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Runge pairs of Φ-like domains

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Abstract. We prove that if $E \subseteq \mathbb{C}^n$ is a Φ-like domain and $D \subseteq E$ is a $\Phi\big|_D$-like domain, then (D, E) is a Runge pair. Certain applications, examples and questions are also provided.

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1. Introduction

Every starlike domain in $\mathbb{C}^n$ is a Runge domain. According to [27, p. 410], this observation goes back at least to Almer [3] and it has been rediscovered several times (cf. [7], [9]). Some of the proofs use the envelopes of holomorphy and/or a result due to Docquier and Grauert [10] (see e.g. [7], [29]). A simple proof has been given by El Kasimi [11]. Hamada [17] has adapted this proof to prove that every spirallike domain in $\mathbb{C}^n$ is a Runge domain.

In this paper, we want to exploit further El Kasimi's ideas [11], in order to develop a criterion for two domains in $\mathbb{C}^n$ to form a Runge pair, in terms of Φ-likeness, a notion introduced by Brickman [8], for one complex variable, and later extended by Gurganus [16], for several complex variables. Certain general results that give sufficient conditions for two pseudoconvex domains in $\mathbb{C}^n$ to form a Runge pair are given in e.g. [10], [23, Theorem 4.25] and [29] (cf. [19, Proposition 3.1.22]). However, in our case, the domains are not necessarily pseudoconvex. The following is our main result.

Theorem 1.1. *Let $E \subseteq \mathbb{C}^n$ be a Φ-like domain. If $D \subseteq E$ is a $\Phi\big|_D$-like domain, then (D, E) is a Runge pair.*

For the proof, we combine the ideas from the proof of [11, Proposition 1] with some results from the theory of semigroups of holomorphic self-mappings, extended by Abate [2] for domains in $\mathbb{C}^n$.

We shall provide some examples that point out various aspects of our main result. For example, the domains in Theorem 1.1 are not necessarily Runge domains,

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even though they form a Runge pair. As an application to Theorem 1.1, we prove that every Φ-like domain admits a Runge exhaustion - see [12]. Also, we obtain a result related to [4, Proposition 5.1]. Further, we shall take a look at the invariant domains of a semigroup of holomorphic mappings on a Φ-like domain. Furthermore, we deduce the Runge property of spirallike domains after generalizing [17, Theorem 3.1]. Finally, we point out a version of our main result for taut domains.

2. Preliminaries

For every open sets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ we denote by $\mathcal{H}(D, E)$ the space of holomorphic mappings from D into E. For every open set $D \subseteq \mathbb{C}^n$, we denote by $\mathcal{O}(D)$ the space of holomorphic functions from D into $\mathbb{C}$. We consider the topology of locally uniform convergence on these spaces. Also, we denote by $L(\mathbb{C}^n, \mathbb{C}^m)$ the space of complex linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$.

We present the definition of Runge pairs (see e.g. [24]).

Definition 2.1. *Let $D \subseteq E \subseteq \mathbb{C}^n$ be open sets. We say that (D, E) is a Runge pair, if, for every $f \in \mathcal{O}(D)$ and every compact set $K \subset D$, there exists a sequence in $\mathcal{O}(E)$ that converges uniformly on K to f. We say that an open set $D \subseteq \mathbb{C}^n$ is Runge, if $(D, \mathbb{C}^n)$ is a Runge pair.*

Remark 2.2. Let $D \subseteq E \subseteq \mathbb{C}^n$ be open sets. We note that (D, E) is a Runge pair if and only if the family of functions in $\mathcal{O}(E)$ restricted to D is dense in $\mathcal{O}(D)$. In particular, D is Runge if and only if the family of complex polynomial functions on $\mathbb{C}^n$ is dense in $\mathcal{O}(D)$.

In the case $n = 1$, (D, E) is a Runge pair if and only if $E \setminus D$ has no nonempty relatively open, compact subsets (see e.g. [21, Theorem 4.9]), i.e., each connected component of $E \setminus D$ is not compact. In particular, we have the well known Runge theorem: $D \subseteq \mathbb{C}$ is a Runge domain if and only if D is simply connected.

Next, we consider the definition of a Φ-like domain. It was introduced by Brickman [8], in dimension one, as a generalization of starlike and spirallike domains in $\mathbb{C}$. Later, Gurganus [16] extended the definition to higher dimensions.

In the following, we use the notation $m(A) = \min\{\Re\langle A(z), z\rangle : \|z\| = 1\}$, for $A \in L(\mathbb{C}^n, \mathbb{C}^n)$, where $\|\cdot\|$ is the Euclidean norm.

Definition 2.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. If $0 \in \Omega$ and there exists $\Phi \in \mathcal{H}(\Omega, \mathbb{C}^n)$ such that $\Phi(0) = 0$, $m(D\Phi(0)) > 0$ and, for every $z \in \Omega$, the initial value problem*

$$\frac{\partial w}{\partial t}(z, t) = -\Phi(w(z, t)), \quad t \geq 0, \qquad w(z, 0) = z, \tag{2.1}$$

has a solution $w(z, \cdot)$ on $[0, \infty)$ such that $w(z, t) \in \Omega$, $t \geq 0$, and $w(z, t) \to 0$, as $t \to \infty$, then we say that Ω is a Φ-like domain.

The initial value problem (2.1) is related to the study of one-parameter semigroups of holomorphic self-mappings (see e.g. [1], [25]). We consider below the definition of a one-parameter semigroup on a domain in $\mathbb{C}^n$ (see e.g. [2]).

Definition 2.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We say that $w : \Omega \times [0, \infty) \to \Omega$ is a one-parameter semigroup (or, simply, a semigroup) on Ω if $t \mapsto w_t = w(\cdot, t)$ is a* *continuous map from* $[0,\infty)$ *into* $\mathcal{H}(\Omega,\Omega)$, $w_0 = \mathrm{id}_\Omega$ *and* $w_{s+t} = w_s \circ w_t$, *for all* $s,t \geq 0$.

Remark 2.5. For every one-parameter semigroup w on a domain Ω, w_t is univalent on Ω, for all $t \geq 0$ (see [2, Proposition 1]).

It is well known that there is a one-to-one correspondence between one-parameter semigroups and infinitesimal generators. To be more precise, we consider first the following definition (see e.g. [1], [25]).

Definition 2.6. *Let* $\Omega \subseteq \mathbb{C}^n$ *be a domain. We say that* $\Phi \in \mathcal{H}(\Omega, \mathbb{C}^n)$ *is an infinitesimal generator on* Ω *if, for every* $z \in \Omega$, *the initial value problem* (2.1) *associated to* Φ *has a solution* $w(z,\cdot)$ *on* $[0,\infty)$. *We mention that, if these solutions exist, then they are necessarily unique* (*see e.g.* [2]).

Remark 2.7. For every semigroup w on a domain $\Omega \subseteq \mathbb{C}^n$, there is a unique infinitesimal generator Φ, which is given by $\Phi(z) = \lim\limits_{t\searrow 0} \frac{1}{t}(z - w(z,t))$, locally uniformly with respect to $z \in \Omega$, such that $w(z,\cdot)$ is the solution on $[0,\infty)$ of the initial value problem (2.1) associated to Φ, for every $z \in \Omega$ (see [2, Theorem 5]). Conversely, let Φ be an infinitesimal generator on a domain $\Omega \subseteq \mathbb{C}^n$ and let $w : \Omega \times [0,\infty) \to \Omega$ be such that, for every $z \in \Omega$, $w(z,\cdot)$ is the solution of the initial value problem (2.1) associated to Φ. Then w is a one-parameter semigroup on Ω (see [2, p. 169]). In particular, $w_t = w(\cdot,t) \in \mathcal{H}(\Omega,\Omega)$, for all $t \geq 0$.

Remark 2.8. Taking into account Definitions 2.3 and 2.4, it is clear that if $\Omega \subseteq \mathbb{C}^n$ is a Φ-like domain, then Φ is an infinitesimal generator on Ω.

The following family of infinitesimal generators on the Euclidean unit ball $\mathbb{B}^n$ plays an important role in the geometric function theory in several complex variables (see [15], [25]).

Definition 2.9. *Let*

$$\mathcal{N} = \{h \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n) : h(0) = 0, \Re\langle h(z), z\rangle > 0, z \in \mathbb{B}^n \setminus \{0\}\}.$$

By [16, Lemma 2], $\mathbb{B}^n$ is a Φ-like domain with respect to every $\Phi \in \mathcal{N}$. In particular, every mapping in $\mathcal{N}$ is an infinitesimal generator on $\mathbb{B}^n$.

3. Main result

The following lemmas are useful in our proof of the main result.

The first lemma points out that in every Φ-like domain there is a sufficiently small ball which is invariant with respect to the semigroup generated by Φ (see the discussion in [16, p. 393]).

Lemma 3.1. *Let* $\Omega \subseteq \mathbb{C}^n$ *be a domain with* $0 \in \Omega$ *and let* $\Phi \in \mathcal{H}(\Omega, \mathbb{C}^n)$ *be an infinitesimal generator on* Ω *with* $\Phi(0) = 0$ *and* $m(D\Phi(0)) > 0$. *Also, let* w *be the semigroup on* Ω *generated by* Φ. *Then there exists* $\delta > 0$ *such that* $\delta\mathbb{B}^n \subset \Omega$ *and* $w_t(\delta\mathbb{B}^n) \subseteq \delta\mathbb{B}^n$, *for all* $t \geq 0$.

Proof. Let $A = D\Phi(0)$ and let $\omega(z) = \Phi(z) - Az$, for $z \in \Omega$. In view of the definition of the Fréchet differential of Φ at 0, we have $\lim_{z\to 0} \frac{\|\omega(z)\|}{\|z\|} = 0$. Let $\delta > 0$ be such that $\delta\mathbb{B}^n \subset \Omega$ and $\|\omega(z)\| \le \frac{m(A)}{2}\|z\|$, for all $z \in \delta\mathbb{B}^n$. Then

$$\Re\langle \Phi(z), z\rangle = \Re\langle Az, z\rangle + \Re\langle \omega(z), z\rangle \ge m(A)\|z\|^2 - \|\omega(z)\|\|z\| \ge \frac{m(A)}{2}\|z\|^2 > 0,$$

for all $z \in \delta\mathbb{B}^n \setminus \{0\}$.

Let $h(z) = \frac{1}{\delta}\Phi(\delta z)$, for $z \in \mathbb{B}^n$. In view of the above inequalities, $h \in \mathcal{N}$. For every $z \in \mathbb{B}^n$, let $v(z, \cdot)$ be the solution on $[0, \infty)$ of the initial value problem

$$\frac{\partial v}{\partial t}(z, t) = -h(v(z, t)), \quad t \ge 0, \qquad v(z, 0) = z,$$

which is given by [16, Lemma 2]. We easily deduce that

$$\frac{\partial}{\partial t}\big(\delta v(\frac{1}{\delta}\zeta, t)\big) = -\Phi(\delta v(\frac{1}{\delta}\zeta, t)), \quad t \ge 0,\ \zeta \in \delta\mathbb{B}^n.$$

For every $\zeta \in \delta\mathbb{B}^n$, $w(\zeta, \cdot)$ is the unique solution on $[0, \infty)$ of the initial value problem (2.1) associated to Φ, and thus $w(\zeta, t) = \delta v(\frac{1}{\delta}\zeta, t) \in \delta\mathbb{B}^n$, for all $t \ge 0$. □

In Definition 2.3, we have that every trajectory in a Φ-like domain, given by the initial value problem (2.1), converges to 0, as time goes to infinity. In the following we prove that the convergence is actually uniform on compact subsets of Ω.

Lemma 3.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a Φ-like domain. Also, let w be the semigroup on Ω generated by Φ. Then $w_t \to 0$, as $t \to \infty$, locally uniformly on Ω.*

Proof. In view of Vitali's theorem, it suffices to show that $\{w_t\}_{t\ge 0}$ is a normal family. Let $K \subset \Omega$ be a compact set. Also, let $\delta > 0$ be such that $w_t(\delta\mathbb{B}^n) \subseteq \delta\mathbb{B}^n$, for all $t \ge 0$, whose existence is ensured by Lemma 3.1. By Definition 2.3, for every $z \in \Omega$, $w_t(z) \to 0$, as $t \to \infty$. For every $z \in \Omega$, let $t_z \ge 0$ be such that $w_{t_z}(z) \in \delta\mathbb{B}^n$, and then let $V_z \subset \Omega$ be an open set with $z \in V_z$ such that $w_{t_z}(V_z) \subseteq \delta\mathbb{B}^n$. Since K is compact, there exist $z_1, \dots, z_m \in K$ such that $K \subset V_{z_1} \cup \ldots \cup V_{z_m}$. Let $T = \max\{t_{z_1}, \dots, t_{z_m}\}$. Then, for every $j \in \{1, \dots, m\}$ and $t \ge T$, we have

$$w_t(V_{z_j}) = w_{t-t_{z_j}}(w_{t_{z_j}}(V_{z_j})) \subseteq w_{t-t_{z_j}}(\delta\mathbb{B}^n) \subseteq \delta\mathbb{B}^n.$$

Hence $w_t(K) \subseteq \delta\mathbb{B}^n$, for all $t \ge T$. Since $t \mapsto w(\cdot, t)$ is a continuous map from $[0, \infty)$ into $\mathcal{H}(\Omega, \Omega)$, we deduce that w is continuous on $\Omega \times [0, \infty)$, and thus $w(K \times [0, T])$ is compact. Therefore we conclude that the family $\{w_t\}_{t\ge 0}$ is bounded on K. □

The next lemma is a consequence of [18, Theorem 1.1], which tells us that every semigroup extends holomorphically in a neighborhood of any nonnegative time.

Lemma 3.3. *Let w be a semigroup on a domain $\Omega \subseteq \mathbb{C}^n$. Then, for every compact $K \subset \Omega$, there exist an open set $V \subseteq \Omega$ that contains K and a domain $U \subseteq \mathbb{C}$ that contains the interval $[0, \infty)$ such that $w\big|_{V\times[0,\infty)}$ has a holomorphic extension to $V \times U$ which takes values in Ω.*

Proof. *Step* 1. Let Φ be the infinitesimal generator of w and let $z_0 \in \Omega$. By [18, Theorem 1.1] (cf. [22, Theorem 1.8.10]), there exists $\varepsilon > 0$ such that

$$V_{z_0} = \left\{z \in \mathbb{C}^n : |z_j - z_{0,j}| < \varepsilon,\ j = \overline{1,n}\right\} \subset \Omega$$

and, for every $z \in V_{z_0}$, the initial value problem

$$\frac{\partial v}{\partial \zeta}(z,\zeta) = -\Phi(v(z,\zeta)), \quad |\zeta| < \varepsilon, \qquad v(z,0) = z, \tag{3.1}$$

has a unique holomorphic solution $v(z,\cdot)$ on $U_{z_0,0} = \{\zeta \in \mathbb{C} : |\zeta| < \varepsilon\}$ which takes values in Ω and depends holomorphically on $z \in V_{z_0}$. Let $z \in V_{z_0}$. In view of (3.1), we have that $v(z,\cdot)$ is a solution on $[0,\varepsilon)$ of the initial value problem

$$\frac{\partial v}{\partial t}(z,t) = -\Phi(v(z,t)), \quad t \in [0,\varepsilon), \qquad v(z,0) = z.$$

Taking into account the uniqueness of this solution, we deduce that $w(z,t) = v(z,t)$, for all $t \in [0,\varepsilon)$. Hence w has a holomorphic extension to $V_{z_0} \times U_{z_0,0}$ which takes values in Ω. Since w_{t_0} is holomorphic on Ω and $w_t = w_{t_0} \circ w_{t-t_0}$, for all $t \geq t_0 \geq 0$, we deduce that w has a holomorphic extension to $V_{z_0} \times U_{z_0,t_0}$, where $U_{z_0,t_0} = \left\{\zeta : \zeta - t_0 \in U_{z_0,0}\right\}$, for all $t_0 \geq 0$, which takes values in Ω.
Step 2. Using the notations from the previous step, let $z_1, z_2 \in \Omega$ and $t_1, t_2 \in [0,\infty)$ be such that $V_{z_1} \cap V_{z_2} \neq \emptyset$ and $U_{z_1,t_1} \cap U_{z_2,t_2} \neq \emptyset$ and let v_j be the holomorphic extension of w to $V_{z_j} \times U_{z_j,t_j}$, $j \in \{1,2\}$. If $(z,\zeta) \in (V_{z_1} \times U_{z_1,t_1}) \cap (V_{z_2} \times U_{z_2,t_2})$, then $v_1(z,\zeta) = v_2(z,\zeta)$. Indeed, since $v_1(z,t) = v_2(z,t) = w(z,t)$, for all $z \in V_{z_1} \cap V_{z_2}$ and $t \in U_{z_1,t_1} \cap U_{z_2,t_2} \cap [0,\infty) \neq \emptyset$, we deduce, by the Identity Principle, that $v_1(z,\zeta) = v_2(z,\zeta)$, for all $z \in V_{z_1} \cap V_{z_2}$ and $\zeta \in U_{z_1,t_1} \cap U_{z_2,t_2}$.
Step 3. For every $z_0 \in \Omega$, let $U_{z_0} = \bigcup_{t_0 \geq 0} U_{z_0,t_0}$ and note that U_{z_0} is a domain in $\mathbb{C}$ that contains $[0,\infty)$ and w has a well defined holomorphic extension to $V_{z_0} \times U_{z_0}$, in view of the previous step, which takes values in Ω.
Step 4. Since K is compact, there exist $z_1, \ldots, z_m \in K$ such that K is a subset of the open set $V = V_{z_1} \cup \ldots \cup V_{z_m}$. $U = U_{z_1} \cap \ldots \cap U_{z_m}$ is a domain that contains $[0,\infty)$. w has a holomorphic extension to $V \times U$ (this extension is well defined in view of the second step) which takes values in Ω. □

The next lemma is a consequence of a result of Laufer [21, Theorem 4.11].

Lemma 3.4. *Let $E \subseteq \mathbb{C}^n$ be a domain and let $K \subset E$ be a compact set. Then there exists an open set $V \subset E$ relatively compact in E (i.e., $\overline{V}$ is a compact subset of E) such that $K \subset V$ and, for every $f \in \mathcal{O}(V)$, there exists a sequence in $\mathcal{O}(E)$ that converges uniformly on K to f.*

Proof. Taking into account that every domain admits a normal exhaustion (see e.g. [24, p. 17, E.1.2]), we deduce that there exists a sequence of open sets $\{V_k\}_{k \in \mathbb{N}}$ relatively compact in E such that $K \subset V_1 \subset V_2 \subset \ldots$ and $E = \bigcup_{k=1}^{\infty} V_k$. By [21, Theorem 4.11], there exists $m \in \mathbb{N}$ such that, for every $f \in \mathcal{O}(V_m)$, there exists a sequence in $\mathcal{O}(E)$ that converges uniformly on K to f. □

Remark 3.5. We mention that the above lemma cannot, in general, be strengthened, in the sense that: if E is a domain and $K \subset E$ is a compact set, then there exists an open set $V \subset E$ relatively compact in E such that $K \subset V$ and (V,E) is a Runge pair (see the example of Fornaess and Zame [12]).

Now, we are ready to prove the main result.

Theorem 3.6. *Let $E \subseteq \mathbb{C}^n$ be a Φ-like domain. If $D \subseteq E$ is a $\Phi\big|_D$-like domain, then (D, E) is a Runge pair.*

Proof. We follow some basic ideas in the proof of [11, Proposition 1] (cf. the proof of [17, Theorem 3.1]). Let $f \in \mathcal{O}(D)$ and let $K \subset D$ be a compact set. We note that $\mathcal{O}(E)$ is a linear subspace of the space $C(K)$ of continuous functions on K with the supremum norm. Hence, by a consequence of the Hahn-Banach theorem (see [26, Theorem 5.19]), f can be approximated uniformly on K by functions in $\mathcal{O}(E)$ if and only if every continuous linear functional on $C(K)$ which vanishes on $\mathcal{O}(E)$ also vanishes at f. Taking into account the Riesz-Markov-Kakutani representation theorem for $C(K)$ (see [26, Theorem 6.19]), it suffices to prove that: if μ is a complex Borel measure on K such that $\int_K g(z)\mathrm{d}\mu(z) = 0$, for all $g \in \mathcal{O}(E)$, then $\int_K f(z)\mathrm{d}\mu(z) = 0$. This strategy of proof has been used in [11] and [17] (cf. the proof of [26, Theorem 13.6]).

Fix a measure μ that satisfies the above assumptions. Let w be the semigroup on E generated by Φ. Let $V \subset E$ be an open set relatively compact in E such that $K \subset V$ and every function in $\mathcal{O}(V)$ can be approximated uniformly on K by functions in $\mathcal{O}(E)$, whose existence is ensured by Lemma 3.4. Since D is a domain that contains the origin, Lemma 3.2 implies that there exists $T \geq 0$ such that $w_t(V) \subseteq D$, for all $t \geq T$. Hence, for every $t \geq T$, $f \circ w_t$ is well-defined and holomorphic on V, and thus, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathcal{O}(E)$ such that $g_k \to f \circ w_t$, as $k \to \infty$, uniformly on K. Therefore, we have

$$\int_K f(w(z,t))\mathrm{d}\mu(z) = 0, \qquad \text{for all } t \geq T. \tag{3.2}$$

Since D is a $\Phi\big|_D$-like domain, we have that w restricted to $D\times[0,\infty)$ is a semigroup on D (in particular, $w_t(D) \subseteq D$, for all $t \geq 0$). In the following, we use the same notation w for this restriction. By Lemma 3.3, there exist an open set $W \subseteq D$ that contains K and a domain $U \subseteq \mathbb{C}$ that contains the interval $[0,\infty)$ such that $w\big|_{W\times[0,\infty)}$ has a holomorphic extension to $W \times U$ which takes values in D. We use the same notation w for this extension. The function $\varphi : U \to \mathbb{C}$ given by $\varphi(\zeta) = \int_K f(w(z,\zeta))\mathrm{d}\mu(z)$, $\zeta \in U$, is well-defined and holomorphic on U. In view of (3.2), we have $\varphi(t) = 0$, for all $t \in [T,\infty)$. By the Identity Principle, we deduce that $\varphi(\zeta) = 0$, for all $\zeta \in U$. In particular, $\varphi(0) = 0$, and thus $\int_K f(z)\mathrm{d}\mu(z) = 0$. Taking into account the discussion at the beginning, the proof is complete. □

4. Applications, examples and questions

Definition 4.1. *We say that $D \subseteq \mathbb{C}^n$ is a starlike domain with respect to 0 if $rz \in D$, for all $z \in D$ and $r \in [0,1]$. We say that $D \subseteq \mathbb{C}^n$ is a starlike domain if there exists $z_0 \in D$ such that $-z_0 + D$ is starlike with respect to 0.*

Remark 4.2. Theorem 3.6 implies that every starlike domain is Runge (cf. [7, p. 666], [11, Proposition 1], [19, Corollary 3.1.23]).

Proof. Let $D \subseteq \mathbb{C}^n$ be a starlike domain with respect to 0 and I be the identity mapping on $\mathbb{C}^n$. D is an $I\big|_D$-like domain and $\mathbb{C}^n$ is an I-like domain (see [16, Section 4]). Theorem 3.6 implies that D is a Runge domain. The general case follows easily from Definitions 2.1 and 4.1. □

The following proposition, related to [16, Theorem 1, Corollary 1], is useful in providing some examples of Φ-like domains.

Proposition 4.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a Φ_1-like domain and let $F : \Omega \to \mathbb{C}^n$ be a univalent mapping with $F(0) = 0$ such that $DF(0)$ and $D\Phi_1(0)$ commute. Then $F(\Omega)$ is a Φ_2-like domain, where $\Phi_2 \in \mathcal{H}(F(\Omega), \mathbb{C}^n)$ is given by*

$$\Phi_2(z) = DF(F^{-1}(z))\Phi_1(F^{-1}(z)), \quad z \in F(\Omega).$$

In particular, if Ω is a starlike domain with respect to 0 and $F : \Omega \to \mathbb{C}^n$ is a univalent mapping with $F(0) = 0$, then $F(\Omega)$ is a Φ-like domain, where $\Phi \in \mathcal{H}(F(\Omega), \mathbb{C}^n)$ is given by

$$\Phi(z) = DF(F^{-1}(z))F^{-1}(z), \quad z \in F(\Omega).$$

Proof. Let v be the semigroup on Ω generated by Φ_1. Let $w : F(\Omega) \times [0, \infty) \to \mathbb{C}^n$ be given by $w(z, t) = F(v_t(F^{-1}(z)))$, for $z \in F(\Omega)$. Then it is not difficult to show that w is a semigroup on $F(\Omega)$, which is generated by Φ_2. Moreover, since $v_t(\zeta) \to 0$, as $t \to \infty$, for all $\zeta \in \Omega$, then $w_t(z) \to 0$, as $t \to \infty$, for all $z \in F(\Omega)$. Also, it is easy to prove that $\Phi_2(0) = 0$ and $D\Phi_2(0) = D\Phi_1(0)$. Hence $F(\Omega)$ is a Φ_2-like domain.

The particular result for a starlike domain with respect to 0 follows by taking $\Phi_1 = \mathrm{id}_\Omega$. □

Question 4.4. *Is every Φ-like domain $E \subseteq \mathbb{C}^n$ biholomorphic to a starlike domain $D \subseteq \mathbb{C}^n$ for $n \geq 2$?*

Remark 4.5. Every Φ-like domain $E \subseteq \mathbb{C}^n$ is simply connected.

Proof. Let w be the semigroup on E generated by Φ. Let $f : \partial\mathbb{U} \to E$ be a continuous closed curve. Let $F : \overline{\mathbb{U}} \to E$ be given by $F(r\zeta) = w(f(\zeta), -\log r)$, for $r \in (0, 1]$, $\zeta \in \partial\mathbb{U}$, and $F(0) = 0$. Since $w(\cdot, 0) = \mathrm{id}_E$, we have $F\big|_{\partial\mathbb{U}} = f$. Since w is jointly continuous on $E \times [0, \infty)$ (it has continuous partial derivatives) and $w_t \to 0$, as $t \to \infty$, uniformly on the compact set $f(\partial\mathbb{U})$ (see Lemma 3.2), we deduce that F is continuous on $\overline{\mathbb{U}}$. Hence f can be continuously contracted inside E to 0. □

The following example points out that the domains $D \subseteq E \subseteq \mathbb{C}^n$ in Theorem 3.6 are not necessarily Runge domains, for $n \geq 2$ (in the case $n = 1$, D and E are always Runge domains, because they are simply connected, see Remarks 2.2 and 4.5). We shall use the example of non-Runge domain biholomorphic to a polydisc due to Wermer [30], as it is presented in [20, Example 6.8].

Example 4.6. Let $\varphi : \mathbb{C}^3 \to \mathbb{C}^3$ be given by $\varphi(z) = (z_1, z_1z_2 + z_3, z_1z_2^2 - z_2 + 2z_2z_3)$, for $z = (z_1, z_2, z_3) \in \mathbb{C}^3$. Let $\varepsilon_1, \varepsilon_2 \in (0, \frac{1}{2})$, $\varepsilon_1 \leq \varepsilon_2$, be sufficiently small such that φ is biholomorphic on the polydiscs

$$P_j = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| < 1 + \varepsilon_j, |z_2| < 1 + \varepsilon_j, |z_3| < \varepsilon_j\}, \ j \in \{1, 2\}.$$

For the existence of such ε_j, $j \in \{1, 2\}$, see [20, Example 8.4].

Let $E = \varphi(P_2)$ and $D = \varphi(P_1)$. Then E and D satisfy the assumptions of Theorem 3.6 (i.e., E is a Φ-like domain and D is a $\Phi\big|_D$-like domain, for some $\Phi \in \mathcal{H}(E, \mathbb{C}^n)$), but neither of these domains is Runge.

Proof. Let $\Phi : E \to \mathbb{C}^3$ be given by $\Phi(\zeta) = D\varphi(\varphi^{-1}(\zeta))\varphi^{-1}(\zeta)$, for $\zeta \in E$. Since P_1 and P_2 are starlike domains with respect to 0, we deduce, by Proposition 4.3, that E is a Φ-like domain and D is a $\Phi\big|_D$-like domain. In view of [20, Example 6.8], neither of these domains is Runge. □

The next example is related to the previous one.

Example 4.7. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a univalent mapping with $\phi(0) = 0$ such that $\phi(\mathbb{C}^2)$ is not a Runge domain (the existence of such a mapping is due to Wold [31]). Then there exists $r > 0$ such that $E = \phi(\mathbb{C}^2)$ and $D = \phi(r\mathbb{B}^2)$ are not Runge domains, but they satisfy the assumptions of Theorem 3.6 with $\Phi(z) = D\phi(\phi^{-1}(z))\phi^{-1}(z)$, for $z \in E$.

Proof. The result follows in view of [5, Example 2.2] (cf. [24, Theorem VI.1.12]) and Proposition 4.3. □

Regarding Theorem 3.6, we consider the following useful proposition.

Proposition 4.8. *Let $E \subseteq \mathbb{C}^n$ be a Φ-like domain and let $D \subseteq E$. Also, let w be the semigroup on E generated by Φ. Then the following are equivalent:*
i) *D is a $\Phi|_D$-like domain.*
ii) *D is a domain with $0 \in D$ and $w_t(D) \subseteq D$, for all $t \geq 0$.*
iii) *there exists an open set $U \subseteq E$ with $0 \in U$ such that $D = w(U, [0, \infty))$.*

Proof. The equivalence i)⇔ii) is straightforward, in view of Definition 2.3. For the implication ii)⇒iii), we just take $U = D$. For the implication iii)⇒ii), we observe that: $D = \bigcup_{t \geq 0} w_t(U)$ is open, $w_t(D) = w(U, [t, \infty)) \subseteq D$, for all $t \geq 0$, and every $z \in D$ is connected with 0 through the path

$$\gamma(s) = \begin{cases} w(z, -\log s), & s \in (0, 1], \\ 0, & s = 0. \end{cases}$$

□

Related to [12] (see also Remark 3.5), we have the following corollary, which tells us that every Φ-like domain has a *Runge exhaustion* (see [12, p. 1]).

Corollary 4.9. *Let E be a Φ-like domain. Then for every compact set $K \subset E$, there exists an open set $V \subset E$ relatively compact in E such that $K \subset V$ and (V, E) is a Runge pair. Moreover, V can be chosen to be a $\Phi\big|_V$-like domain.*

Proof. Let w be the semigroup on E generated by Φ. Since $K \subset E$ is compact, there exists an open set $U \subset E$ relatively compact in E such that $0 \in U$ and $K \subset U$ (one can take U to be a finite union of small balls that cover $K \cup \{0\}$). Let $V = w(U, [0, \infty))$. In view of Lemma 3.2, we have that V is relatively compact in E. By Proposition 4.8, V is a $\Phi\big|_V$-like domain. By Theorem 3.6, (V, E) is a Runge pair. □

Remark 4.10. In [12, p. 1] it is pointed out that every pseudoconvex domain has a Runge exhaustion (see e.g. the proof of [24, Theorem VI.1.17]; see also [23, Theorem 4.25]). Regarding Corollary 4.9, we mention that, in general, not every Φ-like domain is pseudoconvex. For example, let

$$E = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\} \setminus \big\{(\zeta, 0) : \frac{1}{2} \leq |\zeta| < 1\big\}.$$

Then E is a starlike domain with respect to 0. To prove that it is not pseudoconvex, it suffices to observe that every holomorphic function on E is holomorphic on the Hartogs domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, \frac{1}{2} < |z_2| < 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \frac{1}{2}, |z_2| < 1\} \subset E$$

and thus it extends to the whole unit polydisc centered at 0 (see e.g. [24, Theorem II.1.1]).

In view of the above example, we also note that the domains in Theorem 3.6 are not necessarily pseudoconvex.

Question 4.11. *Taking a look at the arguments used in* [7, p. 666] (*see also* [29]) *to prove that every starlike domain is Runge, we ask the following: is it possible to prove Theorem* 3.6 *using envelopes of holomorphy and the result of Docquier and Grauert* [10]*?*

The following corollary of Theorem 3.6, is related to [4, Proposition 5.1].

Corollary 4.12. *Let $E \subseteq \mathbb{C}^n$ be a Φ-like domain. Let w be the semigroup on E generated by Φ. Then $(w_t(E), E)$ is a Runge pair, for all $t \geq 0$.*

Proof. Let $t \geq 0$ be fixed and let $D = w_t(E)$. Then

$$w_\tau(D) = w_\tau(w_t(E)) = w_{\tau+t}(E) = w_t(w_\tau(E)) \subseteq w_t(E) = D, \text{ for all } \tau \geq 0.$$

Since $\Phi(0) = 0$, we have $w_t(0) = 0$, by [1, Proposition 2.5.23], and thus $0 \in D$. By Remark 2.5 and Proposition 4.8, D is a $\Phi\big|_D$-like domain. Hence, by Theorem 3.6, we deduce that (D, E) is a Runge pair. □

Looking at Theorem 3.6 and Corollary 4.12, one might suspect that: if $E \subseteq \mathbb{C}^n$ is a Φ-like domain and $D \subseteq E$ is a $\Phi\big|_D$-like domain, then there exists $t \geq 0$ such that $D = w_t(E)$, where w is the semigroup on E generated by Φ. The following example shows that this is, in general, false.

Example 4.13. Let $E = \mathbb{B}^n$ and let $D = \mathbb{B}^n \setminus \big\{te_1 : t \in [\frac{1}{2}, 1]\big\}$, where $e_1 = (1, 0, \ldots, 0)$. Let $\Phi = \mathrm{id}_E$. Then E is a Φ-like domain and D is a $\Phi\big|_D$-like domain, but $D \neq w_\tau(E)$, for all $\tau \geq 0$, where w is the semigroup on E generated by Φ.

Proof. Since $w_t(z) = e^{-t}z$, for $z \in \mathbb{B}^n$, $t \geq 0$, we easily deduce that E is a Φ-like domain. We observe that

$$w_\tau(D) = \big(e^{-\tau}\mathbb{B}^n\big) \setminus \left\{te_1 : t \in \left[\frac{e^{-\tau}}{2}, e^{-\tau}\right]\right\} \subset \mathbb{B}^n \setminus \left\{te_1 : t \in \left[\frac{e^{-\tau}}{2}, 1\right]\right\} \subset D,$$

for all $\tau \geq 0$. Since $0 \in D$, we have that D is a $\Phi\big|_D$-like domain, by Proposition 4.8. Since $w_\tau(E) = e^{-\tau}\mathbb{B}^n$, we deduce that $D \neq w_\tau(E)$, for all $\tau \geq 0$. □

Taking into account Proposition 4.8, we shall use the following definition.

Definition 4.14. *Let w be a semigroup on a domain $E \subseteq \mathbb{C}^n$. We say that $D \subseteq E$ is an invariant domain of w, if $w_t(D) \subseteq D$, for all $t \geq 0$.*

We consider the following question (cf. [25, Chapter 7]).

Question 4.15. *Let w be a semigroup on a domain $E \subseteq \mathbb{C}^n$ and $D \subset E$ be a domain. Under what conditions is D an invariant domain of w?*

Remark 4.16. Let $E \subseteq \mathbb{C}^n$ be a Φ-like domain and let w be the semigroup on E generated by Φ. By Theorem 3.6 and Proposition 4.8, a necessary condition for a domain $D \subseteq E$ with $0 \in D$ to be an invariant domain of w is (D, E) to be a Runge pair. In the next example, we shall see that this condition is not sufficient.

Example 4.17. Let E be the unit disc centered at the origin of $\mathbb{C}$ and let

$$D = E \setminus \left\{ \frac{t+i}{\sqrt{2}} : t \in [0,1] \right\}.$$

Let $\Phi = \mathrm{id}_E$. Then E is a Φ-like domain and $D \subset E$ is a domain with $0 \in D$ such that (D, E) is a Runge pair, but D is not an invariant domain of w, where w is the semigroup on E generated by Φ.

Proof. Since $w_t(z) = e^{-t}z$, for $z \in E$, $t \geq 0$, it is easy to prove that D is not an invariant domain of w. However, D is a Runge domain, because it is simply connected (see e.g. Remark 2.2), and thus (D, E) is a Runge pair. □

We mention that the previous example can be easily extended to higher dimensions, in view of [22, Corollary 1.7.8].

Next, we ask the following question.

Question 4.18. *Let w be a semigroup on a domain $E \subseteq \mathbb{C}^n$ and let $D \subseteq E$ be an invariant domain of w. Under what conditions is (D, E) a Runge pair?*

By Theorem 3.6 and Proposition 4.8, we have that: if $E \subseteq \mathbb{C}^n$ is a Φ-like domain and $D \subseteq E$ is an invariant domain of the semigroup w generated by Φ such that $0 \in D$, then (D, E) is a Runge pair. In the following example, we show that the condition $0 \in D$ is essential.

Example 4.19. Let E be the unit disc centered at the origin of $\mathbb{C}$ and let $D = E \setminus \{0\}$. Let $\Phi = \mathrm{id}_E$. Then E is a Φ-like domain and $D \subseteq E$ is an invariant domain of w, where w is the semigroup on E generated by Φ, but (D, E) is not a Runge pair.

Proof. Since $w_t(z) = e^{-t}z$, for $z \in E$, $t \geq 0$, it is easy to show that D is an invariant domain of w. Since D is not simply connected, (D, E) is not a Runge pair (see Remark 2.2). □

Next, we consider an example in higher dimension, related to the previous one.

Example 4.20. Let $E \subset \mathbb{C}^2$ be the unit polydisc centered at the origin of $\mathbb{C}^2$. Let $D = \{(z, w) \in \mathbb{C}^2 : |z| < |w| < 1\}$ (this domain is known as the Hartogs triangle, see e.g. [19]). Let $\Phi = \mathrm{id}_E$, and let w be the semigroup on E generated by Φ. Then D is an invariant domain of w, but (D, E) is not a Runge pair.

Proof. We have $w_t(z,w) = (e^{-t}z, e^{-t}w)$, for $(z,w) \in E$, $t \geq 0$. Clearly, we have that D is an invariant domain of w. However, since E is a Runge domain and D is not a Runge domain, we have that (D,E) is not a Runge pair. For the sake of clarity, we give here an elementary argument for the fact that D is not Runge. Let $f \in \mathcal{O}(D)$ be given by $f(z,w) = \frac{1}{w}$, for $(z,w) \in D$. Suppose that there is a sequence $(p_k)_{k\in\mathbb{N}}$ of polynomial functions such that $p_k \to f$, as $k \to \infty$, locally uniformly on D. In particular, $p_k(0,\cdot) \to f(0,\cdot)$, as $k \to \infty$, uniformly on the circle

$$\Gamma = \left\{(0,\zeta) : |\zeta| = \frac{1}{2}\right\}.$$

Hence $0 = \displaystyle\int_\Gamma p_k(0,\zeta)\mathrm{d}\zeta \to \int_\Gamma f(0,\zeta)\mathrm{d}\zeta = 2\pi i$, as $k \to \infty$, which is a contradiction. □

We note that in both of the previous two examples the invariant domain (of the corresponding semigroup) is not simply connected, hence we consider the following question.

Question 4.21. *Let $n \geq 2$. Does there exist simply connected domains $D \subset E \subseteq \mathbb{C}^n$ such that D is an invariant domain of a semigroup on E and (D,E) is not a Runge pair?*

From now on, we consider a slight modification of the definition of a Φ-like domain in $\mathbb{C}^n$, by dropping the condition $m(D\Phi(0)) > 0$ in Definition 2.3.

Definition 4.22. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. If $0 \in \Omega$ and there exists $\Phi \in \mathcal{H}(\Omega,\mathbb{C}^n)$ such that $\Phi(0) = 0$, and, for every $z \in \Omega$, the initial value problem*

$$\frac{\partial w}{\partial t}(z,t) = -\Phi(w(z,t)), \quad t \geq 0, \qquad w(z,0) = z, \tag{4.1}$$

has a solution $w(z,\cdot)$ on $[0,\infty)$ such that $w(z,t) \in \Omega$, $t \geq 0$, and $w(z,t) \to 0$, as $t \to \infty$, then we say that Ω is a Φ-like domain.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$ and let B be the unit ball of $\mathbb{C}^n$ with respect to this norm. Then, $\mathbb{C}^n$ may be regarded as a finite dimensional complex Banach space with respect to this norm. For each $z \in \mathbb{C}^n \setminus \{0\}$, let

$$T(z) = \big\{l_z \in L(\mathbb{C}^n,\mathbb{C}) :\ l_z(z) = \|z\|,\ \|l_z\| = 1\big\}.$$

This set is nonempty by the Hahn-Banach theorem.

For a domain $\Omega \subseteq \mathbb{C}^n$ with $0 \in \Omega$, let

$$\mathcal{N}(\Omega) = \{h \in \mathcal{H}(\Omega,\mathbb{C}^n) :\ h(0) = 0, \Re l_z(h(z)) > 0, z \in \Omega \setminus \{0\}, l_z \in T(z)\}.$$

If $\Phi \in \mathcal{N}(rB)$ for some $r \in (0,\infty)$, then we have $\Re l_z(D\Phi(0)z) > 0$ for $z \in rB\setminus\{0\}$ and $l_z \in T(z)$ by [28, Lemma 3]. Therefore, in view of [14, Theorem 3.1] and considering the map $\Psi(z) = r^{-1}\Phi(rz)$ for $z \in B$, we obtain that B is a Ψ-like domain, so rB is a Φ-like domain (use Proposition 4.3 with $F = rI$) and Lemma 3.2 holds for every mapping $\Phi \in \mathcal{N}(rB)$ for arbitrary $r \in (0,\infty)$.

In the case of $\infty B = \mathbb{C}^n$, if $\Phi \in \mathcal{N}(\mathbb{C}^n)$ and w_r is the semigroup generated by $\Phi\big|_{rB}$ for $r \in (0,\infty)$, then, in view of the uniqueness of the solution of the initial value problem (4.1), there exists a semigroup generated by Φ such that $w(z,\cdot) = w_r(z,\cdot)$, for all $z \in rB$ and $r \in (0, \infty)$, and, in view of the above, $\mathbb{C}^n$ is a Φ-like domain and Lemma 3.2 holds also for every $\Phi \in \mathcal{N}(\mathbb{C}^n)$.

Thus, as in the proof of Theorem 3.6, we obtain the following theorem.

Theorem 4.23. *Let B be the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$ and let $D \subseteq \mathbb{C}^n$ be a domain. If $D \subseteq rB$ for some $r \in (0, \infty]$ and there exists a $\Phi \in \mathcal{N}(rB)$ such that D is a $\Phi\big|_D$-like domain, then D is a Runge domain.*

Next, we consider the definition of a spirallike domain (see [13]).

Definition 4.24. *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$. We say that a domain $D \subseteq \mathbb{C}^n$ is A-spirallike if $0 \in D$ and $e^{-tA}z \in D$, for all $z \in D$ and $t \geq 0$. In the case $A = I$, we obtain the definition of a starlike domain with respect to 0.*

We obtain the following corollary from Theorem 4.23.

Corollary 4.25 ([17, Theorem 3.1])**.** *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$ and let $D \subseteq \mathbb{C}^n$ be an A-spirallike domain. Then D is a Runge domain.*

Proof. Let $\Phi = A$. Then the semigroup w on $\mathbb{C}^n$ generated by Φ is given by $w(z, t) = e^{-tA}z$, for $z \in \mathbb{C}^n$, $t \geq 0$. By Proposition 4.8, D is a $\Phi\big|_D$-like domain. $m(A) > 0$ implies $A \in \mathcal{N}(\mathbb{C}^n)$, with respect to the Euclidean norm. So, Theorem 4.23 implies that D is a Runge domain. □

In view of [6, Theorem 2.1], we ask the following question.

Question 4.26. *Let $E \subset \mathbb{C}^n$ be a Φ-like domain. Under what conditions on Φ do we have the following extension of the Andersén-Lempert theorem: if $f : E \to \mathbb{C}^n$ is a biholomorphic mapping whose image $f(E)$ is a Runge domain, then f can be approximated by automorphisms of $\mathbb{C}^n$ locally uniformly on E?*

In the case $\Phi = A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $k_+(A) < 2m(A)$, where $k_+(A)$ is the largest real part of the eigenvalues of A, Hamada [17, Theorem 4.2] proved that the theorem holds.

Finally, we give a version of Theorem 3.6 for taut domains. For various properties of the taut domains, see e.g. [1].

Definition 4.27. *Let $\mathbb{U} \subset \mathbb{C}$ be the unit disc. A domain $\Omega \subset \mathbb{C}^n$ is said to be taut, if for every sequence $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{H}(\mathbb{U}, \Omega)$ there exists a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ that either converges locally uniformly on $\mathbb{U}$ to a map $f \in \mathcal{H}(\mathbb{U}, \Omega)$ or diverges locally uniformly on $\mathbb{U}$ (i.e., for any two compact sets $K \subset \mathbb{U}$ and $L \subset \Omega$ there exists $k_0 \in \mathbb{N}$ such that $f_{j_k}(K) \cap L = \emptyset$ for $k \geq k_0$).*

Theorem 4.28. *Let $E \subseteq \mathbb{C}^n$ be a taut domain with $0 \in E$ that has an infinitesimal generator Φ such that $\Phi(0) = 0$ and the spectrum of $D\Phi(0)$ lies in the right half-plane $\{\zeta \in \mathbb{C} : \Re\zeta > 0\}$. If $D \subseteq E$ is a $\Phi\big|_D$-like domain, then (D, E) is a Runge pair.*

Proof. The proof follows from the proof of Theorem 3.6, once we have proved that $w_t \to 0$, as $t \to \infty$, locally uniformly on E, where w is the semigroup on E generated by Φ. By [1, Theorem 2.5.21, Proposition 2.5.23] (see also [1, Corollary 2.1.17]), we deduce that there exists $\rho \in \mathcal{H}(E, E)$ such that $w_t \to \rho$, as $t \to \infty$, locally uniformly on E. By [1, Corollary 2.4.2, Proposition 2.5.23 (ii)], we have that $\rho \equiv 0$. □

Remark 4.29. i) If $E \subseteq \mathbb{C}^n$ is a domain that satisfies the assumptions in Theorem 4.28, then, using the same arguments, we deduce that Corollaries 4.9 and 4.12 hold also in this case.

ii) Let B be the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$. If $r \in (0, \infty)$, then rB is a taut domain (see [1, Corollary 2.1.11]), and, if $\Phi \in \mathcal{N}(rB)$, then Φ is an infinitesimal generator and the real part of each eigenvalue of $D\Phi(0)$ is positive, because $\Re l_z(D\Phi(0)z) > 0$ for $z \in rB \setminus \{0\}$ and $l_z \in T(z)$ (see the discussion before Theorem 4.23). So, Theorem 4.28 implies Theorem 4.23 in the case $r \in (0, \infty)$ and the case $r = \infty$ follows from the finite cases, in view of Definition 2.1.

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