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AN EXTENSION OF THE BANACH FIXED-POINT THEOREM AND SOME APPLICATIONS IN THE THEORY OF DYNAMICAL SYSTEMS

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Abstract. In this paper we present an extension of the Banach Fixed-Point Theorem and we apply this new result to find the attractors of some classes of discrete dynamical processes. By associating a convergent sequence of Iterated Function Systems (IFS) to a dynamical process, we derive some applications in the approximation of (IFS) attractors.

1. Introduction

Let's remember the celebrated Banach Fixed-Point Theorem:

Theorem 1.1. *Each contraction f of a complete metric space (X,d) has an unique fixed-point.*

It is well-known that this fixed-point, ξ, is the limit of the sequence $(x_n)_{n\in\mathbf{N}}$, $x_n = f(x_{n-1})$ with an arbitrary $x_0 \in X$ (Picard's method) .

In Section 2 we propose an extension of this result: the contraction f is replaced by a sequence of contractions, $(f_n)_{n\in\mathbf{N}}$. We analyse three cases:

- the sequence $(f_n)_{n\in\mathbf{N}}$ is convergent.
- all the applications f_n , $n \in \mathbf{N}$ have the same fixed-point.
- the sequence $(f_n)_{n\in\mathbf{N}}$ is k-periodic.

In each case we obtain a similar result to Theorem 1.1. (Theorem 2.1., 2.2. and 2.3.).

Banach's classical theorem has some important applications in the theory of Dynamical Systems, namely in the theory of Iterated Function Systems (IFS). The

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existence of (IFS) attractors and of the Hutchinson measure attached to an (IFS), for example, are consequences of Theorem 1.1. Let's see more details.

If (X,d) is a metric space, one can consider $dist_X : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbf{R}_+$ by

$$dist_X(A,B) = \sup_{x\in A}\inf_{y\in B} d(x,y).$$

This application is not quite a metric because $dist_X(A,B) \neq dist_X(B,A)$ for many $A, B \in \mathcal{P}(X)$, but the celebrated Pompeiu-Hausddorff metric can be obtained by

$$h : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbf{R}_+ ,\ h(A,B) = \max(dist_X(A,B), dist_X(B,A))$$

It is clear that $h(\{x\},\{y\}) = d(x,y)$.

(see [HS] for details)

Using this metric, one can see, by Picard's method, that $\lim_{n\to\infty} h(\{x_n\},\{\xi\}) = 0$ for the recurrent sequence $x_{n+1} = f(x_n)$ with arbitrary $x_0 \in X$.

If we should consider the discrete dynamical system (X,f), the previous relation means that $\{\xi\}$ is the global attractor of the system.

The results presented in the second paragraph of our paper may be applied to the theory of dynamical processes (a kind of dynamical systems' generalization).

One can consider that the pair $\left(X,(f_n)_{n\in\mathbf{N}}\right)$ may be thought of as a discrete dynamical process and the corresponding recurrent equation, $x_n = f_n(x_{n-1})$, is used to define the process attractor (a good survey on this problem is [Vis]). If $f_n = f$, for all $n \in \mathbf{N}$ we obtain the classical case. Using the above mentioned results we obtain some characterisations of the dynamical processes' attractors (Theorem 3.1., 3.2.).

This way, we extend to dynamical processes some well-known results.

One can obtain, as a particular case, some well-known results in (IFS) theory and some important applications in the approximation of an (IFS) attractor.

In order to approximate the attractor of an (IFS) using computer facilities, we associate the sequence of truncated (IFS) to a dynamical process and we prove that the initial (IFS) attractor, which is in fact the attractor of the associated process, is the limit of the truncated attractors (Theorem 3.4), so it may be approximated as deep as we want by choosing an appropriate number of decimals for the truncation operator.

2. Some extensions of Banach's Fixed-Point Theorem

The next results are not generalizations of Banach's Fixed-Point Theorem, because we sometimes use the classical result in the proofs.

For punctually convergent generating sequences we can prove:

Proposition 2.1. *Let (X, d) be a metric space and $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contractions, punctually convergent on X to f. Then f is an s-contraction.*

Proof. Because the inequalities

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \leq \\ &\leq sd(x, y) + d(f_n(x), f(x)) + d(f_n(y), f(y)) \end{aligned}$$

hold for every $n \in \mathbf{N}$ and every $x, y \in X$ it is clear that

$$d(f(x), f(y)) \leq \lim_{n\to\infty} [sd(x, y) + d(f_n(x), f(x)) + d(f_n(y), f(y))] = sd(x, y).$$

□

Proposition 2.2. *Let (X, d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contractions punctually convergent on X to f , ξ_n the fixed points of $f_n, n \in \mathbf{N}$, and $\xi \in X$. Then $\xi_n \to \xi$ if and only if $f(\xi) = \xi$.*

Proof. " $\Longleftarrow$" From

$$\begin{aligned} d(\xi_n, \xi) &= d(f_n(\xi_n), f(\xi)) \leq d(f_n(\xi_n) f_n(\xi)) + d(f_n(\xi), f(\xi)) \leq \\ &\leq sd(\xi_n, \xi) + d(f_n(\xi), f(\xi)) \end{aligned}$$

results that

$$d(\xi_n, \xi) \leq \frac{1}{1-s} d(f_n(\xi), f(\xi)) \to 0$$

so $d(\xi_n, \xi) \to 0$. It is clear now that $\xi_n \to \xi$.

" $\Longrightarrow$ " $d(f(\xi), \xi) \leq d(f(\xi), f_n(\xi)) + d(f_n(\xi), f_n(\xi_n)) + d(f_n(\xi_n), \xi) \leq$

$\leq d(f(\xi), f_n(\xi)) + (s+1)\, d(\xi_n, \xi) \to 0$.

Hence $f(\xi) = \xi$. □

The next Lemma (a classical result in mathematical analysis) will be used in the proof of Theorem 2.1.

Lemma 2.1. *If $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are sequences of positive numbers and there is $s \in (0,1)$ such that $a_{n+1} - sa_n \leq b_n$ for all $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

Theorem 2.1. *Let (X, d) be a metric space, $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contraction of X, punctually convergent on X to f and $\xi \in X$. Let also consider the recurrent sequence $x_n = f_n(x_{n-1})$, $n \in \mathbf{N}^*$ with arbitrary $x_0 \in X$. Then $x_n \to \xi$ if and only if ξ is the fixed point of f.*

Proof. "$\Longrightarrow$" From

$$d(f(\xi), \xi) \leq d(f(\xi), f_n(\xi)) + d(f_n(\xi), \xi) \leq$$

$$\leq d(f(\xi), f_n(\xi)) + d(f_n(\xi), f_n(x_{n-1})) + d(f_n(x_{n-1}), \xi) \leq$$

$$d(f(\xi), f_n(\xi)) + sd(\xi, x_{n-1}) + d(x_{n-1}, \xi)$$

for all $n \in \mathbf{N}$, it results that $d(f(\xi), \xi) \leq \lim\limits_{n\to\infty} d(f(\xi), f_n(\xi)) + sd(\xi, x_{n-1}) + d(x_{n-1}, \xi) = 0$.

So $f(\xi) = \xi$.

"$\Longleftarrow$" Let us notice that

$$d(x_n, \xi) = d(f_n(x_{n-1}), f(\xi)) \leq d(f_n(x_{n-1}), f_n(\xi)) + d(f_n(\xi), f(\xi)) \leq$$

$$\leq s \cdot d(x_{n-1}, \xi) + d(f_n(\xi), f(\xi)),$$

so $d(x_n, \xi) - s \cdot d(x_{n-1}, \xi) \leq d(f_n(\xi), f(\xi))$. One can now apply Lemma 2.1. for $a_n = d(x_n, \xi)$ and $b_n = d(f_n(\xi), f(\xi))$. It results that $\lim\limits_{n\to\infty} d(x_n, \xi) = 0$, so $\lim\limits_{n\to\infty} x_n = \xi$. □

The previous result etablishes that $\lim_{n\to\infty} h(x_n,\xi)=0$. It can be formulated in terms of dynamical systems theory:

Corollary 2.1. *Let (X,d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contraction of X, punctually convergent on X to f, and let be $\xi\in X$ the unique fixed-point of f. Then $\{\xi\}$ is the global attractor of the dynamical process $\mathcal{P}=(X,(f_n)_{n\in\mathbf{N}})$.*

It is a natural result and it has some interesting applications.

For periodic generating sequences we can prove:

Proposition 2.3. *If (X,d) is a complete metric space, $(f_n)_{n\in\mathbf{N}}$ is a k-periodic sequence ($f_{n+k},=f_n$ for all $n\in\mathbf{N}$) and $\xi_1,\xi_2,...\xi_k$ are the fixed points of $f_1,f_2,...,f_k$ then $\{\xi_1,...\xi_k\}$ is Lyapunov stable.*

Proof. We must find $U\in\mathcal{V}(\{\xi_1,...\xi_k\})$ such that, for every $x\in X$ there is $n_x\in\mathbf{N}$ with the property $\{f_n(x), n\geq n_x\}\subset U$.

Let's consider $a=\max\{d(\{\xi_1,\xi_2), d(\xi_2,\xi_3)...d(\xi_{k-1},\xi_k), d(\xi_k,\xi_1)\}$.

It is quite simple to see that

$$d(x_j,\xi_j)\leq s^j d(x_1,\xi_1)+\frac{1}{1-s}a$$

But $\xi_{k+j}=\xi_j$ for all $j\in\mathbf{N}$, so

$$d(x_{nk+j},\xi_j)\leq s^{nk+j}d(x_1,\xi_1)+\frac{1}{1-s}a \text{ for all } n\in\mathbf{N} \text{ and } j\in\mathbf{N}$$

We now choose $U=\bigcup_{j=1}^{k}B\left(\xi_j,\frac{2}{1-s}a\right)$,

Because $\lim_{n\to\infty}s^{nk+j}=0$ there is $n_0\in\mathbf{N}$ such that, for all $n\geq n_0$ the inequality $s^{nk}d(x_1,\xi_1)\leq\frac{a}{1-s}$ should hold. Then

$$d(x_{nk+j},\xi_j)\leq\frac{2}{1-s}a \text{ for all } n\geq n_0 \text{ and } j\in\{1,2,...,k\}$$

If $n_x\overset{not}{=}n_0\cdot k$, then $x_n\in U$ for all $n\geq n_x$.

If $(f_n)_{n\in\mathbf{N}}$is a k-periodic sequence of s-contraction on X then the application $f_k\circ f_{k-1}\circ...\circ f_1$ is a s^k- contraction on X and has an unique fixed point, namely ξ.

The sequence $(x_n)_{n\in\mathbf{N}}$, $x_n(\xi) = f_n(x_{n-1})$, with $x_0(\xi) = \xi$ is also k-periodic, so $x_n = x_{n \mod k}$ for all $n \in \mathbf{N}$. □

Theorem 2.2. *Let (X,d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ a k-periodic sequence of s-contractions, ξ the fixed point of $f_k \circ ... \circ f_1$ and $x_n = f_n(x_{n-1})$ with arbitrary $x_0 \in X$. Then*

$$\lim_{n\to\infty} h(\{x_n\}, \{f_1(\xi), (f_2 \circ f_1)(\xi), ..., (f_k \circ f_{k-1} \circ ... \circ f_1)(\xi)\}) = 0.$$

Proof. Because $f_1, f_2, ..., f_k$ are s-contractions it is clear that

$$\lim_{n\to\infty} d(x_n, (f_n \circ ... \circ f_1)(\xi)) = 0,$$

so

$$\lim_{n\to\infty} h(\{x_n\}, \{(f_n \circ f_{n-1} \circ ..., f_1)(\xi) \ , n \in \mathbf{N}^*\}) = 0.$$

We may use now the periodicity of $(f_n)_{n\in\mathbf{N}}$ to obtain that

$$\lim_{n\to\infty} h(\{x_n\}, \{f_1(\xi), (f_2 \circ f_1)(\xi), ..., (f_k \circ f_{k-1} \circ ... \circ f_1)(\xi)\}) = 0$$

□

Corollary 2.2. *Let $S = (\mathbf{N}, X, f)$ be a contractive dynamical system on the complete metric space X , ξ the fixed point of f and $x_n = f(x_{n-1})$ with arbitrary $x_0 \in X$. Then $\lim_{n\to\infty} h(\{x_n\}, \{\xi\}) = 0$.*

Proof. In Theorem 2.2. we choose $k = 1$. □

Now let's see what happens when all $f_n, n \in \mathbf{N}$ have the same fixed-point.

Theorem 2.3. *Let (X,d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ sequences of s-contractions of X and $x_n = f_n(x_{n-1})$ with arbitrary $x_0 \in X$. If the applications $f_n, n \in \mathbf{N}$ have the same fixed point ξ, then $\lim_{n\to\infty} h(\{x_n\}, \{\xi\}) = 0$ and each sphere centered in $\{\xi\}$ is Lyapunov stable.*

Proof. Because $d(x_n, \xi) \leq s^n d(x_0, \xi)$ and $h(\{x_n\}, \{\xi\}) = d(x_n, \xi)$ we obtain immediately the results of the Theorem.

Let's notice that, even if $f_n, n \in \mathbf{N}$ have the same fixed point, we know nothing about the convergence or periodicity of $(f_n)_{n\in\mathbf{N}}$.

For example, the applications $f_n(x) = \frac{1}{n}x$ have the same fixed-point, 0, for all $n \in \mathbf{N}$ and $f_n \overset{punctually}{\underset{\mathbf{R}}{\rightarrow}} 0$, still $(f_n)_{n\in\mathbf{N}}$ is not periodic.

The applications $g_n(x) = \frac{2+(-1)^n}{4}x$ have also the same fixed-point, 0, for all $n \in \mathbf{N}$, but the sequence $(g_n)_{n\in\mathbf{N}}$ is punctually convergent only on $\{0\}$ and it is periodic ($k = 2$). □

It is clear now that the situations analyzed in the previous theorems are different.

3. Applications to the Theory of Dynamical Systems

We shall apply the previous results to the theory of Iterated Function Systems (IFS), which are classical examples of chaotic dynamical systems (in the sense of the Devaney definition) and whose attractors are fractals (see [Hut]).

An IFS on the complete metric space (X, d) is

$S = (X, w_1, w_2, ..., w_n)$ where $w_1, w_2, ..., w_n : X \to X$ are s-contractions of X.

On the family of compact subsets of X with $\mathcal{H}(X)$, we consider

$h : \mathcal{H}(X) \times \mathcal{H}(X) \to \mathbf{R}_+$, Pompeiu-Hausdorff's metric.

It is well-known that $(\mathcal{H}(X), h)$ is a complete metric space if (X, d) is so.

Using the s-contractions $w_1, w_2, ..., w_n$ one can obtain another s-contraction, namely $\overline{w} : \mathcal{H}(X) \to \mathcal{H}(X)$, $\overline{w}(B) = w_1(B) \cup w_2(B) \cup ... \cup w_n(B)$ for each $B \in \mathcal{H}(X)$ which has (see Banach's Fixed Point Theorem) a single fixed point $A \in \mathcal{H}(X)$, so $A = w_1(A) \cup w_2(A) \cup ... \cup w_n(A)$.

The Iterated Function System S is associated to the contractive dynamical system $\widetilde{S} = (\mathcal{H}(X), \overline{w})$.

The single fixed-point of $\overline{w}$, $A \in \mathcal{H}(X)$, is in fact the global attractor of $\widetilde{S}$ (it is a compact set and $\lim_{n\to\infty} h(\overline{w}^n(x), A) = 0$ for every $x \in X$). It is called the attractor of S and it is interesting to prove that S exhibits chaotic dynamics on A (see [Ba] for details).

We associate now an (IFS) sequence to a discrete dynamical process.

Definition 3.1. Let's consider $k \in \mathbf{N}$ and $(S_n)_{n\in\mathbf{N}}$, $S_n = (X, w_{1,n}, w_{2,n}, \ldots, w_{k,n})$a sequence of s-IFS . $\overline{w}_n : \mathcal{H}(X) \to \mathcal{H}(X)$ is defined by

$$\overline{w}_n(B) = w_{1,n}(B) \cup w_{2,n}(B) \cup \ldots \cup w_{k,n}(B)$$

for all $B \in \mathcal{H}(X)$ then $\mathcal{P} = (\mathcal{H}(X), (\overline{w}_n)_{n\in\mathbf{N}})$ is the contractive dynamical process associated to the sequence $(S_n)_{n\in\mathbf{N}}$.

Let's notice that, if $S_n = S$ for all $n \in \mathbf{N}$, then $\mathcal{P} = (\mathcal{H}(X), \overline{w})$ is the contractive dynamical system associated to S.

We shall study the properties of the dynamical process' attractor if $(S_n)_{n\in\mathbf{N}}$ is a convergent or a periodic sequence.

Proposition 3.1. *Let $(w_n)_{n\in\mathbf{N}}$ be a sequence of s-contractions of the compact metric space (X,d). Then $w_n \underset{(X,d)}{\overset{p}{\to}} w$ if and only if $\overline{w}_n \underset{(\mathcal{H}(X),h)}{\overset{p}{\to}} \overline{w}$.*

Proof. "$\Rightarrow$" From the previous definitions, it results that

$$\begin{aligned} d(\bar{w}_n(B), \overline{w}(B)) &= \max_{y\in\bar{w}_n(B)}(\min_{z\in\overline{w}(B)} d(y,z)) = \\ &= \max_{x\in B}(\min_{x'\in B} d(w_n(x), w(x'))) \end{aligned}$$

Suppose that $d(\overline{w}_n(B), \overline{w}(B)) \not\to 0$. Then there is $\epsilon > 0$ and $n_k \to \infty$ so that $d(\overline{w}_{n_k}(B), \overline{w}(B)) > \epsilon$. For this $\epsilon > 0$ and for every $k \in \mathbf{N}$ there is $x_{n_k} \in B$ such that

$$d(w_{n_k}(x_{n_k}), w(x_{n_k})) > \epsilon.$$

But $(x_{n_k})_{k\in\mathbf{N}} \subset B$ and B is a compact set, so it has a convergent subsequence, equally denoted by $(x_{n_k})_{k\in\mathbf{N}}$ for the simplicity of writing.

So there are $\epsilon > 0$, a sequence of natural numbers $(n_k)_{k\in\mathbf{N}}$ tending to ∞ and a sequence $(x_{n_k})_{k\in\mathbf{N}} \subset X$ convergent to $x \in X$ such that

$$d(w_{n_k}(x_{n_k}), w(x_{n_k})) > \epsilon,$$

for all $k \in \mathbf{N}$. Then

$$\begin{aligned}
\epsilon &< d\left(w_{n_k}\left(x_{n_k}\right), w\left(x_{n_k}\right)\right) < \\
&< d\left(w_{n_k}\left(x_{n_k}\right), w_{n_k}(x)\right) + d\left(w_{n_k}(x), w(x)\right) + d\left(w(x), w\left(x_{n_k}\right)\right) < \\
&< sd\left(x_{n_k}, x\right) + d\left(w_{n_k}(x), w(x)\right) + sd\left(x_{n_k}, x\right)
\end{aligned}$$

This is a contradiction, because

$$\lim_{n_k \to \infty} d\left(x_{n_k}, x\right) = 0$$

and

$$\lim_{n_k \to \infty} d\left(w_{n_k}(x), w(x)\right) = 0.$$

It results that $d\left(\overline{w}_n(B), \overline{w}(B)\right) \to 0$. In the same way we can prove that

$$d\left(\overline{w}(B), \overline{w}_n(B)\right) \to 0.$$

We may now see that

$$\lim_{n \to \infty} h\left(\overline{w}_n(B), \overline{w}(B)\right) = 0$$

for every $B \in \mathcal{H}(X)$. It means that $\overline{w}_n \underset{(\mathcal{H}(X),h)}{\overset{p}{\to}} \overline{w}$.

"$\Leftarrow$" Because $\{x\} \in \mathcal{H}(X)$ for every $x \in X$ and $\overline{w}_n(\{x\}) \to w(\{x\})$ it results that

$$\lim_{n \to \infty} d\left(w_n(x), w(x)\right) = \lim_{n \to \infty} h\left(\overline{w}_n(x), \overline{w}(x)\right) = 0$$

so $w_n \underset{(X,d)}{\overset{p}{\to}} w$. □

Corollary 3.1. *Let (X, d) be a compact metric space and*

$$\left(S_n\right)_{n \in \mathbf{N}} = \left(\left(X, w_{i,n}, w_{2,n}, \ldots, w_{k,n}\right)\right)_{n \in \mathbf{N}}$$

a sequence of s-(I.F.S.) such that $w_{i,n} \underset{X}{\overset{p}{\to}} u_i$ for every $i \in \{1, 2, \ldots k\}$ and let's denote $S = (X, u_1, u_2, \ldots, u_k)$. Then the sequence of the associated contractive dynamical systems $\widetilde{S}_n = \left(\mathcal{H}(X), \overline{w}_n\right)$, $n \in \mathbf{N}$ is convergent to $S = \left(\mathcal{H}(X), \overline{u}\right)$ in the Pompeiu-Hausdorff metric.

Using this result and a previous theorem we can prove

Theorem 3.1. *Let (X, d) be a compact metric space and*

$$(S_n)_{n\in\mathbf{N}} = ((X, w_{i,n}, w_{2,n}, ..., w_{k,n}))_{n\in\mathbf{N}}$$

a sequence of s-(I.F.S.) such that $w_{i,n} \underset{X}{\overset{p}{\to}} u_i$ *for every* $i \in \{1, 2, ...k\}$ *and let's note* $S = (X, u_1, u_2, ..., u_k)$. *Then the attractor of the contractive dynamical process associated to* $(S_n)_{n\in\mathbf{N}}$ *is the very attractor of* S.

Proof. Let $\widetilde{S}_n = (\mathcal{H}(X), \overline{w}_n)$ be the contractive dynamical system associated to S_n.

In the theorem's hypothesis it is clear that A_n, the attractor of $\widetilde{S}_n$, is the fixed point of the s-contraction $\overline{w}_n : \mathcal{H}(X) \to \mathcal{H}(X)$.

From Proposition 3.1. it results that $\overline{w}_n \underset{(\mathcal{H}(X),h)}{\overset{p}{\to}} \overline{u}$ (here $\widetilde{S} = (\mathcal{H}(X), \overline{u})$ is the contractive dynamical system associated to S). Let's notice that A is the fixed point of $\overline{u}$ and Corollary 2.1. shows that A is the attractor of $\mathcal{P} = (\mathcal{H}(X), (\overline{w}_n)_{n\in\mathbf{N}})$, the contractive dynamical process associated to $(S_n)_{n\in\mathbf{N}}$. □

A direct method to obtain the attractor A is the following:

- we choose $A_0 \in \mathcal{H}(X)$ (usually with a single element).
- we construct the sequence $A_n = \overline{w}_n(A_{n-1})$ and we see that $\lim\limits_{n\to\infty} A_n = A$, so A may be approximated by A_n for $n \in \mathbf{N}$ large enough.

One may say that the attractor of the approximating system is the approximation of the attractor. The random procedure presented in [Ba] can be easily adapted to this situation.

If $(\overline{w}_n)_{n\in\mathbf{N}}$ is a periodic sequence we may apply Theorem 2.3. in order to prove:

Theorem 3.2. *If* $(S_n)_{n\in\mathbf{N}} = ((X, w_{1,n}, w_{2,n}, ..., w_{k,n}))_{n\in\mathbf{N}}$ *is a k-periodic sequence of s-iterated function systems (so* $w_{i,n} = w_{i,n+k}$ *for all* $i \in \{1, 2, ..., k\}$ *and all* $n \in \mathbf{N}$*) and* $\overline{w}_n : \mathcal{H}(X) \to \mathcal{H}(X)$ *is* $\overline{w}_n = \overline{w}_{1,n} \cup \overline{w}_{2,n} \cup ... \cup \overline{w}_{k,n}$ *then the contractive dynamical process associated to* $(S_n)_{n\in\mathbf{N}}$ *, namely* $\mathcal{P} = \left(\mathcal{H}(X), (\overline{w}_n)_{n\in\mathbf{N}}\right)$ *is k-periodic and its attractor is the orbit of the unique fixed point of the application* $\overline{w}_k \circ ... \circ \overline{w}_1$.

More precisely there is an unique set $A \in \mathcal{H}(X)$ *such that* $\{\overline{w}_1(A), (\overline{w}_2 \circ \overline{w}_1)(A), ..., (\overline{w}_k \circ ... \circ \overline{w}_1)(A)\}$ *is the attractor of* $\mathcal{P}$.

For $k = 1$ this is a well known result in the theory of iterated function systems.

Theorem 3.1. also contains the basic ideas of the approximation of an IFS attractor using computer facilities. In this case, the repeated truncations can dramatically modify the attractor's properties.

If T_k is a 10^{-k}-truncation operator on the metric space (X, d) then $d(T_k(x), T_k(y)) \leq 2 \cdot 10^{-k}$ if $d(x, y) \leq 10^{-k}$ and $\lim_{k \longrightarrow \infty} T_k(x) = x$ for all x, y in X.

Let's consider $S = (X, w_1, w_2, ...w_n)$ an IFS and T_k a 10^{-k}-truncation operator on X.

Let's denote $S_k = (\mathbf{N}, X, T_k \circ w_1, ..., T_k \circ w_n)$.

Simple computations show that $T_k \circ w_i \underset{X}{\overset{p}{\rightarrow}} w_i$. Unfortunately, the previous result may not be applied, because $T_k \circ w_i$ is not a contraction but, using the mentioned properties of T_k, we can easily obtain a result similar to Theorem 3.1.

Theorem 3.3. *Let w be an s-contraction in the compact metric space (X, d) and $(T_k)_{k \in \mathbf{N}}$ a sequence of $10^{-k}, k \in \mathbf{N}$ truncation operators on X.*

Then $T_k \circ w \underset{(X,d)}{\overset{p}{\rightarrow}} w$ if and only if $\overline{T_k \circ w} \underset{(\mathcal{H}(X),h)}{\overset{p}{\rightarrow}} \overline{w}$.

Using this result we can prove

Theorem 3.4. *Let's consider $S = (X, w_1, w_2, ...w_n)$ an IFS on the complete metric space (X, d) and $(T_k)_{k \in \mathbf{N}}$ a sequence of (10^{-k})-truncation operators on X. If A_k, A are the attractors of $S_k = (\mathbf{N}, X, T_k \circ w_1, ..., T_k \circ w_n)$ and S, respectively, then, in $(\mathcal{H}(X), h)$, we have*

$$\lim_{k \to \infty} A_k = A$$

From Theorem 3.4. it results that A can be well approximated, by choosing an appropriate number of decimals for the truncation operator.

It is very important, because there are no general relations between the real attractor and the truncated one.

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ABSOLUTELY F/U-PURE MODULES

IULIU CRIVEI

Abstract. Let R be an associative ring with non-zero identity. A submodule A of a right R-module B is said to be F/U-pure if $f \otimes_R 1_{F/U}$ is a monomorphism for every free left R-module F and for every cyclic submodule U of F, where $f : A \to B$ is the inclusion monomorphism. A right R-module D is said to be absolutely F/U-pure if D is F/U-pure in every right R-module which contains it as a submodule. We characterize absolutely F/U-purity by injectivity with respect to a certain monomorphism. We also prove that the class of absolutely F/U-pure right R-modules is closed under taking direct products, direct sums and extensions. Moreover, we consider absolutely F/U-pure right modules over right noetherian rings and regular (von Neumann) rings.

1. Introduction

In this paper we denote by R an associative ring with non-zero identity and all R-modules are unital. By a homomorphism we understand an R-homomorphism. The category of right R-modules is denoted by $Mod-R$. The injective envelope of a right R-module A is denoted by $E(A)$.

Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of right R-modules and homomorphisms. The monomorphism f is said to be F/U-pure if the tensor product $f \otimes 1_{F/U} : A \otimes_R F/U \to B \otimes_R F/U$ is a monomorphism for every free left R-module F and for every cyclic submodule U of F [1, Definition 2.1]. If f is F/U-pure, then the short exact sequence (1) is called

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F/U-pure. If A is a submodule of B and f is the inclusion monomorphism, then A is said to be an F/U-pure submodule of B.

Let $M \in Mod-R$. Then M is said to be projective with respect to the short exact sequence (1) if the natural homomorphism $Hom_R(M, B) \to Hom_R(M, C)$ is surjective. The right R-module M is said to be injective with respect to the short exact sequence (1) (or with respect to the monomorphism f) if the natural homomorphism $Hom_R(B, M) \to Hom_R(A, M)$ is surjective.

Following Maddox [3], a right R-module M is said to be absolutely pure if M is pure in every right R-module which contains M as a submodule.

In the present paper we introduce the notion of absolutely F/U-pure right R-module and we establish some properties for such modules.

2. Basic results

We shall begin with two results which will be used later in the paper.

Theorem 2.1. [1, Theorem 2.8] *Let A be a submodule of a right R-module B. Then the following statements are equivalent:*

(i) A is F/U-pure in B;

(ii) If $a_1, \ldots, a_n \in R$, $r_1, \ldots, r_n \in R$ and the system of equations $a_i = xr_i$, $i = 1, \ldots, n$ has a solution $b \in B$, then it has a solution $a \in A$.

Theorem 2.2. [2, Theorem 2.3] *A short exact sequence (1) is F/U-pure if and only if for every finitely generated right ideal of R the right R-module R/I is projective with respect to the short exact sequence (1).*

We shall give now the following definition.

Definition 2.3. A right R-module A is said to be *absolutely F/U-pure* if A is F/U-pure in each right R-module which contains A as a submodule.

In the sequel we shall denote by $\mathcal{A}$ the class of absolutely F/U-pure right modules.

Theorem 2.4. *Let $A \in Mod - R$. Then the following statements are equivalent:*

(i) $A \in \mathcal{A}$;

(ii) A is F/U-pure in $E(A)$;

(iii) If A is a finitely generated right ideal of R and $i : I \to R$ is the inclusion monomorphism, then A is injective with respect to i.

Proof. Let I be a finitely generated right ideal of R and consider the short exact sequence of right R-modules

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{p} R/I \longrightarrow 0 \tag{2}$$

where i is the inclusion monomorphism and p the natural epimorphism. Since R is projective, we have $Ext^1_R(R, A) = 0$. Hence the short exact sequence (2) induces the following short exact sequence of abelian groups:

$$Hom_R(R, A) \xrightarrow{Hom_R(i, 1_A)} Hom_R(I, A) \longrightarrow Ext^1_R(R/I, A) \longrightarrow 0 \tag{3}$$

Let $D \in Mod - R$ such that A is a submodule of D and consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{j} E(D) \xrightarrow{q} E(D)/A \longrightarrow 0 \tag{4}$$

where j is the inclusion monomorphism and q the natural epimorphism. By injectivity of $E(D)$, we have $Ext^1_R(R/I, E(D)) = 0$. Hence the short exact sequence (4) induces the following short exact sequence of abelian groups:

$$Hom_R(R/I, E(D)) \xrightarrow{Hom_R(1_{R/I}, q)} Hom_R(R/I, E(D)/A) \longrightarrow$$

$$\longrightarrow Ext^1_R(R/I, A) \longrightarrow 0 \tag{5}$$

$(i) \Longrightarrow (ii)$ This is clear.

$(ii) \Longrightarrow (iii)$ Suppose that A is F/U-pure in $E(A)$ and consider $D = A$ in the short exact sequence (4). By Theorem 2.2, $Hom_R(1_{R/I}, q)$ is surjective. Hence $Ext^1_R(R/I, A) = 0$, because the sequence (5) is exact. By the exactness of the sequence

(3), it follows that $Hom_R(i, 1_A)$ is surjective. Therefore A is injective with respect to i.

$(iii) \implies (i)$ Suppose that A is injective with respect to i. Then $Hom_R(i, 1_A)$ is surjective. Since the short exact sequence (3) is exact, it follows that $Ext^1_R(R/I, A) = 0$. By the exactness of the sequence (5), $Hom_R(1_{R/I}, q)$ is surjective. By Theorem 2.2, A is F/U-pure in $E(D)$. By Theorem 2.1, A is F/U-pure in D. Therefore $A \in \mathcal{A}$. □

Remark. Every injective right R-module is absolutely F/U-pure.

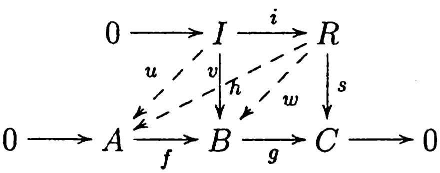
Corollary 2.5. *The class $\mathcal{A}$ is closed under taking direct products and direct summands.*

Lemma 2.6. *The class $\mathcal{A}$ is closed under taking direct sums.*

Proof. Let $(A_j)_{j \in J}$ be a family of absolutely F/U-pure right R-modules and let $A = \oplus_{j \in J} A_j$. Let I be a finitely generated right ideal of R, $i : I \to R$ the inclusion monomorphism and $f : I \to A$ an homomorphism. Since $f(I)$ is finitely generated, there exists a finite subset $K \subseteq J$ such that $f(I) \subseteq \oplus_{k \in K} A_k = B$. By Corollary 2.5, $B \in \mathcal{A}$. Therefore by Theorem 2.4, there exists a homomorphism $g : R \to B$ such that $gi = v$, where $v : I \to B$ is the homomorphism defined by $v(r) = f(r)$ for every $r \in I$. Let $u : B \to A$ be the inclusion monomorphism. Then $ugi = uv = f$. By Theorem 2.4, $A \in \mathcal{A}$. □

Theorem 2.7. *Let (1) be a short exact sequence of right R-modules and let $A, C \in \mathcal{A}$. Then $B \in \mathcal{A}$.*

Proof. Let I be a right ideal of R, $i : I \to R$ the inclusion monomorphism and $h : I \to B$ a homomorphism. Consider the following diagram of right R-modules with exact rows:



where u, v, w, s are homomorphisms which will be defined. Since $C \in \mathcal{A}$, by Theorem 2.4 there exists a homomorphism $s : R \to C$ such that $si = gh$. By projectivity of R, there exists a homomorphism $w : R \to B$ such that $gw = s$. We have $gwi = si = gh$, hence $g(wi - h) = 0$. Let $r \in I$. Then $g((wi - h)(r)) = 0$, therefore $(wi - h)(r) \in Ker\, g = Im\, f$. Since f is a monomorphism, there exists a unique element $a \in A$ such that $(wi - h)(r) = f(a)$. Hence we can define a homomorphism $u : I \to A$ by $u(r) = a$. We have also $h(r) = (wi)(r) - f(a)$. Since $A \in \mathcal{A}$, there exists a homomorphism $v : R \to A$ such that $vi = u$. Then

$$((w - fv)i)(r) = (wi)(r) - (fu)(r) = (wi)(r) - f(a) = h(r)\,.$$

Hence there exists the homomorphism $w - fv : R \to B$ such that $(w - fv)i = h$. By Theorem 2.4, $B \in \mathcal{A}$. □

3. Absolutely F/U-pure modules over particular rings

In this section we shall consider absolutely F/U-pure R-modules over right noetherian rings and regular(von Neumann) rings.

Theorem 3.1. *The following statements are equivalent:*

(i) R is right noetherian;

(ii) If $A \in \mathcal{A}$, then A is injective.

Proof. $(i) \Longrightarrow (ii)$ Suppose that R is noetherian. Let $A \in \mathcal{A}$, let I be a right ideal of R and let $i : I \to R$ be the inclusion monomorphism. Since R is noetherian, I is finitely generated. By Theorem 2.4, A is injective with respect to i.Therefore by Baer's criterion, A is injective.

$(ii) \Longrightarrow (i)$ Suppose that every absolutely F/U-pure right R-module is injective. Let $(A_j)_{j \in J}$ be a family of injective right R-modules and let $A = \oplus_{j \in J} A_j$. Then $A_j \in \mathcal{A}$ for every $j \in J$. By Lemma 2.6, $A \in \mathcal{A}$, hence A is injective. Since every direct sum of injective right R-modules is injective, it follows that R is right noetherian [5, Chapter 4, Theorem 4.1]. □

Remark. If R is not right noetherian, there exist absolutely F/U-pure right R-modules which are not injective.

Lemma 3.2. *Let I be a finitely generated right ideal of R. If $I \in \mathcal{A}$, then I is a direct summand of R.*

Proof. Suppose that $I \in \mathcal{A}$ and let $i : I \to R$ be the inclusion monomorphism. By Theorem 2.4, there exists a homomorphism $p : R \to I$ such that $pi = 1_I$ Therefore I is a direct summand of R. □

Theorem 3.3. *The following statements are equivalent:*

(i) $A \in \mathcal{A}$ for every $A \in Mod - R$;

(ii) $I \in \mathcal{A}$ for every finitely generated right ideal I of R;

(iii) R is regular (von Neumann).

Proof. $(i) \Longrightarrow (ii)$ This is clear.

$(ii) \Longrightarrow (iii)$ It follows by Lemma 3.2, because R is regular if and only if every finitely generated right ideal I of R is a direct summand of R [4, Chapter I, Theorem 14.7.8 and Proposition 4.6.1].

$(iii) \Longrightarrow (i)$ Suppose that R is regular. Let $A \in Mod - R$, let I be a finitely generated right ideal of R and let $f : I \to A$ be a homomorphism. Then I is a direct summand of R. Hence there exists a finitely generated right ideal J of R such that $R = I \oplus J$. Then there exist a unique $r \in I$ and a unique $s \in J$ such that $1 = r + s$. Therefore we can define a unique homomorphism $h : R \to A$ such that $h(1) = f(r)$. It follows that $hi = f$. By Theorem 2.4, $A \in \mathcal{A}$. □

Corollary 3.4. *Let R be regular (von Neumann) and let I be a right ideal of R which is not finitely generated. Then $I \in \mathcal{A}$, but I is not injective.*

Example 3.5. Let $\mathbb{Z}$ be the ring of integers and let $\mathcal{P}$ be the set of all primes. Then $R = \prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$ is a commutative regular (von Neumann) ring and $R = \oplus_{p\in\mathcal{P}}\mathbb{Z}/p\mathbb{Z}$ is an ideal of R. Since I is not finitely generated, it follows that $I \in \mathcal{A}$, but I is not injective.

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A SUFFICIENT CONDITION FOR UNIVALENCE

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Abstract. In this paper we obtain an univalence criterion for holomorphic mappings in the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$$

and norm $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$. The unit ball $\{z \in \mathbb{C}^n : \|z\| < 1\}$ is denoted B^n.

We let $\mathcal{L}(\mathbb{C}^n)$ denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \ \|z\| < 1\}, \quad A \in \mathcal{L}(\mathbb{C}^n).$$

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbb{C}^n)$.

We denote by $H(B^n)$ the class of holomorphic mappings

$$f(z) = (f_1(z), \ldots, f_n(z)), z \in B^n$$

from B^n into $\mathbb{C}^n$. We say that $f \in H(B^n)$ is *locally biholomorphic* in B^n if f has a local inverse at each point in B^n or equivalently if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j,k \le n}$$

is nonsingular at each point $z \in B^n$.

The second derivative of a function $f \in H(B^n)$ is a symmetric bilinear operator $D^2 f(z)(\cdot, \cdot)$ on $\mathbb{C}^n \times \mathbb{C}^n$. $D^2 f(z)(z, \cdot)$ is the linear operator obtained by restricting $D^2f(z)$ to $\{z\} \times \mathbb{C}^n$ and has the matrix representation

$$D^2f(z)(z,\cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \partial z_m} z_m \right)_{1 \le j,k \le n}$$

A mapping $v \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \le \|z\|, z \in B^n$. If $f, g \in H(B^n)$ we say that f is *subordinate* to g $(f \prec g)$ in B^n, if there exists a Schwarz function v such that $f(z) = g(v(z)), z \in B^n$.

A function $L : B^n \times [0,\infty) \to \mathbb{C}^n$ is an *univalent subordination chain* if $L(\cdot,t) \in H(B^n), L(\cdot,t)$ is univalent in B^n for all $t \in [0,\infty)$ and $L(\cdot,s) \prec L(\cdot,t)$, whenever $0 \le s < t < \infty$.

We shall use only normalized functions in an univalent subordination chain, i.e $DL(0,t) = e^t I$, for all $t \ge 0$.

The following theorem is due to J.A. Pfaltzgraff and we shall use it to prove our results.

Theorem 1. [3] *Let $L(z,t) = e^t z + \ldots$, be a function from $B^n \times [0,\infty)$ into $\mathbb{C}^n$ such that:*

(i) For each $t \ge 0$, $L(\cdot,t) \in H(B^n)$.

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in B^n$.

Let $h(z,t)$ be a function from $B^n \times [0,\infty)$ into $\mathbb{C}^n$ such that:

(iii) For each $t \ge 0, h(\cdot,t) \in H(B^n)$, $h(0,t), h(0,t) = 0$, $Dh(0,t) = I$ and $\mathrm{Re} < h(z,t), z \ge 0$, $z \in B^n$.

(iv) For each $T > 0$ and $r \in (0,1)$ there is a number $K = K(r,T)$ such that $\|h(z,t)\| \le K(r,T)$, where $\|z\| \le r$ and $t \in [0,T]$.

(v) For each $z \in B^n$, $h(z,t)$ is a measurable function of t on $[0,\infty)$.

Suppose $h(z,t)$ satisfies

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\,h(z,t) \quad a.e \quad t \ge 0, \quad for\ all \quad z \in B^n. \tag{1}$$

Further, suppose there is a sequence $(t_m)_{m\geq 0}$, $t_m > 0$ increasing to ∞ such that

$$\lim_{m\to\infty} e^{-t_m} L(z, t_m) = F(z) \tag{2}$$

locally uniformly in B^n.

Then for each $t \geq 0$, $L(\cdot, t)$ is univalent on B^n.

2. Main results

Theorem 2. *Let $f, g \in H(B^n)$ such that $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ and g is locally univalent in B^n. If*

$$\left\| (Dg(z))^{-1} Df(z) - I \right\| < 1 \tag{3}$$

and

$$\left\| \|z\|^2 \left[(Dg(z))^{-1} Df(z) - I \right] + \left(1 - \|z\|^2\right) (Dg(z))^{-1} D^2 g(z)(z, \cdot) \right\| < 1 \tag{4}$$

for all $z \in B^n$, then f is an univalent function in B^n.

Proof. We define

$$L(z,t) = f\left(e^{-t} z\right) + \left(e^t - e^{-t}\right) Dg\left(e^{-t} z\right)(z), \quad (z,t) \in B^n \times [0, \infty)$$

We shall prove that $L(z,t)$ satisfies the conditions of Theorem 1 and hence $L(\cdot, t)$ is univalent in B^n, for all $t \in [0, \infty)$. Since $f(z) = L(z, 0)$ we obtain that f is an univalent function in B^n.

We have $L(z,t) = e^t z +$ (holomorphic term). Thus $\lim_{t\to\infty} e^{-t} L(z,t) = z$, locally uniformly with respect to B^n and hence (2) holds for $F(z) = z$.

Clearly $L(z,t)$ satisfies the absolute continuity requirements of Theorem 1.

From (5) we obtain

$$DL(z,t) = e^t Dg\left(e^{-t} z\right)[I - E(z,t)] \tag{5}$$

where, for all $(z,t) \in B^n \times [0, \infty)$, $E(z,t)$ is the linear operator defined by

$$\begin{aligned} E(z,t) &= e^{-2t} \left[\left(Dg\left(e^{-t} z\right)\right)^{-1} Df\left(e^{-t} z\right) - I \right] - \\ &\quad - \left(1 - e^{-2t}\right) \left(Dg\left(e^{-t} z\right)\right)^{-1} D^2 g\left(e^{-t} z\right)\left(e^{-t} z, \cdot\right). \end{aligned} \tag{6}$$

We consider

$$A\left(e^{-t}z\right) = \left(Dg\left(e^{-t}z\right)\right)^{-1} Df\left(e^{-t}z\right) - I$$

$$B\left(e^{-t}z\right) = \left(Dg\left(e^{-t}z\right)\right)^{-1} D^2g\left(e^{-t}z\right)\left(e^{-t}z,\cdot\right) \quad \text{and}$$

$$F(z,t,\lambda) = \lambda A\left(e^{-t}z\right) + (1-\lambda)B\left(e^{-t}z\right), \quad \lambda \in [0,1]$$

From (3) and (4) it results $\|A(e^{-t}z)\| < 1$ and $\|F(z,t,\lambda_z)\| < 1$, where $\lambda_z = e^{-2t}\|z\|^2, z \in B^n, t \geq 0$. Since $1 \geq e^{-2t} > \lambda_z$, for all $z \in B^n$ and $t \geq 0$ we can write $e^{-2t} = u + (1-u)\lambda_z$, where $u \in [0,1)$. Then

$$-E(z,t) = uA\left(e^{-t}z\right) + (1-u)F(z,t,\lambda_z), \quad u \in [0,1).$$

We obtain

$$\|E(z,t)\| \leq u\left\|A\left(e^{-t}z\right)\right\| + (1-u)\|F(z,t,\lambda_z)\| < 1, \quad (z,t) \in B^n \times [0,\infty)$$

and hence $I - E(z,t)$ is an invertible operator.

Further calculation shows that

$$\frac{\partial L(z,t)}{\partial t} = e^t Dg\left(e^{-t}z\right)[I + E(z,t)](z) = DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z).$$

It results that $L(z,t)$ satisfies the differential equation (1) for all $t \geq 0$ and $z \in B^n$, where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z). \tag{7}$$

We shall show that $h(z,t)$ satisfies the condition (iii), (iv) and (v) of Theorem 1. Clearly, $h(z,t)$ satisfies the holomorphy and measurability requirements, $h(0,t) = 0$ and $Dh(0,t) = I$. The inequality

$$\|h(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\| \cdot \|h(z,t) + z\| \leq \|h(z,t) + z\|$$

implies $\mathrm{Re}\langle h(z,t), z\rangle \geq 0$, for $z \in B^n$ and $t \geq 0$.

For a fixed $t \geq 0$, $E(\cdot, t)$ defined by (7) is an holomorphic function from B^n into $\mathcal{L}(\mathbb{C}^n)$, $E(0,t) = 0$ and $\|E(z,t)\| < 1, z \in B^n$.

By using Schwarz lemma for $\mathbb{C}^n$ we obtain $\|E(z,t)\| \leq \|z\|, z \in B^n$.

It follows

$$\|h(z,t)\| \leq \|z\| \frac{1 + \|z\|}{1 - \|z\|}, \quad \text{for all} \quad \in B^n.$$

The conditions of Theorem 1 being satisfied we obtain that the functions $L(z,t)$, $t \geq 0$ are univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

□

Remark. If $g = f$, then Theorem 2 becomes the n-dimensional version of Becker's univalence criterion [3].

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ON A SUBCLASS OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

TÜNDE DOMOKOS

Abstract. This work presents the class of functions, note by $P(n,\lambda,\alpha)$, which contain univalent functions with negative coefficients, satisfying:

$$Re\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\} > \alpha.$$

If $f_j(z) \in P(n,\lambda,\alpha)$, $j=\overline{1,m}$, then the convolution of theese functions, $h(z)$, lies to the class $P(n,\lambda,\beta)$, where we have β.

The author obtain the order of starlikeness of a convex function of order α, with negative coefficients. The theorems **2,3,4** and corrolaris **1,2,4,5** are original results of the author.

Let A(n) denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k,$$

$a_k \geq 0, n \in N = \{1,2,...,n\}$, which are analytic in the unite disk:

$$U = \{z \in C : |z| < 1\}.$$

The function $f(z) \in A(n)$ is said to be in the class $P(n,\lambda,\alpha)$ if it satisfies:

$$Re\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\} > \alpha,$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda \leq 1)$, and for all $z \in U$.

The classes $P(n,0,\alpha) \equiv T_\alpha(n)$ and $P(n,1,\alpha) \equiv C_\alpha(n)$ were studied by Srivastava,Owa and Chatterjea in [3], and the classes $P(1,0,\alpha) \equiv T^\star(\alpha)$ and $P(1,1,\alpha) \equiv C(\alpha)$ by Silvermann in [2].

Theorem 1 ([1]). *The function $f(z) \in A(n)$ is in the class $P(n, \lambda, \alpha)$ if and only if:*

$$\sum_{k=n+1}^{\infty} (k-\alpha)(\lambda k - \lambda + 1)a_k \leq 1 - \alpha.$$

For $\lambda = 0$ and $\lambda = 1$ we obtain two Lemmas in [3], and if $n = 1$ too, we obtain two Lemmas in [2]. We have the following theorem :

Theorem 2. *If the function $f \in C_\alpha(n)$, then $f \in P(n, \lambda, \beta)$, where:*

$$\beta = 1 - \frac{n(1-\alpha)(\alpha n + 1)}{(n+1)(n+1-\alpha) - (1-\alpha)(\lambda n + 1)}.$$

The result is sharp, the extremal function is:

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}.$$

Proof. We know that:

$$f \in C_\alpha(n) \Leftrightarrow \sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1 - \alpha.$$

and

$$f \in P(n, \lambda, \beta) \Leftrightarrow \sum_{k=n+1}^{\infty} (k-\beta)(\lambda k - \lambda + 1)a_k \leq 1 - \beta.$$

We have to find the largest β such that

$$\frac{(k-\beta)(\lambda k - \lambda + 1)}{1-\beta} \leq \frac{k(k-\alpha)}{1-\alpha}. \tag{1}$$

The inequality (1) is equivalent to

$$\beta \leq \frac{k(k-\alpha) - k(1-\alpha)(\lambda k - \lambda + 1)}{k(k-\alpha) - (1-\alpha)(\lambda k - \lambda + 1)} = 1 - \frac{(k-1)(1-\alpha)(\lambda k - \lambda + 1)}{k(k-\alpha) - (1-\alpha)(\lambda k - \lambda + 1)}.$$

We define the function $g(k)$by:

$$g(k) = 1 - \frac{(k-1)(1-\alpha)(\lambda k - \lambda + 1)}{k(k-\alpha) - (1-\alpha)(\lambda k - \lambda + 1)}.$$

Therefore $g(k) \leq g(k+1)$ we have that the function $g(k)$ is an increasing function on k, $k \geq n+1$.

Finaly we have :

$$\beta = g(n+1) = 1 - \frac{n(1-\alpha)(\lambda n + 1)}{(n+1)(n+1-\alpha) - (1-\alpha)(\lambda n + 1)},$$

which completes the proof of our theorem. □

Convolution of functions

Let the functions $f_j(z)$ be defined by :

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{j,k}\, z^k,$$

$a_{j,k} \geq 0$, $j = 1, 2, \ldots, m$. Then we define the function $h(z)$ by:

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2 + \cdots + a_{m,k}^2) z^k. \tag{2}$$

Theorem 3. *If $f_j(z) \in P(n, \lambda, \alpha)$, $j = 1, 2, \ldots, m$, then the function $h(z)$ given by (2) is in the class $P(n, \lambda, \beta)$, where:*

$$\beta = 1 - \frac{mn(1-\alpha)^2}{(n+1-\alpha)^2(\lambda n + 1) - m(1-\alpha^2)}.$$

The result is sharp, the extremal functions are:

$$f_j(z) = z - \frac{1-\alpha}{(n+1-\alpha)(\lambda n + 1)} z^{n+1}, \qquad j = 1, 2, \ldots, m.$$

Proof. By using Theorem 1 we have

$$\sum_{k=n+1}^{\infty} [\frac{(k-\alpha)(\lambda k - \lambda + 1)}{1-\alpha}]^2 a_{j,k}^2 \leq [\sum_{k=n+1}^{\infty} \frac{(k-\alpha)(\lambda k - \lambda + 1}{1-\alpha)} a_{j,k}]^2 \leq 1, \tag{3}$$

$j = 1, 2, \ldots, m$. (3) implies:

$$\frac{1}{m} \sum_{k=n+1}^{\infty} [\frac{(k-\alpha)(\lambda k - \lambda + 1)}{1-\alpha}]^2 (a_{1,k}^2 + \cdots + a_{m,k}^2) \leq 1.$$

We have to find the largest β such that:

$$\frac{(k-\beta)(\lambda k - \lambda + 1)}{(1-\beta)} \leq \frac{1}{m} \frac{(k-\alpha)^2(\lambda k - \lambda + 1)^2}{(1-\alpha)^2}. \tag{4}$$

The inequality (4) is equivalent to

$$\beta \leq \frac{(k-\alpha)^2(\lambda k - \lambda + 1) - mk(1-\alpha)^2}{(k-\alpha)^2(\lambda k - \lambda + 1) - m(1-\alpha)^2} =$$

$$= 1 - \frac{m(k-1)(1-\alpha)^2}{(k-\alpha)^2(\lambda k - \lambda + 1) - m(1-\alpha)^2}.$$

Let the function $s(k)$ be :

$$s(k) = 1 - \frac{m(k-1)(1-\alpha)^2}{(k-\alpha)^2(\lambda k - \lambda + 1) - m(1-\alpha)^2},$$

We prove that $s(k) \le s(k+1)$ for k, $k \ge n+1$, inequality wich is equivalent to

$$g(k) \ge 0,$$

where

$$g(k) = 2\lambda k^3 + (1 - \lambda - 2\alpha\lambda)k^2 + (-1 - \lambda + 2\alpha\lambda)k + (m-1)(1-\alpha)^2.$$

We have

$$g(2) = 6\lambda + 4\lambda(1-\alpha) + 2 + (m-1)(1-\alpha)^2 \ge 0.$$

By calculating the derivate of the $g(k)$, we obtain :

$$g'(k) = 6\lambda k^2 + 2(1 - \lambda - 2\alpha\lambda)k - 1 - 1 + 2\alpha\lambda.$$

We also have :

$$g'(2) = 13\lambda + 6\lambda(1-\alpha) + 3 > 0 \qquad (5)$$

$$g''(k) = 12\lambda k + 2(1 - \lambda - 2\alpha\lambda) \qquad (6)$$

$$g''(2) = 18\lambda + 4\lambda(1-\alpha) + 2 > 0 \qquad (7)$$

$$g'''(k) = 12\lambda > 0, \; for \; 0 < \lambda \le 1 \qquad (8)$$

For $\lambda = 0$ we have $g(k) = k(k-1) + (m-1)(1-\alpha)^2 \ge 0$

So that (8) implies that the function $g''(k)$ is an increasing function on k, and by using (7) we have $g''(k) > 0$. This implies that the function $g'(k)$ is increasing on k. Using (5) we have $g'(k) > 0$ so that the function $g(k)$ is increasing on k. But $g(2) \ge 0$ so $g(k) \ge 0$ for $k \ge n+1$.

Therefore $s(k) \le s(k+1)$, the function $s(k)$ is an increasing function in k , $k \ge n+1$, and this implies that :

$$\beta \le s(n+1) = 1 - \frac{mn(1-\alpha)^2}{(n+1-\alpha)^2(\lambda n + 1) - m(1-\alpha)^2}.$$

For the functions :

$$f_j(z) = z - \frac{1-\alpha}{(n+1-\alpha)(\lambda n+1)} z^{n+1}, \qquad j = 1, 2, \ldots, m,$$

the result is sharp. □

Corollary 1. *If $f_j(z) \in P(n, \lambda, \alpha)$, $j = 1, 2$, then the function :*

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2) z^k$$

is in the class $P(n, \lambda, \beta)$, where:

$$\beta = 1 - \frac{2n(1-\alpha)^2}{(n+1-\alpha)^2)(\lambda n+1) - 2(1-\alpha)^2}.$$

The result is sharp for the functions :

$$f_1(z) = f_2(z) = z - \frac{1-\alpha}{(n+1-\alpha)(\lambda n+1)} z^{n+1}.$$

Corollary 2. *Let $f_j(z) \in T_\alpha(n)$, $j = 1, 2, \ldots, m$. Then the function $h(z)$ given by (2) is in the class $T_\beta(n)$, where*

$$\beta = 1 - \frac{mn(1-\alpha)^2}{(n+1-\alpha)^2 - m(1-\alpha)^2}.$$

The result is sharp, the extremal functions are :

$$f_j(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1} \qquad j = 1, 2, \ldots, m.$$

Corollary 3. *Let $f_j(z) \in C_\alpha(n), j = 1, 2, \ldots, m$. Then the function $h(z)$ given by (2) lies to the class $C_\beta(n)$, where:*

$$\beta = 1 - \frac{mn(1-\alpha)^2}{(n+1)(n+1-\alpha)^2 - m(1-\alpha)^2}.$$

The result is sharp for the functions :

$$f_j(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1} \qquad j = 1, 2, \ldots, m.$$

The order of starlikeness of a convex function of order α from the class $A(n)$

We know that the class $P(n,1,\alpha) \equiv C_\alpha(n)$ contain convex functions of order α, with :

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in U,$$

and the class $P(n,0,\beta) \equiv T_\beta(n)$ contain starlike functions of order β, with :

$$Re\frac{zf'(z)}{f(z)} > \beta, \qquad z \in U.$$

Theorem 4. *If $f \in C_\alpha(n)$, then $f \in T_\beta(n)$, where :*

$$\beta = \frac{n(n+1)}{(n+1)(n+1-\alpha)-(1-\alpha)}.$$

The result is sharp for the function :

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)}z^{n+1}.$$

Proof. Using the Theorem 1. for $\lambda = 1$ we have:

$$\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \le 1-\alpha. \tag{9}$$

From the Theorem 1. for $\lambda = 0$ we have:

$$f \in T_\beta(n) \Leftrightarrow \sum_{k=n+1}^{\infty} (k-\beta)a_k \le 1-\beta. \tag{10}$$

We have to find the largest β such that:

$$\frac{k-\beta}{1-\beta} \le \frac{k(k-\alpha)}{1-\alpha}. \tag{11}$$

The inequality (11) is equivalent to:

$$\beta \le \frac{k(k-1)}{k(k-\alpha)-(1-\alpha)}.$$

Let the function $g(k)$ be:

$$g(k) = \frac{k(k-1)}{k(k-\alpha)-(1-\alpha)}.$$

Therefore $g'(k) \geq 0$ for k, $k \geq n+1$, the function $g(k)$ is an increasing function on k, $k \geq n+1$, we have :

$$\beta \leq g(n+1) = \frac{n(n+1)}{(n+1)(n+1-\alpha)-(1-\alpha)},$$

which completes the proof of our theorem.

The inequality in (9) and (10) are attained for the function:

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}.$$

□

Corollary 4. *For $\alpha = 0$ we obtain $\beta = \frac{n+1}{n+2}$. Thus a convex function from class $A(n)$ is starlike of order $\beta = \frac{n+1}{n+2}$.*

Corollary 5. *For $n = 1$ we have $\beta = \frac{2}{3-\alpha}$. If $\alpha = 0$, then we have $\beta = \frac{2}{3}$, so a convex function of the form:*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

is starlike of order $\frac{2}{3}$, and $\frac{2}{3} > \frac{1}{2}$.

We know, that in case of the functions of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

not necessary with negative coefficients, the theorem of Marx and Strohhacker tell us that a convex function is starlike of order $\frac{1}{2}$.

The same theorem, for $n = 2$, tell us that a convex function of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k,$$

is starlike of order $\frac{2}{\pi}$.

From Theorem 4., for $n = 2$ we have $\beta = \frac{3}{4-\alpha}$, and if $\alpha = 0$, we obtain $\beta = \frac{3}{4}$.

Finally, a convex function of the form :

$$f(z) = z - \sum_{k=3}^{\infty} a_k z^k$$

is starlike of order $\frac{3}{4}$, and $\frac{3}{4} > \frac{2}{\pi}$.

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COUNTERPARTS OF ARITHMETIC MEAN-GEOMETRIC MEAN-HARMONIC MEAN INEQUALITY

S.S. DRAGOMIR

Abstract. Some converse inequalities for the celebrated arithmetic mean-geometric mean-harmonic mean inequality are given.

1. Introduction

Recall the means

1) *weighted arithmetic mean* $A_n(\mathbf{w}, \mathbf{a})$,

$$A_n(\mathbf{w}, \mathbf{a}) := \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i;$$

2) *weighted geometric mean* $G_n(\mathbf{w}, \mathbf{a})$,

$$G_n(\mathbf{w}, \mathbf{a}) := \left(\prod_{i=1}^{n} a_i^{w_i} \right)^{\frac{1}{W_n}}$$

and

3) *weighted harmonic mean* $H_n(\mathbf{w}, \mathbf{a})$,

$$H_n(\mathbf{w}, \mathbf{a}) := \frac{W_n}{\sum_{i=1}^{n} \frac{w_i}{a_i}}$$

where

$$\mathbf{a} = (a_1, ..., a_n), \mathbf{w} = (w_1, ..., w_n), a_i, w_i > 0 \, (i = 1, ..., n)$$

and $W_n := \sum_{i=1}^{n} w_i$.

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The following inequality is well known in the literature as *arithmetic mean - geometric mean - harmonic mean* inequality

$$A_n(\mathbf{w},\mathbf{a}) \geq G_n(\mathbf{w},\mathbf{a}) \geq H_n(\mathbf{w},\mathbf{a}). \tag{1.1}$$

The equality holds in (1.1) if and only if $a_1 = ... = a_n$. Note that (1.1) is equivalent to

$$1 \leq \frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})}, \quad 1 \leq \frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}. \tag{1.2}$$

The main aim of this note in to point out upper bounds for the quotients

$$\frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})}, \quad \frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}.$$

2. The Results

In the recent paper [1], Dragomir and Goh, by the use of an inequality for convex functions, have proved the following analytic inequality for the logarithmic mapping.

Lemma 1. *Let $\xi, p_i > 0$ $(i = 1, ..., n)$ where $\sum_{i=1}^n p_i = 1$. Then*

$$0 \leq \sum_{i=1}^n p_i \ln \xi_i - \ln\left(\sum_{i=1}^n p_i \xi_i\right) \tag{2.1}$$

$$\leq \sum_{i=1}^n \frac{p_i}{\xi_i} \sum_{i=1}^n p_i \xi_i - 1 = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \frac{\left(\xi_i - \xi_j\right)^2}{\xi_i \xi_j}$$

$$= \sum_{1 \leq i < j \leq n} p_i p_j \frac{\left(\xi_i - \xi_j\right)^2}{\xi_i \xi_j}.$$

The equalities hold iff $\xi_1 = ... = \xi_n$

Using this result, we can state the following theorem containing a converse of A.-G.-H. inequalities.

Theorem 2. *Let* $\mathbf{w}, \mathbf{a}$ *be as in Introduction. Then*

$$1 \le \frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})} \le \exp\left[\frac{A_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})} - 1\right] \tag{2.2}$$

$$= \exp\left[\frac{1}{2W_n^2}\sum_{i,j=1}^{n} w_i w_j \frac{(a_i - a_j)^2}{a_i a_j}\right]$$

$$= \exp\left[\frac{1}{W_n^2}\sum_{1\le i<j\le n} w_i w_j \frac{(a_i - a_j)^2}{a_i a_j}\right] =: B_n(\mathbf{w},\mathbf{a})$$

and

$$1 \le \frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})} \le B_n(\mathbf{w},\mathbf{a}). \tag{2.3}$$

The equalities hold in both inequalities iff $a_1 = \ldots = a_n$.

Proof. The proof of (2.2) follow by (2.1) choosing $p_i = \frac{w_i}{W_n}$ and $\xi_i = a_i$ $(i = 1, \ldots, n)$.

The proof of (3) follows by (2.2) choosing $\frac{1}{\mathbf{a}}$ instead of $\mathbf{a}$ and taking into account that $B_n\left(\mathbf{w}, \frac{1}{\mathbf{a}}\right) = B_n(\mathbf{w},\mathbf{a})$. □

We point out another results which does not use the concavity property of log -mapping, but an inequality between the geometric and logarithmic mean of two positive numbers.

Theorem 3. *Let* $\mathbf{w}, \mathbf{a}$ *be as in Introduction. Then*

$$1 \le \frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})} \le \exp\left[\frac{1}{[A_n(\mathbf{w},\mathbf{a})]^{1/2}} \cdot \frac{1}{W_n}\sum_{i=1}^{n} \frac{w_i}{\sqrt{a_i}} |A_n(\mathbf{w},\mathbf{a}) - a_i|\right]$$

$$\le \exp\left[\frac{\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i [A_n(\mathbf{w},\mathbf{a}) - a_i]^2\right]^{1/2}}{[A_n(\mathbf{w},\mathbf{a}) H_n(\mathbf{w},\mathbf{a})]^{1/2}}\right]. \tag{2.4}$$

The equality holds iff $a_1 = \ldots = a_n$.

Proof. We recall the following well known inequality between the *geometric mean* $G(a,b) := \sqrt{ab}\,(a,b>0)$ and the *logarithmic mean* (see for example [2, p. 346])

$$L(a,b) := \begin{cases} a \text{ if } b=a \\ \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a \end{cases} \quad (a,b>0)$$

i.e.,

$$G(a,b) \le L(a,b) \text{ for all } a,b>0. \tag{2.5}$$

Note that (2.5) is equivalent to

$$|\ln b - \ln a| \le \frac{|b-a|}{\sqrt{ab}}, a,b>0. \tag{2.6}$$

The equality holds in (2.6) iff $a=b$.

Now, choose in (2.6)

$$b := A_n(\mathbf{w},\mathbf{a}), a = a_i \; (i=1,...,n)$$

to get

$$|\ln A_n(\mathbf{w},\mathbf{a}) - \ln a_i| \le \frac{|A_n(\mathbf{w},\mathbf{a}) - a_i|}{\sqrt{A_n(\mathbf{w},\mathbf{a})\, a_i}} \tag{2.7}$$

for all $i \in \{1,...,n\}$.

Multiplying by $w_i > 0$ and summing over $i \in \{1,...,n\}$, we deduce

$$\left| W_n \ln A_n(\mathbf{w},\mathbf{a}) - \sum_{i=1}^{n} w_i \ln a_i \right|$$

$$\le \sum_{i=1}^{n} w_i |\ln A_n(\mathbf{w},\mathbf{a}) - \ln a_i|$$

$$\le \sum_{i=1}^{n} w_i \frac{|A_n(\mathbf{w},\mathbf{a}) - a_i|}{\sqrt{A_n(\mathbf{w},\mathbf{a})\, a_i}}$$

$$= \frac{1}{[A_n(\mathbf{w},\mathbf{a})]^{1/2}} \sum_{i=1}^{n} \frac{w_i}{\sqrt{a_i}} |A_n(\mathbf{w},\mathbf{a}) - a_i|$$

from where results the first inequality in (4).

Using the Cauchy-Buniakowski-Schwarz's discrete inequality, we get

$$\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{\sqrt{a_i}}\left|A_n(\mathbf{w},\mathbf{a})-a_i\right|$$

$$\leq\left[\frac{1}{W_n}\sum_{i=1}^{n}w_i\left[A_n(\mathbf{w},\mathbf{a})-a_i\right]^2\right]^{1/2}\left(\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{a_i}\right)^{1/2}$$

$$=\frac{\left[\frac{1}{W_n}\sum_{i=1}^{n}w_i\left[A_n(\mathbf{w},\mathbf{a})-a_i\right]^2\right]^{1/2}}{\left[H_n(\mathbf{w},\mathbf{a})\right]^{1/2}}$$

and the second inequality in (4) also holds.

The case of equality is obvious. □

The following corollary which provides an upper bound for the quotient

$$\frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}$$

holds.

Corollary 4. *Let* $\mathbf{w},\mathbf{a}$ *be as in Introduction. Then*

$$1\leq\frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}\leq\exp\left[\frac{1}{\left[H_n(\mathbf{w},\mathbf{a})\right]^{1/2}}\cdot\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{\sqrt{a_i}}\left|H_n(\mathbf{w},\mathbf{a})-a_i\right|\right]$$

$$\leq\exp\left[\left[\frac{A_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}\right]^{1/2}\cdot\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{a_i^2}\left[H_n(\mathbf{w},\mathbf{a})-a_i\right]^2\right].$$

The equality holds iff $a_1=\ldots=a_n$.

The proof follows by the inequality (4) putting $\frac{1}{\mathbf{a}}$ instead of $\mathbf{a}$ and taking into account that

$$A_n\left(\mathbf{w},\frac{1}{\mathbf{a}}\right)=H^{-1}(\mathbf{w},\mathbf{a}).$$

We omit the details.

For an extensive literature on weighted means and their inequalities, the author recommends the monograph [2].

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NON-ANALYTIC n-STARLIKE AND n-SPIRALLIKE FUNCTIONS

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Abstract. In this paper properties of geometric kind for non-analytic n-starlike and n-spirallike functions, $n \in \mathbb{N} \cup \{0\}$, are obtained. They are extensions of some results in the case of non-analytic (usual) starlike and convex functions proved in [3] and in the case of analytic n-starlike functions in [2].

1. Introduction

A well-known method in complex analysis by which can be introduced new classes of functions is by using differential inequalities of the form

$$F(f, D(f)(z), \ldots, D^n(f)(z)) > 0, z \in U = \{z \in \mathbb{C};\ |z| < 1\},$$

where $D(f)(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}$, $D^n(f)(z) = D[D^{n-1}(f)](z)$ (if f is analytic then $D(f)(z) = zf'(z)$).

In this sense let us mention the following two examples:

(i) the class of analytic n-starlike ($n \in \mathbb{N} \cup \{0\}$) functions on U, introduced in [6];

(ii) the class of analytic logarithmically n-starlike functions on U, introduced in [2].

In Section 2 we extend to non-analytic n-starlike functions some properties of non-analytic usual starlike and convex functions in [3].

Section 3 is concerned with the case of non-analytic n-spirallike functions.

2. Non-analytic n-starlike functions

We say that $f : U \to \mathbb{C}$ is in $\mathbb{C}^n(U)$, $n \in \mathbb{N}$ fixed, if it is continuous and has continuous all partial derivatives of order n with respect $x = \text{Re}z$ and $y = \text{Im}z$. For

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$f \in C^{n+1}(U)$, $n \in \mathbb{N} \cup \{0\}$, we consider the operator (see [3])

$$D(f)(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}},$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right], \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right], \quad i = \sqrt{-1},$$

and the iterates $D^{n+1}(f) = D[D^n(f)]$, $D^0(f) \equiv f$.

Definition 2.1. Let $f \in C^{n+1}(U)$. We say that f is n-starlike in U if $f(0) = 0$, f is univalent on U, $D^n(f)(z) \neq 0$, $\forall\ z \in U \setminus \{0\}$ and

$$\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} > 0, \ \forall\ z \in U \setminus \{0\}. \tag{1}$$

Remarks.

1) For $n = 0$ and $n = 1$, the condition (1) means the geometric condition of starlikeness and of convexity in the case of non-analytic functions, respectively, considered for the first time in [3].
2) If f is analytic then $D(f)(z) = zf'(z)$ and the classes of functions satisfying (1) were considered for the first time in [6].

The following result can be considered, in a certain sense, a property of geometric kind of the left hand-side in (1).

Lemma 2.2. *For fixed $r \in (0,1)$, let us denote $C_r = \partial U_r$, $U_r = \{z \in U;\ |z| < r\}$, $\gamma_r(\theta) = f(\partial U_r)$, $z = re^{i\theta}$, $\theta \in [0, 2\pi)$. If $f \in C^{n+1}(U)$ and $D^n(f)(z) \neq 0$ for all $z \in U \setminus \{0\}$, then*

$$\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} = \frac{\partial}{\partial \theta}[\arg \gamma_r^{(n)}], \quad z = re^{i\theta}, \quad \theta \in [0, 2\pi).$$

Proof. Let $f = A + iB$. We have

$$D(f)(z) = x\frac{\partial B}{\partial y} - y\frac{\partial B}{\partial x} + i\left[y\frac{\partial A}{\partial x} - x\frac{\partial A}{\partial y}\right],$$

$$z = x + iy = r[\cos\theta + i\sin\theta],$$

$$\gamma_r'(\theta) = \frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial \theta} + i\left[\frac{\partial B}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial B}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right] =$$

$$= x\frac{\partial A}{\partial y} - y\frac{\partial A}{\partial x} + i\left[x\frac{\partial B}{\partial y} - y\frac{\partial B}{\partial x}\right] = iD(f)(z).$$

By the general formula $\frac{\partial g}{\partial \theta} = iD(g)$ (see [3]), we get

$$\gamma_r''(\theta) = \frac{\partial}{\partial \theta}[iD(f)(z)] = i^2 D^2(f)(z) = -D^2(f)(z),$$

$$\gamma_r'''(\theta) = \frac{\partial}{\partial \theta}[-D^2(f)(z)] = -iD^3(f)(z),$$

$$\gamma_r^{(4)}(\theta) = \frac{\partial}{\partial \theta}[-iD^3(f)(z)] = D^4(f)(z),$$

and finally,

$$\frac{\partial}{\partial \theta}[\arg \gamma_r^{(n)}] = \frac{\partial}{\partial \theta}[\arg(cD^n(f)(z)] = (\text{see [3]}) = \mathrm{Re}\frac{D[cD^n(f)(z)}{cD^n(f)(z)} = \mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)},$$

where $c \in \{-1, +1, i, -i\}$, which proves the theorem.

In the analytic case, in [6] was proved that the condition (1) and $f(0) = f'(0) - 1 = 0$, imply the univalence of f and that as function of n, the classes of n-starlike functions form a decreasing sequence (in respect with the inclusion).

In the non-analytic case, as was pointed out in [3] for $n = 0$ and $n = 1$, these conditions do not imply the univalence of f and additional conditions are required for that. □

In this order of ideas, concerning the classes introduced by Definition 2.1, we have the followiong extension to non-analytic case of Corollary 2.1 in [6].

Theorem 2.3. *Let $m, n \in \mathbb{N} \cup \{0\}$, $m < n$. If $f \in C^{n+1}(U)$ satisfies:*

$$f(0) = 0, \quad f(z)\prod_{i=1}^{n} D^i(f)(z) \neq 0, \quad z \in U \setminus \{0\}, \quad J(f)(z) > 0, \quad z \in U$$

(here $J(f)$ means the Jacobian of f), then the condition $\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} > 0$, $\forall\ z \in U \setminus \{0\}$, implies $\mathrm{Re}\frac{D^{m+1}(f)(z)}{D^m(f)(z)} > 0$, $\forall\ z \in U \setminus \{0\}$ and the fact that f is univalent on U.

Proof. A direct consequence of the proofs of Theorem 1 and of Lemmas 1, 2, 3 in [4], is the following:

if $F \in C^2(U)$, $F(0) = 0$, $J(F)(0) > 0$, $F(z)D(F)(z) \neq 0$, $\forall\ z \in U \setminus \{0\}$ and $\mathrm{Re}\frac{D^2(F)(z)}{D(F)(z)} > 0$, $\forall\ z \in U \setminus \{0\}$, then $\mathrm{Re}\frac{D(F)(z)}{F(z)} > 0$, $\forall z \in U \setminus \{0\}$. Also, we have $f(0) = D(f)(0) = \cdots = D^n(f)(0) = 0$.

On the other hand, we have $J[D(f)](0) = J(f)(0)$, $J[D^k(f)](0) = [J(f)(0)] > 0$, $k = \overline{1, n}$.

Applying the above result for $F = D^{n-1}(f)$, $F = D^{n-2}(f), \ldots, F = f$, we easily obtain $\mathrm{Re} \dfrac{D^{m+1}(f)(z)}{D^m(f)(z)} > 0$, $\forall\, z \in U \setminus \{0\}$, for any $m \in \{0, 1, \ldots, n-1\}$.

Taking $m = 0$ and applying Theorem 1 in [3] it follows that f is univalent on U, which proves the theorem. □

3. Non-analytic n-spirallike functions

Keeping the notations in Section 2, we introduce the following.

Definition 3.1. Let $f \in C^{n+1}(U)$, $n \in \mathbb{N} \cup \{0\}$. We say that f is logarithmically n-spirallike of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $f(0) = 0$, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$, f is univalent on U and

$$\mathrm{Re}\left[e^{i\gamma} \frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0, \ \forall\, z \in U \setminus \{0\}. \tag{2}$$

We say that f is Archimedean n-spirallike on U if $f(0) = 0$, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$, f is univalent on U and

$$\mathrm{Re}\left[(1 - i|D^n(f)(z)|) \frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0, \ \forall\, z \in U \setminus \{0\}. \tag{3}$$

We say that f is hyperbolic n-spirallike on U if $f(0) = 0$, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$, f is univalent on U and

$$\mathrm{Re}\left[(|D^n(f)(z)| + i) \frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0, \ \forall\, z \in U \setminus \{0\}. \tag{4}$$

Remarks.

1) Taking $n = 0$ in Definition 3.1, we obtain the classes of usual non-analytic spirallike functions studied in [1].

2) For $n = 1$ and f analytic on U, the class of functions defined by (2) was for the first time considered in [5]. In this case, the situation is different from the starlike case, because as was pointed out in [5], even for analytic functions f, the conditions (2), $f(0) = 0$, $f'(z) \neq 0$, $z \in U$, do not imply in general the univalence of f. However, in e.g. the paper [5], was proved that for $0 < \cos\gamma \leq 0.2315$, these above conditions imply the univalence of f.

3) The analytic logarithmically n-spirallike functions were considered in [2].
4) For $\gamma = 0$ in (2) we obtain the classes of n-starlike functions introduced by Definition 2.1.

The following result can be considered in a certain sense, of geometric kind for the relations (2), (3), (4), respectively.

Theorem 3.2. *For fixed $r \in (0,1)$, let us denote $C_r = \partial U_r$, $U_r = \{z \in \mathbb{C};\ |z| < r\}$, $\gamma_r(\theta) = f(\partial U_r)$, $z = re^{i\theta}$, $\theta \in [0, 2\pi)$. Let $f \in C^{n+1}(U)$ be with $D^n(f)(z) \neq 0$, $\forall\ z \in U \setminus \{0\}$, $D(f)(z) = z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}$.*

(i) Let us consider the family of logarithmically spirals

$$w_\phi(t) = e^{t\cos\gamma}e^{i(\phi - t\sin\gamma)}, \quad t \in (-\infty, +\infty),$$

where $\phi \in [0, 2\pi)$ is a parameter. Then (2) is equivalent with

$$\frac{\partial \phi}{\partial \theta} > 0,\ \forall\ \theta \in [0, 2\pi), \tag{5}$$

where $\phi = \phi(\theta, r)$ is the solution of the equation

$$w_\phi(t) = \gamma_r^{(n)}(\theta) \tag{6}$$

(ii) Let us consider the family of Archimedean spirals

$$w_\phi(t) = te^{i(t+\phi)}, \quad t \in (0, +\infty),$$

where $\phi \in [0, 2\pi)$ is a parameter. Then (3) is equivalent with (5), where ϕ is given by (6).

(iii) Let us consider the family of hyperbolic spirals

$$w_\phi(t) = e^{i(t+\phi)}/t, \quad t \in (0, +\infty), \quad \phi \in [0, 2\pi).$$

Then (4) is equivalent with (5), where ϕ is given by (6).

Proof. From the proof of Lemma 2.2 we have

$$\gamma_r^{(n)}(\theta) = cD^n(f)(z), \quad c \in \{-1, +1, i, -i\}.$$

(i) By (6) we get (as in the proof of Theorem 1 in [1])

$$t\cos\gamma = \log|cD^n(f)(z)|$$

$$\phi - t\sin\gamma = \arg(cD^n(f)(z)),$$

and consequently

$$\phi = \arg[cD^n(f)(z)] + \operatorname{tg}\gamma \log|cD^n(f)(z)|,$$

$$\frac{\partial\phi}{\partial\theta} = \frac{1}{\cos\gamma}\mathrm{Re}\left[e^{i\gamma}\frac{cD^{n+1}(f)(z)}{cD^n(f)(z)}\right] = \frac{1}{\cos\gamma}\mathrm{Re}\left[e^{i\gamma}\frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right].$$

(ii) Replacing in the statement (iii) of Theorem 2 in [1] $f(z)$ by $cD^n(f)(z)$, we obtain (5).

(iii) Replace $f(z)$ by $cD^n(f)(z)$ in the statement (iii) of Theorem 3 in [1].

The theorem is proved. □

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ANALYTIC INVARIANTS AND THE RESOLUTION GRAPHS OF THE SINGULARITIES OF THE TYPE *ADE*

ALEXANDRU HORVÁTH

Abstract. In this note we show that the equality of the maximal ideal cycle and the Artin's fundamental cycle for the complex surface singularities of type ADE can be proved directly, using the resolution graphs.

Let (X,x) be a normal surface singularity, and take a good resolution $\phi : \mathcal{Y} \to X$. The combinatorics/topology of ϕ is codified in the dual resolution graph Γ_ϕ. Using the plumbing construction, it is proved, that the information codified in Γ_ϕ is the same as the information codified in the link L_X. In particular Γ_ϕ is completely equivalent to the topology of (X,x).•

We would like to codify numerically some information about the ring of holomorphic functions on (X,x) (actually about the maximal ideal $m_{X,x}$). Therefore, take an arbitrary holomorphic function $f : (X,x) \to (\mathbf{C}, 0)$ (i.e. $f \in m_{X,x}$). Then we take the composed map $f \circ \phi : \mathcal{Y} \to \mathbf{C}$, and denote by $(f \circ \phi)$ its divisor on $\mathcal{Y}$. Recall that $(f \circ \phi)$ is the set of zeros of $f \circ \phi$ with their natural multiplicities.

$$(f \circ \phi) = \sum_i m_i E_i + \sum_j m(\mathrm{St}_j)\mathrm{St}_j .$$

The part of this sum supported by E is $\sum_i m_i E_i$ — where m_i is the vanishing order of $f \circ \phi$ along E_i — and is denoted by $(f \circ \phi)_\Gamma$, while the strict transform $\mathrm{St}(f)$ is $\sum_j m(\mathrm{St}_j)\mathrm{St}_j$.

Therefore, by construction, for any $f \in m_{X,x}$ we get a cycle $(f \circ \phi)_\Gamma = \sum_i m_i(f) E_i$ supported by E.

Definition 1. The set of all these cycles is denoted by

$$\mathcal{Z}_{\mathrm{an}}(\phi) = \{(f \circ \phi)_\Gamma : f \in m_{X,x}\},$$

and is called the set of *analytic cycles.*

Recall, that in the set of cycles we have an ordering. For $Z' = \sum n'_i E_i$ and $Z'' = \sum n''_i E_i$ we write $Z' \leq Z''$ if and only if $n'_i \leq n''_i$ for all i. With these notations obviously $\min(Z', Z'') = \sum_i \min(n'_i, n''_i) \cdot E_i$.

Lemma 2. *If $Z_1, Z_2 \in \mathcal{Z}_{\text{an}}(\phi)$, then*

(1) $Z_1 + Z_2 \in \mathcal{Z}_{\text{an}}(\phi)$,

(2) $\min(Z_1, Z_2) \in \mathcal{Z}_{\text{an}}(\phi)$.

Proof. The proof is easy. For the point (1) just take $f_1 \cdot f_2$. For the point (2), take a generic linear combination $\lambda_1 f_1 + \lambda_2 f_2$. □

The above lemma assures that $\mathcal{Z}(\phi)$ has a unique minimal element with respect to the above ordering. This cycle is denoted by Z_{max}, by S. S.-T. Yau, who introduced it and called it *maximal ideal cycle*; (also it is denoted by Z_f, i.e. *fiber cycle*, by Miles Reid [8]).

Lemma 2 shows that if the linear term of f is sufficiently generic, then $(f \circ \phi) = Z_{\text{max}}$. Actually for any embedding $(X, x) \subset (\mathbf{C}^N, 0)$, a sufficiently general linear function $l : (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$ induces an $l|_X : (X, x) \to (\mathbf{C}, 0)$ with $(l|_X \circ \phi)_\Gamma = Z_{\text{max}}$.

The main goal is to find — up to the extent which is possible — the lattice $\mathcal{Z}_{\text{an}}(\phi)$ from the topology of (X, x), i.e. only from the graph Γ_ϕ.

We recall the most important property of the cycles $(f \circ \phi)_\Gamma$, $(f \in m_{X,x})$. This is, that $(f \circ \phi)_\Gamma \cdot E_k \leq 0$ for any k. This is a consequence of the fact that $(f \circ \phi) \cdot E_k = 0$ and $\text{St}(f) \cdot E_k \geq 0$ for any k.

Notice also that any cycle $Z = (f \circ \phi)_\Gamma$ is a *positive* cycle, i.e. $Z = \sum n_i E_i$ with $n_i \geq 0$ for any i, and $Z \neq 0$, which we will denote by $Z > 0$.

The topological analog (candidate) for $\mathcal{Z}_{\text{an}}(\phi)$ is

$$\mathcal{Z}_{\text{top}}(\phi) = \{Z \text{ is positive cycle } | Z \cdot E_k \leq 0, \text{ for all } k\}.$$

Lemma 3. *(1) If $Z_1, Z_2 \in \mathcal{Z}_{\text{top}}(\phi)$, then $Z_1 + Z_2 \in \mathcal{Z}_{\text{top}}(\phi)$,*

(2) If $Z_1, Z_2 \in \mathcal{Z}_{\text{top}}(\phi)$, then $\min(Z_1, Z_2) \in \mathcal{Z}_{\text{top}}(\phi)$,

(3) If $Z \in \mathcal{Z}_{\text{top}}(\phi)$, $Z = \sum_i n_i E_i$, then $n_i > 0$ for all i.

For the proof, see [2].

The above lemma shows that in $\mathcal{Z}_{\text{top}}(\phi)$ there is a unique minimal cycle, which has only strict positive coefficients. This cycle is denoted by $Z_{\min}$ and is called *minimal cycle* (or *numerical cycle* or *Artin's fundamental cycle*; it was introduced by Artin [1]).

Notice that $\mathcal{Z}_{\text{top}}(\phi)$ and $Z_{\min}$ is completely described by the graph Γ_ϕ. They are the topological candidates for the set $\mathcal{Z}_{\text{an}}(\phi)$ and the cycle $Z_{\max}$.

Obviously, we have also

$$\begin{cases} \mathcal{Z}_{\text{an}}(\phi) & \subseteq \quad \mathcal{Z}_{\text{top}}(\phi), \\ Z_{\min}(\phi) & \leq \quad Z_{\max}(\phi). \end{cases}$$

FACT: In general $Z_{\min}(\phi) \neq Z_{\max}(\phi)$. However for the singularities of type *ADE* these two cycles agree.

The point is, that for this type of singularities the fact that $Z_{\max} = Z_{\min}$ can be verified using the algorithm described in the chapter 6 in [2]. This algorithm constructs the resolution graph of a surface singularity of type $(X,x) = (\{f(x,y) - z^n = 0\}, 0)$, where $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is an isolated plane curve singularity.

Let us show this for the singularities of type A_n.

Proposition 4. *(The case A_n, n odd) Take $f(x,y) = x^{n+1} + y^2$, and (X,x) given by $(\{f(x,y) - z^2 = 0\}, 0)$. Suppose n is odd, $n+1 = 2l$. Then $Z_{\min}(\phi) = Z_{\max}(\phi)$.*

Proof. The embedded resolution graph of f is shown in the figure 1. This is actually the good embedded resolution graph of z. The algorithm described in the chapter

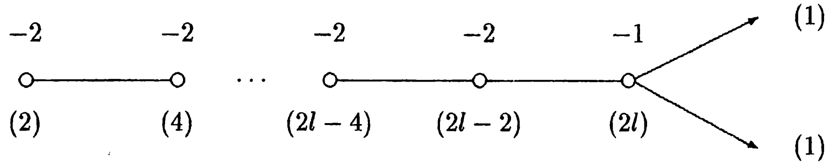

FIGURE 1. The embedded resolution graph of A_n, n odd

6 [2] gives the multiplicities of z, but we need the multiplicities of x, because this corresponds to the generic linear section on $(X, 0)$.

If we follow during the process of blowing up the strict transform of x, we can represent it as the arrow decorated by $(*)$ in figure 2. Notice that we can retain the Euler numbers from the previous graph.

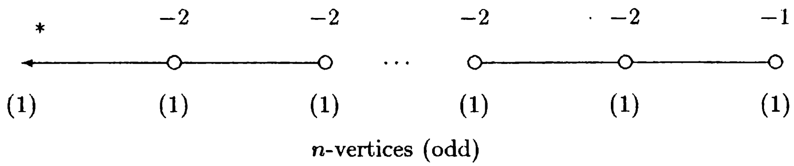

FIGURE 2. The strict transform of x, A_n, n odd

If we apply now the algorithm of chapter 6 [2], we obtain the embedded resolution graph of x, as is shown in the figure 3.

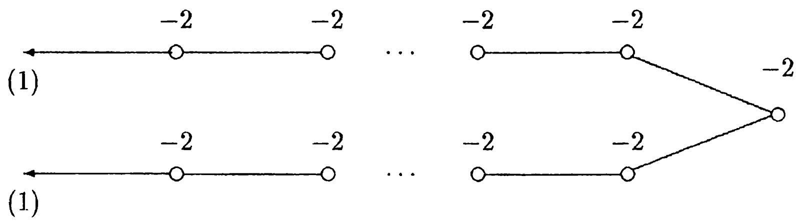

FIGURE 3. The embedded resolution graph of x in the case A_n, n odd

Now, we have the graph of x (see 4), the arrows with multiplicities (1), the Euler numbers, but not all the multiplicities of x in the vertices of the graph.

The trick is, that these multiplicities are uniquely determined by the equations

$$m_w e_w + \sum_{v \in \mathcal{V}_w} m_v = 0, \text{ for every vertex } w,$$

which is a system of linear equations, with nonsingular matrix, hence it has a unique solution. Since the multiplicities $m_v = 1$, for all v, already satisfy the system, they form actually its solution.

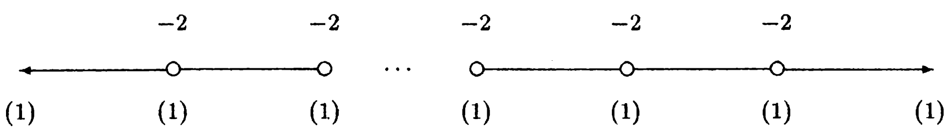

FIGURE 4. The graph of $x : (\{x^{2l} + y^2 + z^2 = 0\}, 0) \to (\mathbf{C}, 0)$

Obviously the cycle given by this set of multiplicities, $Z = E_1 + E_2 + \cdots + E_n$ has the minimal coefficients, and is given by a function (x), hence $Z_{\min} = Z_{\max}$.

The case n even is similar. □

Remark 5. The singularity D_n can be treated analogously.

Remark 6. For the singularities of type E_6, E_7 and E_8, we have to apply in addvance the Laufer's algorithm [3], to get the minimal topological cycle. This is because the multiplicities are not the absolutely minimal one. For all these cases the general linear section represented by the coordinate function which appears on the highest power in the equation of f it turns out to give the minimal cycle given by the Laufer's algorithm.

The computational details are similar for these cases, too. See also [2].

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THE MARKOV PROPERTY FOR THE SOLUTION OF THE STOCHASTIC NAVIER-STOKES EQUATION

HANNELORE LISEI

Abstract. We consider the stochastic Navier-Stokes equation of Navier-Stokes type containing a noise part given by a stochastic integral with respect to a Wiener process. The purpose of this paper is to prove that the solution of this nonlinear equation is a Markov process. We take into consideration the properties of the Galerkin approximations.

1. Introduction

The stochastic Navier-Stokes equation has important physical and technical applications. It describes the behavior of a viscous velocity field of an incompressible liquid. The equation on the domain of flow $G \subset \mathbb{R}^n$ ($n \geq 2$ a natural number) is given by

$$\begin{cases} \dfrac{\partial U}{\partial t} - \nu \Delta U = -(U, \nabla)U + f - \nabla p + \mathcal{C}(U)\dfrac{\partial w}{\partial t} \\ \text{div } U = 0, \quad U(0,x) = U_0(x), \quad U(t,x)\mid_{\partial G} = 0,\ t > 0,\ x \in G, \end{cases} \tag{1}$$

where U is the velocity field, ν is the viscosity, Δ is the Laplacian, ∇ is the gradient, f is an external force, p is the pressure, and U_0 is the initial condition. Realistic models for flows should contain a random noise part, because external perturbations and the internal Browninan motion influence the velocity field. For this reason equation (1) contains a random noise part $\mathcal{C}(U)\dfrac{\partial w}{\partial t}$. Here the noise is defined as the distributional derivative of a Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$, whose intensity depends on the state U.

Throughout this paper we consider strong solutions ("strong" in the sense of stochastic analysis) of a stochastic equation of Navier-Stokes type (we will call it a

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stochastic Navier-Stokes equation) and define the equation in the generalized sense as an evolution equation, assuming that the stochastic processes are defined on a given complete probability space and the Wiener process is given in advance.

An important property in the study of the solutions of stochastic differential equations is the Markov property. This property is used for example in dynamic programming approaches (see [4]) to formulate Bellman's principle, in the theory of random dynamical systems (see [1]) to determine invariant measures, in investigations of the long-time behaviour of the processes (see [8]). In the case of stochastic processes which are also Markov processes we can describe its properties by studying the properties of the corresponding Markov semigroup.

In this paper we prove that the solution of the stochastic Navier-Stokes equation is a Markov process. This property was proved by B. Schmalfuß [6] for the stochastic Navier-Stokes equation, but only for the case of additive noise. Our hypothesis are more general.

The structure of the paper is as follows: In Section 2 we give the assumptions for the Navier-Stokes equation and mention some results concerning the convergence of the Galerkin approximations to the solution of the considered equation. We also prove that the solution depends continuously on the initial data. Section 3 contains the main result of our paper. We prove that the solution of the stochastic Navier-Stokes equation is a Markov process. In Section 4 we give some auxiliary results from stochastic analysis.

Frequently Used Notations

$\rightharpoonup$	weak convergence (in the sense of functional analysis)
I_A	indicator function for the set A
EX	mathematical expectation of the random variable X
$\mathcal{L}^2_V(\Omega)$	space of all $\mathcal{F}$-measurable random variables $u : \Omega \to V$ with $E\|u\|^2_V < \infty$
$\mathcal{L}^2_V(\Omega \times [0,T])$	space of all $\mathcal{F} \times B([0,T])$-measurable processes $u : \Omega \times [0,T] \to V$ that are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $E\int_0^T \|u(t)\|^2_V dt < \infty$

2. Assumptions and formulation of the problem

First we state the assumptions about the stochastic evolution equation that will be considered.

(i): $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ is a right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. $(w(t))_{t\in[0,T]}$ is a real valued standard $\mathcal{F}_t$-Wiener process.

(ii): (V, H, V^*) is an evolution triple (see [10], p. 416), where $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|)$ are separable Hilbert spaces, and the embedding operator $V \hookrightarrow H$ is assumed to be compact. We denote by $(\cdot,\cdot)$ the scalar product in H.

(iii): $\mathcal{A} : V \to V^*$ is a linear operator such that $\langle \mathcal{A}v, v\rangle \geq \nu\|v\|^2_V$ for all $v \in V$ and $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle$ for all $u, v \in V$, where $\nu > 0$ is a constant and $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

(iv): $\mathcal{B} : V \times V \to V^*$ is a bilinear operator such that

$$\langle \mathcal{B}(u,v), v\rangle = 0 \quad \text{for all } u, v \in V$$

and there exists a positive constant $b > 0$ such that

$$|\langle \mathcal{B}(u,v), z\rangle|^2 \leq b\|z\|^2_V \|u\|\|u\|_V \|v\|\|v\|_V$$

for all $u, v, z \in V$.

(v): $\mathcal{C} : [0,T] \times H \to H$ is a mapping such that

(a): $\|\mathcal{C}(t,u) - \mathcal{C}(t,v)\|^2 \le \lambda \|u-v\|^2$ for all $t \in [0,T]$, $u, v \in H$, where λ is a positive constant;

(b): $\mathcal{C}(t,0) = 0$ for all $t \in [0,T]$;

(c): $\mathcal{C}(\cdot, v) \in \mathcal{L}^2_H[0,T]$ for all $v \in H$.

(vi): $\Phi : [0,T] \times H \to H$ is a mapping such that

(a): $\|\Phi(t,u) - \Phi(t,v)\|^2 \le \mu \|u-v\|^2$ for all $t \in [0,T]$, $u, v \in H$, where μ is a positive constant;

(b): $\Phi(t,0) = 0$ for all $t \in [0,T]$;

(c): $\Phi(\cdot, v) \in \mathcal{L}^2_H[0,T]$ for all $v \in H$.

(vii): x_0 is a H-valued $\mathcal{F}_0$-measurable random variable such that $E\|x_0\|^4 < \infty$.

Definition 2.1. We call a process $\Big(U(t)\Big)_{t\in[0,T]}$ from the space $\mathcal{L}^2_V(\Omega \times [0,T])$ with $E\|U(t)\|^2 < \infty$ for all $t \in [0,T]$ a **solution of the stochastic Navier-Stokes equation** if it satisfies the equation:

$$
\begin{aligned}
(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds &= (x_0, v) + \int_0^t \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\
&+ \int_0^t (\Phi(s, U(s)), v) ds + \int_0^t (\mathcal{C}(s, U(s)), v) dw(s)
\end{aligned} \tag{2}
$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$, where the stochastic integral is understood in the Ito sense.

Remark 2.2. If we set $n = 2$, $V = \{u \in \overset{\circ}{W}{}^1_2 (G) : \operatorname{div} u = 0\}$, $\quad H = \bar{V}^{L^2(G)}$ and

$$
\langle \mathcal{A}u, v\rangle = \int_G \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \quad \langle \mathcal{B}(u,v), z\rangle = -\int_G \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i} z_j dx, \quad \Phi(t,u) = f(t)
$$

for $u, v, z \in V, t \in [0,T]$, then equation (2) can be transformed into (1).

Let $h_1, h_2, \ldots, h_n, \cdots \in H$ be the eigenvectors of the operator $\mathcal{A}$, for which we consider the domain of definition $\operatorname{Dom}(\mathcal{A}) = \{v \in V \mid \mathcal{A}v \in H\}$. These eigenvectors form an orthonormal base in H and they are orthogonal in V. For each $n \in \mathbb{N}$ we consider $H_n := \mathrm{sp}\{h_1, h_2, \ldots, h_n\}$ be equipped with the norm induced from H. We write $(H_n, \|\cdot\|_V)$ when we consider H_n equipped with the norm induced from V. We define by $\Pi_n : H \to H_n$ the orthogonal projection of H on H_n

$$\Pi_n h := \sum_{i=1}^{n} (h, h_i) h_i.$$

Let $\mathcal{A}_n : H_n \to H_n$, $\mathcal{B}_n : H_n \times H_n \to H_n$, $\Phi_n, \mathcal{C}_n : [0,T] \times H_n \to H_n$ be defined respectively by

$$\mathcal{A}_n u = \sum_{i=1}^{n} \langle \mathcal{A} u, h_i \rangle h_i, \quad \mathcal{B}_n(u,v) = \sum_{i=1}^{n} \langle \mathcal{B}(u,v), h_i \rangle h_i,$$

$$\mathcal{C}_n(t,u) = \Pi_n \mathcal{C}(t,u), \quad \Phi_n(t,u) = \Pi_n \Phi(t,u), \quad x_{0n} = \Pi_n x_0$$

for all $t \in [0,T]$, $u, v \in H_n$.

The existence of the solution of the Navier-Stokes equation (2) is proved by approximating it by means of the Galerkin method, i.e., by a sequence of solutions of finite dimensional equations $(P_n), n \geq 1$.

For each $n = 1, 2, 3, \ldots$ we consider the sequence of finite dimensional evolution equations

$$(P_n) \qquad (U_n(t), v) + \int_0^t (\mathcal{A}_n U_n(s), v) ds = (x_{0n}, v) + \int_0^t (\mathcal{B}_n(U_n(s), U_n(s)), v) ds$$
$$+ \int_0^t (\Phi_n(s, U_n(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n(s)), v) dw(s),$$

for all $v \in H_n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$.

We use an analogous method as in [9]. Let $\left(\chi_M\right)$ be a family of Lipschitz continuous mappings such that

$$\chi_M(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq M, \\ 0, & \text{if } x \geq M+1, \\ M+1-x, & \text{if } x \in (M, M+1). \end{cases}$$

For each fixed $n \in \mathbb{N}$ we consider the solution U_n of equation (P_n) approximated by $\left(U_n^M\right)$ $(M = 1, 2, \dots)$ which is the solution of the equation

$$(P_n^M) \qquad (U_n^M(t), v) + \int_0^t (\mathcal{A}_n U_n^M(s), v) ds = (x_{0n}, v) + \int_0^t (\chi_M(\|U_n^M(t)\|^2)\mathcal{B}_n(U_n^M(s), U_n^M(s)), v) ds + \int_0^t (\Phi_n(s, U_n^M(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n^M(s)), v) dw(s),$$

for all $v \in H_n$, $t \in [0,T]$, and a.e. $\omega \in \Omega$. For this equation we apply the theory of finite dimensional Ito equations with Lipschitz continuous nonlinearities (see [5], Theorem 3.9, p. 289). Hence there exists $U_n^M \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T])$ almost surely unique solution of (P_n^M) which has continuous trajectories in H.

Theorem 2.1. *For each $n \in \mathbb{N}$, equation (P_n) has a solution $U_n \in \mathcal{L}^2_V(\Omega \times [0,T])$, which is unique almost surely and has almost surely continuous trajectories in H. For each $n \in \mathbb{N}$ it holds*

$$P\left(\lim_{M \to \infty} \sup_{t \in [0,T]} \|U_n^M(t) - U_n(t)\|^2 = 0 \right) = 1.$$

Theorem 2.2. *The Navier-Stokes equation (2) has a solution $U \in \mathcal{L}^2_V(\Omega \times [0,T])$, which is almost surely unique and has almost surely continuous trajectories in H. The following convergence holds*

$$\lim_{n \to \infty} E\|U_n(t) - U(t)\|^2 = 0 \quad \textit{for all } t \in [0,T].$$

Lemma 2.3. *There exists a positive constant c such that*

$$E \sup_{t \in [0,T]} \|U(t)\|^4 + E\Big(\int_0^T \|U(s)\|_V^2 ds\Big)^2 \le cE\|x_0\|^4.$$

The proofs of these results can be found in [2].

Before we investigate the Markov property for the solution of the stochastic Navier-Stokes equation, we prove that the solution U of (2) depends continuously on the initial data x_0.

Theorem 2.4. *Let (x_0^N) be a sequence in H and let $x_0 \in H$ be such that*

$$\lim_{N\to\infty} ||x_0^N - x_0||^2 = 0.$$

Then for each $t \in [0,T]$ it holds

$$\lim_{N\to\infty} E||U_N(t) - U(t)||^2 = 0,$$

where U_N is the solution of (2) satisfying the initial condition $U_N(0) = x_0^N$.

Proof. For all $t \in [0,T]$ and a.e. $\omega \in \Omega$ let

$$e(t) = \exp\left\{ -\frac{b}{\nu}\int_0^t ||U(s)||_V^2 ds - (\lambda + 2\sqrt{\mu})t \right\}.$$

It follows by the Ito formula that

$$\begin{aligned}
& e(t)||U(t) - U_N(t)||^2 + 2\int_0^t e(s)\langle \mathcal{A}U(s) - \mathcal{A}U_N(s), U(s) - U_N(s)\rangle ds \\
= \quad & ||x_0 - x_0^N||^2 + 2\int_0^t e(s)\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_N(s),U_N(s)), U(s) - U_N(s)\rangle ds \\
- \quad & \frac{b}{\nu}\int_0^t e(s)||U(s)||_V^2 ||U(s) - U_N(s)||^2 ds - (\lambda + 2\sqrt{\mu})\int_0^t e(s)||U(s) - U_N(s)||^2 ds \\
+ \quad & 2\int_0^t e(s)(\Phi(s,U(s)) - \Phi(s,U_N(s)), U(s) - U_N(s))ds \\
+ \quad & \int_0^t e(s)||\mathcal{C}(s,U(s)) - \mathcal{C}(s,U_N(s))||^2 ds \\
+ \quad & 2\int_0^t e(s)(\mathcal{C}(s,U(s)) - \mathcal{C}(s,U_N(s)), U(s) - U_N(s))dw(s).
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$
\begin{aligned}
&2\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_N(s),U_N(s)), U(s)-U_N(s)\rangle \\
&\qquad = 2\langle \mathcal{B}(U(s)-U_N(s),U(s)), U(s)-U_N(s)\rangle \\
&\qquad \leq \frac{b}{\nu}\|U(s)\|_V^2\|U(s)-U_N(s)\|^2 + \nu\|U(s)-U_N(s)\|_V^2.
\end{aligned}
$$

Now we use the properties of $\mathcal{A}$, Φ, $\mathcal{C}$, and those of the stochastic integral to obtain

$$
\begin{aligned}
&E\sup_{s\in[0,t]} e(s)\|U(s)-U_N(s)\|^2 \leq \|x_0 - x_0^N\|^2 \\
&+\ 4E\sup_{s\in[0,t]}\left|\int_0^s e(r)(\mathcal{C}(r,U(r)) - \mathcal{C}(r,U_N(r)), U(r)-U_N(r))dw(r)\right| \\
&\leq\ k_1 E\int_0^t \sup_{r\in[0,s]}\left\{e(r)\|U(r)-U_N(r)\|^2\right\}ds \\
&+\ \frac{1}{2}E\sup_{s\in[0,t]} e(s)\|U(s)-U_N(s))\|^2,
\end{aligned}
$$

where k_1 is a positive constant and $t\in[0,T]$. By Gronwall's Lemma we get

$$
E\sup_{s\in[0,t]} e(s)\|U(s)-U_N(s)\|^2 \leq 4e^{2k_1T}\|x_0 - x_0^N\|^2
$$

for all $t\in[0,T]$.

We take $t := \mathcal{T}_M^U$, where $\mathcal{T}_M^U$ is the following *stopping time*

$$
\mathcal{T}_M^U = \begin{cases} T\ , & \text{if } \int_0^T \|U(s)\|_V^2 ds < M \\ \inf\left\{t\in[0,T] : \int_0^t \|U(s)\|_V^2 ds \geq M\right\}, & \text{otherwise.} \end{cases}
$$

Using the hypothesis and the above inequality it follows that for each fixed $M\in\mathbb{N}$ we have

$$
\lim_{n\to\infty} E\|U(\mathcal{T}_M^U) - U_N(\mathcal{T}_M^U)\|^2 = 0.
$$

Applying Proposition 4.1 for $\mathcal{T} := t$, $\mathcal{T}_M := \mathcal{T}_M^U$, $Q_n(\mathcal{T}) := \|U(\mathcal{T}) - U_N(\mathcal{T})\|^2$ we obtain

$$\lim_{n\to\infty} E\|U_N(t) - U(t)\|^2 = 0.$$

□

3. The Markov property

Let us introduce the following σ-algebras

$$\sigma_{[U(s)]} := \sigma\{U(s)\}, \quad \sigma_{[U(r):r\leq s]} := \sigma\{U(r) : r \leq s\}$$

and the event

$$\sigma_{[U(s)=y]} := \{\omega : U(s) = y\}.$$

For the solution U of the Navier-Stokes equation (2) we define the **transition function**

$$\bar{P}(s,x,t,A) := P(U(t) \in A|\sigma_{[U(s)=x]})$$

with $s,t \in [0,T], s < t, x \in H, A \in B(H)$.

In the following theorem we prove that **the solution of the Navier-Stokes equation is a Markov process**. This means that the state $U(s)$ at time s must contain all probabilistic information relevant to the evolution of the process for times $t > s$.

Theorem 3.1. *(i) For fixed $s,t \in [0,T], s < t, A \in B(H)$ the mapping*

$$y \in H \mapsto \bar{P}(s,y,t,A) \in \mathbb{R}$$

is measurable.

(ii) The following equalities hold

$$P(U(t) \in A|\mathcal{F}_s) = P(U(t) \in A|\sigma_{[U(s)]})$$

and

$$P\Big(U(t) \in A\Big|\sigma_{[U(r):r\leq s]}\Big) = P\Big(U(t) \in A\Big|\sigma_{[U(s)]}\Big)$$

for all $s,t \in [0,T], s < t, y \in H, A \in B(H)$.

Proof. **(i)** Let $s, t \in [0,T], s < t, y \in H$. We denote by $\left(\tilde{U}(t,s,y)\right)_{t\in[s,T]}$ the solution of the Navier-Stokes equation starting in s with the initial value y, i.e. $\tilde{U}(s,s,y) = y$ for a.e. $\omega \in \Omega$.

Let $A \in B(H)$. Without loss of generality we can consider the set A to be closed. Let (a_n) be a sequence of continuous and uniformly bounded functions $a_n : H \to \mathbb{R}$, $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} ||a_n(y) - I_A(y)|| = 0 \quad \text{for all } y \in H. \tag{3}$$

By the uniqueness of the solution of the Navier-Stokes equation and from the definition of the transition function we have

$$\bar{P}(s,y,t,A) = E\Big(I_A(U(t))\Big|\sigma_{[U(s)=y]}\Big) = E\Big(I_A(\tilde{U}(t,s,y))\Big).$$

We consider an arbitrary sequence (y_n) in H such that $\lim_{n\to\infty} ||y_n - y|| = 0$. By using Theorem 2.4 (instead of starting in 0 we start in s) it follows that

$$\lim_{n\to\infty} E||\tilde{U}(t,s,y_n) - \tilde{U}(t,s,y)||^2 = 0. \tag{4}$$

Therefore $\left(\tilde{U}(t,s,y_n)\right)$ converges in probability to $\tilde{U}(t,s,y)$. Using (4) and the Lebesgue Theorem it follows that for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} Ea_k\left(\tilde{U}(t,s,y_n)\right) = Ea_k\left(\tilde{U}(t,s,y)\right).$$

We conclude that for each $k \in \mathbb{N}$ the mapping

$$y \in H \mapsto Ea_k\left(\tilde{U}(t,s,y)\right) \in \mathbb{R}$$

is continuous. Hence it is measurable. By the Lebesgue Theorem and (3) we deduce that for all $y \in H$

$$\lim_{k\to\infty} Ea_k\left(\tilde{U}(t,s,y)\right) = EI_A\left(\tilde{U}(t,s,y)\right).$$

Consequently, $\bar{P}(s,\cdot,t,A) = EI_A\left(\tilde{U}(t,s,\cdot)\right)$ is measurable, because it is the pointwise limit of measurable functions.

(ii) First we prove that for each fixed $s,t \in [0,T], s < t, y \in H$ the random variable $\tilde{U}(t,s,y)$ (considered as a H-valued random variable) is independent of $\mathcal{F}_s$. By Theorem 2.1 we have

$$\lim_{M\to\infty} \|\tilde{U}_n^M(t,s,y) - \tilde{U}_n(t,s,y)\| = 0 \quad \text{for each } n \in \mathbb{N} \text{ and a.e. } \omega \in \Omega, \tag{5}$$

and by Theorem 2.2 it follows that there exists a subsequence (n') of (n) such that

$$\lim_{n'\to\infty} \|\tilde{U}_{n'}(t,s,y) - \tilde{U}(t,s,y)\| = 0 \quad \text{for a.e. } \omega \in \Omega, \tag{6}$$

where $\left(\tilde{U}_n^M(t,s,y)\right)_{t\in[s,T]}$ and $\left(\tilde{U}_n(t,s,y)\right)_{t\in[s,T]}$ are the solutions of (P_n^M) and (P_n), respectively, if we start in s with the initial value y. Since for fixed n, M the random variable $\tilde{U}_n^M(t,s,y)$ is approximated by Picard-iteration and each Picard-approximation is independent of $\mathcal{F}_s$ (as a H-valued random variable), it follows by Proposition 4.2 that $\tilde{U}_n(t,s,y)$ is independent of $\mathcal{F}_s$. Using (5), (6), and Proposition 4.2 we conclude that $\tilde{U}(t,s,y)$ is independent of $\mathcal{F}_s$.

Let $A \in B(H)$. Now we apply Proposition 4.3 for $\widehat{\mathcal{F}} := \mathcal{F}_s, f(y,\omega) := I_A\left(\tilde{U}(t,s,y)\right), \xi(\omega) := U(s)$. Hence

$$E\left(I_A\left(\tilde{U}(t,s,U(s))\right)\Big|\mathcal{F}_s\right) = E\left(I_A\left(\tilde{U}(t,s,U(s))\right)\Big|\sigma_{[U(s)]}\right). \tag{7}$$

Since the solution of the Navier-Stokes equation is (almost surely) unique it follows that

$$\tilde{U}(t,s,U(s)) = U(t) \quad \text{for all } t \in [s,T] \text{ and a.e. } \omega \in \Omega.$$

Then relation (7) becomes

$$E\left(I_A\left(U(t)\right)\Big|\mathcal{F}_s\right) = E\left(I_A\left(U(t)\right)\Big|\sigma_{[U(s)]}\right).$$

Consequently,

$$P\left(U(t) \in A\Big|\mathcal{F}_s\right) = P\left(U(t) \in A\Big|\sigma_{[U(s)]}\right). \tag{8}$$

We know

$$\sigma_{[U(s)]} \subseteq \sigma_{[U(r):r\leq s]} \subseteq \mathcal{F}_s.$$

Taking into account the properties of the conditional expectation and (8) we deduce that

$$\begin{aligned} P\Big(U(t) \in A \Big| \sigma_{[U(r):r\leq s]}\Big) &= \Big(E\Big(U(t) \in A \Big| \mathcal{F}_s\Big)\Big| \sigma_{[U(r):r\leq s]}\Big) \\ &= E\Big(E\Big(U(t) \in A \Big| \sigma_{[U(s)]}\Big)\Big| \sigma_{[U(r):r\leq s]}\Big) \\ &= P\Big(U(t) \in A \Big| \sigma_{[U(s)]}\Big). \end{aligned}$$

□

Using results from [3] (Chapter 3, Section 9, p. 59) we deduce the following corollary.

Corollary 3.2. *(i) For fixed* $s, t \in [0,T], s < t, y \in H$ *the mapping*

$$A \in B(H) \mapsto \bar{P}(s, y, t, \cdot) \in \mathbb{R}$$

is a probability measure.

(ii) The Chapman-Kolmogorov equation

$$\bar{P}(s, y, t, A) = \int_H \bar{P}(r, x, t, A)\bar{P}(s, y, r, dx)$$

holds for any $r, s, t \in [0,T], s < r < t, y \in H, A \in B(H)$.

Remark 3.3. We have the **autonomous version** of the stochastic Navier-Stokes equation if for $t \in [0,T]$, $h \in H$ we have $\mathcal{C}(t,h) = \mathcal{C}(h)$ and $\Phi(t,h) = \Phi(h)$. In this case $\Big(U_\Phi(t)\Big)_{t\in[0,T]}$ is a **homogeneous Markov process**, i.e., we have

$$\bar{P}(0, y, t-s, A) = \bar{P}(s, y, t, A) \tag{9}$$

for all $s, t \in [0,T], s < t, y \in H, A \in B(H)$.

Now we prove the above property. Let $s, t \in [0,T], s < t, y \in H$. The solution U_Φ of the Navier-Stokes equation, which starts in s with the initial value y satisfies

$$\begin{aligned} (U_\Phi(t), v) + \int_s^t \langle \mathcal{A}U_\Phi(r), v\rangle dr &= (y, v) + \int_s^t \langle \mathcal{B}(U_\Phi(r), U_\Phi(r)), v\rangle dr \\ &+ \int_s^t (\Phi(U_\Phi(r)), v) dr + \int_s^t (\mathcal{C}(U_\Phi(r)), v) dw(r) \end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$.

We take $\tilde{U}(r) = U_\Phi(s+r), \tilde{w}(r) := w(s+r) - w(s)$ for $r \in [0, t-s]$. Then for $\tilde{U}(t-s)$ we have

$$\begin{aligned}(\tilde{U}(t-s), v) + \int_0^{t-s} \langle \mathcal{A}\tilde{U}(r), v\rangle dr &= (y, v) + \int_0^{t-s} \langle \mathcal{B}(\tilde{U}(r), \tilde{U}(r)), v\rangle dr \\ &+ \int_0^{t-s} (\Phi(\tilde{U}(r)), v) dr + \int_0^{t-s} (\mathcal{C}(\tilde{U}(r)), v) d\tilde{w}(r)\end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$. Since $\left(\tilde{w}(r)\right)_{r\in[0,t-s]}$ and $\left(w(r)\right)_{r\in[s,t]}$ have the same distribution and because of the uniqueness of the solution of the Navier-Stokes equation, it follows that $\tilde{U}(t-s)$ and $U_\Phi(t)$ have the same distribution. Hence (9) holds.

In the case of a homogeneous Markov process we denote

$$\bar{p}(y, t, A) := \bar{P}(0, y, t, A)$$

for all $t \in [0, T], y \in H, A \in B(H)$. The Chapman-Kolmogorov equation (see Corollary 3.2) can be rewritten as

$$\bar{p}(y, s+t, A) = \int_H \bar{p}(x, t, A)\bar{p}(y, s, dx) \tag{10}$$

for each $s, t \in [0, T], y \in H, A \in B(H)$.

We consider the following set of probabilty measures on Ω

$$\mathcal{S} := \left\{\mu \middle| \int_H ||x||^4 \mu(dx) < \infty\right\}$$

and define

$$T_t\mu(\Gamma) := \int_H \bar{p}(x, t, \Gamma)\mu(dx)$$

for each $\mu \in \mathcal{S}$, $t \in [0, T]$. This mapping has the following properties:

(a) $T_t : \mathcal{S} \to \mathcal{S}$,

(b) $T_0\mu = \mu$ for each $\mu \in \mathcal{S}$,

(c) $T_{t+s} = T_t \circ T_s = T_s \circ T_t$ for $s, t, s+t \in [0, T]$.

We deduce the result in (a) by observing that $T_t\mu$ is the distribution of $U(t)$ if the initial condition x_0 has the distribution μ ($\mu \in \mathcal{S}$ because of hypothesis **(vii)**). Then by using Lemma 2.3 we have $T_t\mu \in \mathcal{S}$. For (b) one can make some easy calculations and (c) follows from (10). Hence $(T_t)_t$ satisfies the *semigroup property.*

4. Some results from stochastic analysis

Proposition 4.1. *Let $(\mathcal{T}_M)$ and $\mathcal{T}$ be stopping times, such that*

$$\lim_{M\to\infty} P(\mathcal{T}_M < \mathcal{T}) = 0.$$

Let (Q_n) be a sequence of processes from the space $\mathcal{L}^2_{\mathbb{R}}([0,T]\times\Omega)$ such that for each fixed M we have

$$\lim_{n\to\infty} E|Q_n(\mathcal{T}_M)| = 0$$

and there exists a positive constant c independent of n such that

$$E|Q_n(\mathcal{T})|^2 < c \quad \text{for all} \quad n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$

Proof. Let $\varepsilon, \delta > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$P(\mathcal{T}_{M_0} < \mathcal{T}) \le \frac{\varepsilon}{2}.$$

By the hypothesis it follows that for this M_0 we have $\lim\limits_{n\to\infty} E|Q_n(\mathcal{T}_{M_0})| = 0$. Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta}E|Q_n(\mathcal{T}_{M_0})| \le \frac{\varepsilon}{2}$$

for all $n \ge n_0$. We write

$$\begin{aligned} P\Big(|Q_n(\mathcal{T})| \ge \delta\Big) &\le P\Big(\mathcal{T}_{M_0} < \mathcal{T}\Big) + P\Big(\{\mathcal{T}_{M_0} = \mathcal{T}\} \wedge \Big\{|Q_n(\mathcal{T})| \ge \delta\Big\}\Big) \\ &\le \frac{\varepsilon}{2} + P\Big(|Q_n(\mathcal{T}_{M_0})| \ge \delta\Big) \le \frac{\varepsilon}{2} + \frac{1}{\delta}E|Q_n(\mathcal{T}_{M_0})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $n \ge n_0$. Hence for all $\delta > 0$ we get $\lim\limits_{n\to\infty} P\Big(|Q_n(\mathcal{T})| \ge \delta\Big) = 0$. Therefore, the sequence $\Big(|Q_n(\mathcal{T})|\Big)$ converges in probability to zero. From the hypothesis it follows that this sequence is uniformly integrable (with respect to $\omega \in \Omega$). Hence it converges also in mean to zero

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$

□

Proposition 4.2. *Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, (Q_n) be a sequence of H-valued random variables which converges for a.e. $\omega \in \Omega$ to Q. If each random variable Q_n is independent of $\widehat{\mathcal{F}}$, then Q is independent of $\widehat{\mathcal{F}}$.*

Proof. The random variable Q is independent of $\widehat{\mathcal{F}}$ if

$$P(\{||Q|| < x\} \cap A) = P(||Q|| < x)P(A) \tag{11}$$

for all $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}$. The hypothesis implies that the sequence $\Big(||Q_n||\Big)$ converges in probability to $||Q||$. Therefore, the sequence of their distribution functions is convergent

$$\lim_{n\to\infty} F_{||Q_n||}(x) = F_{||Q||}(x) \tag{12}$$

for each $x \in \mathbb{R}$ which is a continuity point of $F_{||Q||}$.

Let $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}, \delta > 0$. First we consider that $F_{||Q||}$ is continuous in x. Then using the hypothesis and (12) we get

$$\lim_{n\to\infty} P(\{||Q_n|| < x\} \cap A) = \lim_{n\to\infty} P(||Q_n|| < x)P(A) = P(||Q|| < x)P(A). \tag{13}$$

We write

$$\begin{aligned} P(\{||Q|| < x - \delta\} \cap A) &\leq P(\{||Q|| < x - \delta\} \cap \{||Q_n|| < x\} \cap A) \\ &+ P(\{||Q|| < x - \delta\} \cap \{||Q_n|| \geq x\} \cap A) \\ &\leq P(\{||Q_n|| < x\} \cap A) + P\Big(\Big|||Q|| - ||Q_n||\Big| > \delta\Big). \end{aligned}$$

Analogously we have

$$P(\{||Q_n|| < x\} \cap A) \leq P(\{||Q|| < x + \delta\} \cap A) + P\Big(\Big|||Q|| - ||Q_n||\Big| > \delta\Big).$$

Consequently,

$$P(\{||Q|| < x - \delta\} \cap A) - P\Big(\Big|||Q|| - ||Q_n||\Big| > \delta\Big) \leq P(||Q_n|| < x)P(A)$$
$$\leq P(\{||Q|| < x + \delta\} \cap A) + P\Big(\Big|||Q|| - ||Q_n||\Big| > \delta\Big).$$

In the inequalities above we take the limit $n \to \infty$ and use (13) to obtain

$$P(\{||Q|| < x - \delta\} \cap A) \leq P(||Q|| < x)P(A) \leq P(\{||Q|| < x + \delta\} \cap A).$$

Let $\delta \searrow 0$ in the inequalities above. Then

$$P(\{||Q|| \leq x\} \cap A) \leq P(||Q|| < x)P(A) \leq P(\{||Q|| \leq x\} \cap A).$$

Since x is a point of continuity for $F_{||Q||}$ we have

$$P(\{||Q|| \leq x\} \cap A) = P(\{||Q|| < x\} \cap A).$$

Consequently, (11) holds and Q is independent of $\widehat{\mathcal{F}}$.

Now we consider that x is not a point of continuity of $F_{||Q||}$. Let (x_n) be a monotone increasing sequence of continuity points of $F_{||Q||}$ which converges to x. Then

$$\lim_{n \to \infty} F_{||Q||}(x_n) = F_{||Q||}(x),$$

and because x_n is a point of continuity for $F_{||Q||}$, we have

$$P(\{||Q|| < x_n\} \cap A) = P(||Q|| < x_n)P(A).$$

Now we take the limit $n \to \infty$ and conclude that (11) holds. Hence Q is independent of $\widehat{\mathcal{F}}$. □

Proposition 4.3. *Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, $f : H \times \Omega \to H$ be a mapping such that for each $x \in H$ the random variable $f(x, \cdot)$ is bounded, measurable and independent of $\widehat{\mathcal{F}}$. Let ξ be a H-valued $\widehat{\mathcal{F}}$-measurable random variable. Then*

$$E\big(f(\xi, \omega)\big|\widehat{\mathcal{F}}\big) = E\big(f(\xi, \omega)\big|\sigma_{[\xi]}\big),$$

where $\sigma_{[\xi]}$ is the σ-algebra generated by the random variable ξ.

This proposition is proved in [3] (see Lemma 1, p. 63).

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A FIXED POINT THEOREM FOR HICKS-TYPE CONTRACTIONS IN PROBABILISTIC METRIC SPACES

DOREL MIHEŢ

Abstract. A fixed point theorem for mappings with contractive iterate at a point on bounded uniform spaces is proved. As particular cases some fixed point theorems of Hicks - type are obtained.

1. A fixed point theorem

Definition 1.1. Let X be a nonempty set and $\mathcal{B}$ be the class of all functions $\beta : X \times X \to [0, \infty)$ with the properties

$\beta 1)$ $\beta(x, y) = 0 \Leftrightarrow x = y$

$\beta 2)$ $\beta(x, y) = \beta(y, x) (\forall) x, y \in X$

$\beta 3)$ $(\forall)\varepsilon > 0\ (\exists)\delta > 0 : (\beta(x, y) < \delta, \beta(y, z) < \delta) \Rightarrow \beta(x, z) < \varepsilon$

$\beta 4)$ $(\exists) M > 0 : \beta(x, y) \leq M, (\forall) x, y \in X.$

A B - *space* is a pair $(X, \beta)$with $\beta \in \mathcal{B}$.

Proposition 1.2 ([1]). *If β satisfies $\beta 1) - \beta 3)$ then the family*

$$U_\beta = \{S_{\beta,\varepsilon}\}_{\varepsilon > 0}, where S_{\beta,\varepsilon} = \{(x, y) \in X^2 | \beta(x, y) < \varepsilon\}$$

is a base for a metrizable uniformity on X (which we will call the U_β - uniformity).

The proof is easy to be reproduced.

Definition 1.3. If M is a positive number, Φ_M means the class of all functions $\varphi : [0, \infty) \to [0, \infty)$ for which there exists $\alpha > M$ such that $\lim_{n\to\infty} \varphi^n(\alpha) = 0$. ($\varphi^n = \varphi \circ \varphi \circ \cdots \circ \varphi$).

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Definition 1.4. Let (X,β) be a $\mathcal{B}$ space and $\varphi : [0,\infty) \to [0,\infty)$ be a function. We say that the mapping $f : X \to X$ has *a* $(O-\varphi)$ *-contractive iterate at* $x \in X$ if for each $y \in O_f(x)$ there exists
$n = n(y) \in \mathbf{N}$ such that

$$(\varepsilon > 0, z \in O_f(y), \beta(y,z) < \varepsilon) \Rightarrow \beta(f^n y, f^n z) < \varphi(\varepsilon),$$

where $O_f(y)$ denoted the set $\{f^n y | n \in \mathbf{N}\}$.

Theorem 1.5. *Let* (X,β) *be a* U_β *- complete* $\mathcal{B}$ *-space. If* $\varphi \in \Phi_M$ *and* $f : X \to X$ *is a continous mapping which has a* $(O-\varphi)$ *- contractive iterate at* $x \in X$*, then* f *has a fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of* $O_f(x)$.

Proof. We will show that for every $y \in O_f(x)$ the sequence $\{f^n y\}_{n\in\mathbf{N}}$ is a *Cauchy sequence.*

Lemma 1.6. *Let* (X,β) *be a* $\mathcal{B}$ *space,* $\varphi \in \Phi_M$ *and* $f : X \to X$ *be a mapping with* $(O-\varphi)$ *-contractive iterate at* $x \in O_f(x)$*. Then, for every* $\varepsilon > 0$ *there exists* $m_0 = m(y,\varepsilon) \in \mathbf{N}$ *such that*

$$\beta(f^{m_0} y, f^{m_0+m} y) < \varepsilon \ (\forall) m \in \mathbf{N}.$$

Proof of Lemma. First we show that for every $s \in \mathbf{N}$ there exists $u = u(s,y) \in \mathbf{N}$ such that, for every $\alpha > M$,

$$(1) \quad \beta(f^{u(s)} y, f^{u(s)+m} y) < \varphi^s(\alpha),\ (\forall)\ m \in \mathbf{N}.$$

(1) is true for $s = 0$, because β satisfies $\beta 4)$ and $M < \alpha$. Next, for every $j \geq 0$, let us define recursively the numbers $v(i)$ and $u(j)$ by $v(0) = u(0),\ v(j+1) = n(f^{u(j)}x)$ and $u(j+1) = u(j) + v(j+1)$.

Then we have the following implication :
$\beta(f^{u(s)} y, f^{u(s)+m} y) < \varphi^s(\alpha) \Rightarrow \beta(f^{v(s+1)}(f^{u(s)} y), f^{v(s+1)}(f^{u(s)+m} y)) < \varphi^{s+1}(\alpha)$, that is

$$\beta(f^{u(s)} y, f^{u(s)+m} y) < \varphi^s(\alpha) \Rightarrow \beta(f^{u(s+1)} y, f^{u(s+1)+m)} y) < \varphi^{s+1}(\alpha).$$

So, the relation (1) is proved by induction. □

Now let $\varepsilon > 0$ be given. Since $\varphi \in \Phi_M$, then there exists $\alpha > M$ such that $\varphi^n(\alpha) \to 0$. For this α let us consider $s_0 \in \mathbf{N}$ for which $\varphi^{s_0}(\alpha) < \varepsilon$. If we take in (1) $m_0 = u(s_0)$ then we obtain

$$\beta(f^{m_0}y, f^{m_0+m}y) < \varepsilon \ (\forall) \ m \in \mathbf{N}$$

and the lemma is proved.

Now it is easy to show that $\{f^n(y)\}$ is a Cauchy sequence:
For given $\varepsilon > 0$ we consider $\delta(\varepsilon)$ from $\beta 3$).
By *Lemma 1.6* there exists $m_0 \in \mathbf{N}$ such that

$$\beta(f^{m_0}y, f^{m_0+p}y) < \delta, (\forall) p \in \mathbf{N},$$

hence

$$\beta(f^{m_0}y, f^n y) < \delta, \beta(f^{m_0}y, f^{n+m}y) < \delta, \ (\forall) \ n \geq m_0, (\forall) m$$

From $\beta 3$) it follows that there exists $m_0 = m_0(y, \varepsilon) \in \mathbf{N}$ such that

$$\beta(f^n y, f^{n+m}y) < \varepsilon, \ (\forall) n \geq m_0, \ (\forall) m \in \mathbf{N},$$

i.e.$\{f^n(y)\}_{n \in \mathbf{N}}$ is a Cauchy sequence.

Since (X, β) is U_β - complete, then there exists $u_* \in X$, $u_* = \lim_{n\to\infty} f^n y$. From the continuity of f it follows that $f(u_*) = u_*$. □

Corollary 1.7. *If (X, β) is a complete $\mathcal{B}$-space and $f : X \to X$ is a continous mapping with the property that for every $y \in X$ there exists $n = n(y) \in \mathbf{N}$ such that, for every $z \in O_f(y)$,*

$$(\varepsilon > 0, \beta(y, z) < \epsilon) \Rightarrow \beta(f^n y, f^n z) < \varphi(\varepsilon)$$

then f has a fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of X.

2. Applications to PM - spaces

Definition 2.1 ([10]). A PSM space $(S, \mathcal{F})$ is called a *H - space* if the following triangle inequality takes place :
(PM_{3H}) $(\forall)\varepsilon > 0$ $(\exists)\delta > 0 : (F_{pq}(\delta) > 1 - \delta, F_{qr}(\delta) > 1 - \delta) \Rightarrow F_{pr}(\varepsilon) > 1 - \varepsilon.$

Two important examples are

Example 2.2.

a) Every σ - Menger space $(S, \mathcal{F}, T)$ with $\sup_{a<1} T(a, a) = 1$ and $\inf_{b>0}\sigma(b, b) = 0$ is a H - space.
b) If $(S, \mathcal{F}, \tau)$ is a Serstnev space and the t - function τ is continuous at $(\varepsilon_0, \varepsilon_0)$, then $(S, \mathcal{F})$ is a H - space (for the basic notions used here see [6], [14] or [15]).

Lemma 2.3. *Let r be a strictly decreasing and continuous function on $[0, \infty)$ such that $r(0) = 1$ and there exists $\alpha > 0$ such that $r(\alpha) = 0$. If $(S, \mathcal{F})$ is a PSM space and K_r is the mapping defined on $S \times S$ by*

$$K_r(p, q) = \sup\{t > 0 | F_{pq}(t) \leq r(t)\}$$

then

a) K_r satisfies $\beta 4)$, $\beta 1)$ and $\beta 2)$.

b) $K_r(p, q) < t \Leftrightarrow F_{pq}(p, q) > r(t)$

c) If $(S, \mathcal{F})$ is a H - space then K_r satisfies $\beta 3)$ and the uniformity U_{K_r} is the $\mathcal{F}$ - uniformity.

Proof. Let $g(t) = F_{pq}(t) - r(t)$. Then g is a left conntinous and strictly increasing function on $[0, \infty)$. If we denote $A = \{t > 0 | g(t) \geq 0\}$ then because $g(t) > 0$ if $t > \alpha$, we have $A \subset [0, \alpha]$ and we can choose $M = \alpha$. So K_r satisfies $\beta 4)$.

Let now $p = q$. Then from PM1), $F_{pq}(t) = 1$ $(\forall)t > 0$, so $r(0) = 1$ from which it follows $K_r(p,q) = 0$. Conversely, if we suppose that $K_r(p,q) = 0$, then $F_{pq}(t) > r(t)$ $(\forall)t > 0$, so $F_{pq}(0+) = 1$ and, from PM 1) again, $p = q$.

$\beta 2)$ follows immediately from $PM2) : F_{pq} = F_{qp}$ $(\forall)p, q \in S$.

b) Let p, q be fixed and $m = \sup A = K_r(p,q)$.

If $(x_n) \subset A, x_n \nearrow m$, then, by the left continuity of g it follows $g(m) \leq 0$ and by the monotonocity of g we deduce that $g(t) > 0 \Rightarrow t > m$. So we proved the implication $F_{pq}(t) > r(t) \Rightarrow K_r(p,q) < t$. The converse implication immediately follows from the definition of $K_r(p,q)$.

c) Let us suppose that $(S, \mathcal{F})$ is a H - space. We prove that

$(\forall)\varepsilon > 0(\exists)\delta > 0 : (F_{pq}(\delta) > r(\delta), F_{qr}(\delta) > r(\delta)) \Rightarrow F_{pr}(\varepsilon) > r(\varepsilon)$ and then $\beta 3)$ follows from b).

Let $\varepsilon > 0$ be fixed. We choose $\varepsilon_1 = \min(\varepsilon, 1 - r(\varepsilon))$ and let δ_1 be the ε_1 - correspondent from $PM3_H)$. There exists $\delta < \delta_1$ such that $r(\delta) > 1 - \delta_1$. Then, $F_{pq}(\delta) > r(\delta), F_{qr}(\delta) > r(\delta)) \Rightarrow (F_{pq}(\delta_1) > 1 - \delta_1\ F_{qr}(\delta_1) > 1 - \delta_1) \Rightarrow F_{pr}(\varepsilon_1) > 1 - \varepsilon_1 \Rightarrow F_{pr}(\varepsilon) > r(\varepsilon)$.

The fact that the U_{K_r} uniformity is the $\mathcal{F}$ uniformity immediately follows from b).

The lemma is proved. □

The mapping d_{m_1,m_2} bellow has been introduced by V. Radu ([12]). Let $\mathcal{M}$ be the family of all functions $m : [0,\infty) \to [0,\infty)$ such that

m1) $m(t+s) \geq m(t) + m(s)$ $(\forall)t, s \geq 0$

m2) $m(t) = 0 \Leftrightarrow t = 0$

m3) m is continous.

Let $(S, \mathcal{F})$ be a PSM space. If $f : [0,1] \to \mathbf{R}$ is a continuous and strictly decreasing function with $f(1) = 0$ and $m_1, m_2 \in \mathcal{M}$, then d_{m_1,m_2} is the mapping defined on S^2 by

$$d_{m_1,m_2}(p,q) = \sup\{t \geq 0 | m_1(t) \leq f \circ F_{pq}(m_2(t)\}.$$

Lemma 2.4. *a)* $(\exists)$ $M > 0 : d_{m_1,m_2}(p,q) \leq M,$ $(\forall)$ $p, q \in S$.

b) d_{m_1,m_2} *satisfies* $\beta 1), \beta 2)$.

c) $d_{m_1,m_2}(p,q) < t \Leftrightarrow f \circ F_{pq}(m_2(t)) < m_1(t)$

d) If $(S,\mathcal{F})$ *is a* H *- space, then* $\beta = d_{m_1,m_2}$ *satisfies* $\beta 3)$ *and the uniformity* U_β *is the* $\mathcal{F}$ *- uniformity.*

Proof. For a) let us observe that $\lim_{t\to\infty} m_1(t) = \infty$ and so there exists $t_0 > 0$ such that $m_1(t_0) > f(0)$. From this it follows that $d_{m_1,m_2}(p,q) < t_0$ $(\forall)p,q \in S$, because $f \circ F_{pq}(m_2(t_0) \leq f(0) < m_1(t_0)$.

b), c) and d) follows from [12, Theorem 1]. □

From *Lemma 2.3* and *Lemma2.4* it follows that if $(S,\mathcal{F})$ is a H - space, then (S,β) is a $\mathcal{B}$ space (in the first case $\beta = K_r$ and $M = \alpha$ and in the second one $\beta = d_{m_1,m_2}$ and $M = t_0$), and the $\mathcal{U}_\beta$ - uniformity is the $\mathcal{F}$ - uniformity. Thus we can transpose the results from the previous paragraph to contractions on H - space.

Theorem 2.5. *Let* $(S,\mathcal{F})$ *be a complete* H *- space and* $f : S \to S$ *be a continuous mapping with the property that there exists* $p \in S$ *such that*
$(\forall)p \in O_f(p)(\exists)n = n(q) \in \mathbf{N} : (t > 0, r \in O_f(q), F_{qr}(t) > 1-t) \Rightarrow F_{f_q^n, f_r^n}(\varphi(t) > 1 - \varphi(t)$, *where* $\varphi \in \Phi_1$. *Then* f *has a fixed point.*

The proof follows from *Lemma 2.3* $(r(t) = 1-t)$ and *Theorem 1.5.*
For $\varphi(t) = kt$ one obtains *Theorem 1.2* from [11].

Theorem 2.6. *Let* $(S,\mathcal{F})$ *be a complete* H *- space and* $f : S \to S$ *be a continuous mapping with the property that for every* $p \in S$ *there exist* $n = n(p) \in \mathbf{N}$ *such that, for every* $q \in O_f(p)$,
$(t > 0, F_{pq}(t) > 1-t) \Rightarrow F_{f_p^n f_q^n}(\varphi(t)) > 1 - \varphi(t)$, *where* $\varphi \in \Phi_1$.
Then f *has a fixed point.*

The proof follows from *Lemma2.3* with $r(t) = 1-t$ and *Corollary 1.7*

Corollary 2.7 ([7, Theorem 1.a)]). *If* $(S, \mathcal{F}, T)$ *is a complete Menger space under the t - norm* T *satisfying* $\sup_{a<1} T(a,a) = 1$ *and* $f : S \to S$ *is a continuous mapping with the property that for every* $x \in S$ *there exists* $n(x) \in \mathbf{N}$ *such that, for every* $v \in O_f(x)$,

$$(r > 0, F_{xv}(r) > 1 - r) \Rightarrow F_{f^{n(x)}xf^{n(x)}v}(g(r)) > 1 - g(r)$$

where $g : [0,\infty) \to [0,\infty)$ *is a mapping which satisfies* $\lim_{n\to\infty} g^n(r) = 0$ *for every* $r > 0$ *and* $g(u) < u$ $(\forall)u > 0$, *then* f *has a fixed point.*

The proof follows from *Theorem 2.6* and *Example 2.2.*

If T=Min and g satisfies stronger conditions of Browder type, then one obtains *Theorem 3.3* from [3].

Theorem 2.8. *Let* $(S, \mathcal{F})$ *be a complete* H *- space and* $f : S \to S$ *be a continuous mapping with the property that there exists* $p \in S$ *such that*
$(\forall)q \in O_f(p)$ $(\exists)n = n(q) \in \mathbf{N} : (t > 0, r \in O_f(q), f \circ F_{pq}(m_2(t)) < m_1(t)) \Rightarrow$
$f_\circ F_{f_p^n f_q^n}(m_2(\varphi(t)) < m_1(\varphi(t))$
$(m_1, m_2$ *are like in Lemma 2.4 and* $\varphi \in \Phi_{t_0})$, *then* f *has a fixed point which is the limit of the succesive approximations, starting from an arbitrary point of* $O_f(p)$.

The proof follows from *Lemma 2.4* and *Theorem 2.6.*

If $(S, \mathcal{F})$ is a H - space of the type $(S, \mathcal{F}, T)$ - Menger space under the t - norm T with $\sup_{a<1} T(a,a) = 1$, then from the above theorem we obtain *Theorem 1* from [8].

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A FIXED POINT APPROACH OF THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. In an adecquate Banach space the integral operator associated to the initial value problem

$$\begin{cases} u'' + f(t, u, u') = 0, \ t \geq t_0 \\ u(t_0) = U_0 \quad u'(t_0) = U_1 \end{cases} \tag{1}$$

for some $t_0 \geq 1$ (for simplicity) satisfies the requirements of the Schauder-Tychonov theorem if $f(t, u, v)$ is under a Bihari type restriction. The fixed point $u(t)$ of this operator is asymptotic to $at + b$ as $t \to +\infty$, where a, b are real constants.

1. Introduction

Starting with the paper by Bellman [3], functional analysis is frecquently involved in studying the asymptotic behavior of solutions for differential equations. Papers such as those of Massera and Schäffer [5] are now fundamental.

Another important step is made by Corduneanu [4] who uses certain function spaces to analyze those solutions which go to $+\infty$ in the same way as some positive test function g, *i.e.* solutions $x(t)$ such that $|x(t)| = O(g(t))$.

Corduneanu introduces Banach spaces like $(C_g, ||\cdot||_g)$ below:

$$C_g = \{x \in C(\mathbb{R}_+, \mathbb{R}^m) : \lambda_x > 0, |x(t)| \leq \lambda_x g(t), \ t \in \mathbb{R}\}$$

with the norm

$$||x||_g = \sup_{t \geq 0} \frac{|x(t)|}{g(t)}.$$

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Such spaces are used also by Avramescu [1] for solutions $x(t)$ such that $|x(t)| = o(g(t))$.

Following these ideas an adecquate Banach space is introduced herein to study the solutions $u(t)$ of problem (1) which go to some $a_u t + b_u$ as $t \to +\infty$, where a_u, b_u are real constants.

2. The fixed point technique applied to the study of asymptotic behavior

Consider the initial value problem

$$\begin{cases} u'' + f(t, u, u') = 0, t \geq t_0 \\ u(t_0) = U_0 \quad u'(t_0) = U_1 \end{cases}$$

when the following hold true:

(i) The function $f(t, u, v)$ is continuous in $D = \{(t, u, v) : t \in [t_0, +\infty),\ u,\ v \in \mathbb{R}\}$ and $f(t, 0, 0) = 0$ for every $t \geq t_0$.

(ii) There exist three continuous functions h, g_1, $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq h(t) g_1(\frac{|u_1 - u_2|}{t}) g_2(|v_1 - v_2|) \tag{2}$$

where for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are positive and nondecreasing,

$$A = \int_{t_0}^{\infty} s h(s) ds < +\infty \tag{3}$$

and

$$\sup_{r \geq t_0} \frac{r}{g_1(r) g_2(r)} = +\infty \tag{4}$$

and

$$g_1(0) g_2(0) = 0. \tag{5}$$

On the real linear space $X(t_0) = \{u \in C^1(t_0, +\infty; \mathbb{R}) : \lim_{t \to +\infty} u'(t) = a_u,\ \lim_{t \to +\infty} [u(t) - a_u t] = b_u;\ a_u,\ b_u \in \mathbb{R}\}$ we introduce the norm

$$||u|| = \sup_{t \geq t_0} \{|u'(t)| + |u(t) - a_u t| + \frac{|u(t)|}{t}\}.$$

Proposition 2.1. *The space* $(X(t_0), ||.||)$ *is complete.*

Proof. Consider $(f_n)_{n\geq 1}$ a Cauchy sequence in $X(t_0)$. Then the sequence of derivatives, $(f'_n)_{n\geq 1}$, is uniformly convergent on $[t_0, +\infty)$ to a continuous function g while $(f_n)_{n\geq 1}$ is pointwise convergent on $[t_0, +\infty)$ to a certain function f. The Weierstrass theorem regarding sequences of derivable functions (see Niculescu [7], Theorem 6.5.4, p. 283-284) shows that f is a C^1–function on $[t_0, +\infty)$ and $f' = g$. Furthermore, $(f_n)_{n\geq 1}$ has local uniform convergence to f since for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$\left|\frac{f_n(t)}{t} - \frac{f(t)}{t}\right| < \varepsilon,\ t \geq t_0$$

for every $n \geq N(\varepsilon)$. In this way,

$$\sup_{t\in[t_0,T]} |f_n(t) - f(t)| \leq \varepsilon T,\ n \geq N(\varepsilon)$$

for $T > 0$ fixed. The usual $\varepsilon - N(\varepsilon)$ technique shows that $f \in X(t_0)$ and

$$\lim_{n\to+\infty} a_{f_n} = a_f,\ \lim_{n\to+\infty} b_{f_n} = b_f$$

and $\frac{f_n(t)}{t}$ goes uniformly to $\frac{f(t)}{t}$ on $[t_0, +\infty)$ as $n \to +\infty$ and $f_n(t) - a_{f_n}t$ goes uniformly to $f(t) - a_f t$ on $[t_0, +\infty)$ as $n \to +\infty$.

Finally, f_n goes to f in $X(t_0)$ as $n \to +\infty$. □

The operator $T : X(t_0) \to X(t_0)$ is defined by

$$(Tu)(t) = U_1 t + U_0 - \int_{t_0}^{t} (t-s) f(s, u, u')ds.$$

One has the following estimations:

$$\begin{cases} |(Tu_1 - Tu_2)'(t)| \leq g_1(\|u_1 - u_2\|) g_2(\|u_1 - u_2\|) \int_{t_0}^{\infty} h(s)ds \\ \frac{|(Tu_1 - Tu_2)(t)|}{t} \leq g_1(\|u_1 - u_2\|) g_2(\|u_1 - u_2\|) \int_{t_0}^{\infty} h(s)ds \end{cases}$$

and

$$\left|\int_t^{\infty} |f(s,u_1,u_1')|\,ds - \int_t^{\infty} |f(s,u_2,u_2')|\,ds\right| \tag{6}$$
$$\leq \frac{g_1(\|u_1-u_2\|)g_2(\|u_1-u_2\|)}{t}\int_t^{\infty} sh(s)ds$$

for $u_1, u_2 \in X(t_0)$ and $t \geq t_0$. The values of a_{Tu}, b_{Tu} can be computed from

$$\begin{cases} a_{Tu} = U_1 - \int_{t_0}^{\infty} f(s,u,u')ds \\ b_{Tu} = U_0 + \int_{t_0}^{\infty} sf(s,u,u')ds, \end{cases}$$

since $\lim_{t\to+\infty}\{t\int_t^{\infty} f(s,u,u')ds\} = 0$ for every $u \in X(t_0)$. Using $u(t) = 0$, this follows easily from (6) since $f(t,0,0) = 0$ for $t \geq t_0$. We need also the formula $(Tu)(t) - a_{Tu}t = U_0 + \int_{t_0}^{t} sf(s,u,u')ds + t\int_t^{\infty} f(s,u,u')ds$.

All of this shows that $TX_0 \subseteq X_0$ and

$$\|Tu_1 - Tu_2\| \leq 3Ag_1(\|u_1-u_2\|)g_2(\|u_1-u_2\|),\ u_1, u_2 \in X(t_0). \tag{7}$$

A compactness criterion on $X(t_0)$ is the one below.

Proposition 2.2. *Let $M \subset X(t_0)$ satisfy the next properties:*

(i) *For every $\varepsilon > 0$ there exists $L > 0$ such that*

$$|u'(t)| \leq L,\ \ |u(t) - a_u t| \leq L$$

for every $t \geq t_0$ and $u \in M$.

(ii) *For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$|u'(t_1) - u'(t_2)| < \varepsilon,\ \ |u(t_1) - u(t_2) - a_u(t_1 - t_2)| < \varepsilon$$

for every t_1, $t_2 \geq t_0$, with $|t_1 - t_2| < \delta(\varepsilon)$, and $u \in M$.

(iii) *For every $\varepsilon > 0$ there exists $Q(\varepsilon) > 0$ such that*

$$|u'(t) - a_u| < \varepsilon,\ \ |u(t) - a_u t - b_u| < \varepsilon$$

for every $t \geq Q(\varepsilon)$ and $u \in M$.

Then, M is relatively compact in $X(t_0)$.

Proof. A simple consequence of the compactness criterion on $C_n^f = \{u \in C(t_0, +\infty; \mathbb{R}^n) : \lim_{t \to +\infty} u(t) = l_u,\ l_u \in \mathbb{R}^n\}$. See Avramescu [2]. □

We introduce the straight line $x_0(t) = U_1 t + U_0$. Thus, $T(0) = x_0$. According to (4) $\sup_{r \geq t_0} \frac{r}{g_1(||x_0||+r)g_2(||x_0||+r)} = +\infty$ and from (3) there exists $b \geq ||x_0||$ such that $3\int_{t_0}^{\infty} sh(s)ds \leq \frac{b}{g_1(||x_0||+b)g_2(||x_0||+b)}$.

The set $D_0 = \{u \in X(t_0) : ||u - x_0|| \leq b\}$ is closed and convex.

Theorem 2.3. *The requirements below are satisfied:*

(a) $TD_0 \subseteq D_0$.

(b) *If H is bounded in $X(t_0)$ then TH is relatively compact in $X(t_0)$.*

(c) *The operator T is continuous in $X(t_0)$.*

Proof. For (a) one has the estimation

$$||Tu - x_0|| \leq 3g_1(||x_0|| + b)g_2(||x_0|| + b) \int_{t_0}^{\infty} sh(s)ds \leq b.$$

For (b) one has to test the properties (i), (ii) and (iii) from Proposition 2.2. For (i), if $M = \sup_{h \in H} ||h||$ then

$$||Th|| \leq L = ||x_0|| + 3Ag_1(M)g_2(M),$$

according to (7) since $T(0) = x_0$. For (ii), if $t_1 \geq t_2 \geq t_0$ then one has the following estimations:

$$|(Tu)'(t_1) - (Tu)'(t_2)| \leq g_1(M)g_2(M) \int_{t_2}^{t_1} h(s)ds$$

and

$$|(Tu)(t_1) - (Tu)(t_2) - a_{Tu}(t_1 - t_2)|$$

$$\leq g_1(M)g_2(M)\{\int_{t_0}^{\infty} h(s)ds + \int_{t_2}^{t_1} h(s)ds\}(t_1 - t_2).$$

For (iii), again, one has the estimations below:

$$|(Tu)'(t) - a_{Tu}| \leq g_1(M)g_2(M)\int_t^{\infty} h(s)ds$$

and

$$|(Tu)(t) - a_{Tu}t - b_{Tu}| \leq g_1(M)g_2(M)\int_t^{\infty} sh(s)ds + |R(u)|(t),$$

where $R(u)(t) = t\int_t^{\infty} f(s,u,u')ds$. According to (6), $\lim_{t\to+\infty} R(u)(t) = 0$ uniformly with respect to $u \in H$ since $R(0) = 0$ and

$$|R(u)(t)| \leq g_1(M)g_2(M)\int_t^{\infty} sh(s)ds,\ u \in H.$$

The requirement (c) is justified by (5). If u_n goes to u in $X(t_0)$ as $n \to +\infty$ then

$$||Tu_n - Tu|| \leq 3Ag_1(||u_n - u||)g_2(||u_n - u||) \to 0$$

as $n \to +\infty$. □

According to the Schauder-Tychonov theorem (see Rus [9], Theorem 7.42, p. 58-59) the operator T has a fixed point $u(t)$ in $X(t_0)$. This is exactly the desired solution of problem (1).

Note. Whenever (2) is replaced by the Lipschitz type restriction

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq h(t)\left(\frac{|u_1 - u_2|}{t} + |v_1 - v_2|\right)$$

and (3) is valid the operator $T : X(t_0) \to X(t_0)$ becomes a contraction under some Bielecki type norm. See Mustafa [6].

In what regards the term $\frac{|u_1-u_2|}{t}$ in (2) it appears to be a natural requirement. See Rogovchenko [8], Theorems 1-3.

Since (4) implies that

$$\int_{t_0}^{\infty} \frac{ds}{g_1(s)g_2(s)} \geq \sup_{r\geq t_0} \frac{r}{g_1(r)g_2(r)} = +\infty,$$

which is the standard Bihari condition, (4) itself can be properly refer to by a Bihari type condition.

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ON THE RELATION BETWEEN ABSOLUTELY SUMMING OPERATORS AND NUCLEAR OPERATORS

CARMEN PÂRVULESCU AND CRISTINA ANTONESCU

Abstract. It is known that every absolutely summing operator acting between $C(\Omega)$, where Ω is an arbitrary compact set, and a space, F, with the Radon-Nikodym property is nuclear.

The purpose of this paper is to show that composing a weakly compact operator with an absolutely summing one we obtain a nuclear operator even the space, F, has not the Radon-Nikodym property.

We give, also, a proof for the "factorisation" theorem and we put an interesting problem.

1. Preliminaries

1.1. **Notations.** Let E, F be Banach spaces over the field Γ. Γ is the set of real, or complex, numbers.

1) $L(E,F) := \{T : E \to F : T \text{ is linear and bounded}\}$.

2) $E^* := L(E,\Gamma)$.

3) $U_E := \{x \in E : \|x\| \leq 1\}$.

4) Let $e^* \in E^*$ and $e \in E$, $\langle e, e^* \rangle := e^*(e)$.

5) Let $e^* \in E^*$ and $f \in F$. We denote by $e^* \otimes f$ the following operator: $e^* \otimes f : E \to F, (e^* \otimes f)(e) = \langle e, e^* \rangle \cdot f$.

1.2. **Definition [5].** Let E be a Banach space. A subset $A \subset E$ is said to be **weakly compact** if it is compact in the weak topology, $\sigma(E, E^*)$.

1.3. **Definition [5].** Let E, F be Banach spaces and $T \in L(E,F)$. T is said to be **weakly compact** if TU_E is relatively weakly compact.

Key words and phrases. Absolutely summing operator, nuclear operator, weak compact operator, space with Radon-Nikodym property, space with the extension property.

1.4. **Definition [3].** Let E, F be Banach spaces. An operator $T \in L(E,F)$ is called **absolutely summing** ($T \in ABS(E,F)$) if there is a constant $c \geq 0$ such that:

$$\sum_{i=1}^{n} \|Tx_i\| \leq c \cdot \sup \left\{ \sum_{i=1}^{n} |\langle x_i, x^* \rangle| : x^* \in U_{E^*} \right\},$$

for every finite family of elements $x_1, x_2, ..., x_n \in E$.

For every $T \in ABS(E,F)$ we define: $\pi(T) := \inf c$.

1.5. **Definition [3].** Let E, F be Banach spaces. An operator $T \in L(E,F)$ is said to be **nuclear** if there is a representation:

$$T = \sum_{i=1}^{\infty} e_i^* \otimes f_i,$$

where $e_i^* \in E^*$ and $f_i \in F$, for every natural i.

We write $T \in N(E,F)$.

1.6. **Definition [1].** Let $(\Omega, \sum, \mu)$be a finite real measure space and E a Banach space.

We say that E has the **Radon -Nikodym property** with respect to $(\Omega, \sum, \mu)$ if, for each $\mu-$continuously vector measure $\vartheta : \sum \to E$ of bounded variation, there is $g \in L_1(\mu, E)$ such that $\vartheta(A) = \int_A g d\mu$ for every $A \in \sum$:

We say that E has the **Radon -Nikodym property** (E has the **R.N.p**) if E has the Radon -Nikodym property with respect to every finite real measure space.

1.7. **Examples of spaces with the R.N.p.[1].** 1) Every reflexive space.(Phillips' theorem)

2) Let F be a Banach space. If $E = F^*$ and, in addition, E is separable then E has the R.N.p.

3) Let I be an arbitrary set, $I \neq \emptyset$. Then $l_1(I)$ has the R.N.p.

4) Let $1 < p < \infty$ and X be a space with the R.N.p. Then $L_p(X, \mu)$ has the R.N.p.

1.8. **Examples of spaces without the R.N.p.[1].** 1) $(c_0, \|\cdot\|_\infty)$, where $c_0 := \{x = \{x_n\}_n : x_n \in \Gamma \text{ and } x_n \to 0\}$, $\|x\|_\infty := \sup_n |x_n|$.

2) $L_1(\mu)$, where μ is a finite and non-purely atomic measure.

1.9. **Definition [7].** The Banach space $\widetilde{F}$ is said to have **the extension property** if every $T \in L(E, \widetilde{F})$, where E is an arbitrary Banach space, can be extended to any Banach space $\widetilde{E}$ containing E as a suspace, where the extension $\widetilde{T} : \widetilde{E} \to \widetilde{F}$ is linear and bounded.

1.10. **Example [7].** The Banach space $(l_\infty(\Gamma), \|\cdot\|_\infty)$ has the metric extension theory.

1.11. **Theorem (the "Domination" theorem)[3].** Let $T \in L(E,F)$. $T \in ABS(E,F)$ if and only if there is a regular normalized measure μ on U_{E^*} such that:

$$\|Tx\| \leq \pi(T) \cdot \int_{U_{E^*}} |\langle x, x^* \rangle| \, d\mu(x^*),$$

for every $x \in E$.

1.12. **Corolar [3].** Let J be the inclusion from $C(\Omega)$ into $L_1(\mu)$, where Ω is a compact set and μ is a measure with the properties from the "domination" theorem. Then:

$$J \in ABS(C(\Omega), L_1(\mu)).$$

1.13. **Theorem (the "Factorization" theorem) [3].** Let E, F be Banach spaces, F having the extension property, and $T \in ABS(E,F)$. Then there exist the operators:

1) $A \in L(E, C(U_{E^*}))$,

2) $Y \in L(L_1(\mu), F)$, where μ is a regular, positive, normalised, Borel measure on U_{E^*}, likewise in the "domination" theorem,

such that: $T = Y \circ J \circ A$, where J is the inclusion from $C(U_{E^*})$ into $L_1(\mu)$.

Proof. (authors'adaptation)

Construction

1) Let $A : E \to C(U_{E^*})$ be defined, for every $x \in E$, by: $Ax := J_x$, where $J_x : E^* \to \Gamma, J_x(x^*) = \langle x, x^* \rangle$, for every $x^* \in E^*$.

From the definition it follows that $A \in L(E, C(U_{E^*}))$, $\|Ax\| = \|J_x\| = \|x\|$, the corolar of the Hahn-Banach teorem, and further $\|A\| = 1$.

2) We consider now the inclusion,J, from $C(U_{E^*})$ into $L_1(\mu)$. From the corolar 1.12 we obtain that $J \in ABS(C(U_{E^*}), L_1(\mu))$.

3) Let $\widetilde{Y} : Im(J \circ A) \to F$ be defined by $\widetilde{Y}((J \circ A)x) := Tx$.

We prove now that $\widetilde{Y} \in L(Im(J \circ A), F)$.

a) The linearity is obvious

b) $\left\|\widetilde{Y}((J \circ A)x)\right\| = \|Tx\| \leq \pi(T) \cdot \int\limits_{U_{E^*}} |\langle x, x^* \rangle| \, d\mu(x^*) =$

$= \pi(T) \cdot \int\limits_{U_{E^*}} |J_x(x^*)| \, d\mu(x^*) = \pi(T) \cdot \|J_x\| = \pi(T) \cdot \|(J \circ A)x\|.$

So $\widetilde{Y}$ is bounded on $Im(J \circ A)$.

F has the extension property so $\widetilde{Y}$ can be extended to Y defined on $L_1(\mu)$.

In conclusion we obtain the announced factorization of T. □

1.14. **Remark [3].** If F has not the extension property the factorization of an operator

$T \in ABS(E, F)$ is as follows:

$T = Y \circ J \circ A$, where $A \in L(E, C(U_{E^*}))$, J is the inclusion from $C(U_{E^*})$ into $\overline{Im(J \circ A)E} \subset L_1(\mu)$ and $Y \in L\left(\overline{Im(J \circ A)E}, F\right)$.

1.15. **Theorem (Davies, Figiel, Johnson, Pelczynski) [5].** Let E, F be Banach spaces. Every weak compact operator $S : E \to F$ can be factorised through a reflexive Banach space.

1.16. **Theorem [3].** Let Ω be a compact set and F a space with the R.N.p. Then every $T \in ABS(C(\Omega), F)$ is nuclear.

2. Result

2.1. Theorem. Let E, F, G be Banach spaces, F having, in addition, the extension property.

If $S : F \to G$ is weak compact and $T \in ABS(E, F)$ then $S \circ T$ is nuclear.

Proof. From the factorisation theorem it follows that $T = Y \circ J \circ A$, likewise the factorization theorem, and from teorem 1.15 it follows that $S = U \circ V$, where $V \in L(F, R)$, R being a reflexive space, and $U \in L(R, G)$.

Further $S \circ T = U \circ V \circ Y \circ J \circ A$.

From the following facts $V \circ Y \circ J \in ABS\left(C\left(U_{E^*}, R\right)\right)$ and R is a space with the R.N.p. we obtain that $V \circ Y \circ J$ is nuclear.

In conclusion $S \circ T = U \circ V \circ Y \circ J \circ A$ is nuclear. □

3. Open Problem

Let E, F be Banach spaces, F having, in addition, the R.N.p. Any $T \in ABS(E, F)$ admits a factorisation, likewise in the "factorization" theorem. So:

$T = Y \circ J \circ A$, where $A : E \to C\left(U_{E^*}\right)$, $J : C\left(U_{E^*}\right) \to \overline{Im\left(J \circ A\right)} \subset L_1\left(\mu\right)$, $Y : \overline{Im\left(J \circ A\right)} \to F$.

From the facts that $Y \circ J \in ABS\left(C\left(U_{E^*}\right), F\right)$ and F is a space with the R.N.p it follows that $Y \circ J$ is nuclear.

In conclusion $T = Y \circ J \circ A$ must be nuclear.

But it is false because we can give a contraexample.

If we consider the identity from l_1 to l_2, $I : l_1 \to l_2$, this operator is ABS and l_2 , being a Hilbert space, is a space with the R.N.p., it follows that $I : l_1 \to l_2$ must be nuclear. But it is false because $I : l_1 \to l_2$ isn't even compact.

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CLASSES OF MENGER SPACES WITH THE FIXED POINT PROPERTY FOR PROBABILISTIC CONTRACTIONS

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Abstract. We present the most important contributions to the theory of probabilistic contractions of Sehgal type and a new method of obtaining fixed point theorems on Menger spaces under Archimedean triangular norms.

1. Some history

1.1. **Introduction.** The notion of a Probabilistic Metric has been introduced by **K. Menger in 1942** as a function

$$S \times S \ni (p,q) \xrightarrow{\mathcal{F}} F_{pq} \in \mathcal{D}_+$$

where $\mathcal{D}_+$ is the set of all distribution functions F, for which $F(0) = 0$, and the following axioms are imposed:

I. $F_{xy} = \varepsilon_0$ if and only if $x = y$

II. $F_{xy} = F_{yx}$ $\forall x, y \in X$.

III_M. $F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$

Here $T : [0,1] \times [0,1] \to [0,1]$ is supposed to satisfy the conditions: T is nondecreasing in each variable , it is symmetric and $T(1,1) = 1$, $T(a,1) > 0 \; \forall \; a > 0$.

Thus the *the first idea of Menger* was to use distribution functions instead of nonegative real numbers as values of the metric .The second *useful idea* was the formulation of the triangle inequality (III_M) .

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Key words and phrases. Menger spaces, fixed point theorems, probabilistic contractions.

A. Wald in 1943 proposed the following triangle inequality , accepted by Menger himself in subsequent works:

$$(III_w) \quad F_{pq} \geq F_{pr} * F_{rq}$$

which admits a natural interpretation

$$Prob\{dist(p,q) < x\} \geq Prob\{dist(p,r) + dist(r,q) < x\}$$

if the "distances" are considered independent random variables .

B.Schweizer &A.Sklar in 1960 (see also [47]) reconsidered the problem of the triangle ienquality by imposing the associativity for T; thus $([0,1],T)$ is a commutative semigroup with 1 as unity and T(a,b)≤Min{a,b}.Min, Prod and $W = Max\{Sum - 1, 0\}$ are the most important t-norms ; $(S, \mathcal{F}, T)$ is called *PM-space* and T a triangular norm or *t-norm.*

In the same works they introduce two new important notions, *the* (ε,λ)-*topology* ($\mathcal{F}$-topology, strong topology), generated by $\{N_p(\varepsilon,\lambda)\}_{\varepsilon>0,\ \lambda\in(0,1)}$ and *the* (ε,λ)-*uniformity(* $\mathcal{F}$-uniformity),generated by $\{U(\varepsilon,\lambda)\}_{\varepsilon>0,\ \lambda\in(0,1)}$(see 2.1. below).

In the same year Schweizer, Sklar and R.Thorp proved that *if* $\sup_{x<1} T(x,x) = 1$, *then the* (ε,λ) *-uniformity exists and it is metrizable.*

Later on, J.Nagata & B.Morrel and, independently, U.Hohle, proved that the above condition on T is *the weakest one* which ensures the existence of the $\mathcal{F}$-uniformity.

In 1962 -1963, A.N. Šerstnev proposed a new formulation of the triangle inequality , by means of a nondecreasing (semigroup) operation τ on $\mathcal{D}_+$(a *t-function*)

$$III_S \quad F_{pq} \geq \tau(F_{pr}, F_{rq}) \ , \ \tau : \mathcal{D}_+ \times \mathcal{D}_+ \to \mathcal{D}_+ \ , \tau(H_0) = H_0$$

and has formulated explicitly a metric-like function which seemingly agrees with the uniformity , namely

$$d(p,q) = \Sigma \frac{1}{2_n} \cdot \frac{1 - F_{pq}(\frac{1}{n})}{2 - F_{pq}(\frac{1}{n})}$$

Later on I observed that d does not verify yhe triangle inequality , and therefore this is not an adequate example.***Šerstnev*** introduced also the notion of random normed space[1].

Now we present some results concerning the

1.2. **Fixed Point Principles in Menger spaces. In 1966** V.M Sehgal & A.T Bharucha- Reid(see [50] and [51]) have introduced the notion of probabilistic contraction on Menger spaces, namely functions $f : S \to S$ s.t. there is an L , $0 < L < 1$, for which

$$\text{(PC)} \quad F_{fp,fq}(Lx) \geq F_{pq}(x), \forall x > 0,$$

and they proved the following

E1. *Any probabilistic contraction on a complete Menger space (S, F, Min) has a fixed point , which is the limit of the successive approximations, defined by $p_{n+1} = fp_n$, that is a principle of Banach- type holds for the t-norm Min .*

In 1976 , G.L.Cain and R.Kasriel proved the above theorem of Sehgal&Bharucha-Reid by a different method : *If $d_b(p,q) = sup\{x, F_{pq} \leq b\}$ is defined on (S, F, Min), then d_b is a semimetric on S and $\{d_b\}_{b\in(0,1)}$ generates the (ε, λ)-topology ;* and they obtained the Banach's principle from a classical result for $(S, \{d_b\})$.

In 1971 , H.Sherwood gave an example of a complete Menger space $(S, \mathcal{F}, T_m)$ together with a probabilistic contraction with no fixed points(see also[47]).

A fundamental result has been obtained by O. Hadžić **in 1978** (cf [9], [10], [11] and [15])which extended the class of t-norms for which th. BP. holds:

E2. *If T is continuous and T^n are equicontinuous at $x = 1$, then the B.P. holds in every complete Menger space $(S, \mathcal{F}, T)$*

[1] At the West University of Timişoara a Seminar on PM-spaces has been created in 1972. The most part of the contributions to this Seminar are due to D.Barbu,Gh.Bocşan, Gh.Constantin, I.Istrăţescu, D.Mihet, V.Radu and D.Zaharie. A number of 125 papers have been issued in preprints and most of them appeared in well known periodicals. A series of monographs (three volumes until now) has been created. Some of the topics of the Seminar are *Measures of noncompactness , Fixed points , NonArchimedean PM-spaces,Construction of deterministic metrics and Random operators &equations.*

In 1983 I have shown that the condition of continuity of T can be dropped . Also I obtained the following characterization of the *t-norms of Hadžić- type :*

E3. *The following are equivalent , for a continuous t-norm T,*

a) $\{T^n\}$ *is equicontinuous at* $x = 1$

b) $\forall\, a \in [0,1)$, $\exists\, b \in [a,1)$ *such that* $T(b,b) = b$

By using this result one can obtain the following:

E4. *Let T be a continuous t-norm . Then the B.P. holds in every complete Menger space* $(S, \mathcal{F}, T)$ *if, and only if, T is of Hadžić- type.*

E5. *The following are equivalent*

α) *There exists a complete* $(S, \mathcal{F}, T_m)$ *in which the Banach's Principle fails*;

β) *There exists a complete* $(S, \mathcal{F}, Prod)$ *complete, in which the Banach's Principle fails*;

γ) *There exists a complete* $(S, \mathcal{F}, T)$, *where T is not of Hadžić- type t-norm, in which the Banach's Principle fails.*

E6. *E4 is essentially equivalent to the classical Banach Principle.*

Remark. In the general case for equicontinuous T^n (at $x = 1$) one can use a method similar to that proposed by Cain & Kasriel, by using a countable family of pseudo-metrics:

$$b_n \nearrow 1 \ , \ T(b_n, b_n) = b_n$$

$$d_n(p,q) = inf\{t \ , \ F_{pq}(t) \geq b_n\}, n = 1, 2, ...$$

1.3. **Hicks contractions and generalizations. In 1983,** T.L.Hicks has introduced a different condition for contractions

$$(c_h) \quad \exists L < 1 \vdots F_{pq}(t) > 1 - t \ \Rightarrow \ F_{fpfq}(Lt) > 1 - Lt$$

and has shown that

E7. *Every c_h-contraction on a complete Menger space $(S,\mathcal{F},Min)$ has a unique fixed point , which is the limit of the succesive approximations .*

The idea of Hicks' proof is as follows:

- Construct a metric ρ on S, for which
- (S,ρ) is a complete metric space and f is an L strict contraction on (S,ρ).
- Apply the classical B.P.

I essentially extended the above result in two directions(see e.g. [40, 42]).

(a) *E7 remains true for every $T \geq$ W and the proof is similar to that of Hicks ;*

(b) *The result of Hicks is true for every t-norm with the property* $\sup_{t<1} T(t,t) = 1$.

If we observe that the condition (c_h) can be rewritten in the forms :

$$t > 1 - F_{pq}(t) \;\Rightarrow\; Lt > 1 - F_{fpfq}(Lt)$$

$$t > h \circ F_{pq}(t) \;\Rightarrow\; Lt > h \circ F_{fpfq}(Lt)$$

where $h(u) = 1 - u$, $u \in [0,1]$, we can give more extensions.

Let $\mathcal{M}$ be the family of mappings m:$[0,\infty] \to [0,\infty]$, such that

a) $m(t) = 0 \iff t = 0$;

b) m is continuous ;

c) $m(t+s) \geq m(t) + m(s)$.

Lemma. *Let us suppose that*

(i) $m \in \mathcal{M}$;

(ii) $h : [0,1] \to [0,\infty]$ *is a continuous decreasing function, and* $h(1) = 0$;

(iii) $(S,\mathcal{F},T)$ *is a Menger space,with* $T \geq T_h$.

Then

$$\rho(p,q) = k_{mh}(p,q) := sup\{t \;,\; m(t) \leq h \circ F_{pq}(t)\}$$

gives a metric which generates the (ε,λ)*-uniformity.*

Theorem 1.3. *Consider a complete Menger space $(S, \mathcal{F}, T)$ and a self mapping f of S such that*

$$h \circ F_{pq}(t) < m(t) \Rightarrow h \circ F_{fpfq}(Lt)] < m(Lt)$$

where h is as the Lemma , with $h(0) < \infty$. Then f has a unique fixed point (which is the limit of successive approximations).

Remarks.

(1) If $T \geq T_h$, then f is a ρ-strict contraction and the classical BP can be applied . In this case $h(0)$ may be ∞.

(2) The formula

$$\rho(p,q) = sup\{t\ ,\ m_1(t) \leq 1 - F_{pq}(m_2(t))\}$$

where $F_{pq}(t) = Prob(d(p,q) < t)$, gives a metric for the convergence in probability , extending the case $m_1(t) = m_2(t) = t$, when one obtains the Ky Fan metric.

(3) The above ideas and methods have been used and extended by many authors(see [4], [5], [6, 7], [11, 13, 14, 15], [25], [48], [54]).

1.4. **Some comparisons.** The following examples, essentially taken from the very interesting paper [48], clarify the independence of the two types of contractions:

1. Let (S, d) be a metric space, and $f : S \to S$ an $L-$ isometry: $d(fp, fq) = Ld(p,q)$.
If we set $F_{pq}(x) = \frac{x}{x+d(p,q)}$, then we obtain a Menger space (S, F, Min).
   (a) f is a probabilistic contraction, since $F_{fpfq}(L\varepsilon) = F_{pq}(\varepsilon)$.
   (b) It is easily seen that $\rho(p,q) = 2d(p,q)/(d(p,q) + sqr(d^2(p,q) + 4d(p,q)))$;and $\rho(p,q) \to 1$ for $d(p,q) \to 1$,so f cannot be a strict contraction on (S,ρ).
2. Let S={0,1,2,...} and d(p,q)= max$\{L^p, L^q\}, L \in (0,1)$. Define

$$F_{pq}(x) = \begin{cases} 0, & x \leq d(p,q) \\ 1 - d(p,q), & d(p,q) \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

If we take $f(p) = p + 1$, then f is a Hicks contraction . Now if $1 \geq x > L_1$, $x > Ld(p,q)$, then $d(fp, fq) = Ld(p,q) < x \leq 1$,and $F_{fp\ fq}(x) = F_{p+1\ q+1}(x) = 1 - d(fp, fq)$. On the other hand $F_{pq}(\frac{1}{L_1}) = 1$, that is $F_{fp\ fq}(x) < F_{pq}(\frac{x}{L_1})$ and so f is not a probabilistic contraction.

3. Generally , if we have a probabilistic contraction f, then $\rho(p,q) < \varepsilon \Rightarrow \varepsilon > 1 - F_{pq}(\varepsilon) \Rightarrow F_{pq}(\varepsilon) > 1 - \varepsilon \Rightarrow F_{fpfq}(\varepsilon) \geq F_{fpfq}(L\varepsilon) \geq F_{pq}(\varepsilon) > 1 - \varepsilon \Rightarrow \rho(fp, fq) < \varepsilon$, thus $\rho(fp, fq) \leq \rho(p,q)$,that is f is nonexpansive.This explains in some sense the counterexamples of type Sherwood.For more details,examples and counterexamples, see [48], [42] and [15].

2. The "fixed point property" for t-norms

2.1. **Probabilistic (semi)metric spaces.** Let $\mathcal{D}_+$ be the family of all distribution functions F (nondecreasing and left continuous on $\mathbf{R}$, with $\inf F = 0$ and $\sup F = 1$) for which $F(0) = 0$. For every $a \geq 0$, ε_a will be the unique element of $\mathcal{D}_+$ for which $\varepsilon_a(a+) - \varepsilon_a(a) = 1$.

Definition 2.1.1 (cf. [46],[47],[7],[42]). Let X be a nonempty set and $\mathcal{F} : X \times X \longrightarrow \mathcal{D}_+$ a given mapping ($\mathcal{F}(x,y)$ will be denoted by F_{xy}). The pair $(X, \mathcal{F})$ is called a *probabilistic semimetric space* (shortly PSM-space) if

I. $F_{xy} = \varepsilon_0$ if and only if $x = y$

II. $F_{xy} = F_{yx}\ \forall x, y \in X$.

If any kind of "triangle inequality" is verified we use the term *probabilistic metric space* (PM-space). The weakest one is that proposed in [46]:

III_{SS} $[F_{xy}(t) = 1, F_{yz}(t) = 1] \Rightarrow F_{xz}(t+s) = 1$

If there exists a triangular norm T [46] such that

III_M $F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$

then one uses the term *Menger space.* A more general form for III_M, giving σ-*Menger spaces,* has been formulated by using some operations σ on $[0,\infty)$, instead of the addition (cf. [42], [44]).

In [18] is proposed the inequality

$$III_H\ \forall\varepsilon > 0\ \exists\delta > 0\ s.\,t.\ [F_{xy}(\delta) > 1-\delta, F_{yz}(\delta) > 1-\delta] \Rightarrow F_{xz}(\varepsilon) > 1-\varepsilon$$

which has been generalizated as

$$III_h\ \forall\varepsilon > 0\ \exists\delta > 0\ s.t.\ [h\circ F_{xy}(\delta) < \delta\ ,\ h\circ F_{yz}(\delta) < \delta] \Rightarrow h\circ F_{xz}(\varepsilon) < \varepsilon$$

by using additive generators h (cf. [39], [41], [42]).

For every PSM-space $(X,\mathcal{F})$ we can consider the sets of the form

$$U_{\varepsilon,\lambda} = \{(x,y) \in X\times X, F_{xy}(\varepsilon) > 1-\lambda\},\ \varepsilon > 0, \lambda \in (0,1)$$

which generates a semiuniformity denoted by $\mathcal{U}_{\mathcal{F}}$ and a topology $\mathcal{T}_{\mathcal{F}}$. Namely,

$$\mathcal{O} \in \mathcal{T}_{\mathcal{F}} \text{ iff } \forall x \in \mathcal{O}\ \exists\varepsilon > 0,\ \lambda \in (0,1) \text{ s. t. } U_{\varepsilon,\lambda}(x) \subset \mathcal{O}$$

Actually $\mathcal{U}_{\mathcal{F}}$ can also be generated by the family of the sets $V_\delta := U_{\delta,\delta}$.

Proposition 2.1.2. *Let $(X,\mathcal{F})$ be a PSM-space and define the two- place mapping*

(1) $k(x,y) = \sup\{t | t \le 1 - F_{xy}(t)\}$. *Then k is a semi-metric (of Ky Fan type) on X and*

(2) $k(x,y) < \delta \Leftrightarrow F_{xy}(\delta) > 1-\delta\ , \forall\delta > 0,$

which shows that k generates the topology $\mathcal{T}_{\mathcal{F}}$ (and the semiuniformity $\mathcal{U}_{\mathcal{F}}$).

The *proof* is easy to reproduce (cf. [16], [40], [17]).

Examples 2.1.3.

(i) If d is a semi-metric on X and we set $F_{xy} := \varepsilon_{d(x,y)}$ then $(X, \varepsilon_{d(\cdot,\cdot)})$ is a PSM-space and $k(x,y) = \min(d(x,y),1)$.

(ii) Let X be the family of all classes of $\mathbf{R}$-valued random variables on a probability measure space $(\Omega,\mathcal{K},P)$. If we set $\mathcal{F}(x,y) = F_{|x-y|}$, the distribution function of $|x-y|$, then $(X,\mathcal{F},W)$ is a Menger space and k is the Ky Fan metric of the convergence in probability (cf. [19]). Here **R** can be replaced e.g. by any separable metric space (with $|x-y| = dist(x,y)$).

It is to be noted that, generally, k *need not be a metric.* In order to ensure the verification of the triangle inequality for k, T. L. Hicks [17] proposed the following form of the triangle inequality for $(X,\mathcal{F})$:

$$III^1.\ [F_{xy}(t) > 1-t, F_{yz}(s) > 1-s] \Rightarrow F_{xz}(t+s) > 1-(t+s)$$

and he observed that the property III^1 holds for every Menger space $(X,\mathcal{F},T)$ for which $T \geq W$.

As a matter of fact one has the following

Proposition 2.1.4. *Let T be a t-norm such that the property (III^1) holds for every Menger space (X,F,T). Then $T \geq W$.*

Proof. This will follow from the following well known example. Let $X = \{x,y,z\}$, $F_{xy} = F_{yx}, F_{yz} = F_{zy}, F_{xz} = F_{zx}$, where

$$F_{xy}(t) = \begin{cases} 0 & t \leq 0 \\ a & t \in (0,1] \\ 1 & t > 1 \end{cases}, \quad F_{yz}(t) = \begin{cases} 0 & t \leq 0 \\ b & t \in (0,1] \\ 1 & t > 1 \end{cases},$$

$$F_{zx}(t) = \begin{cases} 0 & t \leq 0 \\ T(a,b) & t \in (0,1] \\ 1 & t > 1 \end{cases}$$

and $F_{xx} = F_{yy} = F_{zz} = \varepsilon_0$. Then $(X,\mathcal{F},T)$ is a Menger space (for which T is the best t-norm) and $k(x,y) = 1-a, k(y,z) = 1-b$, while $k(x,z) = 1-T(a,b)$. Thus we see that $k(x,z) \leq k(x,y)+k(y,z) \Leftrightarrow T(a,b) \geq a+b-1$. □

Remark 2.1.5. Let $(X,\mathcal{F},T)$ as in the above proof and suppose that $T(a,b) < a+b-1$. Therefore $0 < a,b < 1$ and there exists $p > 1$ such that $((1-a)^{\frac{1}{p}} + (1-b)^{\frac{1}{p}})^p > 1-T(a,b)$. Thus $(1-a)^{\frac{1}{p}} + (1-b)^{\frac{1}{p}} > (1-T(a,b))^{\frac{1}{p}}$ and we see that k_p, given by $k_p(u,v) = \sup\{t | t^p \leq 1-F_{uv}(t)\}$, is verifying the triangle inequality. This shows that the more general formulae proposed in [39], [41] and [44] can give metrics in many situations.

It is easy to see that for each $m \in \mathcal{M}$ there exists $t_m > 0$ such that $m : [0, t_m) \longrightarrow [0, \infty)$ is strictly increasing and invertible. If we set, for any PSM-space $(X, \mathcal{F})$,

(1_m) $k_m(x,y) = \sup\{t | t \geq 0, m(t) \leq 1 - F_{xy}(t)\}$

then k_m is a semi-metric. Moreover

(2_m) $k_m(x,y) < \delta \Leftrightarrow F_{xy}(\delta) > 1 - m(\delta)$

from which it follows that k_m generates $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{U}_{\mathcal{F}}$.

This suggests the following definition, which extends (III^1):

Definition 2.1.6. A PSM-space $(X, \mathcal{F})$ for which takes place the following triangle inequality

$$III^m. \; [F_{xy}(t) > 1 - m(t), F_{yz}(s) > 1 - m(s)] \Rightarrow F_{xz}(t+s) > 1 - m(t+s)$$

is called *PM-space of type* $\mathcal{M}$.

In [35] there are presented some fixed point theorems in these classes of PM-spaces.

2.2. **On the fpp for triangular norms.** As we have seen,probabilistic contractions have been introduced by V. M. Sehgal [50]. It is now well known that every probabilistic contraction on a complete Menger space (S, $\mathcal{F}$, Min) has a unique fixed point, which is the limit of succesive approximations. In [53] H. Sherwood constructed complete Menger spaces together with probabilistic contractions which do not have fixed points. O. Hadžić [9] introduced a class of t-norms for which the contraction principle holds [10]. In [38] we proved that a continuous t-norm has the fixed point property iff it is of Hadžić-type.

In the present section, we further investigate the fixed point property of t-norms, by using the structure of continuous t-norms as given in [32]. Essentially we prove that *a t-norm does not have the fpp iff in a neighborhood of 1 it has a behavior similar to that of* $W = Max(Sum - 1, 0)$. Thus the counterexample of H. Sherwood can be generally used for all t-norms which are not of Hadžić-type.

Let I denote the closed unit interval. A t-norm is a two-place function $T: I \times I \to I$ such that T is associative, commutative, nondecreasing in each place and such that $T(a,1) = a$ for each $a \in I$. For a fixed t-norm T, T^m is defined inductively on I by

$$(1) \quad T^1(x) = x,\ T^{m+1}(x) = T(T^m(x), x)$$

We say that T *is of h-type* (and write $T \in \mathcal{H}$) if $\{T^m\}$ is equicontinuous at $x = 1$. The following result is a consequence of [37], [38] and [32]:

Lemma 2.2.1. *(i) If T verifies the condition*

$$(2) : \forall a \in (0,1), \exists b \in [a,1),\ s.t\ T(b,b) = b$$

then $T \in \mathcal{H}$.

(ii) If $T \in \mathcal{H}$. and T is continuous, then (2) holds.

Theorem 2.2.2. *Let T be a continuous t-norm. Then $T \notin \mathcal{H}$ iff*

$$(3)\ \exists a \in [0,1) \text{ such that } T(a,a) = a, T(x,x) < x, \forall x \in (a,1)$$

Proof. By Lemma 2.1, $T \notin \mathcal{H}$ iff (2) is false. If $\exists a_0 \in (0,1)$ such that For each $b \in [a_0, 1)$ one has$T(b,b) < b$, then let

$$a = \lim_{m \to \infty} T^m(a_0).$$

Since $T^{m+1}(a_0) \leq T^m(a_0) \leq a_0$, then $a \in [0,1)$ always exists. Moreover, as $T^{2m+1}(a_0) = T(T^m(a_0), T^m(a_0))$ and T is continuous, then $a = T(a,a)$. Let $b \in (a, a_0)$. If $T(b,b) = b$, then $a = T^m(a) < b = T^m(b) \leq T^m(a_0)$, that is $b \leq a$, a contradiction.

Thus, if (2) is false then (3) holds. The converse is obvious. □

Remark 2.2.3. The number a in (3) is uniquely determined and will be denoted by a_T.

From (3) and [32] we obtain the following

Theorem 2.2.4. *Let T be a continuous t-norm. Then $T \notin H$ iff there exist $a_T \in [0,1)$ and an increasing bijection $h_T : [a_T, 1] \to [0,1]$ such that*

$$(4) : T(\alpha,\beta) = h_T^{-1}[T_*(h_T(\alpha), h_T(\beta))], \forall \alpha, \beta \geq a_T$$

where $T_* = W$ or $T_* = Prod$ (T_* depends only on T).

The following lemma is easy to reproduce:

Lemma 2.2.5. *Let T be a continuous t-norm, $T \notin \mathcal{H}$*

(i) If $(S, \mathcal{F}, W)$ is a Menger space, then $(S, e^{\mathcal{F}-1}, Prod)$ is a Menger space with the same (ε, λ)-uniformity;

(ii) If $(S, \mathcal{F}, T_*)$ is a Menger space, then $(S, h_T^{-1} \circ \mathcal{F}, T)$ is a Menger space with the same (ε, λ)-uniformity;

(iii) If $(S, \mathcal{F}, T)$ is a Menger space, then $(S, h_T \circ \mathcal{F}, T_*)$ is a Menger space with the same (ε, λ)-uniformity. (the notations are as in Theorem 2.4).

One says [38] that T *has the fixed point property* (f.p.p.) iff every probabilistic contraction on a complete Menger space $(S, \mathcal{F}, T)$ has a fixed point (it is obvious that this fixed point is unique and it can be obtained as the limit of the succesive approximations).

If $T \in \mathcal{H}$, then it is well known that T has the f.p.p. and this can be proven by different methods([10], [38], [3]) or is a consequence of the classical Banach principle [37].

For t-norms which *are not of h-type* we have the following

Theorem 2.2.6. *Let T be an arbitrary but fixed t-norm such that $T \notin \mathcal{H}$. Then the following are equivalent*

(i) T does not have the f.p.p.;

(ii) Prod does not have the f.p.p.;

(iii) W does not have the f.p.p.

Proof. Firstly we observe that the constructions in Lemma 2.5 do not change the property of f of being a probabilistic contraction. Therefore the equivalence (ii)$\Leftrightarrow$(iii) results for Lemma 2.5 (i), for $Prod \geq W$. The fact that (i)⇒(ii) or (iii) is a consequence of Lemma 2.5. (iii). The implication (ii) and (iii)⇒(i) is a consequence of Lemma 2.5. (ii), and the theorem is proved. □

In [53] it is proved by an example that W does not have the fixed point property. Thus we have the following

Corollary 2.2.7 ([38]). *If T is a continuous t-norm such that $T \notin \mathcal{H}$, then T does not have the fixed point property.*

3. Some "iff" conditions for the f.p.p. in the archimedean case

From the above it seems very clear that in order to can hope to obtain some kind of fixed point theorems in the case of Archimedean(or not of Hadzic type) t-norms, one has to impose supplementary conditions either on the probabilistic contractions or on the probabilistic metrics.

Some positive effort has been made in this sense by H. Sherwood [53], R. M. Tardiff [55], V. Radu [42, 43] and E. Părău & V. Radu [34, 35].

Nevertheless we think that the problem has not yet a satisfactory answer, especially for concrete purposes. This is seen from the following simple case of affine mappings on E-spaces:

Example 3.0. Let $L_0(0,1)$ be the space of all classes of random variables on the Lebesgue measure space $((0,1),\mathcal{L},\lambda)$ and fix the element w defined by the mapping $t \to e^{\frac{1}{t}}$. Let S be any closed (for the convergence in probability) linear subspace of $L_0(0,1)$ which contains w and 1. Now define $f$on S by

$$fp = Lp + (1-L)w$$

when L is fixed in (0, 1). It is easily seen that

$$f^n p_0 = L^n p_0 + (1-L^n)w \to w = fw$$

On the other hand, the distribution function of w has the value 0 for $x \leq e$ and $1-\frac{1}{lnx}$ for $x > e$, such that $\int_1^\infty lnx dF_w(x) = +\infty$. Therefore the conditions of [55] are not verified for $F_{\lambda w \mu w}$, with $\lambda \neq \mu$. At the same time, for every $k > 0$,

$$\int_0^1 w^k(t)dt \geq \int_\epsilon^1 w^k(t)dt \geq \int_\epsilon^1 (1+\frac{k}{t})dt = 1 - \epsilon - k \ln \epsilon \stackrel{\epsilon \to 0}{\longrightarrow} \infty$$

which shows that we cannot hope to work in any Lebesgue space $L_k(0,1)$:

$$p_0 \in L_k(0,1) \Leftrightarrow f^n p \notin L_k(0,1), n \geq 1$$

The aim of this section is to obtain a *characterization* of the probabilistic contractions, on complete Menger spaces under Archimedean t-norms, which have a fixed point. Our method of proof is very simple and is based upon a *new* family of *metrics* which all generate the strong $\mathcal{F}$-uniformity and seem to be appropriate for studying probabilistic contractions.

3.1. **A family of semi-metrics on PM- spaces.** In the following lemma we introduce a family of nonnegative functions which measure the distance between ϵ_0 and the elements of $\mathcal{D}_+$.Let k be a (fixed) positive real number.

Lemma 3.1.1. *The one-place mapping* $\delta_k : \mathcal{D}_+ \to \mathbf{R}_+$, *given by*

$$(1) \quad \delta_k(F) := \sup_{x>0}\{x^k[1 - F(x)]e^{-x}\},$$

has the following properties:

(*i*) $\delta_k(F) = 0 <=> F = \epsilon_0$;

(*ii*) *If* $F_1 \leq F_2$, *then* $\delta_k(F_1) \geq \delta_k(F_2)$;

(*iii*) $\delta_k(\lambda \circ F) \leq \lambda^k \delta_k(F), \forall \lambda \geq 1$;

(*iv*) $\delta^{k+1}e^{-\delta} \leq \delta_k(F) \leq \max\{\delta^k, \delta k^k e^{-k}\}$,

where $\delta = \delta(F) := \sup\{t | t \leq 1 - F(t)\}$ *is the écart of Ky Fan.*

(*v*) $\delta_k(F_n) \to 0 <=> F_n(x) \to 1$, *for each* $x > 0$.

Proof. We will give only the proof of (*iv*):

a) $\delta_k(F) = \sup\{x^k[1 - F(x)]e^{-x}\} \geq \delta^k[1 - F(\delta)]e^{-\delta} \geq \delta^{k+1}e^{-\delta}$;

b) If $0 < x \leq \delta$, then $x^k[1 - F(x)]e^{-x} \leq \delta^k$. If $\delta < x$, then $1 - F(x) \leq 1 - F(\delta + 0) \leq \delta$.

Therefore $x^k[1 - F(x)]e^{-x} \leq \delta x^k e^{-x} \leq \delta k^k e^{-k}$. □

Proposition 3.1.2. *Let $(S, \mathcal{F})$ be a probabilistic metric space and define*

$$(2)\quad e_k(p,q) := \delta_k(F_{pq}) = \sup_{x>0} x^k[1 - F_{pq}(x)]e^{-x}, \forall p,q \in S$$

Then

1^o e_k *is a* ***semi-metric*** *for the strong $\mathcal{F}$-topology;*

2^o e_k *generates the $\mathcal{F}$-uniformity, if the latter exists;*

3^o *If $(S, \mathcal{F}, W)$ is a Menger space, then*

$$(3)\quad (p,q) \to \theta_k(p,q) := \{e_k(p,q)\}^{\frac{1}{k+1}}$$

gives a ***metric*** *on S. Moreover, $(S, \mathcal{F})$ is complete if and only if (S, θ_k) is complete.*

Proof. 1^o and 2^o follow from Lemma 1.1. and the definitions. In order to prove 3^o, let us recall that $(S, \mathcal{F}, W)$ is a Menger space iff the following inequality holds

$$(4)\quad 1 - F_{pq}(x) \le 1 - F_{pr}(tx) + 1 - F_{rq}[(1-t)x], \forall p,q,r \in S, \forall x \in \mathbf{R}, \forall t \in [0,1]$$

If we fix $p, q, r \in S$, then we have, for each $x > 0$:

$$x^k[1 - F_{pq}(x)]e^{-x} \le x^k[1 - F_{pr}(tx)]e^{-tx} + x^k[1 - F_{rq}[(1-t)x]e^{-(1-t)x}, \forall t \in (0,1)$$

Then

$$x^k[1 - F_{pq}(x)]e^{-x} \le \frac{1}{t^k} e_k(p,r) + \frac{1}{(1-t)^k} e_k(r,q), \forall t \in (0,1)$$

This implies the inequality

$$e_k(p,q) \le \frac{1}{t^k} e_k(p,r) + \frac{1}{(1-t)^k} e_k(r,q), \forall t \in (0,1)$$

and we easily obtain that

$$\{e_k(p,q)\}^{\frac{1}{k+1}} \le \{e_k(p,r)\}^{\frac{1}{k+1}} + \{e_k(r,q)\}^{\frac{1}{k+1}}$$

that is θ_k verifies the triangle inequality.

The last part of the proposition follows from the inequality (iv) of the Lemma 1.1. □

Remark 3.1.3. The above proof shows that if, instead of (4), we have

$$(4') \quad 1 - F_{pq}(x) \leq 1 - F_{pr}(x) + 1 - F_{rq}(x), \forall p, q, r \in S, \forall x \in \mathbf{R},$$

that is $(S, \mathcal{F}, W)$ is nonArchimedean, then e_k **itself is a metric** which generates the $\mathcal{F}$ - uniformity.

3.2. **An iff condition for probabilistic contractions to have a fixed point.** We are in position to give a slight improvement of the results from [42, 43] and [34]:

Theorem 3.2.1. *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T \geq W$. If $f : S \to S$ is a probabilistic contraction, that is*

$$(5) \quad F_{fpfq}(x) \geq F_{pq}(\frac{x}{L}), \forall x \in \mathbf{R}$$

for some $L \in (0, 1)$ and all pairs $(p, q) \in S \times S$, then ***the following are equivalent***

(5.1) f has a fixed point

(5.2) There exist $p \in S$ and $k \in (0, \infty)$ such that

$$E_k(p) := \sup_{x>0}\{x^k[1 - F_{pfp}(x)]\} < \infty$$

Proof. The implication (5.1) $\Rightarrow$ (5.2) is obvious:

$$p = fp \;\Rightarrow\; F_{pfp}(x) = 1, \; \forall x > 0 \Rightarrow \; E_k(p) = 0$$

Now suppose that $E_k(p) < \infty$ for some $p \in S$ and $k \in (0, \infty)$. From the definition of δ_k we see that $\delta_k(F_{pfp}) \leq E_k(p)$. If we take into account the inequality (5), then we get

$x^k[1-F_{fpf^2p}(x)]e^{-x} \leq x^k[1-F_{pfp}(\frac{x}{L})]e^{-x} = L^k\{(\frac{x}{L})^k[1-F_{pfp}(\frac{x}{L})]\}e^{-x} \leq L^k E_k(p)$,which shows that

$$(6) \quad \theta(fp, f^2p) \leq L^{\frac{k}{k+1}}(E_k(p))^{\frac{1}{k+1}}$$

If we apply (6) for f^n, which verifies (5) with L^n instead of L, then we obtain

$$(8) \quad \sum_{n=0}^{\infty} \theta_k(f^n p, f^{n+1} p) \leq \{\sum_{n=0}^{\infty} (L^{\frac{k}{k+1}})^n\}\{E_k(p)\}^{\frac{1}{k+1}} < \infty$$

This clearly implies that $(f^n p)_{n \geq 0}$ is a Cauchy sequence in the complete metric space (S, θ_k), thus it converges to some point $p_* \in S$. Since (5) implies also the continuity of f ,then p_* is a fixed point which is uniquely determined and globally attractive: $F_{f^n p p_*}(x) \geq F_{p p_*}(\frac{x}{L^n}) \to 1$. □

Remarks 3.2.2. a) Simple examples show that f is generally not contractive relatively to θ_k(or e_k).

b)The suppremum in (5.2) may be infinite for some different values of k or for different points in S. This can be seen from the simple case of the Example3.0. and $fp = Lp + w$. Let $a \in S$ such that

$$\sup x^k [1 - F_{|a|}(x)] < \infty$$

and take $p = \lambda a + \frac{1}{L-1} w$. Our condition (5.2) is verified, for $p - fp = \lambda(1 - L)a$. Clearly f has a fixed point $p_* = \frac{1}{1-L} w$ and it is easily seen that $E_k(p_*) = \infty$.

On the other hand the inequality

$$(10) \quad \int_1^{\infty} ln x dF_{pq}(x) < +\infty$$

does not hold for pairs $p = \lambda p_*$, $q = \mu p_*$ with $\lambda \neq \mu$. Thus we could not apply the results of Tardiff [55], which imposed (10) for **all pairs** (p, q) in $S \times S$.

c) Our condition (5.2) is verified if there exists an element p such that $F_{pfp}(t_p) = 1$ for some $t_p > 0$ (H. Sherwood in [53], Corollary) imposed this condition **for all** F_{pq})

d) The condition (5.2) is verified if F_{pfp} has a finite k moment. Thus Theorem 2.1. slightly extends our results in [34, 42]:

Corollary 3.2.3. *If $T \geq W$ and $(S, \mathcal{F}, T)$ is a complete Menger space, then a given probabilistic contraction f on S has a fixed point if and only if there exist $k > 0$ and*

$p \in S$ such that

$$(11) \quad \int_{\infty}^{0} x^k dF_{pfp}(x) < +\infty$$

Proof. It is well known and easy to see that $\lim_{x\to\infty} x^k(1 - F_{pAp}(x)) = 0$, if (11) holds. □

Remark 3.2.4. A t-norm T is Archimedean if and only if there exists an increasing homeomorphism $h : [0,1] \to [0,1]$ such that

$$(12) \quad T(a,b) = h^{-1}(T_*(h(a), h(b)))$$

where $T_* = W$ or $T_* = Prod$ (see Theorem 2.2.4).

Since $ab \geq a + b - 1$ for all $a, b \in [0,1]$, then we obtain the foolowing.

Theorem 3.2.5. *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T \geq T_h$ for some increasing homeomorphism $h : [0,1] \to [0,1]$. Then a probabilistic contraction f of S has a fixed point if and only if there exist $k > 0$ and $p \in S$ such that*

$$(13) \quad \sup_{x>0} x^k[1 - h \circ F_{pfp}(x)] < +\infty$$

*The **proof** follows from the fact that $(S, h \circ \mathcal{F}, W)$ is seen to be a complete Menger space, Q.E.D.*

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BOOK REVIEWS

Jean-Pierre Aubin, *Mutational and Morphological Analysis– Tools for Shape Evolution and Morphogenesis*, Systems & Control: Foundations & Applications, Birkhäuser Verlag, Basel-Boston-Berlin 1999, xxxvii+425 pp., ISBN 0-8176-3935-7 and 3-7643-3935-7.

The aim of this book is to develop a kind of differential calculus on metric spaces (without any linear structure) both for single-valued maps between two metric spaces E, F (mutational analysis) and for maps between their power spaces $\mathcal{P}(E), \mathcal{P}(F)$ (morphological analysis). This last kind of maps includes set-valued maps (i.e. from E to $\mathcal{P}(F)$) as well as set-defined maps (i.e. maps from $\mathcal{P}(E)$ to F). This new approacch allows the treatment in a unified way of the "multiverse" of various kinds of differentials, codifferentials and derivatives–graphical, contingent, adjacent, paratingent, circatangent– for maps, sets and set-valued maps, which evolved from the harmonious universe of Fermat, Newton and Leibniz. Let's us quote from the Introduction–" The loss of differentiability paradise was the punishment inflicted on those exploring the set-valued purgatory, depriving the sinner of the grace of differential and integral calculus". The aim of the author is to regain some of this "lost paradise" by considering maps as graphs, an idea going back to Fermat and Descartes, instead of pointwise approach which consists in regarding set-valued maps as single-valued from E to $\mathcal{P}(F)$.

Author's approach is motivated by numerous applications to nonsmooth problems in viability theory, image processing, shape optimization, visual control, interval analysis, dynamic economic theory, biological morphogenesis, front propagation.

The basic notions are those of transition on a metric space, of mutational structure, and of mutation of a map. On a normed space E one can take as mutations the applications $\vartheta_v(h,x) = x + hv$, $(h,x) \in [0,1] \times E$, $v \in E$, yielding the set of directional derivatives as the mutation set of a single valued map. Despite the loss of the linearity, the author succeeds to transfer most of basic results of differential calculus and differential geometry in vector spaces to mutational calculus.

The prototypes for the concept of the mutation of a map are those of shape derivatives and velocities in tubes developed by J. Céa and J.-P. Zolesio.

The basic tools of mutational analysis, including Cauchy-Lipschitz type existence theorems for mutational equations, inverse function theorems on metric spaces, Newton's mutational method, a.o., are treated in the first two chapters, which form the first part of the book-- Mutational Analysis in Metric Spaces. The second part, Morphological and Set-Valued Analysis, contains three chapters. The third part of the book, Geometrical and Algebraic Morphology, contains two chapters.

There are also an Appendix dealing with set topologies, coarea formula and variational equations, Gronwall and Filippov estimates, differential inclusions.

The author, a well known specialist in the field, has published a lot of research papers and several monographs (from which two at Birkhäuser– *Viability Theory* (1991) and *Set-Valued Analysis* (1990), the last jointly with H. Frankowska).

Providing new tools for the study of shapes and and images, problems appearing in many fields of current research, this new monograph will get a large audience, including mathematicians, computer scientists, physicists, biologists (as stated in the Introduction, the author intends to treat biological applications in a forthcomming monograph).

S. Cobzaş

L o k e n a t h D e b n a t h, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser Verlag, Basel-Boston-Berlin 1997, 616 pp., ISBN: 3-7643-3902-0

The book provides an introduction to nonlinear partial differential equations and to the basic methods for finding the solutions of these equations. In order to make the book self-contained, the first chapter deals with linear partial differential equations and their methods of solution with examples and applications. The book is intended to serve as a reference work for those seriously interested in advance study and research in the subject, for its applications to other fields of applied mathematics, mathematical physics and engineering science.

The book is designated as a new source for modern topics dealing with nonlinear phenomena and their applications for future development of this subject. Its main features are:

- thorough coverage of derivation and methods of solutions for all fundamental nonlinear model equations which include Korteweg-de Vries, Boussinesq, Burgers, Fisher, nonlinear reaction-diffusion, Euler-Lagrange, nonlinear Klein-Gordon, sine-Gordon, nonlinear Schrödinger, Whitham equations,
- systematic presentation and explanation of conservation laws, weak solutions and shock waves,
- several nonlinear real-world models that include traffic flow, flood waves, chromatographic models, sediment transport in rivers, glacier flow, roll waves,
- solitons and the Inverse Scattering Transform,
- nonlinear instability of dispersive waves with applications to water waves.

The book also contains 450 worked examples, examples of applications and exercises, from the areas of partial differential equations, geometry, vibration and wave propagation, heat conduction, electric circuits, dynamical systems, fluid mechanics, plasma physics, quantum mechanics, nonlinear optics, physical chemistry, mathematical modeling, population dynamics, mathematical biology.

I strongly recommend this monograph as a reference book for a diverse readership, for graduates and professionals in mathematics, physics, science and engineering.

Damian Trif

N i c o l a e D i n c u l e a n u, *Vector Integration and Stochastic Integration in Banach Spaces,* John Wiley & Sons, Inc., New York-Singapore-Toronto 2000, xv+424 pp., ISBN 0-471-377738-4.

The aim of the present book is to develop stochastic integration in Banach spaces. Extensions of stochastic integration to Hilbert space setting has been considered by H. Kunita in 1970, who used an isometry between some L^2-type spaces of processes. This approach was not based, like in the classical real-valued stochastic analysis, on measure theory and vector integration, and cannot be extended to Banach spaces (the method essentially relies on inner product techniques). The first attempt to an approach based on integration with respect to a vector measure belongs to J. Pellaumail (1973), but due to the lack of a satisfactory vector integration theory, it works only for real-valued processes. The main idea of the author of the present book is to use integration of vector-valued functions with respect to vector measures of finite semivariation, as developed by J. K. Brooks and N. Dinculeanu, J. Math. Anal. Appl. vol. 54 (1976), 348-389, and appearing here for the first time in book form.

The book consists of two parts. The first part , Chapter 1. Vector integration (121 pp.) and the paragraphs 18, 20, 29 and 31, concern vector integration. It contains a brief and clear exposition of basics of vector measures and vector integration. Here the proofs of some theorems are only sketched or even skipped, the reader being refered to other texts as, e.g., author's treatise N. Dinculeanu, Vector Measures, Pergamon Press, Oxford 1967, or those by N. Dunford and J. Schwartz, or by J. Diestel and J.

Uhl. The core of this chapter is §5, dealing with the integration of vector-valued functions with respect to vector measures with finite semivariation. The highlights are: the Riesz representation theorem, the integral representation of continuous linear operators between L^p-spaces, the Stieltjes integral with respect to vector-valued functions (of one or two real variables) with finite semivariation.

For the part of stochastic integration the reader is assumed to be familiar with the calssical theory of real-valued stochastic processes. The key notion is that of summable process which play in this theory the role played by the square integrable martingales in the classical theory. In this context a new class of stochastic processes emerges– the processes with integrable semivariation. The stochastic integral with respect to such a process can be computed pathwise, as a Stieltjes integral with respect to a function with finite semivariation. A special attention is paid to Itô integral formula (and entire chapter, Chapter 6, is devoted to this formula).

The book is clearly written and contains a lot of material, some appearing for the first time in book form, which is of interest mainly for researchers in stochastic processes and vector integration. The book can be used as a reference text as well as a text-book for advanced graduate or post-graduate level courses.

Stefan Cobzaş

J o s é F e r r e i r o s, *Labyrinth of Tough–A History of Set Theory and its Role in Modern Mathematics,* Historical Studies–Science Networks, Birkhäuser Verlag, Basel-Boston-Berlin 1999, xxi + 440 pp. ISBN: 3-7643-5749-5.

Set theory is generally considered as the work of a single man, Georg Cantor, who developed alone this basic discipline, which deeply affected the shape of modern mathematics. This excessive concentration upon the work of G. Cantor has led to another misconception, namely that set theory has its roots in the needs of mathematical analysis, and that the successfull application of set-teoretical concepts in algebra, geometry, and all other branches of mathematics, came afterwards in the early 20th century. It is the aim of this book to show that the things are not quite so and, contrary to some general ideas on the development of modern mathematics, concepts of set theory were crucial for emerging new ideas in algebra, arithmetic and geometry. Moreover, all these developments antedate Cantor's earliest investigations in set theory, and it is likely that some motivated his work. First of all, one emphasizes the role played by R. Dedekind in the birthplace of set theory by his work on ideal theory (e.g. Dedekind introduced the actual infinite unambiguously and influentially before Cantor).

The book is divided into three parts. Part I, *The Emergence of Sets Within Mathematics,* emphasizes the flux of ideas between different domains leading to the use of set theoretical concepts in the foundation of various mathematical disciplines, culminating with the discovery around 1868-72 by Cantor of the realm of transfinite set theory. Part II, *Entering the Labyrinth-Toward Abstract Set Theory,* analyzes the crucial contributions to abstract set theory made in the last quarter of the 19th century, Cantor's path-breaking contributions and the foundational work of Dedekind, with a detailed discussion of the complex interraction between these two great mathematicians. This part ends with a discussion of the paradoxes which emerged in the newly created theory, especially connected with the notion of the "set of all sets". Part III, *In Search of an Axiom System,* contains a synthetic account of the evolution of set theory up to 1950, concentrating on foundational questions and on gradual emergence of modern axiomatization, including an analysis of the foundational crisis, constructivist alternatives, Gödel's work on the independence of the Axiom of Choice and of the Continuum Hypothesis.

The book is a fully revised and expanded version of a book published originally in Spanish "*El naciemento de la teoria de conjuntos*", Madrid

1993, which in turn was based on the doctoral dissertation of the author.

The book is of interst for a large audience, first of all researchers in the history and foundation of mathematics, but also for mathematicians working in various areas and for philosophers.

S. Cobzaş

J.-L. B r y l i n s k i, R. B r y l i n s k i, V. N i s t o r, B. T s y g a n, P. X u (Eds.) – *Advances in Geometry*, Birkhäuser, Boston - Basel - Berlin (Progress in Mathematics, vol.172), 1999, 399 + IX pp., Hardback, ISBN 0-8176-4044-4, ISBN 3-7643-4044-4

The seventeen articles included in this book are elaborated versions of communications presented at the Center for Geometry and Mathematical Physics at Penn State University between 1996 and 1998.

The spectra of communication is rather large, but they can be grouped into several fields, including:

- symplectic geometry, quantization and quantum groups (deformation quantization of Kähler manifolds, operatorial methods, Yang-Baxter equations, completely integrable Hamiltonian systems a.o.),
- geometry of holomorphic bundles over algebraic manifolds and their moduli spaces (symplectic cobordism, Witten's formula for volumes, moduli spaces of arrangements of lines in the projective plane and of linkages in the Euclidean plane, differential graded Lie algebra modeling moduli spaces of flat bundles over open Kähler manifolds),
- secondary characteristic classes for holomorphic bundles (Quillen metrics on determinant line bundles, differential geometry of bundles over some special classes of manifolds),
- quantum cohomology ring of complex flag manifolds and its algebraic properties.

I am not going to say who are the authors, because I should enumerate a list of twenty mathematicians (including some of the authors). All of them are well known for some important results in their fields of research. The articles were refereed in a journal style such they should get all the credit.

The subjects touched in the communications included in the book are among the most active in modern mathematics. The articles include both new results and review material. Of course, the book is not for beginners, the reader is supposed to have a solid background in aalgebra and differential geometry, but if this is the case, then he will benefit from the reading of the book, both for the information and for the new lights shaded on some classical topics, suggesting, maybe, new directions of research.

Paul A. Blaga