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Quasilinear differential inclusions driven by degenerated p-Laplacian with weight

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Dedicated to the memory of Professor Csaba Varga

Abstract. The main result of the paper provides the existence of a solution to a quasilinear inclusion problem with Dirichlet boundary condition which exhibits a term with full dependence on the solution and its gradient (convection term) and is driven by the degenerated p-Laplacian with weight. The multivalued term in the differential inclusion is in form of the generalized gradient of a locally Lipschitz function expressed through the primitive of a locally essentially bounded function, which makes the problem to be of a hemivariational inequality type. The novelty of our result is that we are able to simultaneously handle three major features: degenerated leading operator, convection term and discontinuous nonlinearity. Results of independent interest regard certain nonlinear operators associated to the differential inclusion.

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1. Introduction

The aim of this paper is to study the quasilinear differential inclusion

$$\begin{cases} -\Delta_p^a u \in f(x,u,\nabla u) + [\underline{g}(u), \overline{g}(u)] & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, for $N \geq 1$, with a Lipschitz boundary $\partial\Omega$. Here $-\Delta_p^a$ denotes the (negative) degenerated p-Laplacian with the positive weight $a \in L^1_{\mathrm{loc}}(\Omega)$ (see Section 3 for the precise definition). In the right-hand side of equation (1.1) there is the convection term $f(x,u,\nabla u)$, i.e., it depends on the solution u and its gradient ∇u, which is described by a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, that is,

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$f(\cdot, s, \xi)$ is measurable on Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$. The multivalued term in (1.1) is expressed by means of a function $g \in L^\infty_{\text{loc}}(\mathbb{R})$ for which we set

$$\underline{g}(t) = \lim_{\delta \to 0} \operatorname{essinf}_{|\tau - t| < \delta} g(\tau), \ \forall\, t \in \mathbb{R}, \tag{1.2}$$

(i.e., the essential infimum of g at t) and

$$\overline{g}(t) = \lim_{\delta \to 0} \operatorname{esssup}_{|\tau - t| < \delta} g(\tau), \ \forall\, t \in \mathbb{R} \tag{1.3}$$

(i.e., the essential supremum of g at t). Since $g \in L^\infty_{\text{loc}}(\mathbb{R})$, it is clear that the expressions in (1.2) and (1.3) are well defined. If the function g is continuous, then the interval $[\underline{g}(u(x)), \overline{g}(u(x))]$ collapses to the singleton $g(u(x))$. Consequently, in this case (1.1) reduces to the quasilinear Dirichlet equation

$$\begin{cases} -\Delta_p^a u = f(x, u, \nabla(u)) + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The multivalued term $[\underline{g}(u), \overline{g}(u)]$ in (1.1) is actually the generalized gradient of a locally Lipschitz function that will be explicitly identified in Section 4. This fact qualifies problem (1.1) as a hemivariational inequality, which is of a special type due to the degenerated operator in the principal part and the presence of the convection term inducing a full gradient dependence. Besides their substantial mathematical interest in passing from convex nonsmooth potentials to nonconvex nonsmooth potentials, the hemivariational inequalities represent a powerful tool to model phenomena with various contact laws in mechanics and engineering. For theoretical developments in the study of hemivariational inequality based on nonsmooth variational methods, we refer to [4, 6, 7, 11, 12, 14, 15]. Nonvariational techniques, such as theoretic operator methods and sub-supersolution, have also been implementing in the nonsmooth multivalued setting of hemivariational inequalities, for instance, in [1, 8, 9, 10, 13, 16]. Problems (1.1) and (1.4) do not have variational structure due to the presence of the convection term, so the variational methods are not applicable. To overcome this difficulty we are going to apply in Section 5 the main theorem for multivalued pseudomonotone operators. Notice that if $f = 0$, problem (1.1) becomes a nonsmooth variational problem with discontinuous nonlinearities extending statements in [1, 2, 12]) to the case where the driving operator is degenerated exhibiting weights. In the situation of (1.4) with $f = 0$, we have a quasilinear elliptic equation that can be treated by using the smooth critical point theory. If $g = 0$, problems (1.1) and (1.4) extend previous statements to formulations involving degenerated operators with weights (see [10]).

The most significant contribution of the paper is to resolve problem (1.1) (and implicitly (1.4)) that incorporates in the same statement three challenging aspects: degenerated leading operator, convection term and discontinuous nonlinearity. This main result is stated as Theorem 5.1. It is the first available result encompassing the three relevant features mentioned before. The solutions to problems (1.1) and (1.4) are sought in a suitable Sobolev space $W_0^{1,p}(a, \Omega)$ that corresponds to the positive weight $a \in L^1_{\text{loc}}(\Omega)$ as discussed in Section 2. By a (weak) solution to problem (1.1) we mean any $u \in W_0^{1,p}(a, \Omega)$ for which it holds $f(x, u, \nabla u)$, $\underline{g}(u)$, $\overline{g}(u) \in L^{p/(p-1)}(\Omega)$

and

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx-\int_\Omega f(x,u,\nabla u)v\,dx\\&\geq\int_\Omega \min\{\underline{g}(u(x))v(x),\overline{g}(u(x))v(x)\}\,dx\ \text{ for all } v\in W_0^{1,p}(a,\Omega).\end{aligned}\tag{1.5}$$

Replacing $v\in W_0^{1,p}(a,\Omega)$ with $-v$ it is seen that (1.5) is equivalent to

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx-\int_\Omega f(x,u,\nabla u)v\,dx\\&\leq\int_\Omega \max\{\underline{g}(u(x))v(x),\overline{g}(u(x))v(x)\}\,\mathrm{d}x\ \text{ for all } v\in W_0^{1,p}(a,\Omega).\end{aligned}\tag{1.6}$$

As it is apparent from (1.5) (or (1.6)), (1.2) and (1.3), for the Dirichlet equation (1.4) the usual notion of weak solution is retrieved. Indeed, if $g:\mathbb{R}\to\mathbb{R}$ is continuous, then the interval $[\underline{g}(u),\overline{g}(u)]$ reduces to the singleton $g(u)$, thus $u\in W_0^{1,p}(\Omega)$ is a (weak) solution to equation (1.4) provided $f(x,u,\nabla u)$, $g(u)\in L^{p/(p-1)}(\Omega)$ and

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx-\int_\Omega f(x,u,\nabla u)v\,dx\\&=\int_\Omega g(u(x))v(x)\,\mathrm{d}x \text{ for all } v\in W_0^{1,p}(a,\Omega).\end{aligned}\tag{1.7}$$

In addition to the existence result, the paper contains propositions of independent interest establishing properties of certain nonlinear operators associated to problem (1.1).

The rest of the paper is structured as follows. Section 2 collects needed preliminaries regarding multivalued pseudomonotone operators and nonsmooth analysis. Section 3 focuses on the degenerated p-Laplacian with weight driving (1.1). Section 4 investigates nonlinear operators related to problem (1.1). Section 5 presents our main result and its proof.

2. Prerequisites on multivalued pseudomonotone operators and nonsmooth analysis

This section provides necessary mathematical background for our results on problem (1.1), in particular (1.4).

We start by briefly reviewing the multivalued pseudomonotone operators. More details can be found in [1, 11, 17]. Let X be a reflexive Banach space with the norm $\|\cdot\|$, its dual X^* and the duality pairing $\langle\cdot,\cdot\rangle$ between X and X^*. The norm convergence in X and X^* is denoted by $\to$, while the weak convergence by $\rightharpoonup$. A multivalued map $A:X\to 2^{X^*}$ is called bounded if it maps bounded sets into bounded sets. It is said to be coercive if there is a function $\psi:\mathbb{R}_+\to\mathbb{R}$ with $\psi(t)\to+\infty$ as $t\to+\infty$ such that

$$\langle u^*,u-u_0\rangle\geq\psi(\|u\|)\|u\|$$

for all $u^* \in A(u)$ and a fixed element $u_0 \in X$. A multivalued map $A : X \to 2^{X^*}$ is called pseudomonotone if

(i) for each $v \in X$, the set $Av \subset X^*$ is nonempty, bounded, closed and convex;
(ii) A is upper semicontinuous from each finite dimensional subspace of X to X^* endowed with the weak topology;
(iii) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with

$$u_n \rightharpoonup u \text{ in } X,\ u_n^* \in Au_n \text{ for all } n \text{ and } \limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \le 0,$$

and for each $v \in X$, there exists $u^*(v) \in Au$ such that

$$\langle u^*(v), u - v\rangle \le \liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle.$$

We recall the main theorem for pseudomonotone operators (see, e.g., [1, Theorem 2.125]).

Theorem 2.1. *Let X be a reflexive Banach space, let $A : X \to 2^{X^*}$ be a pseudomonotone, bounded and coercive operator, and let $\eta \in X^*$. Then there exists at least a $u \in X$ with $\eta \in Au$.*

Next we outline some basic elements of nonsmooth analysis related to locally Lipschitz functions. An extensive study of this topic is available in [2, 3]). A function $\Phi : X \to \mathbb{R}$ on a Banach space X is called locally Lipschitz if for every $u \in X$ there is a neighborhood U of u in X and a constant $L_u > 0$ such that

$$|\Phi(v) - \Phi(w)| \le L_u \|v - w\|, \quad \forall\, v, w \in U.$$

The generalized directional derivative of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at $u \in X$ in the direction $v \in X$ is defined as

$$\Phi^0(u; v) := \limsup_{w\to u,\ t\to 0^+} \frac{1}{t}(\Phi(w + tv) - \Phi(w))$$

and the generalized gradient of Φ at $u \in X$ is the subset of the dual space X^* given by

$$\partial\Phi(u) := \left\{u^* \in X^* \ :\ \langle u^*, v\rangle \le \Phi^0(u; v), \quad \forall\, v \in X\right\}.$$

A continuous and convex function $\Phi : X \to \mathbb{R}$ is locally Lipschitz and its generalized gradient $\partial\Phi : X \to 2^{X^*}$ coincides with the subdifferential of Φ in the sense of convex analysis. As another important example, if $\Phi : X \to \mathbb{R}$ is continuously differentiable, the generalized gradient of Φ is just the differential $D\Phi$ of Φ.

The preceding notions of subdifferentiability theory for locally Lipschitz functions are needed to handle the multivalued term $[\underline{g}(u), \overline{g}(u)]$ in problem (1.1). Given $g : \mathbb{R} \to \mathbb{R}$ satisfying $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, we introduce

$$G(t) = \int_0^t g(t)\,\mathrm{d}t \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

The function $G : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and one can show that the generalized gradient $\partial G(t)$ of G at any $t \in \mathbb{R}$ is the compact interval

$$\partial G(t) = [\underline{g}(t), \overline{g}(t)], \tag{2.2}$$

where $\underline{g}(t)$ and $\overline{g}(t)$ are the functions in (1.2) and (1.3), respectively (see, e.g., [3, Example 2.2.5]).

3. The degenerated p-Laplacian with weight

Here we provide basic facts on the underlying space and driving operator in problem (1.1). An extensive related material can be found in [5]. The notation $|\cdot|$ will stand for the absolute value and Euclidean norm.
We assume the following hypothesis formulated in [5, p. 26] on the weight $a \in L^1_{\mathrm{loc}}(\Omega)$:

$$\text{(H1)} \qquad a^{-s} \in L^1(\Omega) \ \text{ for some } s \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right).$$

Given a real number $p \in (1, +\infty)$, a positive function $a \in L^1_{\mathrm{loc}}(\Omega)$ satisfying condition (H1), and a bounded domain $\Omega \subset \mathbb{R}^N$ of Lebesgue measure $|\Omega|$, with a Lipschitz boundary $\partial\Omega$, we introduce the weighted space

$$W^{1,p}(a, \Omega) := \{u \in L^p(\Omega) : \int_\Omega a(x)|\nabla u(x)|^p dx < \infty\}, \tag{3.1}$$

which is a Banach space endowed with the norm

$$\|u\|_{W^{1,p}(a,\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall\, u \in W^{1,p}(a, \Omega).$$

Noticing that $C_c^\infty(\Omega) \subset W^{1,p}(a, \Omega)$, the space $W_0^{1,p}(a, \Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(a, \Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(a,\Omega)}$. Hence $W_0^{1,p}(a, \Omega)$ is a separable Banach space. The dual space of $W_0^{1,p}(a, \Omega)$ is denoted $W_0^{1,p}(a, \Omega)^*$.
With the number s in hypothesis (H1) we set

$$p_s = \frac{ps}{s+1}. \tag{3.2}$$

By hypothesis (H1) it holds $s \geq 1/(p-1)$. From (3.2) it follows that $p_s \geq 1$, $p_s < p$ and $p_s/(p - p_s) = s$. Then Hölder's inequality and hypothesis (H1) yield

$$\begin{aligned} \int_\Omega |\nabla u(x)|^{p_s} dx &= \int_\Omega (a(x)^{\frac{p_s}{p}} |\nabla u(x)|^{p_s}) a(x)^{-\frac{p_s}{p}} dx \\ &\leq \|a^{-s}\|^{\frac{1}{s+1}}_{L^1(\Omega)} \|u\|^{p_s}, \quad \forall\, u \in W^{1,p}(a, \Omega). \end{aligned}$$

This implies that $W_0^{1,p}(a, \Omega)$ is continuously embedded into the classical (unweighted) Sobolev space $W_0^{1,p_s}(\Omega)$,

$$W_0^{1,p}(a, \Omega) \hookrightarrow W_0^{1,p_s}(\Omega). \tag{3.3}$$

In view of the Rellich-Kondrachov embedding theorem there is the compact embedding

$$W_0^{1,p_s}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

The preceding assertion is true because with the critical exponent p_s^* (corresponding to p_s), that is,

$$p_s^* := \begin{cases} \frac{Np_s}{N-p_s} & \text{if } N > p_s, \\ \\ +\infty & \text{if } N \le p_s, \end{cases}$$

the assumption $s > N/p$ in (H1) implies $p_s^* > p$. Consequently, due to (3.3), there is the compact embedding

$$W_0^{1,p}(a,\Omega) \hookrightarrow\hookrightarrow L^p(\Omega). \tag{3.4}$$

Thanks to (3.4) we can conclude that

$$\|u\| := \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall\, u \in W_0^{1,p}(a,\Omega), \tag{3.5}$$

is an equivalent norm on $W_0^{1,p}(a,\Omega)$. The norm on $W_0^{1,p}(a,\Omega)$ introduced in (3.5) will be used throughout the rest of the paper.
By assumption (H1) it is known that $a^{-s} \in L^1(\Omega)$ when $s \ge 1/(p-1)$. This gives $a^{-\frac{1}{p-1}} \in L^1(\Omega)$ by noting that

$$\begin{aligned} &\int_\Omega a(x)^{-\frac{1}{p-1}} dx = \int_{\{a(x)<1\}} a(x)^{-\frac{1}{p-1}} dx + \int_{\{a(x)\ge 1\}} a(x)^{-\frac{1}{p-1}} dx \\ &\le \int_\Omega a(x)^{-s} dx + |\Omega| < \infty. \end{aligned}$$

Then [5, Theorem 1.3]) ensures that the space $W_0^{1,p}(a,\Omega)$ is uniformly convex. In particular, $W_0^{1,p}(a,\Omega)$ is a reflexive space.
The (negative) degenerated p-Laplacian with the positive weight $a \in L^1_{\mathrm{loc}}(\Omega)$ is the nonlinear operator $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ given by

$$\langle -\Delta_p^a u, v\rangle := \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \ \forall\, u, v \in W_0^{1,p}(a,\Omega). \tag{3.6}$$

The operator $-\Delta_p^a$ is well defined as seen through Hölder's inequality that

$$\begin{aligned} &\left|\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx\right| \qquad (3.7) \\ &\le \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)|\nabla v(x)|^p dx\right)^{\frac{1}{p}} < \infty \end{aligned}$$

for all $u, v \in W_0^{1,p}(a,\Omega)$. The positive number

$$\lambda_1 := \inf_{u \in W_0^{1,p}(a,\Omega),\ u \ne 0} \frac{\int_\Omega a(x)|\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{3.8}$$

is the first eigenvalue of $-\Delta_p^a$ (refer to [5, Lemma 3.1]). The following proposition addresses essential properties of the operator $-\Delta_p^a$ introduced in (3.6).

Proposition 3.1. *Assume that condition (H1) for the weight $a \in L^1_{\text{loc}}(\Omega)$ positive almost everywhere is satisfied. Then the (negative) degenerated p-Laplacian $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ has the properties:*

(a) *The operator $-\Delta_p^a$ is bounded.*

(b) *The operator $-\Delta_p^a$ is strictly monotone, that is,*

$$\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle > 0 \tag{3.9}$$

for all $u, v \in W_0^{1,p}(a,\Omega)$ with $u \neq v$. Moreover, it holds

$$\langle -\Delta_p^a(u) - (-\Delta_p^a v), u - v\rangle \geq (\|u\| - \|v\|)(\|u\|^{p-1} - \|v\|^{p-1}) \tag{3.10}$$

for all $u, v \in W_0^{1,p}(a,\Omega)$.

(c) *The operator $-\Delta_p^a$ has the S_+ property, that is, any sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and*

$$\limsup_{n\to\infty}\langle -\Delta_p^a u_n, u_n - u\rangle \leq 0 \tag{3.11}$$

fulfills $u_n \to u$ in $W_0^{1,p}(a,\Omega)$.

(d) *The operator $-\Delta_p^a$ is continuous.*

Proof. (a) It turns out from (3.7) that if $\|u\| \leq M$, then

$$\| - \Delta_p^a u\|_{W_0^{1,p}(a,\Omega)^*} \leq M^{p-1},$$

so $-\Delta_p^a$ is a bounded operator.

(b) Let us first prove (3.10). Given $u, v \in W_0^{1,p}(a,\Omega)$, by Hölder's inequality and (3.5) we find that

$$\begin{aligned}
&\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle \\
&= \int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u-v)dx \\
&\geq \int_\Omega a(x)|\nabla u|^p dx + \int_\Omega a(x)|\nabla v|^p dx \\
&- \int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla u|^{\frac{p-1}{p}})(a(x)^{\frac{1}{p}}|\nabla v|)dx - \int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla v|^{\frac{p-1}{p}})(a(x)^{\frac{1}{p}}|\nabla u|)dx \\
&\geq \|u\|^p + \|v\|^p - \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla v|^p dx\right)^{\frac{1}{p}} \\
&- \left(\int_\Omega a(x)|\nabla v|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}} \\
&= \|u\|^p + \|v\|^p - \|u\|^{p-1}\|v\| - \|v\|^{p-1}\|u\| \\
&= (\|u\| - \|v\|)(\|u\|^{p-1} - \|v\|^{p-1}).
\end{aligned}$$

Therefore (3.10) holds true.

From (3.10) we note that $\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle \geq 0$ whenever $u, v \in W_0^{1,p}(a,\Omega)$. Suppose that

$$\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle = 0 \tag{3.12}$$

for some $u, v \in W_0^{1,p}(a,\Omega)$. By (3.10) we have $\|u\| = \|v\|$. Furthermore, (3.10) and (3.12) imply

$$0 = \langle -\Delta_p^a u - (-\Delta_p^a)(\frac{1}{2}(u+v)), \frac{1}{2}(u-v)\rangle$$

$$+\langle (-\Delta_p^a)(\frac{1}{2}(u+v)) - (-\Delta_p^a v), \frac{1}{2}(u-v)\rangle.$$

Again by (3.10), this leads to

$$\|u\| = \|v\| = \|\frac{1}{2}(u+v)\|.$$

The space $W_0^{1,p}(a,\Omega)$ being uniformly convex, it is strictly convex. Consequently, the equality above ensures that $u = v$, thus (3.9) is proven.

(c) Consider a sequence $\{u_n\}$ in $W_0^{1,p}(a,\Omega)$ complying with the conditions required in the statement. From $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and (3.11), we derive

$$\limsup_{n\to\infty} \langle -\Delta_p^a u_n - (-\Delta_p^a u), u_n - u\rangle \leq 0. \tag{3.13}$$

Then (3.9) and (3.13) yield

$$\lim_{n\to\infty} \langle -\Delta_p^a u_n - (-\Delta_p^a u), u_n - u\rangle = 0.$$

Since the right-hand side of inequality (3.10) is nonnegative, we infer from the preceding equality and (3.10) that $\lim_{n\to\infty} \|u_n\| = \|u\|$. We deduce that $u_n \to u$ in $W_0^{1,p}(a,\Omega)$ because the space $W_0^{1,p}(a,\Omega)$ is uniformly convex (see Section 2), thus reaching the desired conclusion.

(d) We now check the continuity of the operator

$$-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*.$$

To this end, let $u_n \to u$ in $W_0^{1,p}(a,\Omega)$. Using the Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\langle -\Delta_p^a u_n - (-\Delta_p^a) u, v\rangle\right| \\
&= \left|\int_\Omega (a(x)^{\frac{p-1}{p}}(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v))(a(x)^{\frac{1}{p}}\nabla v)dx\right| \\
&\leq \left(\int_\Omega a(x)\left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \|v\|
\end{aligned}$$

for all $v \in W_0^{1,p}(a,\Omega)$. This amounts to saying that

$$\begin{aligned}
&\left\|-\Delta_p^a u_n - (-\Delta_p^a u)\right\|_{W_0^{1,p}(a,\Omega)^*} \\
&\leq \left(\int_\Omega a(x)\left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.14}$$

The strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$, in conjunction with the compact embedding (3.4), shows that $u_n \to u$ in $L^p(\Omega)$, so along a relabeled subsequence one has $u_n(x) \to u(x)$ almost everywhere in Ω. In addition, the strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$ provides that $a(x)^{\frac{1}{p}}|\nabla u_n| \to a(x)^{\frac{1}{p}}|\nabla u|$ in $L^p(\Omega)$, which permits to find an $h \in L_+^p(\Omega)$ such that

$$a(x)^{\frac{1}{p}}|\nabla u_n(x)| \leq h(x) \quad \text{for a.e. } x \in \Omega.$$

Through a well-known convexity inequality, this reflects in

$$a(x)\left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} \leq 2^{\frac{1}{p-1}}(h(x)^p + a(x)|\nabla u(x)|^p) =: q(x),$$

with $q \in L^1(\Omega)$. We have checked that we are allowed to apply the Lebesgue's dominated convergence theorem to the integral in (3.14). We infer that $-\Delta_p^a u_n \to -\Delta_p^a u$ in $W_0^{1,p}(a,\Omega)^*$, which completes the proof. $\square$

4. Nemytskii type and multivalued operators associated to problem (1.1)

In order to simplify the notation, we pose $p' := p/(p-1)$. We assume that the nonlinearity $f(x,t,\xi)$ satisfies the growth condition:

(H2) There exist $\sigma \in L^{p'}(\Omega)$ and constants $b_1 \geq 0$ and $b_2 \geq 0$ such that

$$|f(x,t,\xi)| \leq \sigma(x) + b_1|t|^{p-1} + b_2 a(x)^{\frac{1}{p'}}|\xi|^{p-1} \text{ for a.e. } x \in \Omega,\ \forall\, t \in \mathbb{R},\ \forall\, \xi \in \mathbb{R}^{\mathrm{N}}.$$

Consider the weighted space

$$L^p(a,\Omega,\mathbb{R}^N) := \{w : \Omega \to \mathbb{R}^N \text{ measurable} : \int_\Omega a(x)|w(x)|^p dx < \infty\}, \tag{4.1}$$

which is a Banach space endowed with the norm

$$\|w\|_{L^p(a,\Omega,\mathbb{R}^N)} := \left(\int_\Omega a(x)|w(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall\, w \in L^p(a,\Omega,\mathbb{R}^N).$$

The multiplication operator $M_a : L^p(a,\Omega,\mathbb{R}^N) \to L^p(\Omega,\mathbb{R}^N)$ defined by

$$M_a(w) := a^{\frac{1}{p}} w, \quad \forall\, w \in L^p(a,\Omega,\mathbb{R}^N), \tag{4.2}$$

is an isometry, i.e.,

$$\|w\|_{L^p(a,\Omega,\mathbb{R}^N)} = \|M_a(w)\|_{L^p(\Omega,\mathbb{R}^N)}, \quad \forall\, w \in L^p(a,\Omega,\mathbb{R}^N).$$

Lemma 4.1. *Assume that conditions (H1) and (H2) are satisfied. Then the Nemytskii type operator* $N_f : L^p(\Omega) \times L^p(a,\Omega,\mathbb{R}^N) \to L^{p'}(\Omega)$ *given by*

$$N_f(u,w) := f(\cdot,u,w), \quad \forall\, (u,w) \in L^p(\Omega) \times L^p(a,\Omega,\mathbb{R}^N), \tag{4.3}$$

is well defined, bounded and continuous.

Proof. Given $(u,w) \in L^p(\Omega) \times L^p(a,\Omega,\mathbb{R}^N)$, we have from (4.3), hypothesis (H2) and a well-known convexity inequality that

$$\int_\Omega |N_f(u,w)|^{p'} dx \le \int_\Omega \left(\sigma(x) + b_1|u|^{p-1} + b_2 a(x)^{\frac{1}{p'}}|w|^{p-1}\right)^{p'} dx$$

$$\le C\left(\|\sigma\|^{p'}_{L^{p'}(\Omega)} + \|u\|^p_{L^{p'}(\Omega)} + \|w\|^p_{L^p(a,\Omega,\mathbb{R}^N)}\right),$$

with a constant $C > 0$. It follows that the map N_f is well defined and bounded. For proving the continuity of the mapping N_f, we introduce the Carathéodory function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$F(x,t,\xi) := f(x,t,a(x)^{-\frac{1}{p}}\xi), \quad \forall\ (x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

Based on hypothesis (H2), we obtain the estimate

$$|F(x,t,\xi)| = |f(x,t,a(x)^{-\frac{1}{p}}\xi)|$$

$$\le \sigma(x) + b_1|t|^{p-1} + b_2 a(x)^{\frac{1}{p'}}(a(x)^{-\frac{1}{p}}|\xi|)^{p-1}$$

$$= \sigma(x) + b_1|t|^{p-1} + b_2|\xi|^{p-1} \text{ for a.e. } x \in \Omega,\ \forall\ t \in \mathbb{R},\ \forall\ \xi \in \mathbb{R}^{\mathrm{N}}.$$

This estimate guarantees that Krasnoselkii's theorem can be applied to F ensuring that the Nemytskii operator $N_F : L^p(\Omega) \times L^p(\Omega,\mathbb{R}^N) \to L^{p'}(\Omega)$ given by

$$N_F(u,w) := F(\cdot,u,w), \quad \forall\ (u,w) \in L^p(\Omega) \times L^p(\Omega,\mathbb{R}^N),$$

is continuous. From (4.2) and (4.3) we note

$$N_F(u,M_a(w)) = N_f(u,w), \quad \forall\ (u,w) \in L^p(\Omega) \times L^p(\Omega,\mathbb{R}^N).$$

Hence N_f is a composition of continuous mappings, whence its continuity. □

Proposition 4.2. *Assume that conditions (H1) and (H2) are satisfied. Then the Nemytskii type operator* $\mathcal{N}_f : W_0^{1,p}(a,\Omega) \to L^{p'}(\Omega)$ *given by*

$$\mathcal{N}_f(u) := N_f(u,\nabla u), \quad \forall\ u \in W_0^{1,p}(a,\Omega), \tag{4.4}$$

is well defined, bounded and continuous.

Proof. If $u \in W_0^{1,p}(a,\Omega)$, by embedding (3.4) we have that $u \in L^p(\Omega)$ and by (3.1) and (4.1) that $\nabla u \in L^p(a,\Omega,\mathbb{R}^N)$. Therefore the definition of $\mathcal{N}_f(u)$ in (4.4) makes sense. The boundedness and continuity of the mapping $u \in W_0^{1,p}(a,\Omega) \mapsto (u,\nabla u) \in L^p(\Omega) \times L^p(a,\Omega,\mathbb{R}^N)$ follow directly from (3.4) and

$$\|u\| = \|\nabla u\|_{L^p(a,\Omega,\mathbb{R}^N)}, \quad \forall\ u \in W_0^{1,p}(a,\Omega)$$

(refer to (3.5)). Taking into account (4.4) and Lemma 4.1, the desired conclusion is achieved. □

Next we focus on the multivalued term in problem (1.1). To this end we formulate the assumption:

(H3) The function $g:\mathbb{R}\to\mathbb{R}$ satisfies $g\in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and there exists a constant $c>0$ such that
$$\max\{|\underline{g}(t)|,|\overline{g}(t)\})|\le c(1+|t|^{p-1})\ \text{ for a.e. } t\in\mathbb{R},$$
with $\underline{g}$ and $\overline{g}$ in (1.2) and (1.3), respectively. If g is continuous, the above condition reduces to
$$|g(t)|\le c(1+|t|^{p-1})\ \text{ for all } t\in\mathbb{R}.$$

The function $G:\mathbb{R}\to\mathbb{R}$ in (2.1) corresponding to $g\in L^\infty_{\mathrm{loc}}(\mathbb{R})$ is locally Lipschitz. Then, by Lebourg's mean value theorem (see [3, Theorem 2.3.7]) and hypothesis (H3), the functional $\Phi:L^p(\Omega)\to\mathbb{R}$ given by
$$\Phi(v)=\int_\Omega G(v(x))dx\ \text{ for all } v\in L^p(\Omega)\tag{4.5}$$
is Lipschitz continuous on the bounded subsets of $L^p(\Omega)$, thus locally Lipschitz. The generalized gradient $\partial\Phi(u)$ is a nonempty, closed and convex subset of $L^{p'}(\Omega)$ for every $u\in L^p(\Omega)$. Therefore the multivalued mapping $\partial\Phi:L^p(\Omega)\to 2^{L^{p'}(\Omega)}$ is well defined. Since $W_0^{1,p}(a,\Omega)$ is continuously and densely embedded in $L^p(\Omega)$, it can be regarded as a multivalued mapping $\partial\Phi:W_0^{1,p}(a,\Omega)\to 2^{L^{p'}(\Omega)}$ (see [3, p. 47]).

Proposition 4.3. *Assume that conditions (H1) and (H3) are satisfied. Then the multivalued mapping $\partial\Phi:W_0^{1,p}(a,\Omega)\to 2^{L^{p'}(\Omega)}$ is bounded. Moreover, it is sequentially weakly upper semicontinuous in the following sense: if the sequences $\{u_n\}\subset W_0^{1,p}(a,\Omega)$ and $\{\zeta_n\}\subset L^{p'}(\Omega)$ satisfy $u_n\rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ for some $u\in W_0^{1,p}(a,\Omega)$ and $\zeta_n\in\partial\Phi(u_n)$ for all n, then along a relabeled subsequence one has $\zeta_n\rightharpoonup\zeta$ in $L^{p'}(\Omega)$ with some $\zeta\in\partial\Phi(u)$.*

Proof. Let $u\in W_0^{1,p}(a,\Omega)$ and $w\in\partial\Phi(u)$. By applying the Aubin-Clarke theorem (see [3, Theorem 2.7.5]), we derive from (4.5) and (2.2) that
$$w(x)\in\partial G(u(x))=[\underline{g}(u(x)),\overline{g}(u(x))]\quad\text{for a.e. } x\in\Omega.\tag{4.6}$$
Then (4.6), (3.8), and hypothesis (H3) yield
$$\begin{aligned}\|w\|^{p'}_{L^{p'}(\Omega)}&\le\int_\Omega(\max\{|\underline{g}(u(x))|,|\overline{g}(u(x)|\})^{p'}dx\\&\le c^{p'}\int_\Omega(1+|u(x)|^{p-1})^{p'}dx\\&\le 2^{\frac{1}{p-1}}c^{p'}(|\Omega|+\|u\|^p_{L^p(\Omega)})\le 2^{\frac{1}{p-1}}c^{p'}(|\Omega|+\lambda_1^{-1}\|u\|^p).\end{aligned}$$
Hence the multivalued mapping $\partial\Phi$ is bounded.
For the second part of the statement, let $\{u_n\}\subset W_0^{1,p}(a,\Omega)$ and $\{\zeta_n\}\subset L^{p'}(\Omega)$ be sequences satisfying $u_n\rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ with a $u\in W_0^{1,p}(a,\Omega)$ and $\zeta_n\in\partial\Phi(u_n)$ for all n. The compact embedding (3.4) renders $u_n\to u$ in $L^p(\Omega)$. As known from the first part, the sequence $\{\zeta_n\}$ is bounded in $L^{p'}(\Omega)$, whence due to the reflexivity we have along a relabeled subsequence $\zeta_n\rightharpoonup\zeta$ in $L^{p'}(\Omega)$ with some $\zeta\in L^{p'}(\Omega)$. The fact that the multifunction $\partial\Phi$ is weak*-closed (see [3, Proposition 2.1.5]) implies that $\zeta\in\partial\Phi(u)$, which completes the proof. $\square$

5. Existence of solutions to problem (1.1)

In order to prove the solvability of problem (1.1), a new hypothesis linking (H2) and (H3) is needed:

(H4) There holds

$$b_1\lambda_1^{-1} + (b_2 + c)\lambda_1^{-\frac{1}{p}} < 1,$$

where the constants b_1 and b_2 enter (H2), and c is the constant in (H3).

Our existence result on problems (1.1) and (1.4) is as follows.

Theorem 5.1. *Assume that conditions (H1)-(H4) hold. Then problem* (1.1) *admits at least one solution. In particular, if the function g is continuous, then a solution to problem* (1.4) *exists.*

Proof. The proof is conducted by applying Theorem 2.1. Towards this, we introduce the multivalued operator $A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$ by

$$Au := -\Delta_p^a u - \mathcal{N}_f(u) - \partial\Phi(u) \quad \text{ for all } u \in W_0^{1,p}(a,\Omega). \tag{5.1}$$

Since one has $L^{p'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$, the multifunction A in (5.1) is well defined. We verify that all the hypotheses of Theorem 2.1 are fulfilled.

Proposition 3.1 (a) ensures that $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is a bounded operator. By Proposition 4.2 we get that $\mathcal{N}_f : W_0^{1,p}(a,\Omega) \to L^{p'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$ is bounded, while by virtue of Proposition 4.3 we know that the multivalued mapping $\partial\Phi : W_0^{1,p}(a,\Omega) \to 2^{L^{p'}(\Omega)}$ is bounded. In view of (5.1), we infer that the multivalued operator $A : W_0^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ is bounded.

The next step in the proof is to show that the multivalued operator

$$A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$$

is pseudomonotone. In line with this, let sequences

$$\{u_n\} \subset W_0^{1,p}(a,\Omega) \text{ and } \{u_n^*\} \subset W_0^{1,p}(a,\Omega)^*$$

satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$, $u_n^* \in Au_n$ for all n, and

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leq 0. \tag{5.2}$$

Take an arbitrary subsequence of $\{u_n\}$ still denoted $\{u_n\}$ and the corresponding subsequence of $\{\zeta_n\}$. According to (5.1) it holds

$$\zeta_n \in \partial\Phi(u_n), \quad \forall\, n, \tag{5.3}$$

with

$$u_n^* = -\Delta_p^a u_n - \mathcal{N}_f(u_n) - \zeta_n. \tag{5.4}$$

Exploiting the fact that the values of $\mathcal{N}_f$ belong to $L^{p'}(\Omega)$, we have

$$|\langle \mathcal{N}_f(u_n), u_n - u\rangle| \leq \|\mathcal{N}_f(u_n)\|_{L^{p'}(\Omega)}\|u_n - u\|_{L^p(\Omega)}.$$

Due to the compact embeddings of $W_0^{1,p}(a,\Omega)$ into $L^p(\Omega)$ and the boundedness of $\{\mathcal{N}_f(u_n)\}$ in $L^{p'}(\Omega)$, the above estimate entails

$$\lim_{n\to\infty} \langle \mathcal{N}_f(u_n), u_n - u\rangle = 0. \tag{5.5}$$

Then it stems from (5.2), (5.4) and (5.5) that

$$\limsup_{n\to\infty} \langle -\Delta_p^a u_n - \zeta_n, u_n - u\rangle \le 0. \tag{5.6}$$

Based on hypothesis (H3) we can invoke Proposition 4.3 that provides a subsequence of $\{\zeta_n\}$ (so, a fortiori, a subsequence of $\{u_n\}$) along which $\zeta_n \rightharpoonup \zeta$ in $L^{p'}(\Omega)$, with some $\zeta \in \partial\Phi(u)$, whereas $u_n \to u$ in $L^p(\Omega)$. Using that the values of the multifunction $\partial\Phi$ are in $L^{p'}(\Omega)$, along the relabeled subsequence we obtain

$$\lim_{n\to\infty} \langle \zeta_n, u_n - u\rangle = 0. \tag{5.7}$$

Combining (5.6) and (5.7) results in (3.11). This enables us to apply Proposition 3.1 (c). Hence, up to a subsequence, $u_n \to u$ in $W_0^{1,p}(\Omega)$. Actually, the preceding reasoning shows that every subsequence of $\{u_n\}$ contains a subsequence strongly converging to u in $W_0^{1,p}(\Omega)$, which ensures for the entire sequence that $u_n \to u$ in $W_0^{1,p}(\Omega)$. By the continuity of the mappings

$$-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^* \text{ and } \mathcal{N}_f : W_0^{1,p}(a,\Omega) \to L^{p'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$$

(see Proposition 3.1 (d) and Proposition 4.2) we have $-\Delta_p^a u_n \to -\Delta_p^a u$ in $W_0^{1,p}(a,\Omega)^*$ and $\mathcal{N}_f(u_n) \to \mathcal{N}_f(u)$ in $W_0^{1,p}(a,\Omega)^*$.
Let $v \in W_0^{1,p}(a,\Omega)$. From (5.3) and (5.4), in conjunction with the preceding comments, we note

$$\begin{aligned}
&\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle \qquad (5.8)\\
&= \liminf_{n\to\infty} \langle -\Delta_p^a u_n - \mathcal{N}_f(u_n) - \zeta_n, u_n - v\rangle\\
&= \langle -\Delta_p^a u - \mathcal{N}_f(u), u - v\rangle + \liminf_{n\to\infty} \langle -\zeta_n, u_n - v\rangle\\
&= \langle -\Delta_p^a u - \mathcal{N}_f(u), u - v\rangle - \limsup_{n\to\infty} \langle \zeta_n, u_n - v\rangle.\\
&\ge \langle -\Delta_p^a u - \mathcal{N}_f(u), u - v\rangle - \max_{\zeta\in\partial\Phi(u)} \langle \zeta, u - v\rangle.
\end{aligned}$$

Recall that the set $\partial\Phi(u)$ in weak*-compact in $W_0^{1,p}(a,\Omega)^*$ (refer to [3, Proposition 2.1.2]), so there exists $\zeta(v) \in \partial\Phi(u)$ for which it holds

$$\max_{\zeta\in\partial\Phi(u)} \langle \zeta, u - v\rangle = \langle \zeta(v), u - v\rangle.$$

On the basis of (5.8), this confirms that the multivalued operator

$$A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$$

is pseudomonotone.

We now turn to show that $A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$ defined in (5.1) is coercive. From (4.6) and by applying the Aubin-Clarke theorem (see [3, Theorem 2.7.5]) to (4.5), which is possible thanks to hypothesis (H3), we find that

$$\begin{aligned}\langle \zeta, v\rangle &= \int_\Omega \zeta(x)v(x)\,dx \le \int_\Omega |\zeta(x)||v(x)|\,dx \\ &\le \int_\Omega c(1+|v(x)|^{p-1})|v(x)|\,dx \\ &\le c\lambda_1^{-\frac{1}{p}}\|v\|^p + c_0\|v\|\end{aligned} \tag{5.9}$$

whenever $v \in W_0^{1,p}(a,\Omega)$ and $\zeta \in \partial\Phi(v)$, with a positive constant c_0.
Let $v \in W_0^{1,p}(a,\Omega)$ and $v^* \in Av$. Due to (5.1), we can write

$$v^* = -\Delta_p^a v - \mathcal{N}_f(v) - \zeta,$$

with $\zeta \in \partial\Phi(v)$. Then, by (3.4), hypothesis (H2), Hölder's inequality, (5.9), and (3.8), it turns out

$$\begin{aligned}\langle v^*, v\rangle &= \langle -\Delta_p^a v - \mathcal{N}_f(v) - \zeta, v\rangle \\ &\ge \|v\|^p - \|\sigma\|_{L^{p'}(\Omega)}\|v\|_{L^p(\Omega)} - b_1\|v\|^p_{L^p(\Omega)} - b_2\|v\|^{p-1}\|v\|_{L^p(\Omega)} \\ &\quad - c\lambda_1^{-\frac{1}{p}}\|v\|^p - c_0\|v\| \\ &\ge (1 - b_1\lambda_1^{-1} - (b_2+c)\lambda_1^{-\frac{1}{p}})\|v\|^p - (\|\sigma\|_{L^{p'}(\Omega)}\lambda_1^{-1} + c_0)\|v\|.\end{aligned}$$

Hypothesis (H4) postulates that $1 - b_1\lambda_1^{-1} - (b_2+c)\lambda_1^{-\frac{1}{p}} > 0$. Therefore, owing to $p > 1$, the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\psi(t) = (1 - b_1\lambda_1^{-1} - (b_2+c)\lambda_1^{-\frac{1}{p}})t^{p-1} - \|\sigma\|_{L^{p'}(\Omega)}\lambda_1^{-1} - c_0, \quad \forall\, t \in \mathbb{R}_+,$$

satisfies $\psi(t) \to +\infty$ as $t \to +\infty$. Furthermore, it holds $\langle v^*, v\rangle \ge \psi(\|v\|)\|v\|$ for all $v \in W_0^{1,p}(a,\Omega)$ and $v^* \in Av$. This means that the multivalued operator

$$A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$$

is coercive (with $u_0 = 0$ in the definition of coerciveness in Section 2).
Since the multivalued operator $A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$ defined in (5.1) is pseudomonotone, bounded and coercive, Theorem 2.1 is applicable, which provides (choosing $\eta = 0$ in the statement of Theorem 2.1) the existence of a $u \in W_0^{1,p}(a,\Omega)$ solving the equation $Au = 0$, or equivalently

$$\langle -\Delta_p^a u - \mathcal{N}_f(u) - \zeta, v\rangle = 0, \quad \forall\, v \in W_0^{1,p}(a,\Omega),$$

with some $\zeta \in \partial\Phi(u)$. Inserting the expressions of the operators $-\Delta_p^a$ and $\mathcal{N}_f$, and for $\partial\Phi$ refering to (4.6) with $w = \zeta$, we get (1.5) (equivalently, (1.6)). The fact that $\underline{g}(u),\ \overline{g}(u) \in L^{p'}(\Omega)$ follows from hypothesis (H3) and $u \in L^p(\Omega)$. We conclude that $u \in W_0^{1,p}(a,\Omega)$ is a solution of problem (1.1). If the function $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ is continuous, we have that $\underline{g}(u(x)) = \overline{g}(u(x))$ almost everywhere in Ω, so (1.5) (equivalently, (1.6)) becomes (1.7). The proof is thus complete. $\square$

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