = h_m^2 + p_m^2. \tag{2.14}$$

Now, if we add (2.10) and (2.12), we get the following relation

$$m^2\gamma \left[ 2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2 \right] a_{m+1}^2 = h_{2m} + p_{2m}. \tag{2.15}$$

Therefore, from (2.14) and (2.15), we have

$$a_{m+1}^2 = \frac{h_m^2 + p_m^2}{2[m\gamma(1 + \lambda m)]^2} \tag{2.16}$$

and

$$a_{m+1}^2 = \frac{h_{2m} + p_{2m}}{m^2\gamma[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}, \tag{2.17}$$

respectively.

Therefore, we find from the equations (2.16) and (2.17) that

$$|a_{m+1}|^2 \leq \frac{|h_m|^2 + |p_m|^2}{2\gamma^2 m^2 (1 + \lambda m)^2}$$

and

$$|a_{m+1}|^2 \leq \frac{|h_{2m}| + |p_{2m}|}{m^2\gamma[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]},$$

respectively. We have thus derived the desired bound on the coefficient  $|a_{m+1}|$ .

The proof is completed by finding the bound on the coefficient  $|a_{2m+1}|$ . Upon subtracting (2.12) from (2.10), we get

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4\gamma m(1 + 2\lambda m)} + \frac{(m + 1)}{2} a_{m+1}^2. \tag{2.18}$$

Putting the value of  $a_{m+1}^2$  from (2.16) into (2.18), it follows that

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4\gamma m(1 + 2\lambda m)} + \frac{(m + 1)h_{2m} + p_{2m}}{m^2\gamma[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}.$$

Therefore, we conclude the following bound:

$$|a_{2m+1}| \leq \frac{|h_{2m}| + |p_{2m}|}{4\gamma m(1 + 2\lambda m)} + \frac{(m + 1)(|h_m|^2 + |p_m|^2)}{4[\gamma m(1 + \lambda m)]^2}. \tag{2.19}$$

By substituting the value of  $a_{m+1}^2$  from (2.17) into (2.18), we obtain

$$a_{2m+1} = \frac{m[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2](h_{2m} - p_{2m}) + (m + 1)(1 + 2\lambda m)}{4m^2\gamma(1 + 2\lambda m)[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]} \tag{2.20}$$

$$\frac{(h_{2m} + p_{2m})}{4m^2\gamma(1 + 2\lambda m)[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}$$

which readily yields

$$|a_{2m+1}| \leq \frac{|2m(1 + \lambda m) + 2(m + 1)(1 + 2\lambda m) + m(\gamma - 1)(1 + \lambda m)^2|}{4m^2\gamma(1 + 2\lambda m)[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]} |h_{2m}| + \tag{2.21}$$

$$\frac{|2(m + 1)(1 + 2\lambda m) - 2m(1 + \lambda m) - m(\gamma - 1)(1 + \lambda m)^2|}{4m^2\gamma(1 + 2\lambda m)[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]} |p_{2m}|. \tag{2.22}$$

Finally, from (2.19) and (2.21), we get the desired estimate on the coefficient  $|a_{2m+1}|$  as asserted in Theorem 2.4. The proof of Theorem 2.4 is thus completed.  $\square$

### 3. Corollaries and Consequences

If we put

$$h(z) = p(z) = \left( \frac{1 + z^m}{1 - z^m} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \dots,$$

in Theorem 2.4, then it can be obtained the following result.

**Corollary 3.1.** *Let the function  $f$  given by (1.3) be in the class  $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ . Then*

$$|a_{m+1}| \leq \min \left\{ \frac{2\alpha}{m\gamma(1 + \lambda m)}, \frac{2\alpha}{m\sqrt{\gamma[2(1 + \lambda m) + (\gamma - 1)(1 + \lambda m)^2]}} \right\}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{\alpha^2}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2}, \right. \\ \left. \left| \frac{2m(1+\lambda m) + 2(m+1)(1+2\lambda m) + m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]} \right| \alpha^2 + \right. \\ \left. \left| \frac{2(m+1)(1+2\lambda m) - 2m(1+\lambda m) - m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]} \right| \alpha^2 \right\}.$$

**Remark 3.2.** For the coefficient  $|a_{2m+1}|$  it is easily seen that

$$\frac{\alpha^2}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2} \leq \frac{\alpha}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2}.$$

Therefore, clearly, Corollary 3.1 provides an improvement over Theorem 1.2.

By setting  $\gamma = 1$  in Corollary 3.1, we conclude the following result.

**Corollary 3.3.** Let the function  $f$  given by (1.3) be in the subclass  $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$ . Then

$$|a_{m+1}| \leq \min \left\{ \frac{2\alpha}{m(1+\lambda m)}, \frac{2\alpha}{m\sqrt{2(1+\lambda m)}} \right\} = \begin{cases} \frac{2\alpha}{m\sqrt{2(1+\lambda m)}}, & 0 \leq \lambda \leq \frac{1}{m} \\ \frac{2\alpha}{m(1+\lambda m)}, & \frac{1}{m} \leq \lambda < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{m(1+\lambda m)^2 + 2(m+1)(1+2\lambda m)}{m^2(1+2\lambda m)(1+\lambda m)^2} \alpha^2, \frac{(m+1)}{m^2(1+\lambda m)} \alpha^2 \right\}.$$

By setting  $\lambda = 0$  in Corollary 3.3, we conclude the following result.

**Corollary 3.4.** Let the function  $f$  given by (1.3) be in the subclass  $\mathcal{S}_{\Sigma_m}^\alpha$ . Then

$$|a_{m+1}| \leq \min \left\{ \frac{2\alpha}{m}, \frac{\sqrt{2}\alpha}{m} \right\} = \frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{(3m+2)\alpha^2}{m^2}, \frac{(m+1)\alpha^2}{m^2} \right\} = \frac{(m+1)\alpha^2}{m^2}.$$

**Remark 3.5.** The bounds on  $|a_{m+1}|$  and  $|a_{2m+1}|$  given in Corollary 3.4 are better than those given by Altinkaya and Yalcin [1, Corollary 6], because of

$$\frac{\sqrt{2}\alpha}{m} \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2} \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

By setting  $\gamma = 1$  and  $m = 1$  in Corollary 3.1, we conclude the following result.

**Corollary 3.6.** *Let the function  $f$  given by (1.1) be in the subclass  $\mathcal{M}_\Sigma(\alpha, \lambda)$ . Then*

$$|a_2| \leq \min \left\{ \frac{2\alpha}{1+\lambda}, \alpha \sqrt{\frac{2}{1+\lambda}} \right\} = \alpha \sqrt{\frac{2}{1+\lambda}}$$

and

$$|a_3| \leq \min \left\{ \frac{\lambda^2 + 10\lambda + 5}{(1+2\lambda)(1+\lambda)^2} \alpha^2, \frac{2\alpha^2}{1+\lambda} \right\} = \frac{2\alpha^2}{1+\lambda}.$$

**Remark 3.7.** The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 3.6 are better than those given by Li and Wang [9, Theorem 2.2].

By setting  $\lambda = 0$  in Corollary 3.6, we conclude the following result.

**Corollary 3.8.** *Let the function  $f$  given by (1.1) be in the subclass  $S_\Sigma^*(\alpha)$ . Then*

$$|a_2| \leq \sqrt{2}\alpha \quad \text{and} \quad |a_3| \leq 2\alpha^2.$$

**Remark 3.9.** The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 3.10 are better than those given by Brannan and Taha [3].

By setting  $\lambda = 1$  in Corollary 3.6, we conclude the following result.

**Corollary 3.10.** *Let the function  $f$  given by (1.1) be in the subclass  $\mathcal{M}_\Sigma(\alpha, 1)$ . Then*

$$|a_2| \leq \alpha \quad \text{and} \quad |a_3| \leq \alpha^2.$$

**Remark 3.11.** The bound on  $|a_3|$  given in Corollary 3.8 are better than those given by Li and Wang [9, Theorem 2.2] for  $\lambda = 1$ .

By letting

$$h(z) = p(z) = \frac{1 + (1 - 2\beta)z^m}{1 - z^m} = 1 + 2(1 - \beta)z^m + 2(1 - \beta)z^{2m} + \dots \quad (0 \leq \beta < 1).$$

in Theorem 2.4, we deduce the following corollary.

**Corollary 3.12.** *Let the function  $f$  given by (1.3) be in the class  $f \in \mathcal{AS}^*_{\Sigma_m}(\gamma, \lambda; \beta)$ . Then*

$$|a_{m+1}| \leq \min \left\{ \frac{2(1-\beta)}{m\gamma(1+\lambda m)}, \frac{2}{m} \sqrt{\frac{(1-\beta)}{\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}} \right\}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{1-\beta}{m\gamma(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{\gamma^2 m^2 (1+\lambda m)^2}, \left| \frac{2m(1+\lambda m) + 2(m+1)(1+2\lambda m) + m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]} \right| (1-\beta) + \left| \frac{2(m+1)(1+2\lambda m) - 2m(1+\lambda m) - m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]} \right| (1-\beta) \right\}.$$

**Remark 3.13.** Clearly, Corollary 3.12 provides an improvement over Theorem 1.4.

By setting  $\gamma = 1$  in Corollary 3.12, we conclude the following result.

**Corollary 3.14.** *Let the function  $f$  given by (1.3) be in the subclass  $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$ . Then*

$$|a_{m+1}| \leq \min \left\{ \frac{2(1-\beta)}{m(1+\lambda m)}, \frac{2}{m} \sqrt{\frac{(1-\beta)}{2(1+\lambda m)}} \right\}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{1-\beta}{m(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{m^2(1+\lambda m)^2}, \frac{m+1}{m^2(1+\lambda m)}(1-\beta) \right\}.$$

By setting  $\lambda = 0$  in Corollary 3.14, we conclude the following result.

**Corollary 3.15.** *Let the function  $f$  given by (1.3) be in the subclass  $S_{\Sigma_m}^\beta$ . Then*

$$|a_{m+1}| \leq \min \left\{ \frac{2(1-\beta)}{m}, \frac{\sqrt{2(1-\beta)}}{m} \right\} = \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}, & 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{m}, & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{m(1-\beta) + 2(m+1)(1-\beta)^2}{m^2}, \frac{m+(1-\beta)}{m^2} \right\} \\ = \begin{cases} \frac{m+(1-\beta)}{m^2}, & 0 \leq \beta \leq \frac{1+2m}{2(1+m)} \\ \frac{m(1-\beta) + 2(m+1)(1-\beta)^2}{m^2}, & \frac{1+2m}{2(1+m)} \leq \beta < 1. \end{cases}$$

**Remark 3.16.** Clearly, the bounds on  $|a_{m+1}|$  and  $|a_{2m+1}|$  given in Corollary 3.15 are better than those given by Altinkaya and Yalcin [1, Corolary 7].

By setting  $\gamma = 1$  and  $m = 1$  in Corollary 3.12, we conclude the following result.

**Corollary 3.17.** *Let the function  $f$  given by (1.1) be in the subclass  $\mathcal{B}_\Sigma(\beta, \lambda)$ . Then*

$$|a_2| \leq \min \left\{ \frac{2(1-\beta)}{1+\lambda}, \sqrt{\frac{2(1-\beta)}{1+\lambda}} \right\} = \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}}, & 0 \leq \beta \leq \frac{1-\lambda}{2} \\ \frac{2(1-\beta)}{1+\lambda}, & \frac{1-\lambda}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \min \left\{ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}, \frac{2(1-\beta)}{1+\lambda} \right\} \\ = \begin{cases} \frac{2(1-\beta)}{1+\lambda}, & 0 \leq \beta \leq \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \\ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}, & \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \leq \beta < 1. \end{cases}$$

**Remark 3.18.** The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 3.17 is better than that given by Li and Wang [9, Theorem 3.2].

By setting  $\lambda = 0$  in Corollary 3.17, we conclude the following result.

**Corollary 3.19.** *Let the function  $f$  given by (1.1) be in the subclass  $S_{\Sigma}^*(\beta)$ . Then*

$$|a_2| \leq \min \left\{ 2(1 - \beta), \sqrt{2(1 - \beta)} \right\} = \begin{cases} \sqrt{2(1 - \beta)}, & 0 \leq \beta \leq \frac{1}{2} \\ 2(1 - \beta), & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \min \left\{ (1 - \beta)(5 - 4\beta), 2(1 - \beta) \right\} = \begin{cases} 2(1 - \beta), & 0 \leq \beta \leq \frac{3}{4} \\ (1 - \beta)(5 - 4\beta), & \frac{3}{4} \leq \beta < 1. \end{cases}$$

**Remark 3.20.** The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 3.19 are better than those given by Brannan and Taha [3].

By setting  $\lambda = 1$  in Corollary 3.17, we conclude the following result.

**Corollary 3.21.** *Let the function  $f$  given by (1.1) be in the subclass  $B_{\Sigma}(\beta, 1)$ . Then*

$$|a_2| \leq \min \left\{ 1 - \beta, \sqrt{1 - \beta} \right\} = 1 - \beta$$

and

$$|a_3| \leq \min \left\{ \frac{(1 - \beta) + 3(1 - \beta)^2}{3}, 1 - \beta \right\} = \begin{cases} 1 - \beta, & 0 \leq \beta \leq \frac{1}{3} \\ \frac{(1 - \beta) + 3(1 - \beta)^2}{3}, & \frac{1}{3} \leq \beta < 1. \end{cases}$$

**Remark 3.22.** The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 3.21 are better than those given by Li and Wang [9, Theorem 3.2] for  $\lambda = 1$ .

## 4. Conclusions


In this paper, we introduce a new subclass  $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$  of analytic functions, characterized by  $m$ -fold symmetric as a foundational framework. It is worth noting that this subclass is a generalization of many well-known or new subclasses, mentioned in section 2. Moreover, by Theorem 2.4, we obtained sharp bounds of the coefficients for many well-known subclasses as consequences. That in certain cases our data has improved the results of others.


## Declaration of authorship


All authors contributed equally to this work and share first authorship.


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Seyed Hadi Hosseini   
“Shahrood University of Technology”,  
Faculty of Mathematical Sciences  
P. O. Box 316-36155, Shahrood, Iran  
e-mail: [hosseini.hadi2012@gmail.com](mailto:hosseini.hadi2012@gmail.com)

Ahmad Motamednezhad   
“Shahrood University of Technology”,  
Faculty of Mathematical Sciences  
P. O. Box 316-36155, Shahrood, Iran  
e-mail: [a.motamedne@gmail.com](mailto:a.motamedne@gmail.com)

Safa Salehian   
”Islamic Azad University”,  
Department of Mathematics  
Go. C., Gorgan, Iran  
e-mail: [s.salehian84@iau.ac.ir](mailto:s.salehian84@iau.ac.ir)

Ágnes Orsolya Páll-Szabó   
“Babeş-Bolyai” University,  
Department of Statistics- Forecasts- Mathematics,  
FSEGA, Cluj-Napoca, Romania  
e-mail: [agnes.pallszabo@econ.ubbcluj.ro](mailto:agnes.pallszabo@econ.ubbcluj.ro)