Stud. Univ. Babeş-Bolyai Math. (2020), No. , 1–24

Global existence and blow-up of a Petrovsky equation with general nonlinear dissipative and source terms

Mosbah Kaddour and Farid Messelmi

Abstract. We consider in this work the nonlinearly damped semilinear Petrovsky equation with general nonlinear dissipation and source term

$$\frac{\partial^2 u}{\partial t^2}+\Delta^2 u-\Delta u'+|u|^{p-2}\,u+\alpha g\left(u'\right)=\beta f\left(u\right)\ \text{ in }\Omega\times[0,+\infty[$$

where Ω is open and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega=\Gamma,\alpha,\beta>0$. For the nonlinear continuous term $f\left(u\right)$ and for g continuous, increasing, satisfying $g\ (0)=0$, we prove the global existence of its solutions by means the Faedo-Galerkin procedure combined with the stable set method in $H_0^2\left(\Omega\right)$. Furthermore, we show that this solution blows up in finite time, when the energy is negative.

Mathematics Subject Classification (2010): Primary:93C20 ; 35A01 Secondary: 35B44 .

Keywords: Global Existence, Blow-up, Nonlinear Dissipation, Petrovsky Equation.

1. Introduction

This paper devoted to the global existence, uniqueness, and the blow-up of solution for the nonlinear general Petrovsky equations

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}+\Delta^2 u\left(t\right)-\Delta u'\left(t\right)+|u|^{p-2}\,u\left(t\right)+\alpha g\left(u'\left(t\right)\right)=\beta f\left(u\left(t\right)\right),\ \text{in }\Omega\times\mathbb{R}^+,\\ u=\partial_\eta u=0,\ \text{on }\Gamma\times[0,+\infty[\,,\\ u(x,0)=u_0(x),\ u'(x,0)=u_1(x)\text{ in }\Omega.\end{cases}\tag{1}$$

Recently, in the absence of the strong damping term $-\Delta u'\left(t\right)$ and in the case where $\beta f\left(u\left(t\right)\right)=-q\left(x\right)u\left(x,t\right)+|u|^{p-2}\,u\left(t\right)$ for g continuous, increasing, satisfying $g\left(0\right)=0$, and $q:\Omega\to\mathbb{R}^+$, a bounded function, the problem (1) becomes the following

$$\frac{\partial^2 u}{\partial t^2}+\Delta^2 u(t)+q(x)\,u(x,t)+g(u'(t))=0,\ \text{in } \Omega\times\mathbb{R}^+,$$

This equation together with initial and boundary conditions of Dirichlet type was considered by Guesmia in [9], he proved a global existence and a regularity result of solution, the author under suitable growth conditions on g showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. Without the strong damping term $-\Delta u'(t)$ withe $\alpha g(u'(t))=|u'(t)|^{\sigma-2}u'(t)$ and $\beta f(u(t))=(b+1)|u(t)|^{p-2}u(t)$, $b>0$, the problem (1) reduced to the following problem

$$\frac{\partial^2 u}{\partial t^2}+\Delta^2 u(t)+|u'(t)|^{\sigma-2}u'(t)=b\,|u(t)|^{p-2}u(t),\ \text{in } \Omega\times\mathbb{R}^+,$$

this problem has been considered by Messaoudi in [12], where he investigated the global existence and blow up of solution.

More precisely, he showed that solutions with any initial data continue to exist globally in time if $\sigma\geq p$ and blow up in finite time if $\sigma<p$ and the initial energy is negative. He used a new method introduced by Georgiev and Todorova [8] based on the fixed point theorem for the proof.

in [13], Wu and Tsai showed that the solution of the problem considered in [12] is global under some conditions. Also, Chen and Zhou [4] studied the blow up of the solution of the same problem as in [12].

In the presence of the strong damping, in the case where $\beta f(u(t))=(b+1)|u(t)|^{p-2}u(t)$, $g(u'(t))=|u'(t)|^{\sigma-1}u'(t)$, $b>0$, general Petrovsky problem as in (1) becomes

$$\frac{\partial^2 u}{\partial t^2}+\Delta^2 u(t)-\Delta u'(t)+|u'(t)|^{\sigma-1}u'(t)=b\,|u(t)|^{p-1}u(t), \tag{2}$$

this problem was considered by Li et al. [7], [6] and in [2], the authors obtained global existence, uniform decay of solutions without any interaction between p and σ, the blow up of the solution result was established when $\sigma<p$. Very recently, Pişkin and Polat [6] studied the decay of the solution of the problem (2).

In this paper, our aim is to extend the result of [12], [13] and other's established in a bounded domains to general problem as in (1). The nonlinear term f in (1) likes $f(u(x,t))=a(x)|u(t)|^{r-2}u(t)-b(x)|u(t)|^{q-2}u(t)$ with $r>q\geq 1$ and $a(x)$, $b(x)>0$, and g in (1) likes $g(u'(x,t))=\alpha(x)|u'(t)|^{\sigma-2}u'(t)$ with $\sigma\geq 2$ for $\alpha\ :\Omega\to\mathbb{R}^+$ a function, satisfying $\alpha_1\geq\alpha(x)\geq\alpha_0>0$. For these purposes, we must establish the global existence of solution for (1), we use the variational approach of Faedo-Galerkin approximation combined with the monotonous, compactness, stable set methods as in [12], [3] and in [6] with some modification in some passages to drive the blow-up in infinite time of the solution.

2. Hypotheses

Let us state the precise hypotheses on p, g and f. Let p be a number with

$$2 < p \leq \frac{2n-6}{n-4}\,(n \geq 5)\,(2 \leq p < \infty \text{ if } n = 1,2,3,4), \tag{H1}$$

g is an odd increasing C^1 function and

$$\begin{cases} xg(x) \geq d_0 |x|^{\sigma}, \ \forall x \in \mathbb{R},\ p > \sigma \geq 2, \\ |g(x)| \leq d_1 |x| + d_2 |x|^{\sigma-1}, \ \forall x \in \mathbb{R},\ p > \sigma \geq 2,\ d_i \geq 0. \end{cases} \tag{H2}$$

Let $f(x,s) \in C^1(\Omega \times \mathbb{R})$, satisfies:

$$sf(x,s) + k_1(x)|s| \geq pF(x,s),\ p > 2, \tag{H3}$$

where $F(x,s) = \int_0^s f(x,\zeta)\,d\zeta$, and the growth conditions

$$\begin{cases} |f(x,s)| \leq l_1\left(|s|^{\theta} + k_2(x)\right), \\ |f_s(x,s)| \leq l_1\left(|s|^{\theta-1} + k_3(x)\right) \text{ in } \Omega \times \mathbb{R}, \end{cases} \tag{H4}$$

with some l_0, $l_1 > 0$ and the non-negative functions $k_1(x), k_2(x), k_3(x) \in L^{\infty}(\Omega)$, a.e. $x \in \Omega$, and $1 < \theta \leq \frac{\sigma}{2} < \frac{p}{2}$.

3. Local Existence

In this section, we establish a local existence result for (1) under the assumptions on f, g and p.

Theorem 1. *Assume (*H1*)-(*H4*) hold. Then given any* $(u_0, u_1) \in W \cap L^p(\Omega) \times H_0^2(\Omega) \cap L^{2\sigma-2}(\Omega)$, *the problem (*1*) admits a unique solution* $u(t)$ *satisfying:*

$$u \in L^{\infty}(0,T; W \cap L^p(\Omega)), \tag{3}$$

$$u' \in L^{\infty}(0,T; H_0^2(\Omega)), \tag{4}$$

$$g(u'(t)).u'(t) \in L^1\left(0,T; L^1(\Omega)\right), \tag{5}$$

$$u'' \in L^{\infty}(0,T; L^2(\Omega)). \tag{6}$$

where

$$H_0^2(\Omega) = \left\{\varphi \in H^2(\Omega) : \varphi = \partial_{\eta}\varphi = 0 \ \textit{on}\ \partial\Omega\right\},$$

and

$$W = \left\{\varphi \in H^4(\Omega) \cap H_0^2(\Omega) : \Delta\varphi = \partial_{\eta}\Delta\varphi = 0 \ \textit{on}\ \partial\Omega\right\}.$$

Not that throughout this paper, C denotes a generic positive constant depending of Ω and of all other's given constants, which may be different from line to line and is capable of being examined and modified.

Proof. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_m the space generated by $\{\varphi_1,\ \varphi_2, ...,\ \varphi_m\}$, where the set $\{\varphi_m;\ m \in \mathbb{N}\}$ is a basis of $L^2(\Omega)$, $H_0^2(\Omega)$ and $H^4(\Omega) \cap H_0^2(\Omega)$. We construct approximate solutions u_m $(m = 1, 2, 3, ...)$ in the form

$$u_m(t) = \sum_{j=1}^{m} K_{jm}(t) w_j,$$

where K_{jm} are determined by the following ordinary differential equations:

$$\begin{aligned} &(u''_m, w_j) + (\Delta u_m, \Delta w_j) + (\nabla u'_m, \nabla w_j) \\ &+ \left(|u_m|^{p-2} u_m, w_j\right) + \alpha \left(g\left(u'_m\right), w_j\right) = \beta \left(f\left(u_m\right), w_j\right) \end{aligned} \tag{7}$$

$$\begin{aligned} u_m(0) &= u_{0m} = \sum_{i=1}^{m} (u_0, w_j) w_j \overset{\text{as } m\longrightarrow\infty}{\longrightarrow} u_0 \\ &\text{in } H^4(\Omega) \cap H_0^2(\Omega) \cap L^p(\Omega)\ , \end{aligned} \tag{8}$$

$$\begin{aligned} u'_m(0) &= u_{1m} = \sum_{i=1}^{m} (u_1, w_j) w_j \overset{\text{as } m\longrightarrow\infty}{\longrightarrow} u_1 \\ &\text{in } H_0^2(\Omega) \cap L^{2\sigma-2}(\Omega), \end{aligned} \tag{9}$$

with u_0, u_1 are a given functions on Ω, by virtue of the theory of ordinary differential equations, the system (7)-(9) has a unique local solution on some interval $[0, t_m)$. We claim that for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$ by using the first a priori estimate below.We denote by C, C_k or c_k the constants which is independent of m and the initial data u_0 and u_1.

Multiplying the equation (7) by $K'_{jm}(t)$ and performing the summation over $j = 1, ..., m$, the integration par parts gives

$$E'_m(t) + |\nabla u'_m(t)|^2 + \alpha \left(g\left(u'_m(t)\right), u'_m(t)\right) = 0,\ \forall t \geq 0, \tag{10}$$

where

$$E_m(t) = \frac{1}{2} |u'_m(t)|^2 + \frac{1}{2} |\Delta u_m(t)|^2 + \frac{1}{p} \|u_m(t)\|_p^p - \beta \int_\Omega F(x, u_m(t))\, dx, \tag{11}$$

by (H3) and Young inequality, we have

$$\begin{aligned} -\int_\Omega F(x, u_m)\, dx &\geq -\frac{1}{p} \int_\Omega k_1(x) |u_m|\, dx - \frac{1}{p} \int_\Omega u_m f(x, u_m)\, dx \\ &\geq -\varepsilon C_*^2 |\Delta u_m(t)|^2 - C_\varepsilon |k_1(x)|^2 - \frac{1}{p} \int_\Omega u_m f(x, u_m)\, dx, \end{aligned} \tag{12}$$

and by using hypotheses (H4), Young's inequality we have

$$\begin{aligned} \frac{1}{p}\int_{\Omega} u_m f(x,u_m)\,dx &\le \frac{1}{p}\left|f(x,u_m)\right|\left|u_m\right| \\ \le \frac{l_1^2}{p}\varepsilon\int_{\Omega}\left(\left|u_m\right|^{2\theta}+\left|k_2(x)\right|^2\right)dx &+ \frac{c(\varepsilon,p)}{p^2}\int_{\Omega}\left|u_m\right|^2 dx \\ = \frac{l_1^2}{p}\varepsilon\left\|u_m\right\|_{2\theta}^{2\theta} + \frac{l_1^2}{p}\varepsilon\left|k_2(x)\right|^2 &+ \frac{c(\varepsilon,p)}{p^2}\left\|u_m\right\|_p^2 \\ \le \frac{l_1^2}{p}\varepsilon\left(\frac{p-2\theta}{p}+\frac{2\theta}{p}\left\|u_m\right\|_p^p\right) &+ \frac{l_1^2}{p}\varepsilon\left|k_2(x)\right|^2 \\ &+C'(\varepsilon,p)+\frac{1}{p^2}\left\|u_m\right\|_p^p \end{aligned} \tag{13}$$

substituting (13) in (12), and chosen $\varepsilon \le C_0 = \min\left(\frac{1}{2C_*^2};\frac{p}{2\theta l_1^2+1}\right)$, (11) becomes

$$E_m(t) \ge \frac{1}{2}\left|u_m'(t)\right|^2 + C_1\left|\Delta u_m(t)\right|^2 + C_2\left\|u_m\right\|_p^p - C_3\left(1+K_1+K_2\right), \tag{14}$$

or

$$\left|u_m'(t)\right|^2 + \left|\Delta u_m(t)\right|^2 + \left\|u_m\right\|_p^p \le C_4\left(E_m(t)+K_1+K_2+1\right), \tag{15}$$

where

$$\begin{aligned} 0 < C_1 \le \left(1-C_0C_*^2\right),\ 0 < C_2 &\le \left(\frac{1}{p}-\frac{2\theta l_1^2+1}{p^2}C_0\right), \\ C_3 = \max&\left(C_\varepsilon;\frac{l_1^2}{p}\varepsilon;C'(\varepsilon,p)+\frac{l_1^2}{p}\varepsilon\frac{p-2\theta}{p}\right), \\ C_4 &= \max\left(\frac{1}{\min\left(\frac{1}{2},C_1,C_2\right)},C_3\right). \end{aligned}$$

Thus, it follows from (10) and (14) that, for any $m = 1,2,...,$ and $t \ge 0$.

$$\begin{aligned} \left|u_m'(t)\right|^2 + \left|\Delta u_m(t)\right|^2 + \left\|u_m(t)\right\|_p^p + \int_0^t \left|\nabla u_m'(s)\right|^2 ds \\ +\alpha\int_0^t \left(g\left(u_m'(s)\right),u_m'(s)\right)ds \le C_4\left(E_m(0)+K_1+K_2+1\right). \end{aligned} \tag{16}$$

By assumption (H2)-(H4) according to the Hölder's inequality, we have

$$\begin{aligned} \left|\int_{\Omega} F(x,u_{0m})\,dx\right| \le \frac{1}{p}\int_{\Omega} k_1(x)\left|u_{0m}\right|dx + \frac{1}{p}\int_{\Omega} u_{0m} f(x,u_{0m})\,dx \\ \le C\left(\left|u_m(0)\right|^2 + \left|k_1(x)\right|^2 + \left\|u_m(0)\right\|_p^p + \left|k_2(x)\right|^2 + \left|u_m(0)\right|^2\right). \end{aligned} \tag{17}$$

Then using (8), (9), (10), (11) we obtain that

$$\begin{aligned} E_m(t) \leq E_m(0) &= \frac{1}{2}|u_{1m}|^2 + \frac{1}{p}\|u_{0m}\|_p^p \\ &+\frac{1}{2}|\Delta u_{0m}|^2 - \beta \int_\Omega F(x, u_{0m})\,dx \\ &\leq C_4\left(|u_{1m}|^2 + \|u_{0m}\|_p^p + |\Delta u_{0m}|^2 + |u_{0m}|^2 + K_1 + K_2\right) \leq C, \end{aligned} \tag{18}$$

for some $C > 0$, where $K_1 = \|k_1\|_\infty^2$, $K_2 = \|k_2\|_\infty^2$.

Hence, for any $t \geq 0$ and $m = 1, 2, ...,$ from (16) and (18) we get

$$|u'_m(t)|^2 + |\Delta u_m(t)|^2 + \int_0^t |\nabla u'_m(s)|^2\,ds + \|u_m(t)\|_p^p + \alpha \int_0^t \int_\Omega g(u'_m(s))\,u'_m(s)\,dxds \leq C. \tag{19}$$

By the growth conditions, the estimate (19) and as $2\theta \leq p$, we have

$$|f(u_m)|^2 \leq Cl_1\left(|u_m|^{2\theta} + |k_2(x)|^2\right) \leq C\left(||u_m||_p^{2\theta} + \|k_2\|_\infty^2\right) \leq C.$$

With this estimate we can extend the approximate solution $u_m(t)$ to the interval $[0, T]$ and the following apriori estimates

$$\begin{cases} u_m \text{ is bounded in } L^\infty(0, T; L^p(\Omega)), \\ u'_m \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \\ \nabla u'_m \text{ is bounded in } L^2\left(0, T; L^2(\Omega)\right), \\ g(u'_m).u'_m \text{ is bounded in } L^1(\Omega \times (0, T)), \\ \Delta u_m(t) \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \\ f(u_m) \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right). \end{cases} \tag{20}$$

hold. □

Lemma 2. *There exists a constant $K > 0$ such that*

$$||g(u'_m(t))||_{L^{\frac{\sigma}{\sigma-1}}(\Omega\times[0,T])} \leq K,$$

for all $m \in \mathbb{N}$.

Proof. From (H2), Holder's, Young's inequalities gives

$$\int_0^T \int_\Omega |g(u_m')|^{\frac{\sigma}{\sigma-1}} dxdt = \int_0^T \int_\Omega |g(u_m')| \, |g(u_m')|^{\frac{1}{\sigma-1}} dxdt$$

$$\leq \int_0^T \int_\Omega |g(u_m'(t))| \left(d_1 |u_m'(t)| + d_2 |u_m'(t)|^{\sigma-1}\right)^{\frac{1}{\sigma-1}} dxdt$$

$$\leq C \int_0^T \int_\Omega |g(u_m'(t))| \left(|u_m'(t)|^{\frac{1}{\sigma-1}} + |u_m'(t)|\right) dxdt$$

$$= C \int_0^T \int_\Omega |g(u_m'(t))| \, |u_m'(t)|^{\frac{1}{\sigma-1}} dxdt$$

$$+ C \int_0^T \int_\Omega |g(u_m'(t))| \, |u_m'(t)| \, dxdt$$

$$\leq \frac{\sigma-1}{\sigma} \int_0^T \int_\Omega |g(u_m')|^{\frac{\sigma}{\sigma-1}} dxdt + C(\sigma) \int_0^T \int_\Omega |u_m'(t)|^{\frac{\sigma}{\sigma-1}} dxdt$$

$$+ C \int_0^T \int_\Omega |g(u_m'(t))| \, |u_m'(t)| \, dxdt,$$

therefore

$$\frac{1}{\sigma} \int_0^T \int_\Omega |g(u_m'(t))|^{\frac{\sigma}{\sigma-1}} dxdt \leq C(\sigma) \int_0^T \int_\Omega |u_m'(t)|^{\frac{\sigma}{\sigma-1}} dxdt$$

$$+ C \int_0^T \int_\Omega |g(u_m'(t))| \, |u_m'(t)| \, dxdt$$

$$\leq C \int_0^T ||u_m'(t)||_2^{\frac{\sigma}{\sigma-1}} dt + C \int_0^T \int_\Omega |g(u_m'(t))| \, |u_m'(t)| \, dxdt,$$

hence, by (20), we conclude that

$$\int_0^T \int_\Omega |g(u_m'(t))|^{\frac{\sigma}{\sigma-1}} dxdt \leq K.$$

□

Lemma 3. *There exists a constant $M > 0$ such that*

$$|u_m''(t)| + |\Delta u_m'(t)| + \int_0^T |\nabla u_m''(t)| \, dt \leq M,$$

for all $m \in \mathbb{N}$.

Proof. From (7) we obtain

$$|u_m''(0)| \leq |u_{0m}|^{p-1} + \left|\Delta^2 u_{0m}\right| + |\Delta u_{1m}| + \alpha |g(u_{1m})| + \beta |f(u_{0m})|,$$

by (H4) we have

$$|f(u_{0m})|^2 \leq l_1 \left(|u_{0m}|^{2\theta} + |k_2(x)|^2\right) \leq C \left(||\Delta u_{0m}||_2^{2\theta} + \|k_2\|_\infty^2\right)$$

Since $g(u_{1m})$ is bounded in $L^2(\Omega)$ by (H2), from (8) and (9) we obtain

$$|u_m''(0)| \leq C.$$

Differentiating (7) with respect to t, we get

$$\left(u_m''', w_j\right) + \left(\Delta^2 u_m', w_j\right) - \left(\Delta u_m'', w_j\right) + (p-1)\left(\left|u_m\right|^{p-2} u_m', w_j\right) \\ +\alpha\left(g'\left(u_m'\right) u_m'', w_j\right) = \beta\left(f'\left(u_m\right) u_m', w_j\right). \tag{21}$$

Multiplying it by $K_{jm}''(t)$ and summing over j from 1 to m , according to the Hölder's inequality, to find

$$\frac{1}{2}\frac{d}{dt}\left(\left|u_m''(t)\right|^2 + \left|\Delta u_m'(t)\right|^2\right) + \left|\nabla u_m''(t)\right|^2 + \alpha\left(g'\left(u_m'\right) u_m'', u_m''\right) \tag{22}$$
$$\leq (p-1)\int_\Omega \left|u_m\right|^{p-2}\left|u_m'\right|\left|u_m''\right| dx + \beta\int_\Omega \left|f'\left(u_m\right)\right|\left|u_m'\right|\left|u_m''\right| dx.$$

Taking λ verifying the inequalities:

$$\begin{cases} \lambda + 1 \leq \min\left(\frac{p}{2(\theta-1)}, \frac{n}{n-4}\right) \text{ if } n \geq 5, \\ \lambda + 1 \leq \frac{p}{2(\theta-1)} \text{ if } n = 1, 2, 3, 4, \end{cases}$$

then by using (H4), estimates (20) and generalized Hölder's inequality, we deduce that

$$\begin{aligned} &\int_\Omega \left|f'\left(u_m\right)\right|\left|u_m'\right|\left|u_m''\right| dx \\ &\leq \left\| l_1\left(\left|u_m\right|^{\theta-1} + k_3(x)\right)\right\|_{2(\lambda+1)}^{\lambda} \left\|u_m'\right\|_{2(\lambda+1)}\left\|u_m''\right\|_2 \\ &\leq C\left(\left\|\left|u_m\right|^{\theta-1}\right\|_{2(\lambda+1)}^{\lambda} + \left\|k_3(x)\right\|_{2(\lambda+1)}^{\lambda}\right)\left\|u_m'\right\|_{2(\lambda+1)}\left\|u_m''\right\|_2 \\ &\leq C\left(\left\|u_m\right\|_p^{\lambda(\theta-1)} + \left\|k_3(x)\right\|_p^{\lambda}\right)\left\|\Delta u_m'\right\|_2\left\|u_m''\right\|_2 \\ &\leq C_5\left(\left|u_m''(t)\right|^2 + \left|\Delta u_m'(t)\right|^2\right), \end{aligned} \tag{23}$$

where C_1 and C_2 are positive constants independent of m and $t \in [0, T]$.

By same manner, using condition (H1), Young's inequality, Sobolev imbedding and estimate (20) we have

$$\int_\Omega \left|u_m\right|^{p-2}\left|u_m'\right|\left|u_m''\right| dx \leq \left\|\left|u_m\right|^{p-2}\right\|_n \left\|u_m'\right\|_{\frac{2n}{n-2}}\left\|u_m''\right\|_2$$
$$\leq C\left\|\Delta u_m'\right\|_2\left\|u_m''\right\|_2 \leq C_5\left(\left|u_m''(t)\right|^2 + \left|\Delta u_m'(t)\right|^2\right). \tag{24}$$

Combining (22), (23) and (24) we deduce

$$\frac{1}{2}\frac{d}{dt}\left(\left|u_m''(t)\right|^2 + \left|\Delta u_m'(t)\right|^2\right) + \left|\nabla u_m''(t)\right|^2 + \alpha\left(g'\left(u_m'\right) u_m'', u_m''\right)$$
$$\leq C_6\left(\left|u_m''(t)\right|^2 + \left|\Delta u_m'(t)\right|^2\right).$$

Integrating the last inequality over $(0; t)$ and applying Gronwall's lemma, we obtain

$$\left|u_m''(t)\right| + \left|\Delta u_m'(t)\right| + \int_0^t \left|\nabla u_m''(t)\right|^2 ds \leq C \text{ for all } t \geq 0.$$

Therefore

$$\begin{aligned} u_m'' &\text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \Delta u_m' &\text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \nabla u_m'' &\text{ is bounded in } L^2\left(0,T;L^2(\Omega)\right), \end{aligned} \tag{25}$$

it follows from (25), (u_m') is bounded in $L^\infty\left(0,T;H_0^2(\Omega)\right)$.

Furthermore, by applying the Lions-Aubin compactness Lemma in [10], we claim that

$$u_m' \text{ is precompact in } L^2\left(0,T;L^2(\Omega)\right). \tag{26}$$

we conclude from (20) and (25) that, we can extract a subsequence noted again (u_m) such that

$$\begin{cases} u_m \longrightarrow u \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\ u_m \longrightarrow u \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right), \\ u_m' \longrightarrow u' \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\ u_m' \longrightarrow u' \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right), \\ u_m'' \longrightarrow u'' \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right), \\ g\left(u_m'\right) \longrightarrow \chi \text{ weak star in } L^{\frac{\sigma}{\sigma-1}}\left(\Omega\times(0,T)\right), \\ f\left(u_m\right) \longrightarrow \zeta \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right). \end{cases} \tag{27}$$

Using the compactness of $H_0^2(\Omega)$ to $L^2(\Omega)$, it is easy to see that

$$\int_0^T\int_\Omega |u_m|^{p-2}\,u_m v\,dxdt \to \int_0^T\int_\Omega |u|^{p-2}\,uv\,dxdt, \text{ for all } v\in L^\sigma\left(0,T;H_0^2(\Omega)\right),$$

as $m\to\infty$.

By (H2) and estimetes (27) we have

$$g\left(u_m'\right) \longrightarrow g\left(u'\right) \text{ a.e.in } \Omega\times(0,T).$$

Therefore, from [10, Chapter1,Lemma1.3], we infer that

$$g\left(u_m'\right) \longrightarrow g\left(u'\right) \text{ weak star in } L^{\frac{\sigma}{\sigma-1}}\left(0,T;L^{\frac{\sigma}{\sigma-1}}\right),$$

as $m\to\infty$, and this implies that

$$\int_0^T\int_\Omega g\left(u_m'\right)v\,dxdt \to \int_0^T\int_\Omega g\left(u'\right)v\,dxdt \text{ for all } v\in L^\sigma\left(0,T;H_0^2(\Omega)\right).$$

By the same manner using the growth conditions in (H4) and estimate (27), we see that

$$\int_0^T\int_\Omega \left|f\left(u_m\right)\right|^{\frac{\theta+1}{\theta}}dxdt$$

is bounded and

$$f\left(u_m\right) \longrightarrow f(u) \text{ a.e.in } \Omega\times(0,T).$$

then

$$f\left(u_m\right) \longrightarrow f(u) \text{ weak star in } L^{\frac{\theta+1}{\theta}}\left(0,T;L^{\frac{\theta+1}{\theta}}\right),$$

as $m\to\infty$, and this implies that

$$\int_0^T\int_\Omega f\left(u_m\right)v\,dxdt \to \int_0^T\int_\Omega f(u)\,v\,dxdt \text{ for all } v\in L^\theta\left(0,T;H_0^2(\Omega)\right).$$

It follows at once from all estimates that for each fixed $v \in L^{\theta}\left(0,T;H_0^2(\Omega)\right)\cap L^{\sigma}\left(0,T;H_0^2(\Omega)\right)$

$$\int_0^T\int_\Omega \left(u_m''+\Delta^2 u_m-\Delta u_m'+\left|u_m\right|^{\rho}u_m+\alpha g\left(u_m'\right)-\beta f\left(u_m\right)\right)vdxdt$$
$$\to \int_0^T\int_\Omega \left(u''+\Delta^2 u-\Delta u'+\left|u\right|^{p-2}u+\alpha g\left(u'\right)-\beta f\left(u\right)\right)vdxdt,$$

as $m\to\infty$. Consequently

$$\int_0^T\int_\Omega \left(u''+\Delta^2 u-\Delta u'+\left|u\right|^{p-2}u+\alpha g\left(u'\right)-\beta f\left(u\right)\right)vdxdt=0,$$
$$\forall v\in L^{\theta}\left(0,T;H_0^2(\Omega)\right)\cap L^{\sigma}\left(0,T;H_0^2(\Omega)\right).$$

This means that the problem admit a weak solution u satisfying (1), and (3)-(6). □

Theorem 4. *Under the hypotheses of the thoerem 1, we have the solution u given by thoerem 1, is unique.*

Proof. Let u and v two solutions, to the sense of the thoerem 1. Then $w = u-v$ verifies

$$w''+\left(\Delta^2 u-\Delta^2 v\right)-\Delta w'+\alpha\left(g\left(u'\right)-g\left(v'\right)\right)$$
$$+(\left|u\right|^{p-2}u-\left|v\right|^{p-2}v)=\beta\left(f\left(u\right)-f\left(v\right)\right), \tag{28}$$

$$w(0)=w'(0)=0 \text{ in } \Omega, \tag{29}$$

$$w=\ \partial_\eta w=0 \text{ on } \Sigma, \tag{30}$$

$$w\in L^p(0,T;W\cap L^p(\Omega)), \tag{31}$$

$$w'\in L^2(0,T;H_0^2\left(\Omega\right)). \tag{32}$$

Let's multiply the two members of (28) by w' and integrate on Ω. According to the Green's formula and of the conditions (30), and integrate par part the result on $[0,t]$, using conditions (29) to find that

$$\frac{1}{2}\left(\left|w'(t)\right|^2+\left|\Delta w\right|^2\right)\le\int_0^t\int_\Omega\left|\left|u\right|^{p-2}u-\left|v\right|^{p-2}v\right|\left|w'\right|dxds \tag{33}$$
$$+\beta\int_0^t\int_\Omega\left|f\left(u\right)-f\left(v\right)\right|\left|w'\right|dxds.$$

According to the Hölder's, Young's inequalities, condition (H1), estimates (27), the first term in right-hand side of (33) can be estimated as follows:

$$\begin{aligned} &\int_0^t \int_\Omega \left| |u|^{p-2} u - |v|^{p-2} v \right| |w'| \, dxds \\ &\leq (p-1) \int_0^t \left(\left\| |u|^{p-2} \right\|_{L^n(\Omega)} + \left\| |v|^{p-2} \right\|_{L^n(\Omega)} \right) \|w\|_{L^{\frac{2n}{n-2}}(\Omega)} \|w'\|_{L^2(\Omega)} \, ds \\ &\leq C \int_0^t \left(\|u\|^{p-2}_{L^{n(p-2)}(\Omega)} + \|v\|^{p-2}_{L^{n(p-2)}(\Omega)} \right) \|\Delta w\|_{L^2(\Omega)} \|w'\|_{L^2(\Omega)} \, ds \qquad (34) \\ &\leq C \int_0^t \left(\|\Delta u\|^{p-2}_{L^2(\Omega)} + \|\Delta v\|^{p-2}_{L^2(\Omega)} \right) \|\Delta w\|_{L^2(\Omega)} \|w'\|_{L^2(\Omega)} \, ds \\ &\leq C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2 \right) ds. \end{aligned}$$

Now setting $U_\varepsilon = \varepsilon u + (1-\varepsilon) v$, $0 \leq \varepsilon \leq 1$, by the growth conditions, for the second term of the right side to (33), we have

$$\begin{aligned} \left| \int_0^t \int_\Omega |f(u) - f(v)| \, |w'| \, dxdt \right| &= \left| \int_0^t \int_\Omega \int_0^1 \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon w' dxds \right| \\ &\leq \int_0^t \int_\Omega \left| \int_0^1 \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon \right| |w'| \, dxds \\ &\leq \int_0^t \int_\Omega \int_0^1 \left| \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon \right| |w'| \, dxds \\ &\leq l_1 \int_0^t \int_\Omega \int_0^1 \left(|U_\varepsilon|^{\theta-1} + |k_3(x)| \right) |u - v| \, |w'| \, d\varepsilon dxds \\ &\leq C \int_0^t \int_\Omega \left(|u|^{\theta-1} + |v|^{\theta-1} + |k_3(x)| \right) |w(s)| \, |w'(s)| \, dxds = I. \end{aligned}$$

Using the generalized Hölder's, Young's inequalities, the estimates (27), and picking λ satisfying;

$$\begin{cases} \lambda + 1 \leq \frac{n}{(\theta-1)(n-4)} \text{ if } n \geq 5, \\ 2 \leq \lambda + 1 < \infty \text{ if } n = 1, 2, 3, 4, \end{cases}$$

we see that

$$\begin{aligned} I &\leq C \int_0^t \left\| |u|^{\theta-1} + |v|^{\theta-1} + |k_3(x)| \right\|^\lambda_{2(\lambda+1)} ||w||_{2(\lambda+1)} \|w'\|_2 \\ &\leq C \int_0^t \left(\left\| |u|^{\theta-1} \right\|^\lambda_{2(\lambda+1)} + \left\| |v|^{\theta-1} \right\|^\lambda_{2(\lambda+1)} + \|k_3(x)\|^\lambda_{2(\lambda+1)} \right) ||w||_{2(\lambda+1)} \|w'\|_2 \, ds \\ &\leq C \int_0^t \left(\|\Delta u\|_2^{\lambda(\theta-1)} + \|\Delta v\|_2^{\lambda(\theta-1)} + \|k_3(x)\|^\lambda_\infty \right) ||\Delta w||_2 \|w'\|_2 \, ds \\ &\leq C \int_0^t ||\Delta w||_2 \|w'\|_2 \, ds \leq C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2 \right) ds. \qquad (35) \end{aligned}$$

Combining (33), (34) and (35) to obtain

$$|w'(t)|^2 + |\Delta w(t)|^2 \leq C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2\right) ds.$$

The integral inequality and Gronwall's lemma show that $w = 0$. □

In order to state and prove our main results, we first introduce the following functions:

$$I(t) = I(u(t)) = |\Delta u(t)|^2 - \beta \int_\Omega f(u(t))\, u(x,t)\, dx - \beta \int_\Omega k_1(x)\, |u(x,t)|\, dx, \tag{36}$$

$$J(t) = J(u(t)) = \frac{1}{2}|\Delta u|^2 - \beta \int_\Omega F(x,u)\, dx, \tag{37}$$

$$E(t) = E(u(t), u'(t)) = J(u(t)) + \frac{1}{2}|u_t(t)|_2^2 + \frac{1}{p}\|u(t)\|_p^p. \tag{38}$$

And the stable set as

$$W = \left\{u : u \in H_0^2(\Omega),\, I(t) > 0\right\} \cup \{0\}. \tag{39}$$

4. Global existence

The next lemma shows that our energy functional (38) is a nonincreasing function along the solution of (1).

Lemma 5. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -|\nabla u'(t)|^2 - \alpha \int_\Omega u'(t)\, g(u'(t))\, dx \leq 0. \tag{40}$$

Proof. Multiplying the equation of (1) by u' and integrate over Ω, using integrate by parts and summing up the product results

$$E(t) - E(0) = -\int_0^t |\nabla u'(s)|^2 ds - \alpha \int_0^t \int_\Omega u'(s)\, g(u'(s))\, dxds \text{ for } t \geq 0.$$

□

Lemma 6. *Suppose that (H1)-(H4) hold, let $u_0 \in W$ and $u_1 \in H_0^2(\Omega)$ such that*

$$\gamma = \beta C_*^{\theta+1}\left(\frac{2p}{p-2}E(0)\right)^{\frac{\theta-1}{2}}\left(l_1 + l_1\,||k_2(x)||_\infty + ||k_1(x)||_\infty\right) < 1. \tag{41}$$

Then $u \in W$ for each $t \geq 0$.

Where C_* is the Sobolev-Poincaré embedding such that for all $2 < p \leq \frac{2n}{n-4}$ $(n \geq 5)$, $(2 \leq p < \infty$ if $n = 1,2,3,4)$ we have

$$\|u(t)\|_p \leq C_* \|\Delta u(t)\|_2\,,\ \forall u \in H_0^2(\Omega).$$

Proof. Since $I(0) > 0$, by the continuity, there exists $0 < T_m < T$ such

$$I(t) \geq 0,\ \forall t \in [0, T_m],$$

this gives from (37) and (H3), that

$$E(t) \geq J(t) = \frac{1}{p} I(t) + \frac{p-2}{2p} |\Delta u|^2$$

$$+\frac{\beta}{p}\left(\int_\Omega f(u)\, u dx + \int_\Omega k_1(x)|u|\, dx - p\int_\Omega F(x,u)\, dx\right) \geq \frac{p-2}{2p}|\Delta u|^2. \tag{42}$$

By using (42), (38) and (40)

$$|\Delta u|^2 \leq \frac{2p}{p-2} J(t) \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0), \tag{43}$$

By recalling (H1), (H2), (43), (41) and Cauchy-Schwartz inequality, Sobolev imbedding we have

$$\begin{aligned}
\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)|u|\, dx &\leq \beta \int_\Omega |f(u)||u|\, dx + \beta \int_\Omega |k_1(x)||u|\, dx \\
&\leq \beta l_1 \int_\Omega |u|^{\theta+1} dx + \beta l_1 \int_\Omega |k_2(x)||u|\, dx + \beta \int_\Omega |k_1(x)||u|\, dx \\
&\leq \beta l_1 \|u(t)\|_{\theta+1}^{\theta+1} + \beta\left(l_1 ||k_2(x)||_\infty + ||k_1(x)||_\infty\right)\|u(t)\|_{\theta+1}^{\theta+1} \\
&\leq \beta l_1 C_*^{\theta+1} |\Delta u(t)|^{\theta+1} + \beta C_*^{\theta+1}\left(l_1 ||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\Delta u(t)|^{\theta+1} \\
&= \beta l_1 C_*^{\theta+1} |\Delta u(t)|^{\theta-1}|\Delta u(t)|^2 \\
&\quad + \beta C_*^{\theta+1}\left(l_1 ||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\Delta u(t)|^{\theta-1}|\Delta u(t)|^2 \\
&\leq \beta C_*^{\theta+1}\left(\frac{2p}{p-2}E(0)\right)^{\frac{\theta-1}{2}}\left(l_1 + l_1 ||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\Delta u|^2 \\
&< |\Delta u|^2 \ \text{ on } \ [0, T_m].
\end{aligned} \tag{44}$$

Therefore, by using (36), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$\lim_{t \to T_m} \beta C_*^{\theta+1}\left(\frac{2p}{p-2}E(t)\right)^{\frac{\theta-1}{2}}\left(l_1 + l_1 ||k_2(x)||_\infty + ||k_1(x)||_\infty\right) \leq D < 1$$

T_m is extended to T. □

Lemma 7. *Let the assumptions (41) holds. then there exists $\eta = 1 - \gamma$ such that*

$$\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)|u|\, dx \leq (1-\eta)|\Delta u|^2 \tag{45}$$

and therefore

$$|\Delta u|^2 \leq \frac{1}{\eta} I(t). \tag{46}$$

Proof. From (44) we have

$$\beta \int_{\Omega} f(u)\, u dx + \beta \int_{\Omega} k_1(x)\,|u|\, dx \leq \gamma\, |\Delta u|^2 .$$

We get (45) by taking $\eta = 1-\gamma > 0$, and by using (45) from (36) we get the result (46). ☐

Theorem 8. *Suppose that (*H1*)-(*H4*) hold. Let* $u_0 \in W$ *satisfying (*41*). Then the solution of problem (*1*) is global.*

Proof. It sufficient to show that $\|u_t\|_2^2 + |\Delta u|^2$ is bounded independently to t. To see this we use (36), (38) and (H3) to obtain

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}\,|\Delta u|^2 - \beta \int_{\Omega} F(x,u)\, dx + \frac{1}{2}\,\|u'(t)\|_2^2 + \frac{1}{p}\,\|u(t)\|_p^p \\ &\geq \frac{1}{2}\,|\Delta u|^2 - \frac{\beta}{p} \int_{\Omega} f(u)\, u dx - \frac{\beta}{p} \int_{\Omega} k_1(x)\,|u|\, dx \\ &+\frac{1}{2}\,\|u'(t)\|_2^2 + \frac{1}{p}\,\|u(t)\|_p^p = \frac{1}{2}\,|\Delta u|^2 + \frac{1}{p}\left(I(t) - |\Delta u|^2\right) + \\ &+\frac{1}{2}\,\|u'(t)\|_2^2 + \frac{1}{p}\,\|u(t)\|_p^p \\ &= \frac{p-2}{2p}\,|\Delta u|^2 + \frac{1}{p} I(t) + \frac{1}{2}\,\|u'(t)\|_2^2 + \frac{1}{p}\,\|u(t)\|_p^p \\ &\geq \frac{1}{2}\,\|u'(t)\|_2^2 + \frac{p-2}{2p}\,|\Delta u(t)|^2 . \end{aligned}$$

since $I(t) \geq 0$ and $p > 2$. Therefore

$$\|u'(t)\|_2^2 + |\Delta u|^2 \leq \max\left(2, \frac{2p}{p-2}\right) E(0),$$

These estimates imply that the solution $u(t)$ exist globally in $[0, +\infty[$. ☐

5. Blow up of Solution

In this section, after some long, tedious calculations, we show that the solution of problem (1) blows up in finite time under the assumption $E(0) < 0$, where

$$E(t) = E(u(t), u'(t)) = \frac{1}{2}\,|u'(t)|^2 + \frac{1}{2}\,|\Delta u(t)|^2 + \frac{1}{p}\,\|u(t)\|_p^p - \beta \int_{\Omega} F(x, u(t))\, dx \tag{47}$$

Remark 9. *We set*

$$H(t) = -E(t), \tag{48}$$

*we multiply Eq.(*1*) by* $-u'$ *and integrate over* Ω*, using (*H2*) to get*

$$H'(t) = |\nabla u'(t)|^2 + \alpha \int_{\Omega} u'(t)\, g(u'(t))\, dx \geq \alpha d_0\, \|u'(t)\|_{\sigma}^{\sigma} \quad a.e.\ t \in [0, T], \tag{49}$$

$H(t)$ is absolutely continuous hence

$$0 < H(0) \leq H(t) \leq \beta \int_{\Omega} F(x,u)\,dx \tag{50}$$

when

$$E(0) < 0.$$

We need the following lemma, what is easy, by using the definition of the energy corresponding to the solution, to show it

Lemma 10. *Let $2 < p \leq \frac{2n}{n-4}$ if $n \geq 5$ and $2 < p$ if $n \leq 4$. Then there exists a positive constant $C > 1$, depending only on Ω, such that*

$$\|u(t)\|_p^s \leq C\left(\|u(t)\|_p^p + |\Delta u(t)|^2\right),\ \text{with } 2 \leq s \leq p \tag{51}$$

for any $u \in H_0^2(\Omega)$. If u is the solution constructed as theorem 1, then

$$\|u(t)\|_p^s \leq C\left(H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta \int_{\Omega} F(x,u(t))\,dx\right), \tag{52}$$

with $2 \leq s \leq p$ on $[0,T)$.

Theorem 11. *Let the conditions of the theorem 1 be satisfied. Assume further that*

$$E(0) < 0. \tag{53}$$

Then the solution (3) blows up in finite time.

Proof. We pose

$$\begin{cases} L(t) = |u(t)|^2 = \int\limits_{\Omega} |u(x,t)|^2\,dx, \\ L'(t) = 2(u(t), u'(t)), \\ L''(t) = 2|u'(t)|^2 + 2(u(t), u''(t)), \end{cases}$$

we define the function

$$\begin{aligned} G(t) = H^{1-a}(t) + \varepsilon L'(t) - 3\varepsilon p e^{T-t} \beta \int_{\Omega} F(x,u(t))\,dx \\ + \gamma_1 \varepsilon t \|k_1(x)\|_\infty + \gamma_2 \varepsilon t \|k_2(x)\|_\infty^\sigma,\ t \geq 0, \end{aligned} \tag{54}$$

where γ_1, γ_2, $\varepsilon > 0$ is a positives constants to be specified later, and

$$0 < a \leq \min\left(\frac{p-2}{2p}, \frac{p-\sigma}{(\theta+1)(\sigma-1)}\right) < 1, \tag{55}$$

derivative the Eq (54) and using Eq. (1), hypotheses (H3) we obtain

$$\begin{aligned}
&\frac{d}{dt}G(t) = (1-a)H^{-a}(t)H'(t) + \varepsilon L''(t) + \gamma_1\varepsilon\|k_1(x)\|_\infty \\
&+\gamma_2\varepsilon\|k_2(x)\|_\infty^\sigma + \frac{d}{dt}\left(-3p\varepsilon e^{T-t}\beta\int_\Omega F(x,u(t))\,dx\right) \\
&= (1-a)H^{-a}(t)H'(t) + 2\varepsilon|u'(t)|^2 + 2\varepsilon(u(t),u''(t)) \\
&+\gamma_1\varepsilon\|k_1(x)\|_\infty + \gamma_2\varepsilon\|k_2(x)\|_\infty^\sigma + \\
&+3p\varepsilon e^{T-t}\beta\int_\Omega F(x,u(t))\,dx - 3p\varepsilon e^{T-t}\beta\int_\Omega f(u(t))u'(t)\,dx \qquad (56) \\
&= (1-a)H^{-a}(t)H'(t) + 2\varepsilon|u'(t)|^2 + 2\beta\varepsilon\int_\Omega u(t)f(u(t))\,dx \\
&-2\varepsilon|\Delta u(t)|^2 - 2\varepsilon\int_\Omega u(t)\Delta u'(t)\,dx - 2\varepsilon\|u(t)\|_p^p \\
&+\gamma_1\varepsilon\|k_1(x)\|_\infty + \gamma_2\varepsilon\|k_2(x)\|_\infty^\sigma \\
&+3p\varepsilon e^{T-t}\beta\int_\Omega F(x,u(t))\,dx - 3p\varepsilon e^{T-t}\beta\int_\Omega f(u(t))u'(t)\,dx \\
&-2\alpha\varepsilon\int_\Omega u(t)g(u'(t))\,dx,
\end{aligned}$$

We then exploit Holder's, Young's inequalities hypotheses on g, to estimate the last term in (56) as

$$\begin{aligned}
&2\alpha\varepsilon\left|\int_\Omega u(t)g(u'(t))\,dx\right| \le 2\alpha\varepsilon d_1\int_\Omega |u'(t)||u(t)|\,dx + 2\alpha\varepsilon d_2\int_\Omega |u'(t)|^{\sigma-1}|u(t)|\,dx \\
&\le 2\alpha\varepsilon d_1\frac{\delta^\sigma}{\sigma}||u(t)||_\sigma^\sigma + 2\alpha\varepsilon d_1\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} \qquad (57) \\
&+2\alpha\varepsilon d_2\frac{\delta^\sigma}{\sigma}||u(t)||_\sigma^\sigma + 2\alpha\varepsilon d_2\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}||u'(t)||_\sigma^\sigma \\
&= 2(d_1+d_2)\frac{\delta^\sigma}{\sigma}\alpha\varepsilon||u(t)||_\sigma^\sigma \\
&+2\alpha\varepsilon\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}\left(d_1||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + d_2||u'(t)||_\sigma^\sigma\right),\ \delta>0.
\end{aligned}$$

because $\frac{\sigma}{\sigma-1} \le \sigma$ then by (49) we have

$$\begin{aligned}
&d_1||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + d_2||u'(t)||_\sigma^\sigma \le C(\Omega)^{\frac{\sigma-2}{\sigma}}d_1||u'(t)||_\sigma^{\frac{\sigma}{\sigma-1}} + \frac{d_2}{\alpha d_0}H'(t) \\
&\le\ C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}||u'(t)||_\sigma^\sigma + \frac{d_2}{\alpha d_0}H'(t) \le \frac{1}{\alpha d_0}\left(C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)H'(t) \qquad (58)
\end{aligned}$$

By the boundary conditions we derive the following estimates

$$\int_\Omega u(t)\Delta u'(t)\,dx = \int_\Omega \Delta u(t)u'(t)\,dx \le \frac{1}{4}|\Delta u(t)|^2 + |u'(t)|^2. \qquad (59)$$

Using hypotheses (H4), Holder's, Young's inequalities, conditions (55) and (49) we have

$$\begin{aligned}
\int_\Omega |f(u(t))|\,|u'(t)|\,dx &\le l_1 \int_\Omega \left(|u|^\theta |u'(t)| + |k_2(x)|\,|u'(t)|\right) dx \\
&\le l_1 \,||u(t)||_{2\theta}^{\theta}\, ||u'(t)||_2 \\
&+ l_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} \,||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + l_1 \frac{\delta^\sigma}{\sigma} \|k_2(x)\|_\infty^\sigma \\
&\le \frac{l_1}{\sigma} C(\delta,\sigma)\,\delta^\sigma \,||u(t)||_{2\theta}^{2\theta} + \frac{1}{\sigma} l_1 \delta^{\frac{\sigma}{1-\sigma}} \,||u'(t)||_2^2 \\
&+ l_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} \,||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + l_1 \frac{\delta^\sigma}{\sigma} \|k_2(x)\|_\infty^\sigma \\
&\le \frac{l_1}{\sigma} C^* C(\delta,\sigma)\, C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} \,||u||_\sigma^\sigma \\
&+ \frac{1}{\sigma} l_1 C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \delta^{\frac{\sigma}{1-\sigma}} \,||u'(t)||_\sigma^\sigma \\
&+ l_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \,||u'(t)||_\sigma^\sigma + l_1 \frac{\delta^\sigma}{\sigma} \|k_2(x)\|_\infty^\sigma \\
&\le \frac{l_1}{\alpha d_0} C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \delta^{\frac{\sigma}{1-\sigma}} H'(t) \\
&+ \frac{l_1}{\sigma} C(\delta,\sigma)\,\delta^\sigma C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} \,||u||_\sigma^\sigma + l_1 \frac{\delta^\sigma}{\sigma} \|k_2(x)\|_\infty^\sigma .
\end{aligned}$$

By the hypotheses (H3), estimate (50) we have

$$\begin{aligned}
2\beta \int_\Omega u(t) f(u(t))\,dx &\ge 2\beta p \int_\Omega F(x)\,dx - 2\beta \int_\Omega k_1(x)\,|u(x)|\,dx \\
&\ge 2pH(t) - 2\beta \int_\Omega k_1(x)\,|u(x)|\,dx \qquad (60)
\end{aligned}$$

and by Holder's, Young's inequalities

$$\int_\Omega k_1(x)\,|u(x)|\,dx \le C(\sigma,\alpha)\,\|k_1(x)\|_\infty + 2\alpha \frac{\delta^\sigma}{\sigma} \,||u(t)||_\sigma^\sigma \qquad (61)$$

By substituting in (56) and using (57)-(61), yields,

$$\begin{aligned}&\frac{d}{dt}G(t)\\ &\geq \begin{pmatrix} (1-a)H^{-a}(t) \\ -\frac{1}{\alpha d_0}\left(3p\varepsilon e^{T-t}\beta C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}+2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}}\left(C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\right)\delta^{\frac{\sigma}{1-\sigma}}\end{pmatrix}H'(t)\\ &+2p\varepsilon H(t)-2\varepsilon\|u(t)\|_p^p-\frac{5}{2}\varepsilon|\Delta u(t)|^2+(\gamma_1-2\beta C(\sigma,\alpha))\varepsilon\|k_1(x)\|_\infty\\ &+\left(\gamma_2-3p\varepsilon e^{T-t}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\varepsilon\|k_2(x)\|_\infty^\sigma+3p\beta\varepsilon\int_\Omega F(x,u(s))\,dx\\ &-\varepsilon\left(3\theta pe^{T-t}\beta l_1C(\delta,\sigma)C^*C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}+2\beta\alpha(d_1+d_2)\right)\frac{\delta^\sigma}{\sigma}||u(t)||_\sigma^\sigma,\ \forall\delta,\ \varepsilon>0\end{aligned} \tag{62}$$

At this point, for a large positive constant λ to be chosen later, picking δ such that $\delta^{\frac{\sigma}{1-\sigma}}=\lambda H^{-a}(t)>0$ in (62) we arrive for all $t>0$ at

$$\begin{aligned}&\frac{d}{dt}G(t)\\ &\geq \begin{pmatrix} (1-a) \\ -\frac{\lambda}{\alpha d_0}\left(3p\varepsilon e^{T}\beta C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}+2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}}\left(C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\right)\end{pmatrix}H^{-a}(t)H'(t)\\ &+3\beta p\varepsilon\int_\Omega F(x,u)\,dx-2\varepsilon\|u(t)\|_p^p-\frac{5}{2}\varepsilon|\Delta u(t)|^2+2p\varepsilon H(t)\\ &+(\gamma_1-2\beta C(\sigma,\alpha))\varepsilon\|k_1(x)\|_\infty\\ &+\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\varepsilon\|k_2(x)\|_\infty^\sigma\\ &-\varepsilon\left(3\theta pe^{T}\beta l_1C(\delta,\sigma)C^*C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}+2\beta\alpha(d_1+d_2)\right)\frac{\lambda^{1-\sigma}}{\sigma}H^{a(\sigma-1)}(t)||u(t)||_\sigma^\sigma,\ \forall\delta,\ \varepsilon>0\end{aligned} \tag{63}$$

By exploiting (50), we have

$$H^{a(\sigma-1)}(t)||u(t)||_\sigma^\sigma\leq\beta^{a(\sigma-1)}\left(\int_\Omega F(x,u)\,dx\right)^{a(\sigma-1)}||u(t)||_\sigma^\sigma, \tag{64}$$

from (H3) we have

$$\begin{aligned}\int_\Omega F(x,u)\,dx&\leq\frac{l_1}{p}\left(\int_\Omega|u(t)|^{\theta+1}dx+(|k_2(x)|+|k_1(x)|)|u|\right)\\ &\leq\frac{l_1}{p}||u(t)||_{\theta+1}^{\theta+1}+C\frac{l_1}{p}(\|k_1(x)\|_\infty+\|k_2(x)\|_\infty)||u(t)||_{\theta+1}^{\theta+1}\\ &\leq C\frac{l_1}{p}||u(t)||_{\theta+1}^{\theta+1}\end{aligned} \tag{65}$$

by condition (55) and the estimates (52) we confirm that

$$\begin{aligned}
&\beta^{a(\sigma-1)}\left|\int_{\Omega} F(x,u)\,dx\right|^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma} \leq \\
&\leq C\frac{l_1}{p}\beta^{a(\sigma-1)}\left(||u(t)||_{\theta+1}^{\theta+1}\right)^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma} \\
&= C\frac{l_1}{p}\beta^{a(\sigma-1)} ||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma} \\
&\leq C\frac{l_1}{p}\beta^{a(\sigma-1)} ||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)} ||u(t)||_{\theta+1}^{\sigma} \\
&= C\frac{l_1}{p}\beta^{a(\sigma-1)} ||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)+\sigma} \\
&\leq \frac{l_1}{p}\beta^{a(\sigma-1)}C\left(H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int_{\Omega}F(x,u)\,dx\right) \\
&\leq C\frac{l_1}{p}\beta^{a(\sigma-1)}\left(\begin{array}{c} H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int_{\Omega}F(x,u)\,dx \\ +\|k_1(x)\|_{\infty}+\|k_2(x)\|_{\infty}^{\sigma}\end{array}\right) \qquad (66)
\end{aligned}$$

substituting (66) in (63) we obtain

$$\begin{aligned}
&\frac{d}{dt}G(t) \\
&\geq \left((1-a)-\frac{\lambda}{\alpha d_0}\left(\begin{array}{c} 3p\varepsilon e^T\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \\ +2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\right) \\
&\times H^{-a}(t)\,H'(t) \\
&+3p\beta\varepsilon\int_{\Omega}F(x,u)\,dx-\frac{5}{2}\varepsilon|\Delta u(t)|^2-2\varepsilon\|u(t)\|_p^p \\
&+\varepsilon(\gamma_1-2\beta C(\sigma,\alpha))\|k_1(x)\|_{\infty} \\
&+\varepsilon\left(\gamma_2-3p\varepsilon e^T\beta l_1\frac{\delta^{\sigma}}{\sigma}\right)\|k_2(x)\|_{\infty}^{\sigma} \qquad (67) \\
&+\varepsilon\left(\begin{array}{c} 2pH(t)-\left(\begin{array}{c} 3\theta pe^T\beta l_1 C(\delta,\sigma)\,C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} \\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)} \\ \times C\left(\begin{array}{c} H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int_{\Omega}F(x,u)\,dx \\ +\|k_1(x)\|_{\infty}+\|k_2(x)\|_{\infty}^{\sigma}\end{array}\right)\end{array}\right)
\end{aligned}$$

or

$$\begin{array}{c}\frac{d}{dt}G(t)\\ \geq\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)H^{-a}(t)H'(t)\\ +3p\beta\varepsilon\int_{\Omega}F(x,u)\,dx-\frac{5}{2}\varepsilon\left|\Delta u(t)\right|^{2}-2\varepsilon\left\|u(t)\right\|_{p}^{p}\\ +\varepsilon\left(\gamma_1-2\beta C(\sigma,\alpha)\right)\left\|k_1(x)\right\|_{\infty}\\ +\varepsilon\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^{\sigma}}{\sigma}\right)\left\|k_2(x)\right\|_{\infty}^{\sigma}\\ +\varepsilon\left(\begin{array}{c}(5p-1)H(t)\\ -\left(\begin{array}{c}3\theta pe^{T-t}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\\ \times C\left(\begin{array}{c}H(t)+\|u(t)\|_{p}^{p}+|u'(t)|^{2}+\beta\int_{\Omega}F(x,u)\,dx\\ +\|k_1(x)\|_{\infty}+\|k_2(x)\|_{\infty}^{\sigma}\end{array}\right)\end{array}\right)\\ -\varepsilon(3p-1)H(t).\end{array} \tag{68}$$

By using the definition (48), the estimate (68) gives

$$\begin{array}{c}\frac{d}{dt}G(t)\\ \geq\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)\\ \times H^{-a}(t)H'(t)\\ \\ +\varepsilon\left[\begin{array}{c}\left(\frac{3p-1}{2}\right)\\ -\left(C\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\\ \times|u'(t)|^{2}\\ \\ +\left(\frac{3p-1}{2}-\frac{5}{2}\right)\varepsilon\left|\Delta u(t)\right|^{2}\\ +\varepsilon\left[\begin{array}{c}(\gamma_1-2\beta C(\sigma,\alpha))\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\|k_1(x)\|_{\infty}\\ +\varepsilon\left[\begin{array}{c}\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^{\sigma}}{\sigma}\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\|k_2(x)\|_{\infty}^{\sigma}\end{array}$$

$$+\varepsilon\left[\begin{array}{c}\left(\frac{3p-1}{p}-2\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1 C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\times\|u(t)\|_p^p$$

$$+\varepsilon\left[\begin{array}{c}3p-(3p-1)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1 C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\beta\int_{\Omega}F(x,u)\,dx$$

$$+\varepsilon\left[\begin{array}{c}(5p-1)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1 C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]H(t)$$

pose

$$C_1=C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1 C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{1}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right),$$

we arrive at

$$\begin{aligned}&\frac{d}{dt}G(t)\\ &\geq\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\varepsilon\left(\begin{array}{c}3pe^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\frac{\sigma-1}{\sigma}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)H^{-a}(t)H'(t)\\ &+\varepsilon\left[\frac{3p-1}{2}-C_1\lambda^{1-\sigma}\right]|u'(t)|^2+\left(\frac{3p-1}{2}-\frac{5}{2}\right)\varepsilon|\Delta u(t)|^2\\ &+\varepsilon\left((\gamma_1-2\beta C(\sigma,\alpha))-C_1\lambda^{1-\sigma}\right)\|k_1(x)\|_\infty\\ &+\varepsilon\left(\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)-C_1\lambda^{1-\sigma}\right)\|k_2(x)\|_\infty^\sigma\\ &+\varepsilon\left[\frac{p-1}{p}-C_1\lambda^{1-\sigma}\right]\|u(t)\|_p^p+\varepsilon\left[1-C_1\lambda^{1-\sigma}\right]\beta\int_{\Omega}F(x,u)\,dx\\ &+\varepsilon\left((5p-1)-C_1\lambda^{1-\sigma}\right)H(t).\end{aligned}\tag{69}$$

chosen $\gamma_1=1+2\beta C(\sigma,\alpha),\gamma_2=1+3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}$ and λ satisfying the following inequality

$$\lambda\geq\lambda_0=\min\left(\sqrt[\sigma-1]{\frac{2C_1}{3p-1}},\ \sqrt[\sigma-1]{\frac{pC_1}{p-1}},\ \sqrt[\sigma-1]{C_1},\ \sqrt[\sigma-1]{\frac{C_1}{5p-1}}\right)$$

so that the coefficients of $H(t)$, $|u'(t)|^2$, $|\Delta u(t)|^2$, $\|u(t)\|_p^p$, $\|k_1(x)\|_\infty$, $\|k_2(x)\|_\infty$ and $\int_\Omega F(x,u)\,dx$ in (69) are strictly positive, hence we get

$$\begin{aligned}&\frac{d}{dt}G(t)\\ \geq&\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\varepsilon\left(\begin{array}{c}3pe^T\beta C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha^{\frac{\sigma-1}{\sigma}}\left(C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)H^{-a}(t)\,H'(t)\\ &+\omega\varepsilon\left(\begin{array}{c}H(t)+|u'(t)|^2+\|u(t)\|_p^p+\int_\Omega F(x,u)\,dx\\ +\|k_1(x)\|_\infty+\|k_2(x)\|_\infty^\sigma\end{array}\right),\end{aligned}\tag{70}$$

where ω is the minimum of these coefficients.

We pick ε small enough, so that

$$0<\varepsilon\leq\varepsilon_0=\min\left(\begin{array}{c}\frac{1-a}{\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3pe^T\beta C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha^{\frac{\sigma-1}{\sigma}}\left(C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)};\\ \frac{H^{1-a}(0)}{-L'(0)+3pe^T\beta\int_\Omega F(x,u_0)\,dx}\end{array}\right)$$

therefore (70) take the form

$$\frac{d}{dt}G(t)\geq\omega\varepsilon\left(\begin{array}{c}H(t)+|u'(t)|^2+\|u(t)\|_p^p\\ +\int_\Omega F(x,u)\,dx+\|k_1(x)\|_\infty+\|k_2(x)\|_\infty^\sigma\end{array}\right),\tag{71}$$

hence

$$G(t)\geq G(0)>0 \text{ for all } t\geq 0.$$

The second term in (54) by applying Young's inequality we can estimate as follows

$$\frac{1}{2}L'(t)=(u(t),u'(t))\leq c\,|u'(t)|\,\|u(t)\|_p\leq c\left(|u'(t)|^{2(1-a)}+\|u(t)\|_p^{\frac{2(1-a)}{1-2a}}\right),$$

so

$$|(u(t),u'(t))|^{\frac{1}{1-a}}\leq C\left(|u'(t)|^2+\|u(t)\|_p^{\frac{2}{1-2a}}\right)$$

using Lemma (10) and the condition (55) we obtain

$$\begin{aligned}&|(u(t),u'(t))|^{\frac{1}{1-a}}\\ \leq&\, C\left(H(t)+|u'(t)|^2+\|u(t)\|_p^p+\int_\Omega F(x,u)\,dx\right),\ \forall t\geq 0,\end{aligned}\tag{72}$$

Consequently we have

$$G(t)^{\frac{1}{1-a}} = \left(H^{1-a}(t) + 2\varepsilon \int_{\Omega} u(x,t)\, u'(t)\, dx + \gamma_1 \varepsilon t \left\| k_1(x) \right\|_{\infty} + \gamma_2 \varepsilon t \left\| k_2(x) \right\|_{\infty}^{\sigma}\right)^{\frac{1}{1-a}}$$

$$\leq c\left(H(t) + \left|2\varepsilon \int_{\Omega} u(x,t)\, u'(t)\, dx\right|^{\frac{1}{1-a}} + \left|\gamma_1 \varepsilon t \left\| k_1(x) \right\|_{\infty}\right|^{\frac{1}{1-a}} + \left|\gamma_2 \varepsilon t \left\| k_2(x) \right\|_{\infty}^{\sigma}\right|^{\frac{1}{1-a}}\right)$$

$$\leq C\left(H(t) + \left|u'(t)\right|^2 + \left\| u(t) \right\|_p^p + \int_{\Omega} F(x,u)\, dx + \left\| k_1(x) \right\|_{\infty} + \left\| k_2(x) \right\|_{\infty}^{\sigma}\right), \tag{73}$$

We then combine (71), (72) and (73), to arrive at

$$\frac{d}{dt} G(t) \geq \rho G(t)^{\frac{1}{1-a}} \tag{74}$$

where ρ is a constant depending on C, ω, and ε only, and not depend of u.

Integrate (74) over $(0,t)$ to get

$$G(t)^{\frac{a}{1-a}} \geq \frac{1}{G^{\frac{a-1}{a}}(0) - t\frac{a}{(1-a)}\rho}.$$

Therefore $G\ (t)$ blows up in a finite time T^* where

$$T^* \leq \frac{1-a}{a\rho G^{\frac{a}{1-a}}(0)}.$$

□

References

[1] Adams, R.A, *Sobolev Spaces.* Academic Press, New York, 1975.

[2] Amroun, N.E., Benaissa, A., *Global existence and energy decay of solution to a petrovesky equation with general dissipation and source term.* Georgian Mathematical Journal, 13(3):397–410, 2006.

[3] Chena, W., Zhoub, Y., *Global nonexistence for a semilinear petrovsky equation.* Nonlinear Analysis, 70:3203–3208, 2009.

[4] Chenand, W., Zhou, Y., *Global nonexistence for a semilinear petrovsky equation.* Nonlinear Analysis, 70, 2009.

[5] Dautry, R., Lions, J., *Analyse mathématique et calcul numérique pour les sciences et les techniques*, volume 3. Masson, Paris, 1985.

[6] Piskin, E., Polat, N., *on the decay of solutions for a nonlinear petrovskyequation.* Mathematical Sciences Letters, 3(1):43–47, 2014.

[7] Li, G, Sun, Y., Liu, W., *Global existence and blow-up of solutions for a strongly damped Petrovsky system with nonlinear damping.* Appl. Anal, 91(3):575–586, 2012.

[8] Georgiev, V., Todorova, G., *Existence of a solution of the wave equation with nonlinear damping and source terms.* J. Differential Equations, 109(2):295–308, 1994.

[9] Guesmia, A., *Existence globale et stabilisation interne non linéaire d'un systèm de petrovsky.* Bell. Belg. Math. Soc, 5:583–594, 1998.

[10] Lions, J. L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1966.

[11] Lions, J. L., Magenes, E., *Problèmes aux limites non homogènes et applications*, volume 1, 2. Dunod, Paris, 1968.

[12] Messaoudi, S.A., *Global existence and nonexistence in a system of petrovsky*. J MathAnal Appl, 265(2):296–308, 2002.

[13] Wu, S.T., Tsai, L.Y., *On global solutions and blow-up of solutions for a nonlinearly damped petrovsky system*. Taiwanese J. Math, 19(2A):545–558, 2009.

Mosbah Kaddour
Faculty of Mathematics and Computer Science,
Mohamed Boudiaf University-M'Sila 28000, Algeria
e-mail: mosbah_kaddour@yahoo.fr

Farid Messelmi
Department of Mathematics and LDMM Laboratory Ziane Achour University, Djelfa 17000, Algeria
e-mail: foudimath@yahoo.fr