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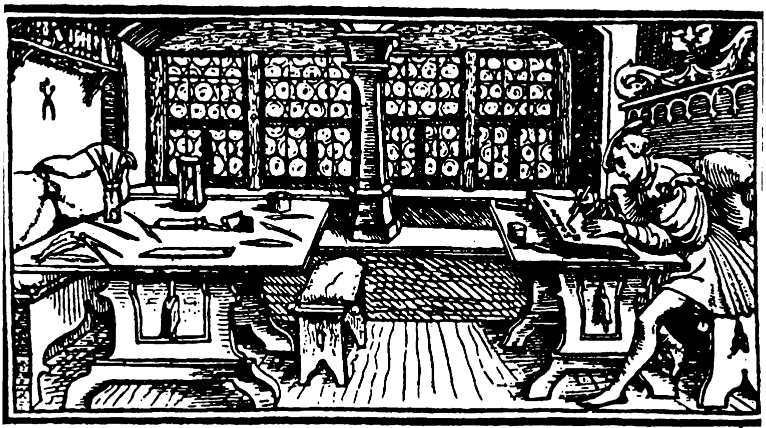

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EXTENSION OF BILINEAR FUNCTIONALS AND BEST APPROXIMATION IN 2-NORMED SPACES

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Abstract. The paper investigates the relations between the extension properties of bounded bilinear functionals and the approximation properties in 2-normed spaces.

1. Introduction

In the sixties S.Gähler ([8] and [9]) introduced and studied the basic properties of 2-metric and 2-normed spaces. Since then these topics have been intensively studied and deve loped.The references given at the end of this paper are far from being complete, containing only the papers related to the problems treated here.

The aim of the present paper is to study the relations between the extension properties of bounded bilinear functionals and the approximation properties in 2-normed spaces. In the case of bounded linear functionals on normed linear spaces the problem was first considered by R.R.Phelps [19]. For other related results see I. Singer's book [20].

In the case of Banach spaces of Lipschitz functions similar results were obtained by the authors (see [1], [18]). The case of bilinear operators on 2-normed spaces has been considered in [2].

Throughout this paper all the linear spaces will be considered over the field $K = \mathbf{R}$ or $K = \mathbf{C}$. A 2-*norm* on a linear space X of algebraic dimension at least 2 , is a functional $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ verifying the axioms:

BN 1) $\|x,y\| = 0$ if and only if x,y are linearly dependent,

BN 2) $\|x,y\| = \|y,x\|$,

BN 3) $\|\lambda x,y\| = \|\lambda\| \cdot \|x,y\|$,

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BN 4) $\|x+y,z\| \le \|x,z\| + \|y,z\|$,

for all $x,y,z \in X$ and $\lambda \in K$ (see [9])

If $\|\cdot,\cdot\|$ is a 2-norm on the linear space X then the function $\rho : X^3 \to [0,\infty)$ defined by $\rho(x,y,z) = \|x-z, y-z\|$, $x,y,z \in X$ is a 2-metric on X, in the sense of S.Gähler [8], which is translation invariant, i.e. $\rho(x+a, y+a, z+a) = \rho(x,y,z)$ for all $x,y,z \in X$ and a fixed element $a \in X$.

For a fixed $b \in X$, the function $p_b(x) = \|x,b\|$, $x \in X$, is a seminorm on X and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X, called the *natural topology induced by the 2-norm* $\|\cdot,\cdot\|$.

A pair $(X, \|\cdot,\cdot\|)$ where X is a linear space and $\|\cdot,\cdot\|$ a 2-norm an X will be called a 2-*normed space.*

Remark 1. S.Gähler [10] considered only 2-normed space over the field $\mathbf{R}$ of real numbers, but his definition automatically extends to the complex scalars too.

2. Continuity and boundedness properties for bilinear functionals.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and X_1, X_2 two subspaces of X. A 2-*functional* is an application $f : X_1 \times X_2 \to K$. The 2-functional f is called *bilinear* if:

BL 1) $f(x+x', y+y') = f(x,y) + f(x,y') + f(x',y) + f(x',y')$

BL 2) $f(\alpha x, \beta y) = \alpha\beta f(x,y)$,

for all $(x,y), (x',y')$ in $X_1 \times X_2$ and all $\alpha, \beta \in K$.

A 2-functional $f : X_1 \times X_2 \to K$ is called *bounded* if there exists a real number $L \ge 0$ (called a *Lipschitz constant* for f) such that

$$|f(x,y)| \le L\|x,y\|, \tag{2.1}$$

for all $(x,y) \in X_1 \times X_2$.

This notion of boundedness was introduced by A.G.White Jr. [20] who defined also the *norm* of a bounded bilinear functional by:

$$\|f\| = \inf\{L \ge 0 : L \text{ is a Lipschitz constant for } f\} \tag{2.2}$$

Some immediate consequences of the definition are given in:

Proposition 2.1. *(A.G.White Jr. [21].) Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, X_1, X_2 two linear subspaces of X and $f : X_1 \times X_2 \to K$ a bounded bilinear functional. Then* *a)* $f(x,y)=0$, *for any pair* $(x,y)\in X_1\times X_2$ *of linear dependent elements;*

b) $f(y,x)=-f(x,y)$ *,i.e.* f *is an alternate bilinear functional;*

c) The norm $\|f\|$ *of* f *can be calculated also by the formulae:*

$$\begin{aligned}\|f\| &= \sup\{|f(x,y)| : (x,y)\in X_1\times X_2, \|x,y\|\le 1\} \\ &= \sup\{|f(x,y)| : (x,y)\in X_1\times X_2, \|x,y\|=1\} \\ &= \sup\{|f(x,y)|/\|x,y\| : (x,y)\in X_1\times X_2, \|x,y\|>0\}.\end{aligned} \tag{2.3}$$

A.G.White Jr. [21] defined a kind of continuity for 2-functionals, called subsequently 2-continuity by S.Gähler [11].

A 2-functional $f : X_1\times X_2 \to K$, where X_1, X_2 are linear subspaces of a 2-normed space $(X, \|\cdot,\cdot\|)$ is called 2-*continuous* at $(x_0,y_0)\in X_1\times X_2$ if for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(x,y)-f(x_0,y_0)|<\varepsilon$ whenever

$$\begin{aligned} &(i) \quad \|x, y-y_0\|<\delta \text{ and } \|x_0-x, y\|<\delta, \text{ or} \\ &(ii) \quad \|x_0-x, y\|<\delta \text{ and } \|x_0, y_0-y\|<\delta \end{aligned} \tag{2.4}$$

A 2-functional f is called 2-*continuous* on $X_1\times X_2$ if it is 2-continuous at every point $(x,y)\in X_1\times X_2$.

An example of 2-continuous 2-functional is given by:

Proposition 2.2. *(A.G.White Jr. [21, Th 2.2]) If* $(X,\|\cdot,\cdot\|)$ *is a 2-normed space then the 2-functional* $\|\cdot,\cdot\|$ *is 2-continuous on* $X\times X$.

It turns out that for bilinear functionals, boundedness and 2-continuity are equivalent and 2-continuity at $(0,0)$ implies 2-continuity on whole $X_1\times X_2$:

Theorem 2.3. *(A.G.White Jr. [21, Theorems 2.3 and 2.4]) a) A bilinear functional* $f : X_1\times X_2\to K$ *is 2-continuous on* $X_1\times X_2$ *if and only if it is bounded;*

b) A bilinear functional $f : X_1\times X_2\to K$ *which is 2-continuous at* $(0,0)$ *is continuous on* $X_1\times X_2$.

S.Gähler [11] remarked that 2-continuity of a 2-functional f on $X\times X$ and its continuity with respect to the product topology on $X\times X$ are different notions. By proposition 2.2 a 2-norm is a 2-continuous functional on $X\times X$, but S.Gähler [11] exhibited an example of a 2-norm which is not continuous on $X\times X$ (with respect to the product topology) and gave conditions ensuring the continuity of a 2-norm on $X\times X$.

There are also examples of 2-functionals which are continuous on $X \times X$ with respect to the product topology but are not 2-continuous (see also S.Gähler [11]).

3. Extension theorems for bounded bilinear functionals.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, X_1, X_2 two linear subspaces of X and $f : X_1 \times X_2 \to K$ a bounded bilinear functional. The extension problem for f consists in finding a bounded bilinear functional $F : X \times X \to K$ such that

$$\begin{aligned} &i)\ F(x,y) = f(x,y), \text{ for all } (x,y) \in X_1 \times X_2, \\ &ii)\ \|F\| = \|f\|. \end{aligned} \tag{3.1}$$

We agree to call such an F a *norm preserving extension* or a *Hahn-Banach extension* of f. As it was remarked by S.Gähler [11], p.345 Korollar zu S.5 und S.6, the norm preserving extension is not always possible. Some Hahn-Banach and Hahn type extension theorems for subspaces of the form $Y \times [b]$, where Y is a linear subspace of X, $b \in X$ and $[b]$ denotes the subspace of X spanned by b, were proved in the case of real 2-normed spaces by A.G.White Jr. [21], S.Mabizela [17] and I.Franić [7].

In the following we shall show that all these extension results can be derived directly from the classical Hahn-Banach theorem. This approach allows to consider simultaneously both the cases of real and complex scalars.

Our methods of proofs rely upon slight extensions of Hahn-Banach and Hahn theorems from normed to seminormed spaces.

In what follows (X,p) will denote a seminormed space (over the field $K = \mathbf{R}$ or $\mathbf{C}$), with p a nontrivial seminorm on X (i.e. $p \neq 0$). It is well known that a linear functional x^* is continuous on X if and only if it is bounded (or Lipschitz) on X, i.e. there exists a number $L \geq 0$ such that

$$|x^*(x)| \leq L \cdot p(x), \text{ for all } x \in X. \tag{3.2}$$

A number $L \geq 0$ verifying (3.2) is called a *Lipschitz constant* for x^*.

Proposition 3.1. *Let (X,p) be a seminormed space, X^* its conjugate space and let $q : X^* \to [0,\infty)$ be defined by*

$$q(x^*) = \sup\{|x^*(x)| : x \in X, p(x) \leq 1\} \tag{3.3}$$

Then

a) $|x^*(x)| \leq q(x^*) \cdot p(x)$, *for all* $x \in X$;

b) $q(x^*) = \inf\{L \geq 0 : L$ *is a Lipschitz constant for* $x^*\}$;

c) The functional q *is a norm on* X^* *and* (X^*, q) *is a Banach space.*

Proof. a) Since $x^* \in X^*$ there exists $L \geq 0$ such that (3.2) holds. Now, if $x \in X$ is such that $p(x) = 0$ then $x^*(x) = 0$ too, and the inequality a) is trivially verified. If $p(x) > 0$ then $p\left(\frac{1}{p(x)} \cdot x\right) = 1$ so that $|x^*\left(\frac{1}{p(x)} \cdot x\right)| \leq q(x^*)$, which is equivalent to a).

b) If $L \geq 0$ verifies (3.2) then $|x^*(x)| \leq L$, for all $x \in X$ with $p(x) \leq 1$, implying $q(x^*) \leq L$. Since $L \geq 0$ is an arbitrary Lipschitz constant it follows

$$q(x^*) \leq \inf\{L \geq 0 : L \text{ is a Lipschitz constant for } x^*\}.$$

Because $q(x^*)$ is a Lipschitz constant for x^* it follows that

$$q(x^*) = \min\{L \geq 0 : L \text{ is a Lipschitz constant for } x^*\}$$

implying the equality b).

c) It is immediate from (3.3) that q is a seminorm on X^*. If $x^* \neq 0$ and $x_0 \in X$ is such that $x^*(x_0) \neq 0$ then by a)

$$0 < |x^*(x_0)| \leq q(x^*) \cdot p(x_0)$$

implying $q(x^*) > 0$ and showing that q is a norm on X^*.

The proof that (X^*, q) is a Banach space is standard and we omit it. □

Theorem 3.2. *(Hahn-Banach Theorem). Let* (X, p) *be a seminormed space (over* $K =$ **R** *or* **C***) with* $p \neq 0$, Y *a linear subspace and* $y^* \in Y^*$ *a continuons linear functional on* Y. *Define* $q_1(y^*)$ *by*

$$q_1(y^*) = \sup\{|y^*(y)| : y \in Y, p(y) \leq 1\}. \tag{3.4}$$

Then there exists a continuous linear functional x^* *on* X*such that*

$$\begin{aligned} &i)\ x^*|_Y = y^* \text{ and} \\ &ii)\ q(x^*) = q_1(y^*) \end{aligned} \tag{3.5}$$

where $q(x^*)$ *is defined by* (3.3).

Proof. The functional $p_1 : X \to [0,\infty)$ defined by $p_1(x) = q_1(y^*) \cdot p(x)$, $x \in X$ is a seminorm on X and $|x^*(y)| \le p_1(y)$ for all $y \in Y$, i.e. y^* is dominated by p_1. By the Hahn-Banach Theorem (see e.g. [6] or [14]) there exists $x^* \in X^*$ such that

$$\begin{aligned} &i) \quad x^* |_Y = y^* \\ &ii) \quad |x^*(x)| \le q_1(y^*) \cdot p(x), \text{ for all } x \in X. \end{aligned} \tag{3.6}$$

By (3.6) ii) and Proposition 3.1 b) we obtain $q(x^*) \le q_1(y^*)$. The reverse inequality follows from

$$\begin{aligned} q(x^*) &= \sup\{|x^*(x)| : x \in X, p(x) \le 1\} \\ &\ge \sup\{|x^*(y)| : y \in Y, p(y) \le 1\} \\ &= q_1(y^*). \end{aligned}$$

□

Hahn's theorem ([6, Lemma II. 3.12) can be transposed to the seminormed case too

Theorem 3.3. *(Hahn Theorem). Let (X,p) be a seminormed space, Y a linear subspace of X and $x_0 \in X \setminus \overline{Y}$. Then there exists a functional $x^* \in X^*$ such that*

$$\begin{aligned} &i) \quad x^*(x_0) = 1 \text{ and } x^*(Y) = \{0\}; \\ &ii) \quad q(x^*) = \delta^{-1} \end{aligned} \tag{3.7}$$

where $\delta = \inf\{p(x_0 - y) : y \in Y\}$.

Proof. Observe that $x_0 \in X \setminus \overline{Y}$ implies $\delta > 0$. Let $Z = Y \dot{+} K x_0$ and let $z^* : Z \to K$ be defined by $z^*(y + \alpha x_0) = \alpha$, for $y \in Y$ and $\alpha \in K$. Obviously that z^* is linear and, for $\alpha \ne 0$,

$$|z^*(y + \alpha x_0)| = |\alpha| \le |\alpha| \cdot \delta^{-1} \cdot p\left(\alpha^{-1} y + x_0\right) = \delta^{-1} \cdot p(y + \alpha x_0)$$

Since, for $\alpha = 0$, $|z^*(y)| = 0 \le \delta^{-1} \cdot p(y)$ it follows the continuity of z^* and $q_1(z^*) \le \delta^{-1}$, where $q_1(z^*) = \sup\{|z^*(z)| : z \in Z, p(z) \le 1\}$. Taking a minimizing sequence $(y_n) \subseteq Y$ (i.e. $p(x_0 - y_n) \to \delta$, for $n \to \infty$), we obtain

$$1 = z^*(x_0 - y_n) = |z^*(x_0 - y_n)| \le q_1(z^*) \cdot p(x_0 - y_n),$$

which for $n \to \infty$ gives $q_1(z^*) \ge \delta^{-1}$, implying $q_1(z^*) = \delta^{-1}$.

Now Theorem 3.3 follows from Theorem 3.2 applied to Z and z^*. □

Remark 2. The functional $x_1^* \in X^*$, $x_1^* = \delta \cdot x^*$, verifies the conditions:

$$\begin{aligned} &i)\ x_1^*(x_0) = \delta \text{ and } x_1^*(Y) = \{0\} \\ &ii)\ q(x_1^*) = 1 \end{aligned} \tag{3.8}$$

Pass now to the extension theorems for bounded bilinear functionals. The reduction to Hahn-Banach and Hahn's theorems for bounded linear functionals on seminormed linear spaces will be based on the following result:

Proposition 3.4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space (over $K = \mathbf{R}$ or $\mathbf{C}$),Z a subspace of X, $b \in X \setminus \{0\}$ and let $[b]$ be the subspace of X spanned by b. Denote by p_b the seminorm on Z given by*

$$p_b(z) = \|z, b\|,\ z \in Z,$$

and let q_b be its conjugate norm on Z^*, in the sense of Proposition 3.1. Then

a) If $f : Z \times [b] \to K$ is a bounded bilinear functional then the functional $z^* : Z \to K$ defined by $z^*(z) = f(z, b)$, $z \in Z$ is a continuous linear functional on Z and

$$q_b(z^*) = \|f\|.$$

b) Conversely, if z^* is a bounded linear functional on Z, then the 2-functional $f : Z \times [b] \to K$ defined by $f(z, \alpha b) = \alpha z^*(z)$, for $(z, \alpha) \in Z \times K$, is a bounded bilinear functional and

$$\|f\| = q_b(z^*).$$

Proof. a) Obviously that, for a given bounded bilinear functional $f : Z \times [b] \to K$, the functional $z^* : Z \to K$ defined by $z^*(z) = f(z, b)$, $z \in Z$, is a linear functional on Z and

$$|z^*(z)| = |f(z, b)| \leq \|f\| \cdot \|z, b\| = \|f\| \cdot p_b(z),$$

for all $z \in Z$, implying that z^* is a continuous linear functional on the seminormed space (Z, p_b) and

$$q_b(z^*) \leq \|f\|.$$

On the other hand

$$|f(z, \alpha b)| = |f(\alpha z, b)| = |z^*(\alpha z)| \leq q_b(z^*) \cdot p_b(\alpha z) = q_b(z^*) \cdot \|\alpha z, b\| = q_b(z^*) \cdot \|z, \alpha b\|$$

implying that $q_b(z^*)$ is a Lipschitz constant for f, so that $||f|| \leq q_b(z^*)$ and, therefore, $||f|| = q_b(z^*)$.

b) Suppose now that z^* is a given continuous linear functional on the seminormed space (Z, p_b) and define $f : Z \times [b] \to K$ by $f(z, \alpha b) = \alpha \cdot z^*(z)$, $(z, \alpha) \in Z \times K$. Obviously that f is a bilinear functional and

$$|f(z, \alpha b)| = |\alpha z^*(z)| = |z^*(\alpha z)| \leq q_b(z^*) \cdot p_b(\alpha z) = \\ = q_b(z^*) \cdot ||\alpha z, b|| = q_b(z^*) \cdot ||z, \alpha b||,$$

for all $(z, \alpha) \in Z \times K$, showing that f is a bounded bilinear functional and that $||f|| \leq q_b(z^*)$.

Taking into account the fact that $p_b(z) = ||z, b||$ we obtain

$$q_b(z^*) = \sup\{|z^*(z)| : z \in Z, ||z, b|| \leq 1\} = \sup\{|f(z, b)| : z \in Z, ||z, b|| \leq 1\} \leq \\ \leq \sup\{|f(z, \alpha b)| : (z, \alpha) \in Z \times K, ||z, \alpha b|| \leq 1\} = ||f||$$

Again the equality $||f|| = q_b(z^*)$ holds. □

Now we are in position to prove the promised extension theorem.

Theorem 3.5. *(Hahn-Banach Extension Theorem, A.G.White Jr. [21, Th.2.7]) Let $(X, ||\cdot, \cdot||)$ be a 2-normed space (over $K = \mathbf{R}$ or $\mathbf{C}$), Y a subspace of X, $b \in X$ and let $[b]$ be the subspace of X spanned by b. If $f : Y \times [b] \to K$ is a bounded bilinear functional then there exists a bounded bilinear functional $F : X \times [b] \to K$ such that*

$$\begin{aligned} &i) \quad F|_{Y \times [b]} = f, \text{ and} \\ &ii) \quad ||F|| = ||f||. \end{aligned} \tag{3.9}$$

Proof. Let $p_b : X \to [0, \infty)$ be the seminorm defined by $p_b(x) = ||x, b||$, $x \in X$, and let $y^* : Y \to K$ be given by $y^*(y) = f(y, b)$. Then by Proposition 3.4 a), y^* is a continuous linear functional on Y and $q_b'(y^*) = ||f||$, where

$$q_b'(y^*) = \sup\{|y^*(y)| : y \in Y, p_b(y) \leq 1\}. \tag{3.10}$$

By Theorem 3.2 there exists a bounded linear functional $x^* \in X^*$ such that $x^*|_Y = y^*$ and $q_b(x^*) = q_b'(y^*)$, where

$$q_b(x^*) = \sup\{|x^*(x)| : x \in X, p_b(x) \leq 1\}. \tag{3.11}$$

Defining now $F : X \times [b] \to K$ by $F(x, \alpha b) = \alpha \cdot x^*(x)$, for $(x, \alpha) \in X \times K$ and applying Proposition 3.4 b) it follows that the bilinear functional F fulfils all the requierements of the Theorem. □

The analogue of Hahn's·theorem for bilinear functionals is:

Theorem 3.6. *(S.Mabizela [17, Th.2]) Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space over $K = \mathbf{R}$ or $\mathbf{C}$, Y a linear subspace of X, $b \in X$ and [b] the subspace of X spanned by b. If $x_0 \in X$ is such that $\delta > 0$, where*

$$\delta = \inf\{\|x_0 - y, b\| : y \in Y\} \tag{3.12}$$

then there exists a bounded bilinear functional $F : X \times [b] \to K$ such that

$$\begin{aligned} &i)\ F(x_0, b) = 1, F(y, b = 0)\ \text{for all } y \in Y,\ \text{and} \\ &ii)\ \|F\| = \delta^{-1} \end{aligned} \tag{3.13}$$

Proof. Consider again the seminormed space (X, p_b), where $p_b(x) = \|x, b\|, x \in X$, and apply Theorem 3.3 to obtain a bounded linear functional x^* on X such that

$$\begin{aligned} &i)\ x^*(x_0) = 1 \text{ and } x^*(Y) = \{0\},\ \text{and} \\ &ii)\ q_b(x^*) = \delta^{-1},\cdot \end{aligned} \tag{3.14}$$

where $q_b(x^*)$ is given by (3.11).

Defining $F : X \times [b] \to K$ by $F(x, \alpha b) = \alpha \cdot x^*(x)$, $(x, \alpha) \in X \times K$, and applying Proposition 3.4 b), it follows that the bounded bilinear functional F verifies the conditions (3.13) of the Theorem. □

Remark 3. S.Mabizela [17, Th.2] requieres for x_0 and b to be linearly independent. Observe that if x_0, b are linearly dependent then, by the axiom BN 1) in Section 1, $\|x_0, b\| = 0$ and *a fortiori* $\delta = 0$, because

$$0 \leq \delta \leq \|x_0 - 0, b\| = \|x_0, b\| = 0$$

Therefore the hypothesis $\delta > 0$ forces x_0 and b to be linearly independent and $x_0 \in X \backslash \overline{Y}$, where $\overline{Y}$ denotes the closure of Y in the seminormed space (X, p_b).

An immediate consequence of Theorem 3.6 is the following result, known also as Hahn's Theorem:

Theorem 3.7. *If $(X, \|\cdot,\cdot\|)$ is a 2-normed space and x_0, b are linearly independent elements in X then there exists a bounded bilinear functional $F : X \times [b] \to K$ such that:*

$$\begin{aligned} &i)\ F(x_0, b) = \|x_0, b\|, \text{ and} \\ &ii)\ \|F\| = 1. \end{aligned} \tag{3.15}$$

Proof. Putting $Y = \{0\}$ in Theorem 3.6 and taking into account the linear independence of x_0 and b, one obtains $\delta = \|x_0, b\| > 0$.

By Theorem 3.6, it follows the existence of a bounded bilinear functional $G : X \times [b] \to K$ such that $G(x_0, b) = 1$ and $\|G\| = \delta^{-1}$. Then $F = \delta \cdot G$ satisfies the conditions (3.15) of the theorem. □

4. Unique extension of bounded bilinear functionals and unique best approximation

For a 2-normed space $(X, \|\cdot,\cdot\|)$, a subspace Y of X and $b \in X$ denote by $Y_b^\sharp$ the linear space of all bounded bilinear functionals on $Y \times [b]$. Equipped with the norm (2.2), $Y_b^\sharp$ is a Banach space (see A.G.White Jr.[20]) The Banach space $X_b^\sharp$ is defined similarly.

For $f \in Y_b^\sharp$ denote by $E(f)$ the set of all norm-preserving extensions of f to $X \times [b]$,i.e.

$$E(f) = \{F \in X_b^\sharp : F\,|_{Y\times[b]} = f\, and\, \|F\| = \|f\|\} \tag{4.1}$$

By Theorem 3.5, $E(f) \neq \phi$ and $E(f)$ is a convex subset of the unit sphere $S(0, \|f\|) = \{G \in X_b^\sharp : \|G\| = \|f\|\}$. Indeed, for $F_1, F_2 \in E(f)$ and $\lambda \in [0, 1]$,

$$(\lambda F_1 + (1-\lambda) F_2)\,|_{Y\times[b]} = f$$

and

$$\|\lambda F_1 + (1-x) F_2\| \leq \lambda\|F_1\| + (1-\lambda)\|F_2\| = \lambda\|f\| + (1-\lambda)\|f\| = \|f\|.$$

Denoting $G = \lambda F_1 + (1-\lambda) F_2$ it follows $G\,|_{Y\times[b]} = f$ and

$$\|G\| = \sup\{|G(x, \alpha b)| : (y, \alpha) \in X \times K, \|x, \alpha b\| \leq 1\} \geq$$
$$\geq \sup\{|G(y, \alpha b)| : (y, \alpha) \in Y \times K, \|y, \alpha b\| \leq 1\} = \|f\|$$

For a subspace Y of a 2-normed space $(X, \|\cdot,\cdot\|)$ let

$$Y_b^{\perp} = \{G \in X_b^{\sharp} : G(Y \times [b]) = \{0\}\} \tag{4.2}$$

be the *annihilator* of Y in $X_b^{\sharp}$.

For a nonvoid subset Z of $X_b^{\sharp}$ the *distance* of an element $F \in X_b^{\sharp}$ to Z is defined by

$$d(F, Z) = \inf\{\|F - G\| : G \in Z\}. \tag{4.3}$$

An element $G_0 \in Z$ such that $\|F - G_0\| = d(F, Z)$ is called an *element of best approximation* (or a *nearest point*) for F in Z.

Let

$$P_Z(F) = \{G \in Z : \|F - G\| = d(F, Z)\} \tag{4.4}$$

denote the set of all elements of best approximation for F in Z. The set Z is called *proximinal* if $P_Z(F) \neq \emptyset$ for all $F \in X_b^{\sharp}$, *Chebyshev* provided $P_Z(F)$ is a singleton for all $F \in X_b^{\sharp}$ and *semi-Chebyshev* if $\mathrm{card} P_Z(F) \leq 1$, for all $F \in X_b^{\sharp}$.

A subspace of the form $Y_b^{\perp}$ of $X_b^{\sharp}$ is always proximinal and we have simple formulae for the distance of an element $F \in X_b^{\sharp}$ to $Y_b^{\perp}$ and for the set of nearest points.

Theorem 4.1. *If $(X, \|\cdot,\cdot\|)$ is a 2-normed space, Y a subspace of X, $b \in X$ and $F \in X_b^{\sharp}$ then*

$$d(F, Y_b^{\perp}) = \|F|_{Y \times [b]}\| \tag{4.5}$$

Moreover, $Y_b^{\perp}$ is a proximinal subspace of $X_b^{\sharp}$ and

$$P_{Y_b^{\perp}}(F) = F - E(F|_{Y \times [b]}) = \{F - H : H \in E(F|_{Y \times [b]})\} \tag{4.6}$$

Proof. Since $(F - G)|_{Y \times [b]} = F|_{Y \times [b]}$, for any $G \in Y_b^{\perp}$ it follows

$$\|F|_{Y \times [b]}\| = \|(F - G)|_{Y \times [b]}\| \leq \|F - G\|,$$

so that

$$\|F|_{Y \times [b]}\| \leq d(F, Y_b^{\perp}).$$

To prove the reverse inequality observe that $f = F|_{Y\times[b]} \in Y_b^{\sharp}$. Now if H is a norm-preserving extension of f to $X \times [b]$ then $F - H \in Y_b^{\perp}$ and

$$\|F|_{Y\times[b]}\| = \|H\| = \|F - (F - H)\| \geq d(F, Y_b^{\perp}),$$

proving the formula (4.5).

For $H \in E(F|_{Y\times[b]})$ we have $F - H \in Y_b^{\perp}$ and $\|F - (F - H)\| = \|H\| = \|F|_{Y\times[b]}\| = d(F, Y_b^{\perp})$, showing that $F - H$ is a nearest point to F in $Y^{\perp}$.

Conversely, if G is a nearest point to F in $Y_b^{\perp}$ then $(F - G)|_{Y\times[b]} = F|_{Y\times[b]}$ and, denoting $H = F - G$, it follows $G = F - H$ and

$$\|H\| = \|F - G\| = d(F, Y_b^{\perp}) = \|F|_{Y\times[b]}\|$$

showing that H is a norm preserving extension for $F|_{Y\times[b]}$. The equality (4.6) is proved and since, by Theorem 3.5, $E(F|_{Y\times[b]}) \neq \emptyset$, for all $F \in X_b^{\sharp}$, it follows the proximinality of the subspace $Y_b^{\perp}$ in $X_b^{\sharp}$. □

Now we are in position to state and prove the duality theorem relating the uniqueness of extension and of best approximation. Recall that for normed linear spaces and bounded linear functionals a similar result was first proved by R.R.Phelps [18].

Theorem 4.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, Y a subspace of X and $b \in X$. Then the following assertions are equivalent:*

1^0 Every $f \in Y_b^{\sharp}$ has a unique norm preserving extension to $X \times [b]$;

2^0 $Y_b^{\perp}$ is a Chebyshev subspace of the Banach space $X_b^{\sharp}$.

Proof. The Theorem is an immediate consequence of the formula (4.6) from Theorem 4.1. □

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ON SOME o-SCHUNCK CLASSES

RODICA COVACI

Abstract. In this paper, Ore's generalized theorems given in [4] are used to study some special o-Schunck classes. Thus we prove that: 1) the equivalence of D, A and B properties (given in [7] and [3]) on a o-Schunck class takes place; 2) the "composite" of two o-Schunck classes with the D property is in turn a o-Schunck class with the D property; 3) the class D of all o-Schunck classes with the D property, ordered by inclusion, forms respect to the operations of "composite" and intersection a complete lattice.

1. Preliminaries

All groups considered in the paper are finite. We denote by o an arbitrary set of primes and by o' the complement to o in the set of all primes.

Definition 1.1. a) A class $\underline{\mathbf{X}}$ of groups is a *homomorph* if $\underline{\mathbf{X}}$ is closed under homomorphisms.

b) A group G is *primitive* if G has a stabilizer, i.e. a maximal subgroup W with $\mathrm{core}_G W = 1$, where

$$\mathrm{core}_G W = 3\{W^g / g\chi G\}.$$

c) A homomorph $\underline{\mathbf{X}}$ is a *Schunck class* if $\underline{\mathbf{X}}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{\mathbf{X}}$, is itself in $\underline{\mathbf{X}}$.

Definition 1.2. Let $\underline{\mathbf{X}}$ be a class of groups, G a group and H a subgroup of G. We say that:

a) H is an $\underline{\mathbf{X}}$-*subgroup* of G if $H\chi\underline{\mathbf{X}}$;

b) H is an $\underline{\mathbf{X}}$-*maximal subgroup* of G if:

(1) $H\chi\underline{\mathbf{X}}$;

(2) from $H[H^*[G, H^*\chi\underline{\mathbf{X}}$ follows $H = H^*$.

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c) H is an $\underline{\mathbf{X}}$-*covering subgroup* of G if :

(1) $H\chi\underline{\mathbf{X}}$;

(2) $H[V[G, V_0 \leftrightarrow V, V/V_0\chi\underline{\mathbf{X}}$ imply $V = HV_0$.

Obviously we have:

Proposition 1.3. *Let $\underline{\mathbf{X}}$ be a homomorph, G a group and H a subgroup of G. If H is an $\underline{\mathbf{X}}$-covering subgroup of G, then H is $\underline{\mathbf{X}}$-maximal in G.*

The converse of 1.3. does not hold generally.

Definition 1.4. a) A group G is *σ-solvable* if any chief factor of G is either a solvable σ-group or a σ'-group. For o the set of all primes we obtain the notion of "solvable group".

b) A class $\underline{\mathbf{X}}$ of groups is said to be *σ-closed* if:

$$G/O\pi'(G) \in \underline{\mathbf{X}} \Rightarrow G \in \underline{\mathbf{X}},$$

where $O\pi(G)$ denotes the largest normal π'-subgroup of G. We shall call *π-homomorph* a π-closed homomorph and *π-Schunck class* a π-closed Schunck class.

In our considerations we shall use the following result of R. Baer given in [1]:

Theorem 1.5. *A solvable minimal normal subgroup of a group is abelian.*

2. Ore's generalized theorems

In [4] we obtained a generalization on π-solvable groups of some of Ore's theorems given only for solvable groups. In this paper we shall use the following of them:

Theorem 2.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Theorem 2.2. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

Theorem 2.3. *Let G be a π-solvable group such that:*

(i) there is a minimal normal subgroup M of G which is a solvable π-group and $C_G(M) = M$;

(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a π'-group. Then G is primitive.

Theorem 2.4. *If G is a π-solvable group satisfying (i) and (ii) from 2.3., then any two stabilizers W_1 and W_2 of G are conjugate in G.*

3. Some special π-Schunck classes

Ore's generalized theorems are a powerful tool in the formation theory of π-solvable groups. This is proved by [5], which we complete here with new results. We first give a new proof, based on Ore's generalized theorems, for the equivalence of D, A and B properties (given in [7] and [3]) on a π-Schunck class.

Definition 3.1. ([7]; [3]) Let $\underline{\mathbf{X}}$ be a π-Schunck class. We say that $\underline{\mathbf{X}}$ has *the D property* if for any π-solvable group G, every $\underline{\mathbf{X}}$-subgroup H of G is contained in an $\underline{\mathbf{X}}$- covering subgroup E of G.

Remark 3.2. Definition 3.1. has sense because of the existence theorem of $\underline{\mathbf{X}}$- covering subgroups in finite π-solvable groups ([5]), where $\underline{\mathbf{X}}$ is a π-Schunck class. Furthermore, any two covering subgroups are conjugate.

Theorem 3.3. *Let $\underline{\mathbf{X}}$ be a π-Schunck class. $\underline{\mathbf{X}}$ has the D property if and only if in any π-solvable group G, every $\underline{\mathbf{X}}$-maximal subgroup is an $\underline{\mathbf{X}}$-covering subgroup.*

Proof. Suppose $\underline{\mathbf{X}}$ has the D property. Let G be a π-solvable group and H an $\underline{\mathbf{X}}$-maximal subgroup of G. Obviously $H \in \underline{\mathbf{X}}$. Applying the D property we obtain that $H \subseteq E$, where E is an $\underline{\mathbf{X}}$-covering subgroup of G. But H is $\underline{\mathbf{X}}$-maximal in G. It follows that $H = E$ and so H is an $\underline{\mathbf{X}}$-covering subgroup of G.

Conversely, suppose that in any π-solvable group G every $\underline{\mathbf{X}}$-maximal subgroup is an $\underline{\mathbf{X}}$-covering subgroup. Let G be a π-solvable group and H an $\underline{\mathbf{X}}$-subgroup of G. If H itself is $\underline{\mathbf{X}}$-maximal in G, we put $E = H$ and E is an $\underline{\mathbf{X}}$-covering subgroup of G. If H is not $\underline{\mathbf{X}}$-maximal in G, let E be an $\underline{\mathbf{X}}$-maximal subgroup of G such that $H \subseteq E$. Then $H \subseteq E$ and E is an $\underline{\mathbf{X}}$-covering subgroup of G. So $\underline{\mathbf{X}}$ has the D property. □

Definition 3.4. ([7];[3])

a) The π-Schunck class $\underline{\mathbf{X}}$ has the *A property* if for any π-solvable group G and any subgroup H of G with $core_G H \neq 1$, every $\underline{\mathbf{X}}$-covering subgroup of H is contained in an $\underline{\mathbf{X}}$-covering subgroup of G.

b) Let G be a group and S a subgroup of G. The subgroup S ***avoids*** the chief factor M/N of G if $S \cap M \subseteq N$. Particularly, if N is a minimal normal subgroup of G, S ***avoids*** N if $S \cap N = 1$.

c) The π-Schunck class $\underline{\mathbf{X}}$ has the B ***property*** if for any π-solvable group G and any minimal normal subgroup N of G, the existence of an $\underline{\mathbf{X}}$-covering subgroup of G which avoids N implies that every $\underline{\mathbf{X}}$-maximal subgroup of G avoids N.

Theorem 3.5. *Let $\underline{\mathbf{X}}$ be a π-Schunck class. The following statements are equivalent:*

(i) $\underline{\mathbf{X}}$ has the A property;

(ii) $\underline{\mathbf{X}}$ has the D property;

(iii) $\underline{\mathbf{X}}$ has the B property.

Proof. A proof of 3.5. is given in [3], using some of R. Baer's theorems from [1]. We consider the same proof like in [3] for (2) $\Rightarrow$ (3) and for (3) $\Rightarrow$ (1).

A new proof is given here for (1) $\Rightarrow$ (2). This proof is based on Ore's generalized theorems. Let $\underline{\mathbf{X}}$ be a π-Schunck class and suppose that $\underline{\mathbf{X}}$ has the A property. In order to prove that $\underline{\mathbf{X}}$ has the D property we use 3.3. Let G be a π-solvable group and H an $\underline{\mathbf{X}}$- maximal subgroup of G. Let now S be an $\underline{\mathbf{X}}$-covering subgroup of G (S exists by 3.2.). We shall prove by induction on $|G|$ that H and S are conjugate in G. Two cases are considered:

1) $G \in \underline{\mathbf{X}}$. Then $H = S = G$.

2) $G \notin \underline{\mathbf{X}}$. Let N be a minimal normal subgroup of G. Applying the induction on G/N, we deduce that $HN = S^gN$, where $g \in G$. Hence $H \subseteq S^gN$. Again two cases are considered:

   a) $S^gN \subset G$. Applying the induction on S^gN, we obtain that H and S^g are conjugate in S^gN. Hence H and S are conjugate in G.

   b) $S^gN = G$. It follows that $G = (SN)^g$, hence $S^gN = G = SN$. If $\text{core}_G S \neq 1$, the induction on $G/\text{core}_G S$ leads to $H^x \text{core}_G S = S$, where $x \in G$. Then $H^x \subseteq S$. So $H^x = S$, which means that H and S are conjugate in G. Let now $\text{core}_G S = 1$. G being π-solvable, N is either a solvable π-group or a π'-group. Supposing that N is a π'-group we have $N \leq O\pi'(G)$ and

$$G/O\pi'(G)\varphi(G/N)/(O\pi'(G)/N),$$

where

$$G/N = SN/N\varphi S/S3N\chi\underline{\mathbf{X}}.$$

So $G/O\pi'(G) \in \underline{\mathbf{X}}$, which implies by the π-closure of $\underline{\mathbf{X}}$ that $G \in \underline{\mathbf{X}}$, a contradiction. It follows that N is a solvable π-group, hence by 1.5., N is abelian. This and $G = SN$ lead to $S \cap N = 1$ and S is a maximal subgroup of G. From $H \in \dot{\underline{\mathbf{X}}}$ and $G \notin \underline{\mathbf{X}}$ we have $H \subset G$. Let M be a maximal subgroup of G such that $H \subseteq M$. Applying the induction on M it follows that H is an $\underline{\mathbf{X}}$-covering subgroup of M. We consider now two possibilities:

b.1) $core_G M \neq 1$. Applying the A property on G, $M < G$, $\mathrm{core}_G M \neq 1$, the $\underline{\mathbf{X}}$-covering subgroup H of M and the $\underline{\mathbf{X}}$-covering subgroup S of G, we obtain $H \subseteq S^x$, where $x \in G$. Hence $H = S^x$. So H and S are conjugate in G.

b.2) $core_G M = 1$. Then S and M are two stabilizers of G. Hence G is primitive. We prove now that G satisfies (i) and (ii) from 2.3.:

(i) There is a minimal normal subgroup M of G which is a solvable π-group and $C_G(M) = M$. Indeed, we put $M = N$. We proved that N is a solvable π-group and by 2.2. we have $C_G(N) = N$.

(ii) There is a minimal normal subgroup L/N of G/N such that L/N is a π'-group. Suppose the contrary, i.e. any minimal normal subgroup L/N of G/N is a solvable π-group. Since N is also a solvable π-group, it follows that L is a solvable π-group. By 2.1., N is the only minimal normal subgroup of G. If L is a minimal normal subgroup of G, obviously follows that $L = N$ and $L/N = 1$, in contradiction with L/N minimal normal subgroup of G/N. If L is not a minimal normal subgroup of G, we have $N \subset L$ and again a contradiction is obtained by $G = SN \subset SL = G$.

So G satisfies (i) and (ii) from 2.3. Then by 2.4., S and M are conjugate in G, i.e. $M = S^x$, where $x \in G$. But $H \subseteq M$, hence $H \subseteq S^x$, where $S^x \in \underline{\mathbf{X}}$. H being $\underline{\mathbf{X}}$-maximal, it follows that $H = S^x$.

□

4. The "composite" of two π-Schunck classes

Let us note by $\underline{\mathbf{D}}$ the class of all π-Schunck classes with the D property.

Definition 4.1. ([3]) If $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are two π-Schunck classes, we define the "composite" $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ as the class of all π-solvable groups G such that $G=<S,T>$, where S is an $\underline{\mathbf{X}}$-covering subgroup of G and T is an $\underline{\mathbf{Y}}$-covering subgroup of G.

In [3] we proved the following result:

Theorem 4.2. *If $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are two π-Schunck classes, then $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ is also a π- Schunck class.*

Using Ore's generalized theorems we can prove now:

Theorem 4.3. *If $\underline{\mathbf{X}}\in\underline{\mathbf{D}}$ and $\underline{\mathbf{Y}}\in\underline{\mathbf{D}}$, then $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle\in\underline{\mathbf{D}}$.*

Proof. By 4.2., $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ is a π-Schunck class. Let us prove that $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ has the D property using 3.3. Let G be a π-solvable group and H an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-maximal subgroup of G. We prove by induction on $|G|$ that H is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G. We consider two cases:

1) $G\in\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$. Then $H=G$ is its own $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup.
2) $G\notin\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$. Applying 3.2., there is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup P of G. We shall prove that $H=P^x$, where $x\in G$.

Let N be a minimal normal subgroup of G. By the induction on G/N, if we take $HN/N\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-maximal in G/N and $PN/N\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G/N, we have $HN/N\subseteq P^gN/N$ for some $g\in G$. Hence $H\subseteq P^gN$. Now two possibilities:

a) $P^gN\subset G$. Applying the induction on P^gN, for H $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-maximal in P^gN and P^g an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of P^gN, it follows that $H=(P^g)^{g'}=P^{gg'}$, where $g'\in P^gN$. So $H=P^{gg'}$ is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G.
b) $P^gN=G$. Then $G=PN$. Again two cases:
   b.1) $\mathrm{core}_GP\neq 1$. By the induction on G/core_GP, we have $H=P^x$, where $x\in G$. So H is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G.
   b.2) $\mathrm{core}_GP=1$. First N is a solvable π-group, for if we suppose that N is a π'-group, we have $N\subseteq O\pi'(G)$ and

$$G/O\pi'(G)\varphi(G/N)/(O\pi'(G)/N);$$

$$G/N=PN/N\varphi P/P\cap N\in\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$$

imply $G/O\pi'(G) \in \langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$, hence $G \in \langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$, a contradiction. By 1.5., N is abelian. From $G = PN$ and N abelian, we deduce that $P \cap N = 1$, hence P is a maximal subgroup of G. So P is a stabilizer of G and G is primitive. Then, by 2.1., we obtain that N is the only minimal normal subgroup of G and by 2.2. that $C_G(N) = N$. It is easy to notice that $HN = G$ and so, like for P, we have $H \cap N = 1$ and H is a maximal subgroup of G. Now we consider two possibilities:

b.2.1) $\mathrm{core}_G H \neq 1$. Applying the induction on $G/\mathrm{core}_G H$, we obtain that $H = P^x$ $(x \in G)$ is an $\langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$-covering subgroup of G.

b.2.2) $\mathrm{core}_G H = 1$. Then H is a stabilizer of G. Let us notice that we are in the hypotheses of theorem 2.4. Indeed, (i) is true, because N is a minimal normal subgroup of G which is a solvable π-group and $C_G(N) = N$. Further, (ii) is also true, for if we suppose the contrary, we obtain that any minimal normal subgroup L/N of G/N is a solvable π-group and in each of the two cases given below we get a contradiction:

(#): If L is a minimal normal subgroup of G, obviously $L = N$ and $L/N = 1$, in contradiction with L/N minimal normal subgroup of G/N.

(##): If L is not a minimal normal subgroup of G, then $N \subset L$ and $G = HN \subset HL = G$, a contradiction.

So we are in the hypotheses of theorem 2.4. It follows that the two stabilizers P and H of G are conjugate in G, i.e. there is $x \in G$ such that $H = P^x$. But this means that H is an $\langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$-covering subgroup of G.

□

An immediate consequence of theorem 4.3. is the following:

Theorem 4.4. *The class D, ordered by inclusion, forms respect to the operations of "composite" and intersection a complete lattice.*

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ON THE CONTINUITY AND DIFFERENTIABILITY OF THE IMPLICIT FUNCTIONS FOR GENERALIZED EQUATIONS

DOMOKOS ANDRÁS

Abstract. The aim of this paper is to show that the existence, continuity and differentiabilty of the implicit functions can be proved at the same time, using one sequence of succesive approximations of a mapping of two variables. The proof from this paper unifies methods used in the study of local stability and sensitivity of the solutions of integral equations [7], variational inequalities and nonsmooth generalized equations [1, 2, 5, 6]. We will prove the continuous differentiability of the solution mapping mapping in a neighborhood of a fixed parameter λ_0.

Throughout this paper X will be a Banach space, Y, Z, Λ will be normed spaces. Let X_0 and Λ_0 be open neighborhoods of the fixed points $x_0 \in X$ and $\lambda_0 \in \Lambda$ respectively. We will study the behaviour of the solutions of the following generalized equation:

$$0 \in f(x,\lambda) + G(x) ,$$

where $f : X_0 \times \Lambda_0 \to Z$ is a single-valued mapping and $G : X_0 \to Z$ is a set-valued mapping.

This problem includes the variational inequalities from the papers [4, 5, 6]. Assumption (iii) of Theorem 2 appears in both nonsmooth [1] and smooth [2] cases and generalize the strong-regularity condition for variational inequalities [4, 5]. Theorem 2 also generalize the classical version of the implicit function theorem [3].

Let $M \subset X$, $N \subset Y$. We will denote by $\mathbf{L}(X,Y)$ the set of the linear and continuous mappings from X to Y, by $\mathbf{C}(M,N)$ the set of the continuous mappings from M to N, by $\mathbf{B}(M,N)$ the set of the bounded mappings from M to N, by $B(x_0,r)$ the closed ball with center at x_0 and radius r.

We will apply the following result:

Theorem 1. [7] *Let (S,d) and (U,ρ) be complete metric spaces and let $A : S \times U \to S \times U$ be a mapping with the following properties:*

(i) A is continuous;

(ii) there exist the mappings $P : S \to S$, $Q : S \times U \to U$ such that $A(s,u) = (P(s), Q(s,u))$ and

- *P is a contraction,*
- *there exists $l \in [0,1)$ such that*

$$\rho(Q(s,u_1), Q(s,u_2)) \leq l\,\rho(u_1,u_2)$$

for all $s \in S$, $u_1, u_2 \in U$.

Then for all $(s,u) \in S \times U$ the succesive approximations $A^n(s,u)$ converge to a unique $(\overline{s},\overline{u}) \in S \times U$, where $\overline{s}$ is the unique fixed point for P.

Theorem 2. *Let us suppose that Λ is finite dimensional and :*

(i) $0 \in f(x_0,\lambda_0) + G(x_0)$;

(ii) f is continuous Fréchet differentiable on $X_0 \times \Lambda_0$:

(iii) there exist an open neighborhood Z_0 of 0_Z and a mapping $g : Z_0 \to X$ which is continuous differentiable on Z_0, $g(0) = 0$ and for all $z \in Z_0$

$$g(z) \in \left(f(x_0,\lambda_0) + \nabla_x f(x_0,\lambda_0)(\cdot - x_0) + G(\cdot)\right)^{-1}(z)\,.$$

Then there exists an open neighborhood Λ_0' of λ_0 and a mapping $x : \Lambda_0' \to X_0$ such that x is continuous differentiable on Λ_0', $x(\lambda_0) = x_0$ and $0 \in f(x(\lambda),\lambda) + G(x(\lambda))$, for all $\lambda \in \Lambda_0'$.

Proof. We can suppose ∇f bounded on $X_0 \times \Lambda_0$, and the mean value inequality for Fréchet differentiable mappings implies that f is Lipschitz continuous on $X_0 \times \Lambda_0$. The same is true for g and let us denote by γ the Lipschitz constant of g.

We can suppose $x_0 = 0$ and we denote by $h(x) = f(0,\lambda_0) + \nabla_x f(0,\lambda_0)(x)$. The continuity of $\nabla_x f$ at $(0,\lambda_0)$ implies that h strongly approximates f at $(0,\lambda_0)$ [6], i. e. for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|f(x_1,\lambda) - h(x_1) - (f(x_2,\lambda) - h(x_2))\| \leq \varepsilon\|x_1 - x_2\|$$

for all $x_1, x_2 \in B(0,\delta)$, $\lambda \in B(\lambda_0,\delta)$.

Let us choose the constants $\varepsilon, \delta, s, r > 0$ such that:

- $\varepsilon \cdot \gamma < 1$;
- $B(0,\delta) \subset X_0$, $B(\lambda_0,\delta) \subset \Lambda_0$;
- $h(x) - f(x,\lambda) \in Z_0$, for all $x \in B(0,\delta)$, $\lambda \in B(\lambda_0,\delta)$;
- $s \leq \delta$, $r \leq \delta$;
- $\gamma\|f(0,\lambda) - f(0,\lambda_0)\| \leq (1-\gamma\varepsilon)r$, for all $\lambda \in B(\lambda_0,s)$;
- $\|\nabla g\,(h(x) - f(x,\lambda)) \circ (\nabla_x f(0,\lambda_0) - \nabla_x f(x,\lambda))\| \leq a < 1$, for all $x \in B(0,r)$, $\lambda \in B(\lambda_0,s)$.

Let us define the mapping $F : B(0,r) \times B(\lambda_0,s) \to X$ by

$$F(x,\lambda) = g(\,h(x) - f(x,\lambda)\,)\,.$$

For all $x \in B(0,r)$, $\lambda \in B(\lambda_0,s)$ we have

$$\|F(x,\lambda)\| = \|g(h(x) - f(x,\lambda))\| = \|g(h(x) - f(x,\lambda)) - g(h(0) - f(0,\lambda_0))\| \leq$$

$$\leq \gamma\|h(x) - f(x,\lambda) - h(0) + f(0,\lambda_0))\| \leq$$

$$\leq \gamma\|h(x) - f(x,\lambda) - h(0) + f(0,\lambda)\| + \gamma\|f(0,\lambda_0) - f(0,\lambda)\| \leq$$

$$\leq \gamma\varepsilon r + (1-\gamma\varepsilon)r = r\,.$$

Hence $F\,(B(0,r) \times B(\lambda_0,s)) \subset B(0,r)$.

We can define now the mapping

$$P : \mathbf{C}\,(B(\lambda_0,s)\,,\, B(0,r)) \to \mathbf{C}\,(B(\lambda_0,s)\,,\, B(0,r))$$

by $P(x)(\lambda) = F(x(\lambda),\lambda)$.

Let $x_1, x_2 \in \mathbf{C}(B(\lambda_0,s)\,,\, B(0,r))$. Then

$$\|P(x_1) - P(x_2)\| = \sup_{\lambda\in B(\lambda_0,s)} \|P(x_1)(\lambda) - P(x_2)(\lambda)\| =$$

$$= \sup_{\lambda\in B(\lambda_0,s)} \|g\,(h(x_1(\lambda)) - f(x_1(\lambda),\lambda)) - g\,(h(x_2(\lambda)) - f(x_2(\lambda),\lambda))\,\| \leq$$

$$\leq \gamma\varepsilon \sup_{\lambda\in B(\lambda_0,s)} \|x_1(\lambda) - x_2(\lambda)\| = \gamma\varepsilon\|x_1 - x_2\|\,.$$

For $x \in B(0,r)$ and $\lambda \in B(\lambda_0,s)$ we have

$$\|\nabla_x F(x,\lambda)\| = \|\nabla g\,(h(x) - f(x,\lambda)) \circ (\nabla_x f(0,\lambda_0) - \nabla_x f(x,\lambda))\,\| \leq a\,.$$

We define now the mapping

$$Q : \mathbf{C}\,(B(\lambda_0,s)\,,\, B(0,r)) \times \mathbf{B}\,(B(\lambda_0,s)\,,\, \mathbf{L}(\Lambda,X)) \to \mathbf{B}\,(B(\lambda_0,s)\,,\, \mathbf{L}(\Lambda,X))$$

by

$$Q(x,y)(\lambda) = \nabla_x F(x(\lambda),\lambda) \circ y(\lambda) + \nabla_\lambda F(x(\lambda),\lambda) .$$

Let $y_1, y_2 \in \mathbf{B}(B(\lambda_0, s), \mathbf{L}(\Lambda, X))$. Then

$$\|Q(x,y_1) - Q(x,y_2)\| = \sup_{\lambda\in B(\lambda_0,s)} \|Q(x,y_1)(\lambda) - Q(x,y_2)(\lambda)\| =$$

$$= \sup_{\lambda\in B(\lambda_0,s)} \|\nabla_x F(x(\lambda),\lambda) \circ (y_1(\lambda) - y_2(\lambda))\| \le$$

$$\le \sup_{\lambda\in B(\lambda_0,s)} \|\nabla_x F(x(\lambda),\lambda)\| \cdot \|y_1(\lambda) - y_2(\lambda)\| \le a\|y_1 - y_2\| .$$

Using the continuity of ∇f and the compactness of $B(\lambda_0, s)$ we deduce that for $x \in \mathbf{C}(B(\lambda_0, s), B(0,r))$, the mappings $\nabla_x F(x(\cdot),\cdot)$ and $\nabla_\lambda F(x(\cdot),\cdot)$ are uniformly continuous on $B(\lambda_0, s)$, which implies the continuity of $Q(\cdot, y)$.

We apply now Theorem 1, to the mapping $A = (P, Q)$ and hence

$$A^n(x,y) \to (\overline{x},\overline{y})$$

for all $x \in \mathbf{C}(B(\lambda_0,s), B(0,r))$ and $y \in \mathbf{B}(B(\lambda_0,s), \mathbf{L}(\Lambda, X))$. Let us choose $x \equiv 0$, $y \equiv 0$. Then

$$x_1(\lambda) = P(0)(\lambda) = F(0,\lambda)$$

and

$$y_1(\lambda) = Q(0,0)(\lambda) = \nabla_\lambda F(0,\lambda) = \nabla x_1(\lambda) .$$

If $y_n = \nabla x_n$, then

$$x_{n+1}(\lambda) = P(x_n)(\lambda) = F(x_n(\lambda), \lambda)$$

and

$$y_{n+1}(\lambda) = \nabla_x F(x_n(\lambda), \lambda) \circ \nabla x_n(\lambda) + \nabla_\lambda F(x_n(\lambda), \lambda) = \nabla x_{n+1}(\lambda) .$$

Hence $y_n = \nabla x_n$ for all $n \in \mathbf{N}$,

$$x_n \to \overline{x} \text{ in } \mathbf{B}(B(\lambda_0,s), B(0,r))$$

and

$$\nabla x_n \to \overline{y} \text{ in } \mathbf{B}(B(\lambda_0,s), \mathbf{L}(\Lambda, X)) .$$

This means that ∇x_n converges uniformly to $\overline{y}$ in $B(\lambda_0, s)$, so $\overline{x}$ is differentiable on $intB(\lambda_0, s)$ and $\nabla\overline{x} = \overline{y}$. Being the limit of a uniformly convergent sequence of continuous functions, $\overline{y}$ is also continuous.

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RATIONAL BÉZIER CURVES AND SURFACES WITH INDEPENDENT COORDINATE WEIGHTS

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Abstract. A generalization of the rational Bézier curves and surfaces was made in [3]. In this paper we make another extending of the possibilities for the modelling curves and surfaces by attaching different weights to each coordinate of the control Bézier points. Derivatives of high orders in the initial and final points of the curves are also deduced. Some figures show the increased flexibility of these partial or total rational Bézier curves and surfaces comparative with the polynomial and classical rational corresponding to the same control Bézier polygon. On observes that we do not always have the convex hull property (Fig.2) and the affine invariance (Fig.3).

1. Introduction

Rational Bézier curves and surfaces are represented by the equations (1) and (2) respectively

$$r(t)=\sum_{i=0}^{n}\frac{w_i b_{n,i}(t)}{\sum\limits_{i=0}^{n} w_i b_{n,i}(t)}b_i,\quad t\in[0,1] \tag{1}$$

and

$$r(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{w_{ij}b_{m,i}(u)b_{n,j}(v)}{\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} w_{ij}b_{m,i}(u)b_{n,j}(v)}b_{ij},\quad (u,v)\in[0,1]^2, \tag{2}$$

where $b_{n,i}(t)=\binom{n}{i}(1-t)^{n-i}t^i$, $i=\overline{0,n}$, $b_i\in R^3$ and $b_{ij}\in R^3$ are given points. The positive real numbers w_i, $i=\overline{0,n}$ from (1), called shape parameters, are used for the remodelling curve. So, as it is known, if one increases, say w_k, then the curve is pulled towards the points b_k and if w_k decreases then the contribution of b_k to the curve is diminished (see Figures 1 and 2; dotted curves and the dash curves are polynomial Bézier, corresponding to the same control polygon). We also mention the possibility to control the curvature and torsion at the points b_0 and b_n respectively, [1] p.180. Thus

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the flexibility of the rational Bézier curves is its characteristic property comparative with the polynomial Bézier. Similar remark is true regarding to the rational Bézier surfaces.

2. Partial and Total Coordinates Rational Bézier Curves

In the equation (1) the parameter w_i affects the all coordinates (x_i, y_i, z_i) of points b_i in the same measure. Next we attach to each coordinate (x_i, y_i, z_i) of the point b_i different shape parameters (independent weights) denoted by w_i^x, w_i^y and w_i^z respectively.

The equation of a rational Bézier curve with coordinate shape parameters is of the following form

$$R(t) = [x(t), y(t), z(t)]^T,$$

where

$$\begin{aligned} x(t) = & \sum_{i=0}^{n} \frac{w_i^x b_{n,i}(t)}{\sum_{i=0}^{n} w_i^x b_{n,i}(t)}, \\ y(t) = & \sum_{i=0}^{n} \frac{w_i^y b_{n,i}(t)}{\sum_{i=0}^{n} w_i^y b_{n,i}(t)}, \\ z(t) = & \sum_{i=0}^{n} \frac{w_i^z b_{n,i}(t)}{\sum_{i=0}^{n} w_i^z b_{n,i}(t)}, \qquad t \in [0,1]. \end{aligned} \tag{3}$$

Consider the folowing sets of positive real numbers

$$W = \{w_0, w_1, \dots, w_n\}, \ U = \{1, 1, \dots, 1\}$$

and

$$W^t = \{w_0^t, w_1^t, \dots, w_n^t\}, \ t \in \{x, y, z\}.$$

Definitions.

1. We name W^t the set of t-coordinate shape parameters.

2. If $W^x = W$ and $W^y = W^z = U$ then we name (3) a **partial x-coordinate rational Bézier curve**. Analogously one defines another partial variable Bézier curve.

3. A curve is called **x,y-rational Bézier** if $W^t \neq U, t \in \{x, y\}$ and $W^z = U$.

4. A curve which is x,y,z-rational will be called a **total coordinates rational Bézier curve.**

Remarks:

1. If $W^t = U$, for any $t \in \{x, y, z\}$, then equations (3) represent a polynomial Bézier curve.

2. If $W^x = W^y = W^z = W$, then (3) are parametric equations of a classical rational Bézier curve.

In the Figures 1 and 2 are illustrated the effects of the coordinates shape parameters on a partial rational Bézier curve (continous curve). As witness curves we have taken the classical rational Bézier - dotted curve and the polynomial Bézier - reare dotted curve. The points $b_i, i = \overline{0,5}$ and the coordinate shape parameters are

$$b_0 = (-6, 6), b_1 = (2, 0), b_2 = (12, 7), b_3 = (-5, 18), b_4 = (3, 23), b_5 = (6, 18);$$

for Figure 1: $W^x = W = \{1, 1, 9, 9, 1, 1\}$, $W^y = U$;
and for Figure 2: $W^x = W = \{1, 1, 2, 4, 1, 1\}$, $W^y = \{1, 3, 1, 1, 5, 1\}$.

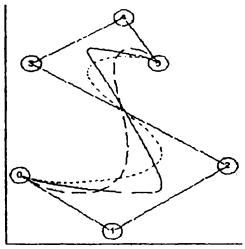

Fig.1

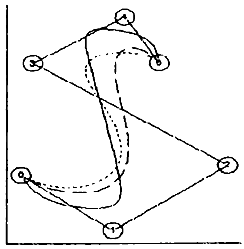

Fig.2.

Remarks:

1. If one increases say w_i^x then the curve is pulled towards the straight line (in R^3 to the plane) $x = x_i$, because the contribution of x_i, to the function $x(t)$ increases.

2. If $W^x = W$ then the partial x-coordinate rational and classical rational Bézier curves have the same function $x(t)$. From geometrical point of view these curves have comon tangent, perpendicular on the Ox axis, as can be seen from the above figures. Similar remarks are valueable if $W^y = W$ or $W^z = W$.

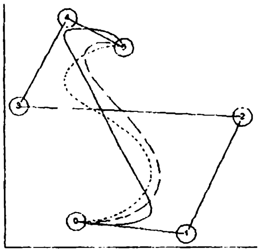

Fig.3.

3. In the case of total coordinates rational Bézier curve we have independent control of the scalar functions $x(t), y(t)$ and $z(t)$.

4. If we make a rotation of the coordinate system then, as it is known, the polynomial and classical rational Bézier curves are invariant [1], p.232, but a coordinate rational curve one modifies as we can see in fig.3. Fig.3 results from fig.2 if one does a rotation of the control polygon with angle $\alpha = \frac{\pi}{6}$.

5. Total rational coodonates Bézier curves do not always have conex hull property (see figure 2). Denoting

$$\min_{i=\overline{0,n}} x_i = a_1, 2cm \qquad \max_{i=\overline{0,n}} x_i = b_1,$$
$$\min_{i=\overline{0,n}} y_i = a_2, \qquad \max_{i=\overline{0,n}} y_i = b_2,$$
$$\min_{i=\overline{0,n}} z_i = a_3, \qquad \max_{i=\overline{0,n}} z_i = b_3,$$

and taking in view that $x(t), y(t)$ and $z(t)$ in (3) are weighted means, one can say that this type of curve lies in the interior of parallelepiped

$$D = \{(x, y, z) | a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3\}.$$

Concerning to the derivatives of $R(t)$ we have

$$R^{(k)}(t) = \left[x^{(k)}(t), y^{(k)}(t), z^{(k)}(t)\right]^T, t \in [0, 1].$$

First we will give the expression of the $x^{(k)}(t)$, proceeding as in [1], p.236. From (3), denoting

$$p(t) = \sum_{i=0}^{n} b_{n,i}(t) w_i^x x_i \quad \text{and} \quad \mathrm{w}(t) = \sum_{i=0}^{n} \mathrm{b}_{n,i}(t) \mathrm{w}_i^x \tag{4}$$

results $p(t) = x(t)w(t)$. Further, using the Leibniz's formula for the computation of $[x(t)w(t)]^{(k)}$ on obtains

$$p^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} x^{(k-j)}(t) w^{(j)}(t) = x^{(k)}(t)w(t) + \sum_{j=1}^{k} \binom{k}{j} x^{(k-j)}(t) w^{(j)}(t).$$

From here results the following recursive formula

$$x^{(k)}(t) = \frac{1}{w(t)} \left[p^{(k)}(t) - \sum_{j=1}^{n} \binom{k}{j} x^{(k-j)}(t) w^{(j)}(t) \right]. \tag{5}$$

Taking in view (5) and formula 4.19 from [1] p.44, we have

$$p^{(k)}(t) = \frac{n!}{(n-k)!} \sum_{i=0}^{n-k} b_{n-k,i}(t) \Delta^k (w_i^x x_i) \tag{6}$$

and

$$w^{(j)}(t) = \frac{n!}{(n-j)!} \sum_{i=0}^{n-j} b_{n-j,i}(t) \Delta^j (w_i^x) \tag{7}$$

where the forward difference of order m has the expression

$$\Delta^m y_p = \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} y_{p+q}. \tag{8}$$

Taking into account (6),(7) and (8), the formula (5) has the following final form

$$\begin{aligned} x^{(k)}(t) = \; & \frac{n!}{w(t)} \left[\frac{1}{(n-k)!} \sum_{i=0}^{n-k} b_{n-k,i}(t) \sum_{q=0}^{k} (-1)^{k-q} \binom{k}{q} w_{i+q}^x x_{i+q} - \right. \\ & \left. - \sum_{j=1}^{k} \binom{k}{j} \frac{x^{(k-j)}(t)}{(n-j)!} \sum_{j=0}^{n-j} b_{n-j,i}(t) \sum_{q=0}^{j} (-1)^{j-q} \binom{j}{q} w_{i+q}^x \right]. \end{aligned} \tag{9}$$

For the particular cases $t = 0$ and $t = 1$ this formula becomes

$$\begin{aligned} x^{(k)}(0) = \; & \frac{n!}{w_0^x} \left[\frac{1}{(n-k)!} \sum_{q=0}^{k} (-1)^{k-q} \binom{k}{q} w_q^x x_q - \right. \\ & \left. - \sum_{j=1}^{k} \binom{k}{j} \frac{x^{(k-j)}(0)}{(n-j)!} \sum_{q=0}^{j} (-1)^{j-q} \binom{j}{q} w_q^x \right] \end{aligned} \tag{10}$$

and

$$x^{(k)}(1) = \frac{n!}{w_n^x}\left[\frac{1}{(n-k)!}\sum_{q=0}^{k}(-1)^{k-q}\binom{k}{q}w_{n-k+q}^x x_{n-k+q} - \right.$$
$$\left. -\sum_{j=1}^{k}\binom{k}{j}\frac{x^{(k-j)}(1)}{(n-j)!}\sum_{q=0}^{j}(-1)^{j-q}\binom{j}{q}w_{n-j+q}^x\right]. \tag{11}$$

Similar formulas we have for $y^{(k)}(t)$, $y^{(k)}(0)$, $y^{(k)}(1)$, $z^{(k)}(t)$, $z^{(k)}(0)$ and $z^{(k)}(1)$.

Special interest present the derivatives $\dot{R}(t)$ and $\ddot{R}(t)$ for $t = 0$ and $t = 1$, respectively.

From (10) and (11) results

$$\dot{R}(0) = n\left[\frac{w_1^x}{w_0^x}\Delta x_0, \frac{w_1^y}{w_0^y}\Delta y_0, \frac{w_1^z}{w_0^z}\Delta z_0\right]^T,$$
$$\dot{R}(1) = n\left[\frac{w_{n-1}^x}{w_n^x}\Delta x_{n-1}, \frac{w_{n-1}^y}{w_n^y}\Delta y_{n-1}, \frac{w_{n-1}^z}{w_n^z}\Delta z_{n-1}\right]^T \tag{12}$$

and

$$\ddot{R}(0) = \left[\ddot{x}(0), \ddot{y}(0), \ddot{z}(0)\right]^T, \ddot{R}(1) = \left[\ddot{x}(1), \ddot{y}(1), \ddot{z}(1)\right]^T.$$

where

$$\ddot{x}(0) = \frac{n}{w_0^x}\left\{(n-1)\left[\Delta^2\left(w_0^x x_0\right) - x_0\Delta^2 w_0^x\right] - 2n\frac{w_1^x}{w_0^x}\Delta x_0 \Delta w_0^x\right\}$$
$$\ddot{x}(1) = \frac{n}{w_n^x}\left\{(n-1)\left[\Delta^2\left(w_{n-2}^x x_{n-2}\right) - x_n\Delta^2 w_{n-2}^x\right] - 2n\frac{w_{n-1}^x}{w_n^x}\Delta w_{n-1}^x\right\} \tag{13}$$

Analogous formulas we have for $\ddot{y}(0), \ddot{y}(1), \ddot{z}(0)$ and $\ddot{z}(1)$.

Remarks.

1. Denoting by $m(t)$ and $m_T(t)$ the slopes of the polynomial (or classical rational) and total coordinates rational Bézier curves, respectively, in virtue of (12), we have

$$m_T(0) = \frac{w_0^x w_1^y}{w_1^x w_0^y}m(0) \text{ and } \mathrm{m_T}(1) = \frac{\mathrm{w_n^x w_{n-1}^y}}{\mathrm{w_{n-1}^x w_n^y}}\mathrm{m}(1), \tag{14}$$

so we can control the slopes in b_0 and b_n by means of the coordinate shape parameters. As a consequence, a total coordinates rational Bézier curve do not always have convex hull property (see Figure 2).

2. Similar remark is valueable relative to the direction cosines of a total coordinates rational Bézier curve in b_0 and b_n, respectively.

3. Partial and Total Coordinates Rational Bézier Surfaces

As for curves, we generalize the rational Bézier surfaces by introduceing independent coordinate shape parameters. The vectorial equation of this generalized surfaces is

$$R(u,v) = [x(u,v),\ y(u,v),\ z(u,v)]^T,$$

where

$$\begin{aligned}
x(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w^x_{ij} b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w^x_{ij} b_{m,i}(u) b_{n,j}(v)} x_{ij},\\
y(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w^y_{ij} b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w^y_{ij} b_{m,i}(u) b_{n,j}(v)} y_{ij},\\
z(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w^z_{ij} b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w^z_{ij} b_{m,i}(u) b_{n,j}(v)} z_{ij},\\
&(u,v) \in [0,1]\mathrm{x}[0,1].
\end{aligned} \tag{15}$$

We denote the matrices of the shape parameters and the coordinate shape parameters, respectively as follows

$$W = [w_{ij}],\ U = [1],\ W^t = [w^t_{ij}],\ t \in \{x,y,z\},\ i = \overline{0,m},\ j = \overline{0,n}.$$

We observe that if $W^x = W^y = W^z = U$ then equations (15) represent a polynomial Bézier surface and for $W^x = W^y = W^z = W$ results a classical rational Bézier surface. The definition 1-4, from curves, one extends to coordinates rational Bézier surfaces.

In Figure 4 is represented the polynomial Bézier surface corresponding to the following control points:

	j=0	j=1	j=2	j=3	j=4	j=5
i=0	(-4,-2,5)	(-3,-1,6)	(-2,1,5)	(-1,3,5)	(-1,5,6)	(0,6,7)
i=1	(-2,-2,10)	(-2,-1,8)	(-1,1,7)	(-2,3,8)	(-3,5,4)	(-2,7,3)
i=2	(2,-2,9)	(3,-1,7)	(2,1,6)	(-1,2,7)	(-6,3,9)	(-5,4,10)
i=3	(6,-2,2)	(5,-1,6)	(4,1,8)	(3,3,9)	(2,5,11)	(3,6,9)
i=4	(12,-2,4)	(10,-1,4)	(8,2,3)	(7,4,3)	(6,5,1)	(6,6,1)
i=5	(8,-3,5)	(9,0,7)	(10,2,6)	(8,4,5)	(10,5,6)	(11,6,7)

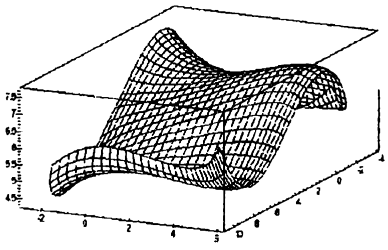

Fig.4.

The coordinate rational surfaces from Fig.5 and Fig.6 have the same control net as the surface presented in Figure 4 and the coordinate shape parameters $\left(w_{ij}^{x},\ w_{ij}^{y},\ w_{ij}^{z}\right)$ are specified for each figure.

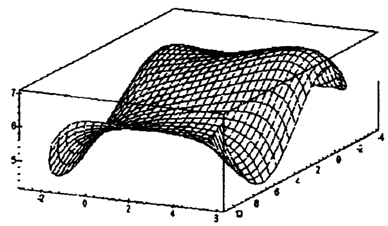

Fig.5.

$$W^x = W^y = U \text{and} \mathrm{W}^z = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0.1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 20 & 1 & 1 & 1 \end{pmatrix}$$

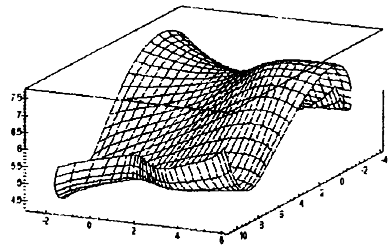

Fig.6.

$$W^z = U \text{and} \mathrm{W}^x = \mathrm{W}^y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0.1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 10 & 1 & 1 & 1 \end{pmatrix}$$

It evidently is that we have more possibilities to control the shape of a coordinate rational Bézier surface.

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INVARIANT SETS IN MENGER SPACES

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Abstract. The purpose of the paper is to extend some results regarding the self-similar sets from the case of the ordinary metric spaces to the case of probabilistic metric spaces, introduced by K. Menger.

1. Introduction

In recent years the interest for sets having non-integer Hausdorff dimension is growing. There were named fractals by Mandelbrot. The most known fractals are invariant sets with respect to a system of contraction maps, especially the so called self-similar sets. In a famous work, Hutchinson [4] first studied the invariant sets systematically in a general framework. He proved among others the following:*Let X be a complete metric space and* $f_1, \ldots, f_m : X \to X$ *be contraction maps. Then there exists a unique compact set* $K \subseteq X$ *such that* $K = \bigcup_{i=1}^{m} f_i(K)$. If the maps f_i are similitudes, this invariant set K is said to be *self-similar.*

Our aim in this work is to generalize the above result for probabilistic metric spaces introduced in 1942 by K. Menger [5] who generalized the theory of metric spaces, to the development of which he already brought a major contribution. He proposed to replace the distance $d(x, y)$ by a distribution function $F_{x,y}$ whose value $F_{x,y}(t)$, for any real number t, is interpreted as the probability that the distance between x and y is less than t. The theory of probabilistic metric spaces was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [8].

The study of contraction mappings for probability metric spaces was initiated by V.M.Sehgal [10],[11], H.Sherwood [13],[14], and A.T.Bharucha-Reid [1], [12]. For more recently papers dealing with generalizations and applications one can consult [2] and [6].

In section 2 we shall recall some fundamental notions from the theory of probabilistic metric spaces and prove some new results on the probabilistic Hausdorff-Pompeiu metric (Propositions 2.4 and 2.5). In section 3 we prove our main result (Theorem 3.1).

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2. Preliminaries

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$. A mapping $F : \mathbf{R} \to [0,1]$ is called a ***distribution function*** if it is non-decreasing, left continuous with $\inf F = 0$. By Δ we shall denote the set of all distribution functions F. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x,y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbf{R}$ by $F_{x,y}(t)$, respectively. The ordered pair $(X, \mathcal{F})$ is a ***probabilistic metric space*** if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbf{R}$;

2) $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3) if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s+t) = 1$.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following conditions are satisfied:

4) $T(a,1) = a$ for every $a \in [0,1]$;

5) $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$

6) if $a \geq c$ and $b \geq d$ then $T(a,b) \geq T(c,d)$;

7) $T(a, T(b,c)) = T(T(a,b), c)$ for every $a, b, c \in [0,1]$.

We list here the simplest:

$T_1(a,b) = max\{a+b-1, 0\}$,

$T_2(a,b) = ab$,

$T_3(a,b) = Min(a,b) = min\{a,b\}$,

A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, T is a t-norm, and

8) $F_{x,y}(s+t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbf{R}_+$.

The (t, ϵ)-topology in a Menger space was introduced in 1960 by B. Schweizer and A. Sklar [7]. The base for the neighbourhoods of an element $x \in X$ is given by

$$\{U_x(t, \epsilon) \subseteq X : t > 0, \epsilon \in]0,1[\},$$

where

$$U_x(t, \epsilon) := \{y \in X : F_{x,y}(t) > 1 - \epsilon\}.$$

If t-norm T satisfies the condition

$$sup\{T(t,t) : t \in [0,1[\} = 1,$$

then the (t, ϵ) -topology is metrizable (see [9]).

In 1966, V.M. Sehgal [10] introduced the notion of a contraction mapping in probabilistic metric spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists a $r \in]0,1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbf{R}_+$.

For example, if (X,d) is a metric space and $G \in \Delta^+$, $G \not= H$, in [7] one defines

$$F_{x,y}(t) = G(\frac{t}{d(x,y)}) \ \ if\ x \not= y,$$

and

$$F_{x,y}(t) = H(t) \ \ if\ x = y,$$

where the distribution function H is defined by $H(t) = 1$ if $t > 0$, and $H(t) = 0$ if $t \leq 0$. If $f : X \to X$ is a contraction with ratio r, then it is a contraction in Sehgal sence with the same ratio. Indeed, we have

$$F_{f(x),f(y)}(rt) = G(\frac{rt}{d(f(x),f(y))}) \geq G(\frac{rt}{rd(x,y)})\ if\ \ f(x) \not= f(y)\ and\ x \not= y,$$

$$F_{f(x),f(y)}(rt) = G(\frac{rt}{rd(x,y)}) \geq H(t)\ if\ \ x \not= y\ and\ f(x) = f(y),$$

$$F_{f(x),f(y)}(rt) = H(t) = F_{x,y}(t) \ \ if\ x = y.$$

A sequence $(x_n)_{n \in \mathbf{N}}$ in X is said to be *fundamental* if

$$\lim_{n,m \to \infty} F_{x_m, x_n}(t) = 1$$

for all $t > 0$. The element $x \in X$ is called *limit* of the sequence, and we write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$, if $\lim_{n \to \infty} F_{x, x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent. If (X,d) is a metric space, then the metric d induces a mapping $\mathcal{F} : X \times X \to \Delta^+$, where $\mathcal{F}(x,y) = F_{x,y}$ is defined by

$$F_{x,y}(t) = H(t - d(x,y)),\ t \in \mathbf{R}.$$

Moreover $(X,\mathcal{F},Min)$ is a Menger space. It is complete if the metric d is complete (see [12]). The space $(X,\mathcal{F},Min)$ thus obtained is called the ***induced Menger space.***

Proposition 2.1. *(V.M. Sehgal [10], see also [2]) Every contraction mapping $f : X \to X$ on a complete Menger space $(X,\mathcal{F},Min)$ has a unique fixed point x_0. Moreover, $f^n(x) \to x_0$ for each $x \in X$.*

Let $(X,\mathcal{F},T)$ be a Menger space with T continuous and let A be a nonempty subset of X. The function $D_A : \mathbf{R} \to [0,1]$ defined by

$$D_A(t) := \sup_{s<t} \inf_{x,y\in A} F_{x,y}(s)$$

is called the ***probabilistic diameter of A***. It is a distribution function from Δ^+. The set $A \subseteq X$ is ***probabilistic bounded*** if $\sup_{t>0} D_A(t) = 1$. If B and C are two subsets of X with $B \cap C \neq \emptyset$, then

$$D_{B\cup C}(s+t) \geq T(D_B(s), D_C(t)); \ s,t \in \mathbf{R} \tag{1}$$

(see [3, Theorem 10]).

Set

$$\mathcal{D}^+ = \{F \in \Delta^+ : \sup_{t\in R} F(t) = 1\}.$$

In the following we suppose that $(X,\mathcal{F},T)$ is a Menger space with $\mathcal{F} : X \times X \to \mathcal{D}^+$and T is continuous. In this case every set with two elements is probabilistic bounded.

Proposition 2.2. *If A is a probabilistic bounded set in $(X,\mathcal{F},T)$ and $b \in X$, then the set $A_1 = A \cup \{b\}$ is also bounded.*

Proof. Let $a \in A$. Then $A_1 = A \cup \{a,b\}$,hence by (1)

$$D_{A_1}(2t) \geq T(D_A(t), F_{a,b}(t)).$$

Since $\sup_{t\in R} D_A(t) = 1$ and $\sup_{t\in R} F_{a,b}(t) = 1$, we have $\sup_{t\in R} D_{A_1}(2t) = 1$. □

Corollary 2.1. *Every finit set in $(X,\mathcal{F},T)$ is probabilistic bounded.*

Corollary 2.2. *If A and B are probabilistic bounded sets in $(X,\mathcal{F},T)$, then $A \cup B$ is also probabilistic bounded.*

An example for probabilistic unbounded set is the following. Let $\mathcal{F} : \mathbf{R}\times\mathbf{R} \to D^+$ be defined by $F_{x,y}(t) = H(t - |x - y|)$. Let N be the set of all natural numbers. Then $D_N(t) = 0$ for every t, hence N is probabilistic unbounded.

In a probabilistic metric space $(X, \mathcal{F})$, the set A is said to be ***precompact*** if for every $t > 0$ and $\epsilon \in]0,1[$ there exists a finite cover $\{C_i\}_{i\in I}$ of A such that $D_{C_i}(t) > 1 - \epsilon$ for all $i \in I$. A precompact set A is ***totally bounded,*** i.e. for every $t > 0$ and $\epsilon \in]0,1[$ there exists a finite subset $B \subseteq A$ such that, for each $x \in A$, there is an $y \in B$ with $F_{x,y}(t) > 1 - \epsilon$ (see [2, Proposition 1.2.3.]). In a Menger space with a t-norm T such that $\sup_{a<1} T(a,a) = 1$ the converse assertion also holds: *a set A* is precompact if and only if it is totally bounded (see [2, Theorem 1.2.1.]).

Let A and B nonempty subsets of X. The ***probabilistic Hausdorff-Pompeiu distance*** between A and B is the function $F_{A,B} : \mathbf{R} \to [0,1]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)).$$

Proposition 2.3. *If C is a nonempty collection of nonempty closed bounded sets in $(X, \mathcal{F}, T)$, then $(C, \mathcal{F}_C, T)$ is also a Menger space, where $\mathcal{F}_C$ is defined by $\mathcal{F}_C(A,B) := F_{A,B}$ for all $A, B \in C$.*

Proof. We have, for all $A, B \in C$,

$$F_{A,B}(x) \geq \sup_{t<x} T(\inf_{p\in A} \inf_{q\in B} F_{p,q}(t), \inf_{q\in B} \inf_{p\in A} F_{p,q}(t)) \geq$$

$$\geq T(D_{A\cup B}(t), D_{A\cup B}(t)).$$

Since by Corollary 2.2, the set $A\cup B$ is probabilistic bounded, it follows $\sup_{x\in\mathbf{R}} F_{A,B}(x) = 1$. Therefore, by [3, Theorem 18] $(C, \mathcal{F}_C, T)$ is a Menger space. □

In the following we suppose that $T = Min$.

Proposition 2.4. *If $(X, \mathcal{F}, Min)$ is a complete Menger space and C is the collection of all nonempty closed bounded subsets of X in $(t,\epsilon)-$ topology, then $(C, \mathcal{F}_C, Min)$ is also a complete Menger space.*

Proof. Let $(A_n)_{n\in\mathbf{N}}$ be a fundamental sequence in C and let

$$A = \{x \in X : \forall n \in \mathbf{N}, \exists x_n \in A_n, \forall t > 0, \lim_{n\to\infty} F_{x_n,x}(t) = 1\}. \tag{2}$$

Let $\overline{A}$ denote the closure of A. By [3, Theorem 15] we have $F_{A_n,A} = F_{A_n,\overline{A}}$, so we must show that (i) $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$, and (ii) $\overline{A} \in \mathcal{C}$.

(i) Let $\epsilon > 0$ and $t > 0$ be given. Then there exists $n_\epsilon(t) \in \mathbb{N}$ so that $n, m \geq n_\epsilon(t)$ implies $F_{A_n,A_m}(\frac{t}{2}) > 1 - \epsilon$. Let $n > n_\epsilon(t)$. We claim that $F_{A_n,A}(t) \geq 1 - \epsilon$.

If $x \in A$ then there is a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in A_k$ and $\lim_{k\to\infty} F_{x_k,x}(\frac{t}{2}) = 1$. So, for large enough $k \geq n_\epsilon(t)$, we have $F_{x_k,x}(\frac{t}{2}) > 1-\epsilon$. Thus, since $F_{A_n,A_k}(\frac{t}{2}) > 1-\epsilon$, for $n \geq n_\epsilon(t)$, there exist $y \in A_n$ and $z \in A_k$ such that

$$Min(F_{x_k,y}(\frac{t}{2}), F_{z,y}(\frac{t}{2})) > 1 - \epsilon,$$

hence $F_{x_k,y}(\frac{t}{2}) > 1 - \epsilon$. By 8) we have $F_{x,y}(t) > 1 - \epsilon$, hence

$$\sup_{s<t} \inf_{x\in A} \sup_{y\in A_n} F_{x,y}(s) > 1 - \epsilon. \tag{3}$$

Now suppose $y \in A_n$. Choose integers $k_1 < k_2 < ... < k_i < ...$ so that $k_1 = n$ and

$$F_{A_k,A_{k_i}}(\frac{t}{2^{i+1}}) > 1 - \frac{\epsilon}{2^{i-1}},$$

for all $k > k_i$. Hence we can choose $s < t$ such that $\inf_{z\in A_{k_i}} \sup_{x\in A_k} F_{x,z}(\frac{s}{2^{i+1}}) > 1-\frac{\epsilon}{2^{i-1}}$. Then define a sequence (y_k) with $y_k \in A_k$ as follows: For $k < n$, choose $y_k \in A_k$ arbitrarily. Choose $y_n = y$. If y_{k_i} has been chosen, and $k_i < k \leq k_{i+1}$, choose $y_k \in A_k$ with $F_{y_{k_i},y_k}(\frac{s}{2^{i+1}}) > 1 - \frac{\epsilon}{2^{i-1}}$. Then, for $k_i < k \leq k_{i+1} < ... < k_j < l \leq k_{j+1}$, we have

$$F_{y_l,y_k}(\frac{s}{2^{i-1}}) \geq T(F_{y_k,y_{k_i}}(\frac{s}{2^i}), T(F_{y_{k_i},y_{k_{i+1}}}(\frac{s}{2^{i+1}}), \ldots,$$

$$T(F_{y_{k_{j-1}},y_{k_j}}(\frac{s}{2^j}), F_{y_{k_j},y_l}(\frac{s}{2^j}))...) > 1 - \frac{\epsilon}{2^{i-1}}.$$

Let $r > 0$, $\eta > 0$, and choose i so that $r > \frac{s}{2^{i-1}}$ and $\frac{\epsilon}{2^{i-1}} < \eta$. We have

$$F_{y_k,y_l}(r) \geq F_{y_k,y_l}(\frac{s}{2^{i-1}}) > 1 - \frac{\epsilon}{2^{i-1}} > 1 - \eta.$$

Hence (y_k) is a fundamental sequence, so it converges. Let x be its limit. Then $x \in A$ and we have

$$F_{x,y}(t) \geq T(F_{x,y_k}(\frac{t}{2}), F_{y_k,y}(\frac{t}{2})).$$

We choose k such that $F_{x,y_k}(\frac{t}{2}) > 1-\epsilon$. Since $F_{y,y_k}(\frac{t}{2}) > 1-\epsilon$, it follows $F_{x,y}(t) > 1-\epsilon$. Therefore we have

$$\sup_{s<t} \inf_{y\in A_n} \sup_{x\in A} F_{x,y}(s) > 1 - \epsilon. \tag{4}$$

By (3) this shows that

$$F_{A_n,A}(t) = \sup_{s<t} T(\inf_{x\in A}\sup_{y\in A_n} F_{x,y}(s), \inf_{y\in A_n}\sup_{x\in A} F_{x,y}(s)) > 1-\epsilon.$$

So $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$.

(ii) Taking $\epsilon = 1$ in the last argument, we have proved that A is nonempty.

We have to show that A is bounded. Since $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $\epsilon > 0$ and $t_0 > 0$ we have $\inf_{x\in A}\sup_{w\in A_n} F_{x,w}(t_0) > 1-\epsilon$ and $\inf_{y\in A_n}\sup_{x\in A} F_{x,y}(t_0) > 1-\epsilon$. A_n being probabilistic bounded, for all $\epsilon > 0$ there is $t_\epsilon > t_0$ such that $\inf_{u,v\in A_n} F_{u,v}(t_\epsilon) > 1-\epsilon$.

For $x, y \in A$ there exist $u, v \in A_n$ such that

$$F_{x,u}(t_0) > 1-\epsilon,\ F_{y,v}(t_0) > 1-\epsilon.$$

We have

$$F_{x,y}(3t_\epsilon) \geq T(F_{x,u}(t_\epsilon), F_{u,y}(2t_\epsilon)) \geq T(F_{x,u}(t_0), T(F_{u,v}(t_\epsilon), F_{v,y}(t_0))) > 1-\epsilon.$$

So $D_A(3t_\epsilon) \geq 1-\epsilon$, consequently $\sup_{t\in\mathbf{R}} D_A(t) = 1$. By [3] it follows that $D_A = D_{\overline{A}}$. Since A is bounded, $\overline{A}$ is also bounded and closed, so $\overline{A} \in \mathcal{C}$. □

Proposition 2.5. *Let $\mathcal{K}$ be the collection of all nonempty compact sets in the complete Menger space $(X, \mathcal{F}, Min)$ and let $\mathcal{C}$ be the collection of all nonempty closed bounded subsets of X in $(t,\epsilon)-$ topology. Then $(\mathcal{K}, \mathcal{F}_{\mathcal{K}}, Min)$ is a closed subspace of $(\mathcal{C}, \mathcal{F}_{\mathcal{C}}, Min)$.*

Proof. First we show that $\mathcal{K} \subseteq \mathcal{C}$. For this we have to show that any nonempty compact set A is probabilistic bounded. Let $\epsilon \in]0,1[$. For every $t > 0$, there exists a finite cover $\{C_i\}_{i\in I}$ of A such that $D_{C_i}(t) > 1-\epsilon$ for all $i \in I$. Let $I = \{1, ..., m\}$ and set $C = \cup_{i=1}^{m} C_i$.

For every $i \in \{1, \cdots, m\}$ choose an element $c_i \in C_i$ and set $B = \{c_1, \cdots, c_m\}$. Then, for $C_i^* = C_i \cup B$, we have $C = \cup_{i=1}^{m} C_i^*$. Let $s > 0$ such that $D_B(s) > 1-\epsilon$. By (1) we have

$$\begin{aligned} D_A(m(t+s)) &\geq D_C(m(t+s)) \geq \\ &\geq T(D_{C_1^*}(t+s), T(D_{C_2^*}(t+s), ..., T(D_{C_{m-1}^*}(t+s), D_{C_m^*}(t+s))...) \\ &\geq T(D_{C_1}(t), T(D_{C_2}(t), \cdots, T(D_{C_{m-1}}(t), T(D_{C_m}(t), D_B(s))...) > 1-\epsilon, \end{aligned}$$

hence A is probabilistic bounded.

Let $(A_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{K}$ converging to $A \in \mathcal{C}$. We shall show that A is totally bounded. Let $\epsilon > 0$, $t > 0$, and choose n so that $F_{A_n,A}(\frac{t}{3}) > 1-\epsilon$. The set A_n being precompact in the (t,ϵ) -topology, there exist $x_1, ...x_m \in A_n$ such that

$$A_n \subseteq \bigcup_{i=1}^{m} U_{x_i}(\frac{t}{3},\epsilon).$$

For each x_i there is $y_i \in A$ with $F_{x_i,y_i}(\frac{t}{3}) > 1-\epsilon$. For $y \in A$ there exists $x \in A_n$ with $F_{x,y}(\frac{t}{3}) > 1-\epsilon$. Let $i \in \{1,...,m\}$ such that $x \in U_{x_i}(t,\epsilon)$. Then

$$F_{y,y_i}(t) \geq T(F_{y,x}(\frac{t}{3}), T(F_{x,x_i}(\frac{t}{3}), F_{x_i,y_i}(\frac{t}{3})) > 1-\epsilon,$$

hence

$$A \subseteq \bigcup_{i=1}^{m} U_{y_i}(t,\epsilon).$$

Therefore A is totally bounded. The (t,ϵ)-topology being metrizable and A being closed, it is compact. □

Corollary 2.3. *If $(X, \mathcal{F}, Min)$ is a complete Menger space and $\mathcal{K}$ is the collection of all nonempty compact subsets of X in $(t,\epsilon)-$ topology, then $(\mathcal{K}, \mathcal{F}_{\mathcal{K}}, Min)$ is also a complete Menger space.*

3. Invariant sets

In this section we will generalize Hutchinson's theorem on invariant sets.

Proposition 3.1. *Let $(X, \mathcal{F}, Min)$ be a Menger space and $\mathcal{C}$ be the collection of all nonempty closed bounded sets in X. Let $f_1, ..., f_m : X \to X$ be contractions with ratios $r_1, ...r_m \in]0,1[$ and let $\phi : \mathcal{C} \to \mathcal{C}$ be defined by*

$$\phi(E) := \cup_{i=1}^{p} f_i(E).$$

Then ϕ is a contraction.

Proof. Let $r = max\{r_i, 1 \leq i \leq m\}$ and $A, B \in \mathcal{C}$. We shall show that

$$F_{\phi(A),\phi(B)}(rt) \geq F_{A,B}(t), \tag{5}$$

for all $t > 0$.

For all $A, B \in \mathcal{C}$ and $s < t$, we have

$$F_{\cup_{i=1}^{m} f_i(A), \cup_{i=1}^{m} f_i(B)}(rt) \geq T(\inf_{u\in\cup_{i=1}^{m} f_i(A)} \sup_{v\in\cup_{i=1}^{m} f_i(B)} F_{u,v}(rs), \inf_{v\in\cup_{i=1}^{m} f_i(B)} \sup_{y\in\cup_{i=1}^{m} f_i(A)} F_{u,v}(rs)).$$

Let i_0 and j_0 be such that

$$\inf_{u\in\cup_{i=1}^m f_i(A)} \sup_{v\in\cup_{i=1}^m f_i(B)} F_{u,v}(rs) = \inf_{u\in f_{i_0}(A)} \sup_{v\in\cup_{i=1}^m f_i(B)} F_{u,v}(rs) \geq \inf_{u\in f_{i_0}(A)} \sup_{v\in f_{i_0}(B)} F_{u,v}(rs),$$

$$\inf_{v\in\cup_{i=1}^m f_i(B)} \sup_{u\in\cup_{i=1}^m f_i(A)} F_{u,v}(rs) = \inf_{y\in f_{j_0}(B)} \sup_{u\in\cup_{i=1}^m f_i(A)} F_{u,v}(rs) \geq \inf_{v\in f_{j_0}(B)} \sup_{u\in f_{j_0}(A)} F_{u,v}(rs).$$

Hence

$$F_{\cup_{i=1}^m f_i(A),\cup_{i=1}^m f_i(B)}(rt) \geq T(\inf_{u\in f_{i_0}(A)} \sup_{v\in f_{i_0}(B)} F_{u,v}(rs), \inf_{v\in f_{j_0}(B)} \sup_{u\in f_{j_0}(A)} F_{u,v}(rs))$$

$$\geq T(\inf_{u\in f_{l_0}(A)} \sup_{v\in f_{l_0}(B)} F_{u,v}(rs), \inf_{v\in f_{l_0}(B)} \sup_{u\in f_{l_0}(A)} F_{u,v}(rs)) =$$

$$= T(\inf_{x\in A} \sup_{y\in B} F_{f_{l_0}(x),f_{l_0}(y)}(rs), \inf_{y\in B} \sup_{x\in A} F_{f_{l_0}(x),f_{l_0}(y)}(rs)) \geq$$

$$\geq T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)),$$

where $l_0 = i_0$ if $\inf_{u\in f_{i_0}(A)} \sup_{v\in f_{i_0}(B)} F_{u,v}(rs) \leq \inf_{v\in f_{j_0}(B)} \sup_{u\in f_{j_0}(A)} F_{u,v}(rs)$, and $l_0 = j_0$ else. Therefore we have (5). □

Theorem 3.1. *Let $(X, \mathcal{F}, Min)$ be a complete Menger space and let $f_1, ..., f_m : X \to X$ be contractions with ratios $r_1, ... r_m \in]0, 1[$, respectively. Then there exists a nonempty compact subset K of X such that*

$$f_1(K) \cup ... \cup f_m(K) = K.$$

Moreover, the set K with this property is unique in the space of all nonempty closed bounded sets in X.

Proof. By Proposition 3.1 the function $\phi : \mathcal{C} \to \mathcal{C}$ defined by

$$\phi(E) = \cup_{i=1}^p f_i(E)$$

is a contraction, and by Proposition 2.4 $(\mathcal{C}, \mathcal{F}_\mathcal{C}, Min)$ is a complete Menger space. Then, by Proposition 2.1. there is a unique set K in $\mathcal{C}$ such that $\phi(K) = K$. Moreover, we have $\lim_{n\to+\infty} \phi^n(K_0) = K$ for any $K_0 \in \mathcal{K}$. Thus, by Proposition 2.5 the set K must be in $\mathcal{K}$. □

Corollary 3.1. *(Hutchinson [4]) Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ be contraction maps with ratios $r_1, \ldots, r_m$, respectively. Then there exists a unique nonempty compact set $K \subseteq X$ such that $K = \bigcup_{i=1}^m f_i(K)$.*

Proof. Let $(X, \mathcal{F}, Min)$ be the induced Menger space by the metric d. Since, for each $t > 0$ and $i \in \{1, ..., m\}$,

$$F_{f_i(x),f_i(y)}(r_i t) = H(r_i t - d(f_i(x), f_i(y))) \geq H(r_i t - r_i d(x,y)) = H(t - d(x,y)) = F_{x,y}(t),$$

the conclusion follows from Theorem 3.1. □

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SUFFICIENT CONDITIONS FOR STARLIKENESS II

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Abstract. In this paper we study a differential subordination of the form:

$$\alpha zp'(z) + \alpha p^2(z) + (\beta - \alpha)p(z) \prec h(z),$$

where

$$h(z) = \alpha nzq'(z) + \alpha q^2(z) + (\beta - \alpha)q(z),$$

with $\alpha > 0$, $\alpha + \beta > 0$, and the function q is convex with $q(0) = 1$, and

$$Re q(z) > \frac{\alpha - \beta}{2\alpha}.$$

Our results are obtained by using the method of differential subordinations developed in [1], [2] and [3]. For $\beta = 1$, $q(z) = 1 + \lambda z$ and $n = 1$ this problem was studied in [4].

1. Introduction and preliminaries

Let A_n denote the class of functions f of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \; z \in U,$$

which are analytic in the unit disc U.

Let $A = A_1$ and let $S^* = \left\{ f \in A, \; Re\frac{zf'(z)}{f(z)} > 0, \; z \in U \right\}$ be the class of starlike functions in U.

We will use the following notions and lemmas to prove our results.

If f and g are analytic functions in U,then we say that f is subordinate to g written $f \prec g$, of $f(z) \prec g(z)$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$. If g is univalent then $f \prec g$, if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma A. *([1], [2], [3]) Let q be univalent in $\overline{U}$ with $q'(\zeta) \neq 0$, $|\zeta| = 1$, $q(0) = a$ and let $p(z) = a + a_n z^n + \dots$ be analytic in U, with $p(z) \not\equiv a$, and $n \geq 1$.*

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If $p(z) \not\prec q(z)$ then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ and there is $m \geq n$ such that:

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

The function $L(z,t)$, $z \in U$, $t \in I = [0, \infty)$ is called a Loewner chain or a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + a_3(t)z^3 + \ldots$ for $z \in U$ is analytic and univalent in U for any $t \in I$ and if $L(z,t_1) \prec L(z,t_2)$ when $0 \leq t_1 \leq t_2$.

Lemma B. *([7]) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in [0,1]$ and $M > 0$ such that:*

(i) $L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for each $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.

(ii) There is a function $p(z,t)$ analytic in U for any $t \geq 0$ measurable in $[0,\infty)$ for any $z \in U$, with $Re\, p(z,t) > 0$ for $z \in U$, $t \geq 0$ such that

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z} p(z,t), \text{ for } |z| < r \text{ and for almost any } t \geq 0.$$

2. Main results

Theorem 1. *Let q be a convex function in U, with $q(0) = 1$,*

$$Re\, q(z) > \frac{\alpha - \beta}{2\alpha}, \quad \alpha > 0, \quad \alpha + \beta > 0 \tag{1}$$

and let

$$h(z) = \alpha nzq'(z) + \alpha q^2(z) + (\beta - \alpha)q(z) \tag{2}$$

If the function $p(z) = 1 + p_n z^n + \ldots$ satisfies the condition:

$$\alpha zp'(z) + \alpha p^2(z) + (\beta - \alpha)p(z) \prec h(z) \tag{3}$$

where h is given by (2) then $p(z) \prec q(z)$ and q is the best dominant.

Proof. Let

$$L(z,t) = \alpha(n+t)zq'(z) + \alpha q^2(z) + (\beta - \alpha)q(z) = \psi(q(z),(n+t)zq'(z)) \tag{4}$$

If $t = 0$ we have $L(z,0) = \alpha nzq'(z) + \alpha q^2(z) + (\beta - \alpha)q(z) = h(z)$. We will show that condition (3) implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

From (4) we easily deduce:

$$\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} = (n+t)\left[1+\frac{zq''(z)}{q'(z)}\right] + 2q(z) + \frac{\beta-\alpha}{\alpha}$$

and by using the convexity of q and condition (1) we obtain:

$$Re\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \geq 0$$

. Hence by Lemma B we deduce that $L(z,t)$ is a subordination chain. In particular, the function $h(z) = L(z,0)$ is univalent and $h(z) \prec L(z,t)$, for $t \geq 0$. If we suppose that $p(z)$ is not subordinate to $q(z)$, then, from Lemma A, there exist $z_0 \in U$, and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ with $|\zeta_0| = 1$, and $z_0p'(z_0) = (n+t)\zeta_0 q'(\zeta_0)$, with $t \geq 0$. Hence

$$\psi_0 = \psi(p(z_0), z_0p'(z_0)) = \psi(q(\zeta_0), (n+t)\zeta_0 q'(\zeta_0)) = L(\zeta_0, t),\ t \geq 0,$$

Since $h(z_0) = L(z_0, 0)$, we deduce that $\psi_0 \notin h(U)$, which contradicts condition (3). Therefore, we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

If we let $p(z) = \frac{zf'(z)}{f(z)}$, where $f \in A$, then Theorem 1 can be written in the following equivalent form:

Theorem 1'. *Let q be a convex function with $q(0) = 1$, and*

$$Re\, q(z) > \frac{\alpha-\beta}{2\alpha},\ \alpha > 0,\ \alpha+\beta > 0.$$

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, $z \in U$, satisfies the condition:

$$\frac{\alpha z^2 f''(z)}{f(z)} + \beta\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in U$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

an q is the best dominant.

3. Particular cases

1) If we let $\alpha = 1$, $\beta \geq 1$, and $q(z) = \frac{1+z}{1-z}$, then

$$h(z) = \frac{2nz}{(1-z)^2} + \left(\frac{1+z}{1-z}\right)^2 + \gamma\frac{1+z}{1-z} \quad \text{with} \quad \gamma = \beta - 1 \geq 0.$$

If $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, then

$$h(e^{i\theta}) = \frac{-n}{2\sin^2\frac{\theta}{2}} - \cot^2\frac{\theta}{2} + \gamma\, i \cot\frac{\theta}{2} = u + iv$$

and the domain *D=h(U)* is the exterior of the parabola $u = -\frac{n}{2} - \frac{n+2}{2\gamma^2}v^2$. If $\gamma = 0$, then D is the complex plane slit along the half-line $v = 0$ and $u \leq -\frac{n}{2}$.

Using Theorema 1' we deduce the following criterion for starlikeness:

$$\text{If } f \in A_n \text{ and } \frac{z^2 f''(z)}{f(z)} + \beta\frac{zf'(z)}{f(z)} \in D \text{ then } f \in S^*.$$

2) If we let $\alpha = 1$, $\beta = 0$, and $q(z) = \frac{1}{1-z}$, then, $h(z) = \dfrac{z(n+1)}{(1-z)^2}$ and *h(U)* is the complex plane slit along the half-line *v=0* and $u \leq -\frac{n+1}{4}$. Using Theorema 1' we deduce the following criterion for starlikeness of order $\frac{1}{2}$: If $f \in A_n$ with $\frac{f(z)}{z} \neq 0$ satisfy the condition:

$$\frac{z^2 f''(z)}{f(z)} \prec \frac{(n+1)z}{(1-z)^2} \quad \text{then} \quad Re\frac{zf'(z)}{f(z)} > \frac{1}{2}.$$

In particular:

$$Re\left[\frac{z^2 f''(z)}{f(z)}\right] > -\frac{n+1}{4} \quad \Rightarrow \quad Re\frac{zf'(z)}{f(z)} > \frac{1}{2} \tag{5}$$

Example 1.

$$\text{If } f(z) = \frac{1}{\lambda}\sin\lambda z \text{ then } f \in A_2, \text{ and } \frac{f''(z)}{f(z)} = -\lambda^2,$$

and by using (5) we deduce that $f_\lambda \in S^*\left(\frac{1}{2}\right)$ for $|\lambda| \leq \frac{\sqrt{3}}{2}$.

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INTEGRATION OF VECTOR FUNCTIONS WITH RESPECT TO VECTOR MEASURES

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Abstract. The algebraic theory of integration with respect to a semi-variation is outlined. It is applied to the integration of vector-valued functions with respect to a vector-valued measure. Different settings are considered (bilinear integration, Dobrakov's integral, tensor integration). Emphasis is put on convergence theorems.

The purpose of this paper is to study the integration of a vector function f with respect to a vector measure m. This problem may be settled in different settings. In his pioneering work [1], Bartle supposes f has values in a Banach space F, m has values in a Banach space E. Furthermore a bilinear map from $F \otimes E$ into a Banach space G is given and the integral of f with respect to m has values in G. A another setting is that of Dobrakov [DO] : m has values in the space $L(X, Y)$ of continuous linear operators from X into Y, f is X-valued and the integral is Y-valued. From the set-theoretical point of view, Dobrakov'setting may be considered as a particular case of Bartle's setting. Conversely it is possible to transform a problem given in the Bartle's setting into a problem in the Dobrakov's setting. But the additionnal assumptions concerning mainly the additivity of m makes the set-theoretical manipulations dangerous. For example Dobrakov supposes only the σ-additivity of $m(.)x$ for every $x \in X$ and not the σ-additivity in the strong sense.

Many other contributions appeared in the literature since the pioneering works of Bartle and Dobrakov. For example, Guessous [GU] supposes that the integral has its values in the completion of $F \otimes E$ with respect to a tensor norm (ε or π). This setting will be called the tensor integration. It is studied more recently by Jefferies and Okada [J.O] and [JE] Chapter 4.

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As many proofs are similar in different settings, the authors often fail to give complete proofs, referring abruptly to previous papers. This makes the contributions less and less legible, especially if the reader wish to construct self-contained proofs. The first motivation of this paper is to avoid this discomfort.

In all the previously quoted works, a generalisation of the ordinary semi-variation of a vector measure is introduced. This semi-variation is defined as a sub-additive set-function. In [PB] under the influence of [TH] we modify the definition so as the semi-variation is defined as a semi-norm on the vector space of simple functions. Or course the restriction of our semi-variation to sets (*i.e.* characteristic functions) is the customary semi-variation. But as two different vector-semi-variations may be equal on sets, our definition is more powerfull. In particular it is possible to define the integrability of a scalar function with respect to a vector-semi-variation without taking into account the objects and setting it comes from. This is very important for the integration of vector functions with respect to vector measure because it is possible to define the integrability of a scalar function with respect to the semi-variation introduced to this special context (this semi-variation is called below "contextual semi-variation"). This enable us to define the Bochner-integrability in a convenient frame and to make easier the connection with the classical Bochner-integrability with respect to a scalar measure (for the Dobrakov integral see [DO2] and [PAN]).

Section 1 is designed to recall definitions and fundamental results. Section 2 is devoted to the algebraic theory of integrability of scalar functions with respect to a semi-variation. This theory is inspired by the work of Wilhelm ([W1,W2]) and is exposed in [PB] with more details.

Section 3 is devoted to the particular case where the semi-variation is defined on a space $(\mathcal{T})$ of $\mathcal{T}$-simple functions, $\mathcal{T}$ being a σ-algebra.

Section 4 is devoted to the integrability of vector functions.

Section 5 is devoted to the definition of the "contextual" semi-variation, that means the semi-variation adapted to the spaces E, F, G where the measure, the function to be integrated and the integral take respectively their values.

Section 6 is devoted to the customary semi-variation of a vector measure that means the semi-variation adapted to the integration of scalar functions.

Section 7 is devoted to the general setting. The bilinear form from $F \times E$ into G is denoted by $y, x \mapsto y \otimes x$ with a view of compatibility with the tensor setting. To cover simultanuously the setting of Bartle and the setting of Dobrakov, we suppose that $A \mapsto y \odot m(A)$ is a G-valued measure for every $y \in F$.

Section 8 is devoted to the case G is the completion of $F \otimes E$ with respect to the tensor norm ε or π. Some special results are derived using the Orlicz-Pettis's theorem. The covering of the classical case of the integration of vector functions with respect to a scalar measure is warranted both for Pettis and Bochner integrability.

1. Preliminaries

We first recall some definitions. A measurable space $(T, \mathcal{T})$ is a couple formed by an arbitrary set T and a σ-algebra of subsets of T. A function f on T with values in a space F is said to be simple if it is of the form $f = \sum_i \mathbf{1}_{A_i} \xi_i$ where $\{A_i\}$ is a finite $\mathcal{T}$-partition of T and $\xi_i \in F$. The vector space of F-valued simple functions will be denoted by ${}_F(\mathcal{T})$ (or by $(\mathcal{T})$ if $F = \mathbb{R}$).

We reserve the term "measure" to additive set functions which satisfie some σ-additivity property. If E is a Banach space by a E-valued measure defined on $(T, \mathcal{T})$ we mean a additive set function m on $\mathcal{T}$ such that $m(T) = \sum_n m(T_n)$ for every countable $\mathcal{T}$-partition $\{T_n\}$ of T. The convergence of the series is supposed to be valid in the norm topology. Sometimes to emphasize this property we will write "(strong) measure" instead of measure.

There are two ways to weaken the property of σ-additivity. The first one is to suppose σ-additivity for a weaker topology that the norm topology. For exemple if E is the space $L(Y, X)$ of all continuous operators from a Banach space Y into a Banach space X we may consider the strong operator topology (defined by the semi-norms $A \mapsto \|Ay\|$ for y running over Y) or the weak operator topology (defined by the semi-norms $A \mapsto |\langle x', Ay\rangle|$ for x' running over X' and y running over Y). Following the Dunford-Pettis theorem this two topologies are identical. This situation will be encountered in section 4.

An another way to weaken the property of σ-additivity is to consider two spaces E and F in duality and a E-valued set function m defined on $\mathcal{T}$ satisfying the property that $\langle m(.), y\rangle$ is a measure for every Y in F. Such a set function will be called a "weak measure" (for the duality (E, F)). We may note that this σ-additivity property may be considered as the σ-additivity for the topology $\sigma(E,F)$. In fact the two ways of weaken the property of σ-additivity differ from the expository point of view but are able to handle the same concrete situations.

An important particular case of vector measure is obtained by taking $T = \mathbb{N}$ and $\mathcal{T} = \mathcal{P}(\mathbb{N})$. With any set function m is associated the sequence $\{m(n)\}$ such that $m(n) = m(\{n\})$. The discussion of various forms of the σ-additivity property is indeed the discussion of various notions of summability of a sequence. If E is a Banach space a E-valued sequence $\{x_i\}$ is said to be summable if there exists $S \in E$ such that for every $\varepsilon > 0$ there is a finite subset of $\mathbb{N}$ such that for every finite J containing I we have $\|S - \sum_{i\in J} x_i\| \leq \varepsilon$. The sequence $\{x_i\}$ is summable iff it satisfies the so called Cauchy criterion: for every $\varepsilon > 0$ there exists a finite subset I of $\mathbb{N}$ such that $\|\sum_{i\in J} x_i\| \leq \varepsilon$ provided J is a finite subset of $\mathbb{N} \setminus I$. The Cauchy criterion can be generalized to an arbitrary locally convex vector space under the following form: for every continuous semi-norm p on E and every $\varepsilon > 0$ there exists a finite subset I of $\mathbb{N}$ such that $p\left(\sum_{i\in J} x_i\right) \leq \varepsilon$ provided J is a finite subset of $\mathbb{N} \setminus I$. This criterion may be adopted as the definition of summability of the sequence $\{x_i\}$ but doesn't imply that the series $\sum_i x_i$ converges in E but only in the completion of E (more precisely in the quasi-completion of E). For instance a sequence $\{x_i\}$ in a Banach space E is weakly summable (that means summable for the topology $\sigma(E,E')$) iff the sequence $\{\langle x_i, y\rangle\}$ is summable for every $y \in E'$. Its "sum" (generally called "weak sum") belongs to the $\sigma(E,E')$-completion of E *i.e.* E''.

To end this section let us recall that a control measure of a vector measure m is a measure μ such that $\forall A \in \mathcal{T} \quad \mu(A) = 0 \iff m(A) = 0$. Any vector measure has a control measure ([PB] theorem V.46).

2. Integrability with respect to a semi-variation

In this section we outline the algebric construction of the space of integrable functions with respect to a semi-variation as developped in [P.B.]. We refer to this treatise for detailed proofs.

Definition 2.1. Let T an arbitrary set and $\mathcal{L}$ a vector sub-lattice of $\mathbb{R}^T$. A semi-norm v on $\mathcal{L}$ is said to be a Riesz-semi-norm iff the two following conditions are satisfied:

$$0 \leq f_1 \leq f_2 \quad \Longrightarrow \quad v(f_1) \leq v(f_2)$$

$$v(f) = v(|f|)$$

A Riesz-semi-norm is said to be a semi-variation if the following property is satisfied (σ-subadditivity):

$$f, f_n \in \mathcal{L}_+ and f \leq \sum_{n=1}^{\infty} f_n \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

or equivalently:

$$f, f_n \in \mathcal{L} and |f| \leq \sum_{n=1}^{\infty} |f_n| \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

The condition $f \leq \sum_{n=1}^{\infty} f_n$ (pour $f, f_n \in \mathcal{L}_+$) means that for every $t \in T$, one have $f(t) \leq \sum_{n=1}^{\infty} f_n(t)$, this last inequality being satisfied in particular if $\sum_{n=1}^{\infty} f_n(t) = +\infty$. Our aim is to extend a semi-variation v on $\mathcal{L}$ to a space $\mathcal{L}^1(v)$ such that the latter space is complete and $\mathcal{L}$ is dense in $\mathcal{L}^1(v)$. The first step is to define the space $\mathcal{L}^*(v)$ de $\mathcal{L}$ of "not to large functions". Let us put:

$$\mathcal{L}^*(v) = \left\{ f \in \mathbb{R}^T \;\middle|\; \exists f_n \in \mathcal{L} : |f| \leq \sum_{n=1}^{\infty} |f_n| and \sum_{n=1}^{\infty} v(f_n) < +\infty \right\}$$

It is easy to prove that $\mathcal{L}^*(v)$ is a vector sublattice of $\mathbb{R}^T$ and contains $\mathcal{L}$. On this space, we define:

$$v^*(f) = \inf \left\{ \sum_{n=1}^{\infty} v(f_n) \;\middle|\; f_n \in \mathcal{L}, |f| \leq \sum_{n=1}^{\infty} |f_n| \right\}$$

A routine checking proves that v^* is a semi-variation on $\mathcal{L}^*(v)$ which extends v.

The second step is to define $\mathcal{L}^1(v)$ as the closure of $\mathcal{L}$ in $\mathcal{L}^*(v)$ (equipped with the semi-norm v^*). The elements of $\mathcal{L}^1(v)$ are called integrable with respect to v or v-integrable.

From the construction, it results the following theorem:

Theorem 2.2. *$\mathcal{L}^1(v)$ is a vector sub-lattice et v^* is a semi-variation on $\mathcal{L}^1(v)$.*

The completeness of $\mathcal{L}^1(v)$ is given by the following theorem whose proof is very simple.

Theorem 2.3. *Let f_n be a sequence $\in \mathcal{L}^1(v)$ such that $\sum_{n=1}^{\infty} v^*(f_n) < \infty$. Put:*

$$f(t) = \begin{cases} \sum_{n=1}^{\infty} f_n(t), \text{ if this series is absolutely convergent} \\ \text{otherwise an arbitrary value} \end{cases}$$

Then $f \in \mathcal{L}^1(v)$ and $\lim_N v^*(f - \sum_{n=1}^{N} f_n) = 0$. In other words: $\sum_{n=1}^{N} f_n$ converges to f in $\mathcal{L}^1(v)$.

Proof. We have $|f| \leq \sum_{n=1}^{\infty} |f_n|$ and therefore (by virtue of the σ-subadditivity of v^*):

$$v^*(f) \leq \sum_{n=1}^{\infty} v^*(f_n)$$

Moreover $\left|f - \sum_{n=1}^{N} f_n\right| \leq \sum_{n>N} |f_n|$, and therefore:

$$v^*(f - \sum_{n=1}^{\infty} f_n) \leq \sum_{n>N} v^*(f_n)$$

As the limit of the latter expression is 0, the proof is completed. □

Let N be the set of all t such that $\sum_{n=1}^{\infty} f_n(t)$ is not absolutely convergent. Il we modify f on N, then we get a function f' such that $v^*(f - f') = 0$. In particular $\mathbf{1}_N \in \mathcal{L}^1(v)$ et $v^*(\mathbf{1}_N) = 0$. This incite to introduce the following definitions:

Definition 2.4. Let v be a semi-variation on the vector lattice $\mathcal{L} \subset \mathbb{R}^T$. A function f is said to be v-negligible iff $v^*(f) = 0$. A set $A \subset T$ is said to be v-negligible iff $v^*(\mathbf{1}_A) = 0$. A property concerning elements of T is said to be true almost everywhere with respect to v (in short v-a.e.) if the set of $t \in T$ where it is false is v-negligible.

If $\mathcal{P}$ is a property concerning functions defined on a subset of T, then a function f defined on T is said to have essentially the property $\mathcal{P}$ if there exists a negligible set A such that the restriction of f to the complement of A has the property $\mathcal{P}$.

The most usual properties of negligible functions and sets are given in the following theorem.

Theorem 2.5. *(1) If f est v-negligible and if $|g| \leq |f|$, then g is v-negligible. In particular if $A \subset T$ is v-negligible and if $B \subset A$, then B is v-negligible.*

(2) Any countable union of negligible sets is negligible.

(3) A function f is v-negligible iff $\{t \in T \mid f(t) \neq 0\}$ is v-negligible.

Theorem 2.3. appears now as a variant of the standard Lebesgue's theorem on series.

Let $\{f_n\}$ be a sequence in $\mathcal{L}^1(v)$ converging to an element f. Then there exists a subsequence $\{f_{\phi(k)}\}$ such that $\sum_k v^*(f_{\phi(k)}) < \infty$. By Theorem 2.3., $\{f_{\phi(k)}\}$ converges v-a.e. to f. Hence every converging sequence in $\mathcal{L}^1(v)$ has a subsequence converging v-a.e.. On the other hand remark that if $\{f_n\}$ is a convergent sequence in $\mathcal{L}^1(v)$ which converges pointwise to a function f, then $\{f_n\}$ converges to f in $\mathcal{L}^1(v)$.

Notation. The quotient of $\mathcal{L}^1(v)$ with respect to v^* is denoted by $L^1(v)$. It is a Banach space.

We shall now define an important class of semi-variations.

Definition 2.6. A semi-variation v on a lattice $\mathcal{L}$ is said to be exhaustive [1] iff it satisfies one of the equivalent following properties:

(1) every increasing and majorized sequence $\{f_n\}$ is a Cauchy sequence with respect to v.

(2)

$$\left[f_n \in \mathcal{L}_+ \ , \ f \in \mathcal{L}_+ \ , \ \sum_{n=1}^{\infty} f_n \leq f\right] \Longrightarrow \lim_n v(f_n) = 0$$

The equivalence between the two properties is easy to prove.

The natural norm on $\mathbf{l}^1$ or $\mathbf{c}_0$ is exhaustive. On the other hand the natural norm on $\mathbf{l}^\infty$ is not exhaustive.

A somewhat technical proof leads to the following result:

Theorem 2.7. *If v is an exhaustive semi-variation, then v^* is exhaustive on $\mathcal{L}^1(v)$.*

The main result concerning exhaustive semi-variations is the theorem of dominated convergence. First remark that if v is a exhaustive semi-variation on $\mathcal{L}$ and $\{f_n\}$ is a positive increasing majorized sequence in $\mathcal{L}^1(v)$, then putting $f(t) = \lim_n f_n(t)$, we have $f \in \mathcal{L}^1(v)$ and f is the limit of f_n in $\mathcal{L}^1(v)$. In fact $\{f_n\}$ is a Cauchy sequence by virtue of the exhaustivity of v . As an easy consequence, if $\{f_n\}$ is a decreasing sequence in $\mathcal{L}^1(v)$ converging pointwise to 0, then this sequence converges to 0 in $\mathcal{L}^1(v)$.

Theorem 2.8. *[Theorem of dominated convergence] Let v be an exhaustive semi-variation on $\mathcal{L}$. Let $f_n, h \in \mathcal{L}^1(v)$ such that $|f_n| \leq h$. If f_n converges pointwise to a function f, then f_n converges to f in $\mathcal{L}^1(v)$.*

Proof. Put $\quad g_n = \sup\{|f_p - f_q| \mid p, q \geq n\}$.

We have: $\quad g_n = \lim_m g_{n,m}$ with: $\quad g_{n,m} = \sup\{|f_p - f_q| \mid n \leq p, q \leq m\}$

[1] The term "exhaustive" has been suggested to me by [KA]. I am far from affirming it is the best one. Wilhelm uses the term "saturable". Swartz uses the term "strongly bounded". Bartle ([?]) speeks of the *-property. Lewis [LE] uses a similar concept under the name "variationnel semiregularity". A Banach lattice is said to be "order bounded" if condition (2) is satisfied. Of course all this terms refers to Sdifferent settings. One can only say that the are used to define similar properties.

On the other hand $0 \leq g_{m,n} \leq 2h$, $g_{n,m} \in \mathcal{L}^1(v)$ and the sequence $m \to g_{n,m}$ is increasing. Therefore (see the preliminary remarks), $g_n \in \mathcal{L}^1(v)$ for all n. Furthermore the sequence $\{g_n\}$ is decreasing and converge pointwise to 0. Now $|f_n - f_k| \leq g_n$ for all $k \geq n$ and therefore $|f_n - f| \leq g_n$. Hence $f_n - f \in \mathcal{L}^*(v)$, $f \in \mathcal{L}^*(v)$ and $v^*(f_n - f))$ converges to 0. □

It is common in Convex Analysis to extend convex functions defined on a convex subset into a function defined on the whole space and taking the value $+\infty$ outside the domain of definition of the original function. Adopting the notation used in Convex Analysis, for every $]-\infty, +\infty]$-valued convex function f, we shall denote by dom(f) the "effective domain" of f, i.e. $\{t \mid f(t) \neq +\infty\}$.

Definition 2.9. Let T be an arbitrary set and $\mathcal{L}$ a vector sub-lattice of $\mathbb{R}^T$. One call extended semi-variation on $\mathcal{L}$ any sub-linear symmetric function v defined on $\mathcal{L}$ with values in $[0, +\infty]$ such that:

$$0 \leq f_1 \leq f_2 \quad \Longrightarrow \quad v(f_1) \leq v(f_2)$$

$$v(f) = v(|f|)$$

$$f, f_n \in \mathcal{L}_+ and f \leq \sum_{n=1}^{\infty} f_n \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

or equivalently

$$f, f_n \in \mathcal{L} and |(|f) \leq \sum_{n=1}^{\infty} |f_n| \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

It is easy to verify that dom(v) is a vector sub-lattice of $\mathcal{L}$ and that the restriction of v to dom(v) is a semi-variation as defined in Definitions 2.1. An extended semi-variation is said to be exhaustive if its restriction to dom(f) is exhaustive.

Theorem 2.10. *Any pointwise supremum v of a family $\{v_i\}$ of extended semi-variations is an extended semi-variation.*

Proof. The two first properties in Definition 2.9 are plain. Now suppose:

$$f, f_n \in \mathcal{L}_+ and f \leq \sum_{n=1}^{\infty} f_n$$

For every i, we have $v_i(f) \leq \sum_{n=1}^{\infty} v_i(f_n)$. But for every f, we have $v(f) = \sup v_i(f)$. Therefore $v_i(f) \leq \sum_n v(f_n)$. Taking the pointwise supremum in the left hand member gives: $v(f) \leq \sum_n v(f_n)$. □

It's worthwhile to note that the pointwise supremum of a family of exhaustive extended semi-variations may fail to be exhaustive. For instance let T be a compact set. For every $t \in T$ the function $f \mapsto |f(t)|$ is an exhaustive semi-variation on the space $C(T)$, but the semi-variation $f \mapsto \sup_{t \in T} |f(t)|$ is not exhaustive unless T is finite.

3. Semi-variations on measurable spaces

Let $(T, \mathcal{T})$ be a measurable space. $\mathcal{E}(\mathcal{T})$ or $\mathcal{E}$ denotes the space of $\mathcal{T}$-simple functions. We shall apply the results of Section 2 to the case $\mathcal{L} = \mathcal{E}$.

Theorem 3.1. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}$. For every function $f \in \mathcal{L}^1(v)$, there exists a function $f' \in \mathcal{L}^1(v)$, $\mathcal{T}$-measurable such that* $v^*(f - f') = 0$

Proof. Let us témporarily denote by $\mathcal{L}^{*\mathcal{T}}$ and $\mathcal{L}^{1,\mathcal{T}}$ respectively the subspaces of $\mathcal{T}$-measurable elements of $\mathcal{L}^*(v)$ and $\mathcal{L}^1(v)$. Let us look at Theoreme 2.3. Suppose $f_n \in \mathcal{L}^{1,\mathcal{T}}$. The set N where the series $\sum_{n=0}^{\infty} f_n(t)$ doesn't converge absolutely belongs to $\mathcal{T}$. If we put $f(t) = 0$ for $t \in N$, then $f \in \mathcal{L}^{1,\mathcal{T}}$. Consequently $\mathcal{L}^{1,\mathcal{T}}$ is complete. Now let $f \in \mathcal{L}^1(v)$. There exists a sequence $f_n \in \mathcal{E}$ such that $\lim_n v^*(f - f_n) = 0$. As $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^{1,\mathcal{T}}$, it converges to a element $f' \in \mathcal{L}^{1,\mathcal{T}}$ and we have $v^*(f - f') = 0$. □

Note that the quotient of $\mathcal{L}^{1,\mathcal{T}}$ by v^* is $L^1(v)$.

Definition 3.2. A measurable space $(T, \mathcal{T})$ is said to be complete with respect to a semi-variation v on $\mathcal{E}(\mathcal{T})$ if

$$A, B \in \mathcal{T} \ , \ B \subset A \ , \ v(\mathbf{1}_A) = 0 \quad \Longrightarrow \quad B \in \mathcal{T}$$

Lemma 3.3. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}$. Let A be v-negligible. Then there exists $B \in \mathcal{T}$ such that $A \subset B$ and $v(\mathbf{1}_B) = 0$*

Proof. Suppose $v^*(\mathbf{1}_A) = 0$. Let $\varepsilon > 0$. According to the definition of v^*, there exists a sequence $\{f_n\}$ of elements of $\mathcal{E}_+(\mathcal{T})$ such that $\mathbf{1}_A \leq \sum_{n \in \mathbb{N}} f_n$ and $\sum_{n \in \mathbb{N}} v(f_n) \leq \varepsilon$. Put

$B = \{t \in T \mid \sum_{n\in\mathbb{N}} f_n(t) \geq 1\}$. Then $B \in \mathcal{T}$ and $\mathbf{1}_B \leq \sum_{n\in\mathbb{N}} f_n$. Therefore according to the σ-subadditivity of v: $v(\mathbf{1}_B) \leq \sum_{n\in\mathbb{N}} v(f_n)$, and $v(\mathbf{1}_B) \leq \varepsilon$.

Taking $\varepsilon = 1/n$ ($n \in \mathbb{N}$), we get a sequence of elements $B_n \in \mathcal{T}$ such that $A \subset B_n$ and $v(\mathbf{1}_B) \leq 1/n$. Taking $B = \bigcap_{n\in\mathbb{N}} B_n$, we get $A \subset B$ and $v(\mathbf{1}_B) = 0$. □

Theorem 3.4. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}$. Define $\mathcal{T}'$ as follows:*

$$A \in \mathcal{T}' \iff \exists B \in \mathcal{T} \text{ such that } v^*(A \bigtriangledown B) = 0$$

Then $\mathcal{T}'$ is a σ-algebra and $(T, \mathcal{T}')$ is complete with respect to v.

Proof. Let $A \in \mathcal{T}'$ and $B \in \mathcal{T}$ such that $v^*(A \bigtriangledown B) = 0$. We have $A \bigtriangledown B = A^c \bigtriangledown B^c$ and therefore $A^c \in \mathcal{T}'$.

Now let $\{A_n\}$ be a sequence of elements of $\mathcal{T}'$ and, for every n, $B_n \in \mathcal{T}$ such that $v^*(A_n \bigtriangledown B_n) = 0$. We have $\bigcup_n A_n \bigtriangledown \bigcup_n B_n \subset \bigcup_n A_n \bigtriangledown B_n$. That implies $\bigcup_n A_n \in \mathcal{T}'$. Hence $\mathcal{T}'$ is a σ-algebra. Plainly this σ-algebra is complete with respect to v. □

Proposition 3.5. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}(\mathcal{T})$. Then we have $\mathcal{T}' = \{A \subset T \mid \mathbf{1}_A \in \mathcal{L}^1(v)\}$*

Proof. If $A \in \mathcal{T}'$ there exists $B \in \mathcal{T}$ such that $v^*(\mathbf{1}_A - \mathbf{1}_B) = 0$. Therefore $\mathbf{1}_A \in \mathcal{L}^1(v)$.

Conversely suppose $\mathbf{1}_A \in \mathcal{L}^1(v)$. Then there exist $N \in \mathcal{T}$ and a $\mathcal{T}$-measurable function g such that $\mathbf{1}_A = g$ on $T \setminus N$. Supposing $g = 0$ on N, we have $g = \mathbf{1}_B$ and $v^*(A \bigtriangledown B) = 0$ *i.e.* $A \in \mathcal{T}'$. □

Definition 3.6. Let $(T, \mathcal{T})$ be a measurable space and v be a semi-variation on (T). A real function f is said to be v-measurable if it is the limit v-a.e. of a sequence of $\mathcal{T}$-simple functions.

Theorem 3.7. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}(\mathcal{T})$. Then f is v-measurable iff f is measurable with respect to the σ-algebras $\mathcal{T}'$ (theorem 3.4.) and* $\mathbf{Bor}(\mathbb{R})$ *(in short $\mathcal{T}'$-measurable).*

Proof. Plainly every $\mathcal{T}'$-simple function coincides v-a.e. with a $\mathcal{T}$-simple function. Suppose f is $\mathcal{T}'$-measurable and therefore the limit v-a.e. of $\mathcal{T}'$-simple functions. Then f is v-a.e. the limit of $\mathcal{T}$-simple functions.

Conversely suppose f is v-a.e. the limit of $\mathcal{T}$-simple functions f_n. Let $N \in \mathcal{T}$ such that $v^*(\mathbf{1}_N) = 0$ and $\mathbf{1}_{T\setminus N} f_n$ converges to $\mathbf{1}_{T\setminus N} f$ on T. As $\mathbf{1}_{T\setminus N} f_n$ is $\mathcal{T}$-measurable, $\mathbf{1}_{T\setminus N} f$ is $\mathcal{T}$-measurable. For every $A \in \mathbf{Bor}(I\!N)$ we have

$$f^{-1}(A) \bigtriangledown (\mathbf{1}_{T\setminus N} f)^{-1}(A) \subset N$$

Therefore $f^{-1}(A) \in \mathcal{T}'$. Hence f is $\mathcal{T}'$-measurable. □

Consequently any pointwise limit of v-measurable functions is v-measurable.

Proposition 3.8. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E})(\mathcal{T})$. For every v-integrable function f and every bounded v-measurable function g the function fg is v-integrable.*

Proof. Let g be v-measurable and bounded. Define g_n as follows:

$$g_n(t) = h2^{-n} \text{ if } f(t) \in \{h2^{-n}, (h+1)2^{-n}\}$$

Then $g_n \in (\mathcal{T}')$ and $|g_n - g_m| \leq 2^{-n}$ if $m > n$. On the other hand let $\{f_n\}$ be a sequence of elements of $\mathcal{E}(\mathcal{T}')$ such that f_n converges v-a.e. and $\lim_n v(f - f_n) = 0$. Pick $k, K \in I\!R$ such that $\forall n \in I\!N \quad v(f_n) \leq K$ and $|g_n| \leq k$. Then we have:

$$f_n g_n - f_m g_m = (f_n - f_m) g_n + f_m (g_n - g_m)$$

and therefore:

$$|f_n g_n - f_m g_m| = k\,|f_n - f_m| + |f_m|\,2^{-n}$$

and

$$v(f_n g_n - f_m g_m) = k v(f_n - f_m) + K 2^{-n}$$

Hence $\lim_n v(f_n f_n - f_m g_m) = 0$. The sequence $\{f_n g_n\}$ is a Cauchy sequence in $\mathcal{L}^1(v)$ and converges v-p.p. to fg. We conclude that $fg \in \mathcal{L}^1(v)$. □

Theorem 3.9. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}(\mathcal{T})$. Suppose $f \in \mathcal{L}^1(v)$ and g v-measurable. If $|g| \leq |f|$ then $g \in \mathcal{L}^1(v)$.*

Proof. We have $g = f(g/f)$ (with $g(t)/f(t) = 1$ if $f(t) = 0$). By the preceding proposition g is v-integrable. □

Definition 3.10. Let $(T, \mathcal{T})$ be a measurable space and v be an exhaustive semi-variation on $(\mathcal{E}, \mathcal{T})$. A sequence $\{f_n\}$ of v-measurable functions is said to converge v-almost uniformly to a function f iff for every $\varepsilon > 0$, there exists a v-measurable set A_ε such that $v^*(T \setminus A_\varepsilon) \leq \varepsilon$ and f_n converges uniformly to f on A_ε.

Theorem 3.11. *[Egorov] Let $(T, \mathcal{T})$ be a measurable space and v be an exhaustive semi-variation on $(\mathcal{E}, \mathcal{T})$. A sequence $\{f_n\}$ of v-measurable functions converges v-a.e. to a function f iff $\{f_n\}$ converges to f v-almost uniformly.*

PB. Thorme II.31. □

4. Integrability of vector functions

Definition 4.1. Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $(\mathcal{T})$. Let E be a Banach space. An E-valued function f is said to be v-measurable if f equals v-a.e. the limit of a sequence of $\mathcal{T}$-simple functions.

If f is E-valued v-measurable function then $||f(.)||_E$ is v-measurable.

Definition 4.2. Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $(\mathcal{T})$. Let E be a Banach space. Then $\mathcal{L}^1_E(v)$ denotes the space of all v-measurable functions such that $||f(.)||_E \in \mathcal{L}^1(v)$.

Theorem 4.3. *$\mathcal{L}^1_E(v)$ is a vector space.*

Proof. Let us prove

$$f, g \in \mathcal{L}^1_E(v) \quad \Longrightarrow \quad f + g \in \mathcal{L}^1_E(v)$$

We have $||(f+g)(.)|| \leq ||f(.)|| + ||g(.)||$. Hence $||(f+g)(.)||$ is v-measurable and majorized by an integrable function and therefore is integrable. □

The space $\mathcal{L}^1_E(v)$ is endowed with the semi-norm $f \mapsto ||\ ||f(.)||_E\ ||_{\mathcal{L}^1(v)}$.

The quotient of $\mathcal{L}^1_E(v)$ by the v-a.e. equality will be denoted by $L^1_E(v)$.

Theorem 4.4. *Let v be a semi-variation on $(\mathcal{T})$. Then the space $\mathcal{L}^1_F(v)$ is complete.*

Proof. Let us prove that every sequence $\{f_n\}$ of members of $\mathcal{L}^1_F(v)$ with:

$$\sum_{n \in \mathbb{N}} ||f_n||_{\mathcal{L}^1_F(v)} < \infty$$

has a sum in $\mathcal{L}^1_F(v)$. Put $g_n = \|f_n(.)\|$. By definition $\|f_n\|_{\mathcal{L}^1_F(v)} = \|g_n\|_{\mathcal{L}^1(v)}$ and therefore:

$$\sum_{n\in\mathbb{N}} \|g_n\|_{\mathcal{L}^1(v)} < \infty$$

Lebesgue's theorem on series yields:

$$\sum_{n\in\mathbb{N}} |g_n(t)| < \infty \qquad v-a.e.$$

Hence there exists a v-negligible set N such that for every $t \notin N$, we have:

$$\sum_{n\in\mathbb{N}} \|f_n(t)\|_F < \infty$$

The space F being complete, for every $t \notin N$ there exists $f(t) \in F$ with

$$f(t) = \sum_{n\in\mathbb{N}} f_n(t)$$

For $t \notin N$, put $k_n(t) = \left\|\sum_{p\leq n} f_p(t)\right\|_F$ and $k(t) = \|f(t)\|_F$. If a $\mathcal{T}$-measurable positive function is majorized by an element of $\mathcal{L}^1(v)$ then this function belongs to $\mathcal{L}^1(v)$. Hence we have $k_n \in \mathcal{L}_1(v)$ because $k_n(t) \leq \sum_{p\leq n} \|f_p(t)\|$. On the other hand $k(t) = \lim_n k_n(t)$ for all $t \notin N$.

For $m \leq n$, we have:

$$\begin{aligned} |k_n(t) - k_m(t)| &= \left| \left\|\sum_{p\leq n} f_p(t)\right\|_F - \left\|\sum_{p\leq m} f_p(t)\right\|_F \right| \\ &\leq \left\|\sum_{p\leq n} f_p(t) - \sum_{p\leq m} f_p(t)\right\|_F \\ &= \left\|\sum_{p=m+1}^{n} f_p(t)\right\|_F \end{aligned}$$

Hence:

$$\|k_n - k_m\|_{\mathcal{L}_1(v)} \leq \left\| \left\|\sum_{p=m+1}^{n} f_p(.)\right\|_F \right\|_{\mathcal{L}_1(v)} = \left\|\sum_{p=m+1}^{n} f_p\right\|_{\mathcal{L}^1_F(v)}$$

That implies the sequence $\{k_n \mid n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^1(v)$. As $k_n(t)$ converges to $k(t)$ for all $t \notin N$, we have $k \in \mathcal{L}^1(v)$, which proves $f \in \mathcal{L}^1_F(v)$). It remains to prove that the sequence $\{f_n\}$ converges to f in $\mathcal{L}^1_F(v)$. We have:

$$\left\|f - \sum_{n=1}^{N} f_n\right\|_{\mathcal{L}^1_F(v)} = \left\|\sum_{n>N} f_n\right\|_{\mathcal{L}^1_F(v)} = \left\|\left\|\sum_{n>N} f_n(.)\right\|\right\|_{\mathcal{L}^1(v)} \leq \left\|\sum_{n>N} g_n\right\|_{\mathcal{L}^1(v)}$$

As the limit of the last expression is null, the theorem is proven. □

Theorem 4.5. *[Theorem of dominated convergence] Let v be an exhaustive semi-variation. Suppose $\{f_n\}$ is a sequence in $\mathcal{L}^1_E(v)$ and $h \in \mathcal{L}^1_+(v)$ such that $\|f_n(.)\| \le h$ v-a.e.. If $f_n(t)$ converges v-a.e. to a function f, then $f \in \mathcal{L}^1_E(v)$ and f_n converges to f in $\mathcal{L}^1_E(v)$.*

Proof. Applying Theorem 2.8 to the function $\|f_n(.)\|$ proves $\|f(.)\|$ is integrable so $f \in \mathcal{L}^1_E(v)$. Applying this theorem to the function $\|f_n(.) - f(.)\|$ proves that f_n converges to f in $\mathcal{L}^1_E(v)$. □

5. The contextual semi-variation

Let $(T, \mathcal{T})$ be a measurable space. We are given three Banach spaces E, F and G and a bilinear continuous application $y, x \mapsto y \odot x$ of $F \times E$ into G. The 4-uple $(E, F, G, \odot)$ will be called a bilinear context. Let us consider a mapping m of $\mathcal{T}$ into E such that for every $y \in F$, $A \mapsto y \odot m(A)$ is a (strong) measure on $\mathcal{T}$ with values in G. A particular case of such a mapping is an E-valued (strong) measure.

If f is a $\mathcal{T}$-simple functions with values in F, then its integral with respect to m is defined by

$$\int f \odot m = \sum_i y_i \odot m(A_i)$$

if $\{A_i\}$ is a finite partition of T and $f = \sum_i \mathbf{1}_{A_i} y_i$. Define $\int_A f \odot m = \int (\mathbf{1}_A f) \odot m$. Then the application $A \mapsto \int_A f \odot m$ is a (strong) measure with values in G which is denoted by $f \odot m$.

Let us first define a semi-variation of m which we shall call "contextual" semi-variation because it depends on the context $(E, F, G, \odot)$. For every $h \in$ put:

$$\underline{w}(h) = \sup\left\{ \left\| \int f \odot m \right\|_G \mid f \in_F, \|f(.)\| \le |h| \right\}$$

and $\underline{w}(A) = \underline{w}(\mathbf{1}_A)$. Then

$$\underline{w}(A) = \sup\left\{ \left\| \sum_j y_j \odot m(B_j) \right\| \mid \{B_j\} \text{ finite partition of } A, y_j \in F, \|y_j\| \le 1 \right\}$$

For every partition $\{A_i \mid i \in \mathbb{N}\}$ of A, one have:

$$\underline{w}(A) \le \sum_i \underline{w}(A_i)$$

Indeed let $\{B_j\}$ be a finite partition of A and $y_j \in \mathfrak{B}(F)$. We have:

$$\begin{aligned}\sum_j \|y_j \odot m(B_j)\| &= \left\|\sum_j \sum_i y_j \odot m(B_j \cap A_i)\right\| \\ &= \left\|\sum_i \sum_j y_j \odot m(B_j \cap A_i)\right\| \\ &\le \sum_i \left\|\sum_j y_j \odot m(B_j \cap A_i)\right\| \\ &\le \sum_i \underline{w}(A_i)\end{aligned}$$

This gives the announced inequality. More precisely the following result holds:

Theorem 5.1. $\underline{w}$ *is an extended semi-variation.*

Proof. (0) An easy checking gives:

$$0 \le f_1 \le f_2 \quad \Longrightarrow \quad v(f_1) \le v(f_2)$$

(1) Let us first prove that $\{A_n\}$ being an increasing sequence in $\mathcal{T}$ whose union is T, then $\underline{w}(h) = \lim_n \underline{w}(\mathbf{1}_{A_n} h)$. We may suppose $h \ge 0$.

(1a) Suppose $\underline{w}(h) < \infty$. Let $\varepsilon > 0$. There exists a function $f = \sum_{i \in I} \mathbf{1}_{B_i} y_i$, where $\{B_i\}$ is a finite partition of T, $\{y_i\}$ a family of members of F, such that $\|\mathbf{1}_{A_i}(.) y_i\| \le h$ et $\underline{w}(h) - \varepsilon \le \|\sum_i y_i \odot m(B_i)\|$. For every $n \in \mathbb{N}$, put $f_n = \mathbf{1}_{A_n} f$, *i.e.* : $f_n = \sum_i \mathbf{1}_{B_i \cap A_n} y_i$. We have $\int f_n \odot m = \sum_i y_i \odot m(B_i \cap A_n)$. For every $i \in I$, we have $\lim_n y_i \odot m(B_i \cap A_n) = y_i \odot m(B_i)$, because $y_i \odot m(.)$ is a strong measure. Therefore there exists N such that

$$n \ge N \quad \Longrightarrow \quad \left\|\int f \odot m - \int f_n \odot m\right\| \le \varepsilon$$

Hence $n \ge N \quad \Longrightarrow \quad \underline{w}(h) - 2\varepsilon \le \int f_n \odot m$ with $\|f_n(.)\| \le h$. The assertion is proved.

(1b) Suppose $\underline{w}(h) = \infty$. Let $M > 0$. There exists a function $f = \sum_{i \in I} \mathbf{1}_{B_i} y_i$, where $\{B_i\}$ is a finite partition of T, $\{y_i\}$ a family of members of F, such that $\|\mathbf{1}_{A_i}(.) y_i\| \le h$ et $\left\|\int f \odot m\right\| \ge M$. Defining $f_n = f \mathbf{1}_{A_n}$, one sees that there exists N such that

$$n \ge N \quad \Longrightarrow \quad \left\|\int f \odot m - \int f_n \odot m\right\| \le \varepsilon$$

Therefore $n \ge N \quad \Longrightarrow \quad \int f_n \odot m \ge M - \varepsilon$. The assertion is proved.

(2) Let us prove that

$$h, h_n \in_+ \ , \ \sum_n h_n \geq h \quad \Longrightarrow \quad \sum_n \underline{w}(h_n) \geq \underline{w}(h)$$

Let $g_n = \sum_{p=0}^n h_p$ et $A_n = \{t \in T \mid g_n(t) \geq (1-\varepsilon)h(t)\}$. The sequence $\{A_n\}$ is increasing and its union is T. We have $g_n \geq (1-\varepsilon)\mathbf{1}_{A_n} h$, and therefore $\underline{w}(g_n) \geq (1-\varepsilon)\underline{w}(\mathbf{1}_{A_n} h)$.

Furthermore $\underline{w}(g_n) \leq \sum_{p=0}^n \underline{w}(h_p)$. Hence

$$\sum_{p=0}^{n} \underline{w}(h_p) \geq (1-\varepsilon)\underline{w}(\mathbf{1}_{A_n} h)$$

Making n converge to $+\infty$, we get:

$$\sum_p \underline{w}(h_p) \geq (1-\varepsilon)\underline{w}(h)$$

As ε is arbitrary:

$$\sum_p \underline{w}(h_p) \geq \underline{w}(h)$$

□

Proposition 5.2. *Suppose m is a (strong) E-valued measure. If $f \in_F$, then*

$$\left\| \int f \odot m \right\| \leq \int \|f(.)\| \operatorname{var}(m)$$

Proof. Let $f = \sum_i \xi_i \mathbf{1}_{A_i}$ with $\xi_i \in F$ and $\{A_i\}$ a finite $\mathcal{T}$-partition of T. We have $\int f \otimes m = \sum_i \xi_i \odot m(A_i)$, and therefore

$$\begin{aligned} \left\| \int f \odot m \right\| &\leq \sum_i \|\xi_i \odot m(A_i)\| \\ &= \sum_i \|\xi_i\| \, \|m(A_i\| \\ &\leq \sum_i \|\xi_i\| \operatorname{var}(m)(A_i) \\ &= \int \|f(.)\| \ \operatorname{var}(m) \end{aligned}$$

□

Corollary 5.3. *Suppose m is a (strong) E-valued measure. The contextual semi-variation $\underline{w}$ is majorized by* $\operatorname{var}(m)$. *In particular if* $\operatorname{var}(m)$ *is σ-finite, then the same is true for $\underline{w}$.*

In the case $\underline{w}$ is exhaustive, we may introduce an important class of integrable functions.

Definition 5.4. Let $\underline{w}$ be exhaustive. A function f from T into F is said to be Bochner-integrable iff $f \in \mathcal{L}^1_F(\underline{w})$.

Note that we have restricted the notion of Bochner-integrability to the case $\underline{w}$ is exhaustive. Without this restriction the definition of Bochner-integrability would lead to pathological facts as we shall see further.

As we have

$$\forall A \in \mathcal{T}, f \in_F \qquad (f \odot m)A \leq \underline{w}(f(.))$$

and as $_F$ is dense in $\mathcal{L}^1_F(\underline{w})$, the map $A \mapsto (f \odot m)(A)$ may be extended by continuity to $\mathcal{L}^1_F(\underline{w})$ and the map $A \mapsto (f \odot m)(A)$ is a vector measure. We may put:

$$\text{(BOCHNER)}\int_A f \odot m = (f \odot m)(A)$$

Theorem 4.5. may be translated into the following theorem.

Theorem 5.5. *[Theorem of dominated convergence] Let $\underline{w}$ be exhaustive. Suppose $\{f_n\}$ is a sequence of Bochner-integrable functions and $h \in \mathcal{L}^1_+(\underline{w})$ such that $\|f_n(.)\| \leq h$ $\underline{w}$-a.e.. If $f_n(t)$ converges $\underline{w}$-a.e.to a function f, then f is Bochner-integrable and f_n converges to f in $\mathcal{L}^1_E(\underline{w})$.*

6. The intrinsic semi-variation of a vector measure

Suppose m is an E-valued measure. Taking $F = \mathbb{R}$ and $G = E$, the contextual semi-variation $\underline{w}$ is called the intrinsic (or scalar) semi-variation of m and is denoted by $m^\bullet$. In other words we put:

$$\forall h \in \qquad m^\bullet(h) = \sup\left\{\left\|\int f\ m\right\| \;\middle|\; f \in_E, |f| \leq |h|\right\}$$

and $m^\bullet(A) = m^\bullet(\mathbf{1}_A)$.

When no context is present, the name "semi-variation" will refer to the intrinsic semi-variation.

The following theorem is fundamental in the theory of vector measures.

Theorem 6.1. *The semi-variation $m^\bullet$ of a measure m is exhaustive.*

Proof. [PB] Theorem VI.10. □

Let v be a semi-variation on $(T,\mathcal{T})$ and j be the canonical mapping of $(T,\mathcal{T})$ into $L^1(v)$. It is easy to prove that if j is a measure then its semi-variation is v and that if v is exhaustive then j is a measure. As a consequence we have the next Proposition. Let us first give a notation. If v be a semi-variation on $(T,\mathcal{T})$, ∇v denotes the set of all real measures (considered as linear forms on $(T,\mathcal{T})$) majorized by v. By the Hahn-Banach theorem the following formula holds:

$$\forall f \in (T,\mathcal{T}) \qquad v(f) = \sup\left\{\int f\mu \mid \mu \in \nabla v\right\}$$

Proposition 6.2. *Let v be a semi-variation on $(T,\mathcal{T})$. The following properties are equivalent:*

(1) v is exhaustive

(2) For every decreasing sequence $\{A_n\}$ of elements of $\mathcal{T}$ with empty intersection, one has $\lim_n v(A_n) = 0$

(3) For every countable $\mathcal{T}$-partition $\{T_n\}$ of T, one has $\lim_n v(T_n) = 0$.

(4) ∇v is relatively weakly compact. [2]

Proof. (1) implies (2) by the theorem of dominated convergence.

Suppose (3) fails. Then we can find $\delta > 0$ and a countable family $\{B_n\}$ of disjoint elements of $\mathcal{T}$ such that $v(B_n) \geq \delta$. By replacing B_0 by $B_0 \cap (T \setminus \cup_n B_n)$ we obtain a partition $\{B_n\}$ such that $v(B_n) \geq \delta$. Putting $A_n = \cup_{p \geq n} B_n$ we contradict (2). Hence (2) implies (3).

Suppose (3) holds. Let $\{A_n\}$ be a sequence of elements of $\mathcal{T}$ with empty intersection. Putting $T_n = A_n \setminus A_{n+1}$, we obtain a partition $\{T_n\}$. Hence $\lim_n v(T_n) = 0$ and a fortiori $\lim_n v(A_n) = 0$. Hence (3) implies (2).

Suppose (2) holds. For every countable $\mathcal{T}$-partition $\{T_n\}$ of T, one has

$$\lim_n v(T \setminus \bigcup_{n \leq N} T_n) = 0$$

That means the canonical mapping j of $(T,\mathcal{T})$ into $L^1(v)$ is a measure and v is its semi-variation. Therefore v is exhaustive. Hence (2) implies (1). The equivalence between (3)

[2] People working with set-defined semi-variations would be interested by the set ∂v of all scalar measures μ such that $\mu(A) \leq v(A)$ for all $A \in \mathcal{T}$. As $\nabla v \subset \partial v$ if ∂v is relatively weakly compact then the same is true for ∇v. Conversely if ∇v is relatively weakly compact then (2) shows that for every decreasing sequence $\{A_n\}$ of elements of $\mathcal{T}$ with empty intersection, one has $\lim_n \mu(A_n) = 0$ uniformly with respect to μ if μ runs over ∂v and that means ∂v is relatively weakly compact.

and (4) is a direct consequence of the Hahn-Banach theorem:

$$v(T \setminus \bigcup_{n \leq N} T_n) = \sup \left\{ \mu(T \setminus \bigcup_{n \leq N} T_n) \;\middle|\; \mu \in \nabla v \right\}$$

and the equivalence between uniform σ-additivity and relative weak compactness in the space of real measures. □

As a byproduct of the proof we have the following result:

Lemma 6.3. *Let v be a semi-variation on $(T, \mathcal{T})$. If v is not exhaustive, then there exists $\delta > 0$ and a countable partition $\{T_n\}$ of T such that $v(T_n) \geq \delta$*

Let us remark that if $\{m_n\}$ is a sequence of vector measures defined on $(T, \mathcal{T})$ then $m_n(A)$ converges to 0 uniformly with respect to A iff $m_n^\bullet(T)$ converges to 0. The following result plays an essential role in the next sections.

Theorem 6.4. *[Vitali-Hahn-Saks] Let $(T, \mathcal{T}, \mu)$ be a measure space, E a Banach space and $\{m_n\}$ a sequence of E-valued measures. It is assumed*

(1) all the m_n are absolutely continuous with respect to μ.

(2) for every $A \in \mathcal{T}$, the sequence $\{m_n(A)\}$ converges.

Then:

$$\forall \varepsilon > 0 \; \exists \eta \text{ such that } \quad \mu(A) \leq \eta \quad \Longrightarrow \quad \|m_n(A)\| \leq \varepsilon \text{ for all } n \in \mathbb{N}$$

Equivalently:

$$\forall \varepsilon > 0 \; \exists \eta \text{ such that } \quad \mu(A) \leq \eta \quad \Longrightarrow \quad m_n^\bullet(A) \leq \varepsilon \text{ for all } n \in \mathbb{N}$$

Proof. Cf [DS] III.7.2. □

Corollary 6.5. *With the hypothesis of Vitali-Hahn-Saks theorem, the sequence $\{m_n\}$ is uniformly σ-additive.*

Corollary 6.6. *With the hypothesis of Vitali-Hahn-Saks theorem, if one puts*

$$\forall A \in \mathcal{T} \qquad m(A) = \lim_n m_n(A)$$

then m is an E-valued measure.

Proof. Cf [DS] IV.10.6. □

Remaark 6.7. The theorem of Vitali-Hahn-Saks and its corollaries may be easily deduced from the case of scalar measures (e.g. [PB] Theorem XV.9). Suppose first E is separable. Then there exists a sequence $\{\xi_h\}$ in $\mathfrak{B}_{E'}$ such that $||x|| = \sup_h \langle x, \xi_h \rangle$ for every $x \in E$. The measures $\langle m_n(.), \xi_h \rangle$ satisfie the hypothesis of the Vitali-Hahn-Saks theorem for scalar measures. Therefore

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad |\langle m_n(A), \xi_h \rangle| \leq \varepsilon$$

and consequently

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad ||m_n(A)|| \leq \varepsilon$$

Consequently (E being supposed to be separable) the vector measures m_n are uniformly σ-additive.

In the case E is not supposed to be separable let $\{A_k\}$ be a countable partition of T and $\mathcal{T}'$ the σ-algebra generated by this partition. The restrictions of the measures $\{m_n\}$ to $\mathcal{T}'$ have their values in a separable space and therefore are uniformly σ-additive. But this implies the $\{m_n\}$ are also uniformly σ-additive as well as the $\{\langle m_n(.), y \rangle \mid n \in I\!N, ||y|| \leq 1\}$. Hence this family of measures is relatively weakly compact. By the Dunford-Pettis theorem ([PB] theorem VII.18) we have

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad \forall n \in I\!N \; ||y|| \leq 1 \quad |\langle m_n(A), y \rangle| \leq \varepsilon$$

and therefore

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad \forall n \in I\!N \quad ||m_n(A|| \leq \varepsilon$$

Example 6.8. Let F and E two Banach spaces and suppose $F \subset E'$ and that F is a norming subspace for E. For $y \in F$, $x \in E$ put $y \odot x = \langle y, x \rangle$. Suppose m is an E-valued measure. Let us compute the contextual semi-variation $\underline{w}$.

By Corollory 5.3., we know that $\underline{w} \leq \text{var}(m)$. Let us prove that the equality holds.

Let $h \in_+$ with $h = \sum_j \mathbf{1}_{B_j} z_j$ where $\{B_j\}$ is a $\mathcal{T}$ finite partition of T and $z_j \in I\!R_+$. We have

$$\underline{w}(h) = \sup \left\{ \sum_{ij} \langle \xi_{ij}, m(A_{ij}) \rangle \right\}$$

where the sup is taken over all finite partitions $\{A_{ij}\}$ of B_j and $\|\xi_{ij}\| \leq z_j$. For every i, j we have

$$\sup\left\{\langle \xi_{ij}, m(A_{ij})\rangle \;\Big|\; \|\xi_{ij}\| \leq z_j\right\} = z_j \,\|m(A_{ij})\|$$

and therefore

$$\begin{aligned}
\underline{w}(h) &= \sup\left\{\sum_{ij} z_j \,\|m(A_{i,j})\| \;\Big|\; \{A_{ij}\} \text{ finite partition of } B_j\right\} \\
&= \sup\left\{\sum_{j} z_j \sum_{i} \|m(A_{ij})\| \;\Big|\; \{A_{ij}\} \text{ finite partition of } B_j\right\} \\
&= \sum_j z_j \text{var}(m)(B_j) = \int h \ \text{var}(m)
\end{aligned}$$

Finally we obtain the announced equality: $\underline{w} = \text{var}(m)$. The integral of a function f with respect to m may be denoted by $\int \langle f \ , \ m\rangle$.

Example 6.9. Dinculeanu considers in his treatise [DIN] the following situation. Two Banach spaces are given as well as a $L(Y, X)$-valued additive set function m. It is supposed that for every $y \in Y$, $m(.)y$ is σ-additive, *i.e.* is a X-valued measure. This property is nothing but the σ-additivity of m for the strong operator topology on $L(Y, X)$ (see section 1).

Given $f \in_F$ its integral with respect to m is given by

$$\int f \odot m = \sum_i m(A_i)\xi_i \quad \text{if} \quad f = \sum \xi_i \mathbf{1}_{A_i}$$

with $\{A_i\}$ being a finite $\mathcal{T}$-partition of T and $\xi_i \in F$. This setting has been intensively studied by Dobrakov([DO1] and [DO2]) so we will refer to it as the Dobrakov setting. We suppose that the mapping $m(.)y$ is σ-additive for every $y \in Y$ so the Dobrakov setting is a particular case of the bilinear one with $E = L(Y, X), F = Y, G = X, y \odot u = uy$.

Following the Orlicz-Pettis theorem the assumption on m is equivalent to the following: for every $y \in Y$ and every $x' \in X'$ the mapping $\langle m(.)y, x'\rangle$ is a measure. That means m is a weak measure for the duality $(L(Y, X), Y \otimes X')$.

Let Z be a subspace of X' norming for X. For every $z \in Z$ the mapping $y, u \mapsto \langle z, uy\rangle$ is a bilinear form. We may consider the measure m_z such that $m_z(A) = zm(A)$ for every $A \in \mathcal{T}$. For every simple function f we have:

$$\int f \ m_z = \left\langle z, \int f \odot m\right\rangle$$

Let us compute the contextual semi-variation $\underline{w}$ of m. We have for $h \in_+$:

$$\begin{aligned}
\underline{w}(h) &= \sup\left\{\left\|\int f \odot m\right\| \;\Big|\; \|f(.)\| \leq h\right\} \\
&= \sup\left\{\left\langle z, \int f \odot m\right\rangle \;\Big|\; \|f(.)\| \leq h, \|z\| \leq 1\right\} \\
&= \sup\left\{\left\|\int \langle f, m_z\rangle\right\| \;\Big|\; \|f(.)\| \leq h, \|z\| \leq 1\right\} \\
&= \sup\left\{\int h \operatorname{var}(m_z) \;\Big|\; \|z\| \leq 1\right\}
\end{aligned}$$

We will now show how a problem of integration in the bilinear setting may be transformed into an equivalent one in the Dobrakov setting. Let $(E, F, G, \odot)$ a bilinear context and m a E-valued set function such that $y \odot m(.)$ is a G-valued measure for every $y \in F$. For every $A \in \mathcal{T}$ let $p(A) \in L(F, G)$ defined by $p(A)y = y \odot m(A)$ for every $y \in F$. For $u \in L(F, G)$ and $y \in f$ put $y \underline{\odot} u = \hat{u}y$ Suppose $f \in_F$ with $f = \sum_i \xi_i \mathbf{1}_{A_i}$ where $\{A_i\}$ is a finite $\mathcal{T}$-partition of T and $\xi_i \in F$. We have:

$$\int f \underline{\odot} p = \sum_i \xi_i \underline{\odot} p(A_i) = \sum_i p(A_i)\xi_i = \sum_i \xi_i \odot m(A_i) = \int f \odot m$$

Let us compare the contextual semi-variations $\underline{w}$ and $\overline{w}$ of m and p. We have for every $h \in_+$:

$$\begin{aligned}
\overline{w}(h) &= \sup\left\{\left\|\sum_i p(A_i)\xi_i\right\| \;\Big|\; \sum_i \|\xi_i\| \mathbf{1}_{A_i} \leq h\right\} \\
&= \sup\left\{\|\xi_i \odot m(A_i)\| \;\Big|\; \sum_i \|\xi_i\| \mathbf{1}_{A_i} \leq h\right\} \\
&= \underline{w}(h)
\end{aligned}$$

Hence $\overline{w} = \underline{w}$.

If m is an E-valued (strong) measure, then p is a $L(F, G)$-valued (strong) measure. Indeed we have

$$\|p(A)\| = \sup\{p(A)y \mid y \in \mathfrak{B}_F\} = \sup\{y \odot m(A) \mid y \in \mathfrak{B}_F\} \leq \|\odot\| \, \|m(A)\|$$

But we can only prove the inequality $p^\bullet \leq \|\odot\| \, m^\bullet$.

Example 6.10. The following example stems from [DO1] (example 7). Put $T = \mathbb{N}$ and $\mathcal{T} = \mathcal{P}(\mathbb{N})$. Use the Dobrakov setting with $X = \mathbf{l}^1$, $Y = \mathbf{c}_0$. Suppose we have a bounded sequence $\{\xi_n\}$ in $\mathbf{c_0}$. For every $A \subset I\!N$ define $m(A) \in L(\mathbf{l^1}, \mathbf{c_0})$ by $m(A)x = \sum_{t\in A} x_t \xi_t$. For every $x \in \mathbf{l^1}$, $m(.)x$ is a $\mathbf{c_0}$-valued measure. Furthermore we have

$$||m(A)|| = \sup\left\{\left\|\sum_{t\in A} x_t\xi_t\right\| \mid x \in \mathbf{l^1}, ||x|| \leq 1\right\} = \sup\{||\xi_t|| \mid t \in A\}$$

Hence if $\lim_n ||\xi_n|| = 0$ then m is a $L(\mathbf{l^1}, \mathbf{c_0})$-valued measure.

By choosing appropriately the sequence $\{\xi_i\}$ we can manage to get the contextual semi-variation $\underline{w}$ being finite but not exhaustive. For example choosing

$$\begin{aligned} \xi_1 &= [1,0,0,\ldots] \\ \xi_2 = \xi_3 &= [0,1/2,0,0,\ldots] \\ \xi_4 = \xi_5 = \xi_6 &= [0,0,1/3,0,0,\ldots] \\ \ldots &= \ldots \end{aligned}$$

Then one finds that for any $h \in_+$ one has:

$$\underline{w}(h) = \sup\left\{\left\|\sum_{t\in I\!N}(f(t))_t\xi_t\right\| \mid f \in_{l^1}\ ,\ \ ||f(.)|| \leq h\right\} = \left\|\sum_{t\in I\!N} h(t)\xi_t\right\|$$

In particular:

$$\underline{w}(A) = \left\|\sum_{t\in A}\xi_t\right\|$$

We get $\underline{w}([k, k+1, k+2, \ldots] = 1$ for every $k \in I\!N$ and therefore the following condition fails: $\lim \underline{w}(A_k) = 0$ whenever $\{A_k\}$ is decreasing and has empty intersection.

7. **Bilinear Integration**

In this section we consider a bilinear context $(E, F, G, \odot)$. Given an E-valued set function such that $y \odot m(.)$ is a G-valued measure for all $y \in F$ we define the integrability and the integral of a F-valued function with respect to m. The contextual semi-variation is denoted by $\underline{w}$.

Lemma 7.1. *Let $\{\nu_n\}$ be a sequence of (strong) vector measure on a measurable space $(T, \mathcal{T})$. There exists a positive measure μ on $(T, \mathcal{T})$ such that $\mu(A) = 0 \Longleftrightarrow \nu_n^\bullet(A) = 0$ for every n.*

Proof. Let μ_n be a control measure of ν_n. It suffices to take $\mu = \sum_n 2^{-n}\mu_n/\,||\mu_n||$. □

Lemma 7.2. *Suppose $\underline{w}$ σ-finite. Let $\{f_n\}$ be a sequence of simple functions converging $\underline{w}$-a.e. to 0. Suppose for every $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges. Then it converges to 0, uniformly with respect to A.*

Proof. Let $N \in \mathcal{T}$ such that $\underline{w}(N) = 0$ and f_n converges on $T \setminus N$ to f. Replacing f_n by $\mathbf{1}_{T\setminus N} f_n$, doesn't modify $f_n \odot m$ and brings us to the case f_n converges everywhere to 0. Let μ be the measure associated with the sequence $\{f_n \odot m\}$ by the preceding lemma. As the measures $f_n \odot m$ are absolutely convergent with respect to μ, following the Vitali-Hahn-Saks theorem, there exists η such that

$$\mu(A) \le \eta \quad \Longrightarrow \quad \|(f_n \odot m)(A)\| \le \varepsilon/2$$

Now by the Egorov's theorem, as f_n converges to 0 $\underline{w}$-a.e. (and therefore μ-a.e.), there exists C_η such that $\mu(T \setminus C_\eta) \le \eta$ and f_n converges uniformly to 0 on C_η.

As $\underline{w}$ is σ-finite we may suppose $\underline{w}(C_\eta) < \infty$. Indeed let us consider a countable partition $\{T_k\}$ of T such that $\underline{w}(T_k) < \infty$. Put $S_n = \bigcup_{h \le k} T_h$ and $R_k = \bigcup_{h>n} T_h$. We first take C'_η such that $\mu(T \setminus C'_\eta) \le \eta/2$ and $\{f_n\}$ converges uniformly to C'_η. For k large enough, one have $\mu(C'_\eta \cap R_k) \le \varepsilon/2$. Taking $C_\eta = C'_\eta \cap S_k$, we obtain $\mu(T \setminus C_\eta) \le \eta$ and $\{f_n\}$ converges uniformly to C_η.

Now have:

$$(f_n \odot m)^\bullet(T) \le (f_n \odot m)^\bullet(C_\eta) + (f_n \odot m)^\bullet(T \setminus C_\eta)$$

But

$$(f_n \odot m)^\bullet(C_\eta) \le \underline{w}(\mathbf{1}_{C_\eta} \|f_n(.)\|)$$

There exists N such that

$$n \ge N \quad \Longrightarrow \quad \|f_n(t)\| \le \varepsilon/2\underline{w}(C_\eta) \quad \text{for all} \quad t \in C_\eta$$

That gives

$$n \ge N \quad \Longrightarrow \quad (f_n \odot m)^\bullet(T) \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

□

Definition 7.3. Suppose $\underline{w}$ is σ-finite. A function f from T into F is said to be integrable iff there exists a sequence $\{f_n\}$ of simple functions such that

(1) f_n converges $\underline{w}$-a.e. to f

(2) For every $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges.

The preceding lemma ensures that if two sequences $\{f_n\}$ and $\{f'_n\}$ fulfill the conditions of the preceding definition, then for every $A \in \mathcal{T}$, the two sequences $\{(f_n \odot m)(A)\}$ and $\{(f'_n \odot m)(A)\}$ have the same limit. If f is an integrable function, one may put:

$$\int_A f \odot m = \lim_n (f_n \odot m)(A) \qquad \int f \odot m = \lim_n (f_n \odot m)(T)$$

A glance at Corollary 6.6. shows that the mapping $A \mapsto \int_A f \odot m$ is a G-valued measure. This measure will be denoted by $f \odot m$.

Any integrable function f is $\underline{w}$-measurable (*i.e.* is $\underline{w}$-a.e. a limit of simple functions) as well as the function $t \mapsto \|f(t)\|$.

Lemma 7.4. *The conditions of the preceding definition are supposed to be satisfied. Then $(f_n \odot m)(A)$ converges uniformly to $(f \odot m)(A)$ for $A \in \mathcal{T}$.*

Proof. Let μ the measure associated with the sequence $\{f_n \odot m\}$ by lemma 7.1.. Let $\varepsilon > 0$. There is η such that

$$\mu(A) \leq \eta \quad \Longrightarrow \quad (f_n \odot m)^\bullet(A) \leq \varepsilon \text{ and } (f \odot m)^\bullet)(A) \leq \varepsilon$$

There exists C_η such that $\mu(T \setminus C_\eta) \leq \eta$ and f_n converges uniformly to f on C_η. As $\underline{w}$ is σ-finite, we may suppose $\underline{w}(C_\eta) < \infty$. Then we have

$$((f_n - f) \odot m)^\bullet(T) \leq ((f_n - f) \odot m)^\bullet(C_\eta) + (f_n \odot m)^\bullet(T \setminus C_\eta) + (f \odot m)^\bullet(T \setminus C_\eta)$$

There exists N such that

$$n > N, t \in C_\eta \quad \Longrightarrow \quad \|f(t) - f_n(t)\| \leq \varepsilon$$

Then we have:

$$((f_n - f) \odot m)^\bullet(T) \leq \varepsilon \underline{w}(C_\eta) + 2\varepsilon$$

and therefore $\lim_n((f_n - f) \odot m)^\bullet(T) = 0$. □ □

Lemma 7.5. *Let f be integrable and $C \in \mathcal{T}$. Then $\mathbf{1}_C f$ is integrable and*

$$\forall A \in \mathcal{T} \qquad (\mathbf{1}_C f \odot m)(A) = (f \odot m)(A \cap C)$$

Proof. Let $\{f_n\}$ be a sequence as considered in Definition 7.3.. One easily checks that $(\mathbf{1}_C f_n \odot m)(A) = (f_n \odot m)(A \cap C)$. Therefore $\lim_n(\mathbf{1}_C f_n \odot m)(A)$ exists for all $A \in \mathcal{T}$. Therefore $\mathbf{1}_C f$ is integrable and passing to the limit gives the announced equality. □ □

Theorem 7.6. *Let $\underline{w}$ be σ-finite and exhaustive. Then every Bochner-integrable function is integrable.*

Proof. (1) Suppose first $\underline{w}$ is finite. Then by [PB],Théorème VIII.13., $_F$ is dense in $\mathcal{L}^1_F(\underline{w})$. Let $f \in \mathcal{L}^1_F(\underline{w})$. There exists $f_n \in_F$ which converges to f in $\mathcal{L}^1_F(\underline{w})$. By extracting a sub-sequence, we may suppose that f_n converges to f $\underline{w}$-p.p.. But for all $A \in \mathcal{T}$, we have:

$$\|(f_n \odot m)(A) - (f_p \odot m)(A)\| \leq \underline{w}(\|f_n(.) - f_p(.)\|) = \|f_n - f_p\|_{\mathcal{L}^1_F(\underline{w})}$$

Hence the sequence $\{(f_n \odot m)(A)\}$ is a Cauchy sequence in F. That proves f is integrable.

(2) Suppose now $\underline{w}$ is σ-finite. Let $\{S_n\}$ a increasing sequence of members of $\mathcal{T}$ such that for every n, $\underline{w}(S_n) < \infty$ et $\cup_n S_n = T$. Let $f \in \mathcal{L}^1_F(\underline{w})$. For every n, there is $f_n \in_F$, null outside S_n such that $\|f - f_n\|_{\mathcal{L}^1_F(\underline{w})} \leq 1/n$. The proof is achieved as in (1) by using the sequence $\{f_n\}$. □

Theorem 5.5. takes the following form:

Theorem 7.7. *[Theorem of dominated convergence] Let $\underline{w}$ be σ-finite and exhaustive. Suppose $\{f_n\}$ is a sequence of Bochner-integrable F-valued functions and $h \in \mathcal{L}^1_+(\underline{w})$ such that $\|f_n(.)\| \leq h$ $\underline{w}$-a.e.. If $f_n(t)$ converges $\underline{w}$-a.e. to a function f, then f is Bochner-integrable and f_n converges to f in $\mathcal{L}^1_E(\underline{w})$. Moreover $\int f_n \odot m$ converges to $\int f \odot m$ and $((f_n - f) \odot m)^\bullet(T)$ converges to 0.*

Example 7.8. Take $T = \mathbb{N}, \mathcal{T} = \mathcal{P}(\mathbb{N})$. Then f is integrable iff the sequence $\{f(k) \odot m(k)\}$ is summable.

Proof. (1) Suppose $\{f(k) \odot m(k)\}$ is summable. For $n \in \mathbb{N}$, define $f_n(k) = f(k)$ if $t \leq n$, $= 0$ otherwise. Then $f_n \in_F$ and $f_n(k)$ converges to $f(k)$ for all k. As $f_n = \mathbf{1}_{[0,n]}f$, we have by lemma 7.5.:

$$(f_n \odot m)(A) = \sum_{k \leq n, k \in A} f(k) \odot m(k)$$

Hence f is integrable and

$$(f \odot m)(A) = \lim_n (f_n \odot m)(A)$$

In particular

$$(f \odot m)(k) = \lim_n (f_n \odot m)(k) = f(k) \odot m(k)$$

(2) Suppose f integrable. Then $f \odot m$ is a summable sequence. By lemma 7.5. we have $(f \odot m)(k) = ((\mathbf{1}_{\{k\}} f) \odot m)(\mathbb{N})$, *i.e.* $(f \odot m)(k) = f(k) \odot m(k)$. Hence the sequence $\{f(k) \odot m(k)\}$ is summable. □

Lemma 7.9. *$\underline{w}$ is supposed to be σ-finite. Let $\{f_n\}$ be a sequence of integrable functions converging $\underline{w}$-a.e. to 0. Suppose the sequence $\{(f_n \odot m)(A)\}$ converges for every $A \in \mathcal{T}$. Then this sequence converges to 0, uniformly with respect to A.*

Proof. As for lemma 7.2.. □

Theorem 7.10. *[First theorem of convergence] The semi-variation $\underline{w}$ is supposed to be σ-finite. Let $\{f_n\}$ be a sequence of integrable functions such that:*

(1) f_n converges $\underline{w}$-a.e. to a function f

(2) For every $A \in \mathcal{T}$, $(f_n \odot m)(A)$ converges in F

Then f is integrable and $(f_n \odot m)(A)$ converges to $(f \odot m)(A)$ uniformly with respect to A. In other words $\lim_n((f_n - f) \odot m)^\bullet(T) = 0$.

Proof. Let $\{h_n\}$ a sequence of simple functions converging $\underline{w}$-a.e. to f. Let μ as defined by lemma 7.1. using the measures $\{f_n \odot m \mid n \in \mathbb{N}\} \bigcup \{h_n \odot m \mid n \in \mathbb{N}\}$. Then $f_n - h_n$ converges to 0 $\underline{w}$-a.e. and therefore μ-a.e..Let $\{D_k\}$ be an increasing sequence of members of $\mathcal{T}$ such that $f_n - h_n$ converges uniformly to 0 on D_k and $\lim_k \mu(T \setminus D_k) = 0$. As $\underline{w}$ is σ-finite, we may suppose $\underline{w}(D_k) < \infty$. Put $D = \bigcup_k D_k$ and $N = T \setminus D$. For every k, choose n_k such that $\sup\{\|f_{n_k}(t) - h_{n_k}(t)\| \mid t \in D_k\} \leq 1/k\underline{w}(D_k)$. Put $g_k = \mathbf{1}_{D_k \cup N} h_{n_k}$. Then $\{g_k\}$ is a sequence of simple functions which converges $\underline{w}$-a.e. to f .

For $A \subset N$ and every $n \in \mathbb{N}$ we have $(f_n \odot m)(A) = 0$ and $(h_n \odot m)(A) = 0$ and therefore:

$$\begin{aligned}
((g_k - f_{n_k}) \odot m)^\bullet(T) &= ((g_k - f_{n_k}) \odot m)^\bullet(D) \\
&\leq ((g_k - f_{n_k}) \odot m)^\bullet(D_k) + ((g_k - f_{n_k}) \odot m)^\bullet(D \setminus D_k) \\
&= ((g_k - f_{n_k}) \odot m)^\bullet(D_k) + (f_{n_k} \odot m)^\bullet(D \setminus D_k) \\
&\leq 1/k + (f_{n_k} \odot m)^\bullet(D \setminus D_k)
\end{aligned}$$

Let us apply the Vitali-Hahn-Saks theorem (Theorem 6.4.) to the sequence of measures $\{f_{n_k} \odot m \mid k \in \mathbb{N}\}$. For every $\varepsilon > 0$, there is η such that $\mu(C) \leq \eta \Longrightarrow (f_{n_k} \odot m)^\bullet(C) \leq \varepsilon$. But there exists K such that $k \geq K \Longrightarrow \mu(D \setminus D_k) \leq \eta$. Then we have $k \geq K \Longrightarrow (f_{n_k} \odot m)^\bullet(D \setminus D_k) \leq \varepsilon$. That proves $\lim_k (f_{n_k} \odot m)^\bullet(D \setminus D_k) = 0$.

It follows that $\lim_k((g_k - f_{n_k}) \odot m)^\bullet(T) = 0$. Then for every $A \in \mathcal{T}$, we have

$$\lim_k((g_k \odot m)(A) - (f_{n_k} \odot m)(A)) = 0$$

The sequence $\{(g_k \odot m)(A)\}$ converges and

$$\lim_k(g_k \odot m)(A) = \lim_k(f_{n_k} \odot m)(A) = \lim_n(f_n \odot m)(A)$$

This proves that f is integrable.

For every integrable function f, put $p(f) = (f \odot m)^\bullet(T)$. One has $p(f - f_{n_k}) \leq p(f - g_k) + p(g_k - f_{n_k})$ and therefore $\lim_k p(f - f_{n_k}) = 0$. But applying lemma 7.9. to the double sequence $\{f_n - f_{n'}\}$, one gets $\lim_{n,n'} p(f_n - f_{n'}) = 0$. Therefore $\lim_n p(f - f_n) = 0$. □ □

Lemma 7.11. *$\underline{w}$ is supposed to be σ-finite. Let f be a measurable function with values in F and $\{f_n\}$ be a sequence of simple functions such that:*

(1) f_n converges $\underline{w}$-a.e. to f

(2) for every $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges.

Let given a sequence of simple real functions $\{\lambda_n\}$ such that $\lambda_n(t) \leq 1$ and $\lim_n \lambda_n(t) = 1$ for every $t \in T$. Then for every $A \in \mathcal{T}$, one has $\lim_n(f_n \odot m)(A) = \lim_n((\lambda_n f_n) \odot m)(A)$

Proof. Let μ as defined by lemma 7.1. using the measures $\{f_n \odot m\}$ Let $\varepsilon > 0$. There is η such that $\mu(A) \leq \eta \Longrightarrow \|f_n \odot m)(A)\| \leq \varepsilon$. Then there exists C_η such that $\mu(C_\eta) \leq \eta$ and on C_η, $\lambda_n(t)$ converges uniformly to 1 and $f_n(t)$ converges uniformly to $f(t)$. Furthermore we may suppose $\underline{w}(C_\eta) < \infty$. Then we have

$$(f_n \odot m)(A) - ((\lambda_n f_n) \odot m)(A) = ((1 - \lambda_n f_n) \odot m)(A)$$

$$= ((1 - \lambda_n)f_n \odot m)(A \cap C_\eta) - (\lambda_n f_n \odot m)(A \cap (T \setminus C_\eta)) + (f_n \odot m)(A \cap (T \setminus C_\eta))$$

The following inequalities hold:

$$\|((1 - \lambda_n)f_n \odot m)(A \cap C_\eta)\| \leq \underline{w}(C_\eta) \sup_{t \in T}(\lambda_n(t) - 1)$$

$$\|(\lambda_n f_n \odot m)(A \cap (T \setminus C_\eta))\| \leq (f_n \odot m)^\bullet(T \setminus C_\eta) \leq \varepsilon$$

$$\|(f_n \odot m)(A \cap (T \setminus C_\eta))\| \leq (f_n \odot m)^\bullet(T \setminus C_\eta) \leq \varepsilon$$

There exists N such that $n \geq N \implies \sup_{t \in T}(\lambda_n(t) - 1) \leq \varepsilon / \underline{w}(C_\eta)$. For $n \geq N$, we get:

$$\|(f_n \odot m)(A) - ((\lambda_n f_n) \odot m)(A)\| \leq 3\varepsilon$$

This allows to conclude. □ □

Lemma 7.12. *$\underline{w}$ is supposed to be σ-finite. For every integrable function f, there is a sequence $\{f_n\}$ of simple functions such that*

(1) f_n converges $\underline{w}$-a.e. to f

(2) for all $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges.

(3) for all n, $\|f_n(.)\| \leq \|f(.)\|$.

Proof. Let $\{f_n(t)\}$ a sequence which fulfills the 2 first conditions. Let us consider a sequence $\{k_n\}$ of simple functions such that $0 \leq k_n(t) \leq \|f_n(t)\|$ et $\lim_n k_n(t) = \|f(t)\|$ m-a.e. and define $\lambda_n(t) = \inf\{1, k_n(t)/\|f_n(t)\|\}$. By replacing $f_n(t)$ with $\lambda_n(t) f_n(t)$, we get a sequence which by the preceding lemma fulfills the 3 conditions. □

Proposition 7.13. *$\underline{w}$ is supposed to be σ-finite. Let f be an integrable function with values in F and $h \in$ such that $\|f(.)\| \leq h$. Then*

$$\left\|\int f \odot m\right\| \leq \underline{w}(h)$$

Proof. Let $\{f_n\}$ a sequence as defined by lemma 7.12.. One has: $\|f_n(.)\| \leq h$, and therefore $\left\|\int f_n \odot m\right\| \leq \underline{w}(h)$ The announced inequality is obtained by passing to the limit. □

Corollary 7.14. *For every $h \in_+$, we have :*

$$\underline{w}(h) = \sup\left\{\left\|\int f \odot m\right\| \;\middle|\; f \text{ integrable}, \|f(.)\| \leq h\right\}$$

Recall that in a Banach space E, a sequence $\{x_n\}$ is said to be weakly summable iff the sequence $\{\langle x_n, y\rangle\}$ is summable for every $y \in E'$. This definition is equivalent to the following property: the linear mapping $t \mapsto \sum_n t_n x_n$ from $\mathbf{c_{00}}$ (the space of eventually null sequences) into E is continuous. Consequently there is a bijection from the space of weakly summable sequence onto the space $L(\mathbf{c_{00}}, E)$. A sequence $\{x_n\}$ and a linear mapping ψ corresponds to each other by this isomorphism iff $\psi(t) = \sum_n t_n x_n$ for every $t \in \mathbf{c_0}$.

Proposition 7.15. *Suppose $\underline{w}$ is finite. Then for every countable $\mathcal{T}$-partition $\{T_n\}$ of T and every sequence $\{y_n\}$ in $\mathfrak{B}_F$, the sequence $\{y_n \odot m(T_n)\}$ is weakly summable.*

Proof. For every finite subset A of $\mathbb{N}$ and every $t \in \mathbf{c_{00}}$ vanishing outside A, we have:

$$\sum_{n \in A} t_n(y_n \odot m(T_n)) = \sum_{n \in A}(t_n y_n) \odot m(T_n) \leq \underline{w}(A) \leq \underline{w}(T) < \infty$$

□

Lemma 7.16. *Suppose $\underline{w}$ is finite but not exhaustive. Then there exists a $\mathcal{T}$-partition $\{T_n\}$ of T and a sequence $\{y_n\}$ in $\mathfrak{B}_F$ such that $\{y_n \odot m(T_n)\}$ is weakly summable but not summable.*

Proof. Following Lemma 6.3. there exists $\delta > 0$ and a $\mathcal{T}$-partition of T such that $\underline{w}(T_n) > \delta$. Following the definition of $\underline{w}$ there exists for every $n \in \mathbb{N}$ a finite family $\{y_{n,k} \mid k \in K_n\}$ of elements of $\mathfrak{B}_F$ and a finite $\mathcal{T}$-partition $\{T_{n,k} \mid k \in K_n\}$ of T_n such that

$$\left\| \sum_{k \in K_n} y_{n,k} \odot m(T_{n,k}) \right\| > \delta$$

The sequence $\{y_{n,k} \odot m(T_{n,k})\}$ is weakly summable but not summable.. □

Recall that if $\underline{w}$ is finite and exhaustive then every bounded $\underline{w}$-measurable function is Bochner integrable hence integrable. The following corollary proves that this property characterizes exhaustivity.

Corollary 7.17. *Suppose $\underline{w}$ is finite but not exhaustive. Then there exists a bounded measurable non integrable function.*

Proof. Put $f(t) = y_n$ for $t \in T_n$. Suppose f is integrable. Then we have $y_n \odot m(T_n) = (f \odot m)(T_n)$ and the sequence $\{y_n \odot m(T_n)\}$ is summable. We get a contradiction. □

Recall that a Banach space is said to have the Bessaga-Pelszyńsky property iff every weakly summable sequence is summable. The classical theorem of Bessaga-Pelszyńsky states that a Banach space has the Bessaga-Pelszyńsky property if and only if it contains no copy of $\mathbf{c_0}$.

Corollary 7.18. *If G has the Bessaga-Pelszyńsky property and if $\underline{w}$ is finite, then $\underline{w}$ is exhaustive.*

We give more characterisations of the Bessaga-Pelszyńsky property in the following theorem.

Theorem 7.19. *For a Banach space G the following properties are equivalent:*

(0) G has the Bessaga-Pelszyńsky property

(1) G contains no copy of $\mathbf{c_0}$

(2) For any linear context $(E, F, G, \odot)$ *such that the contextual semi-variation* $\underline{w}$ *is finite,* $\underline{w}$ *is exhaustive*

(3) For any linear context $(E, F, G, \odot)$ *such that the contextual semi-variation* $\underline{w}$ *is finite, every* $\underline{w}$*-measurable bounded function is integrable.*

Proof. (0) $\iff$ (1) is the Bessaga-Pelszyńsky theorem.

(1) $\Longrightarrow$ (2) : if (2) fails the (3) fails by virtue of corollary 7.17.

(2) $\Longrightarrow$ (3) : indeed every bounded $\underline{w}$-measurable function is Bochner integrable

(3) $\Longrightarrow$ (2) by corollary 7.17.

(2) $\Longrightarrow$ (1) : Example 6.10. shows that if $G = \mathbf{c_0}$, then there exists a bilinear context for which $\underline{w}$ is not exhaustive. This remains true if G contains a copy of $\mathbf{c_0}$. Hence if (1) fails then (2) fails too. □

Swartz ([SW2] Theorem 1) proved the following result: Let X be a arbitrary infinite dimentional space. The there exists a sequence $\{m_n\}$ in $L(X, \mathbf{c_0})$ such that $\{m_n x\}$ is a summable sequence in $\mathbf{c_0}$ for every $x \in X$ and a bounded sequence $\{\xi_n\}$ in X such that the sequence $\{m_n \xi_n\}$ is not summable. This result may be translated easily into a sharpening of the part (2) $\Longrightarrow$ (1) in the preceding theorem.

We now go on to a convergence theorem of Vitali type. The following theorem ([DU] proposition I.1.17 and corollary I.5.4) is usefull.

Theorem 7.20. *Let* $\{\nu_i\}$ *be an arbitrary family of G-valued measures. Suppose that for all* i, ν_i *is absolutely continuous with respect to a positive measure* μ. *Then the following properties are equivalent:*

(1) $\{\nu_i\}$ *is uniformly* σ*-additive*

(2) For every decreasing sequence $\{A_k\}$ *in* $\mathcal{T}$ *whose intersection is empty, the sequence* $\{\nu_i(A_k)\}$ *converges to 0 uniformly with respect to i*

(3) For every $\varepsilon > 0$ *there exists* η *such that*

$$\mu(A) \leq \eta \Longrightarrow \nu_i^\bullet(A) \leq \varepsilon$$

Proposition 7.21. *Suppose $\underline{w}$ is σ-finite. Let $\{f_n\}$ be a sequence of F-valued integrable functions such that f_n converges $\underline{w}$-a.e. to a function f. If the sequence $\{f_n \odot m\}$ is uniformly σ-additive, then the sequence $\{(f_n \odot m)(A)\}$ converges for every $A \in \mathcal{T}$.*

Proof. Let μ the measure associated with the sequence $\{f_n \odot m\}$ by lemma 7.1. Let $\varepsilon > 0$. There exists η such that $\mu(A) \leq \eta \Longrightarrow \|(f_n \odot m)(A)\| \leq \varepsilon$. Then there exists C_η such that $\mu(T \setminus C_\eta) \leq \eta$ and that f_n converges uniformly to f on C_η. As $\underline{w}$ is σ-additive, it may be assumed that $\underline{w}(C_\eta) < \infty$. We have:

$$(f_n \odot m)(A) - (f_p \odot m)(A) = ((f_n - f_p) \odot m)(A \cap C_\eta)$$
$$+(f_n \odot m)(A \cap (T \setminus C_\eta)) + (f_p \odot m)(A \cap (T \setminus (C_\eta))$$

and the following majorations:

$$\|((f_n - f_p) \odot m)(A \cap C_\eta)\| \leq \underline{w}(C_\eta) \sup_{t \in A \cap C_\eta} \|f_n(t) - f_p(t)\|$$
$$\|(f_n \odot m)(A \cap (T \setminus C_\eta)\| \leq (f_n \odot m)^\bullet(T \setminus C_\eta)$$
$$\|(f_p \odot m)(A \cap (T \setminus C_\eta)\| \leq (f_p \odot m)^\bullet(T \setminus C_\eta)$$

There exists N such that $n \geq N \Longrightarrow \sup_{t \in A \cap C_\eta} \|f_n(t) - f_p(t)\| \leq \varepsilon$. Then

$$n \geq N \quad \Longrightarrow \quad \|(f_n \odot m)(A) - (f_p \odot m)(A)\| \leq \underline{w}(C_\eta)\varepsilon + 2\varepsilon$$

Hence the sequence $\{(f_n \odot m)(A)\}$ is a Cauchy sequence for every $A \in \mathcal{T}$. □

Corollary 7.22. *Suppose $\underline{w}$ is σ-finite. Then a F-valued function f is integrable iff there exists a sequence $\{f_n\}$ of simple functions such that:*

(1) f_n converges $\underline{w}$-a.e. to f

(2) the measures $f_n \odot m$ are uniformly σ-additive.

Proof. If f is integrable, then there exists a sequence $\{f_n\}$ of simple functions such that $(f_n \odot m)(A)$ converges for every $A \in \mathcal{T}$. The Vitali-Hahn-Saks theorem asserts that (2) holds.

Conversely if (1) and (2) hold then proposition 7.17. asserts that $(f_n \odot m)(A)$ converges for every $A \in \mathcal{T}$. Hence f is integrable. □

Theorem 7.23. *[Convergence theorem of Vitali type] Suppose $\underline{w}$ is σ-finite. Le $\{f_n\}$ be a sequence of integrable functions such that:*

(1) f_n converges $\underline{w}$.a.e. to a function f

(2) the measures $f_n \odot m$ are uniformly additive

Then f is integrable and $(f_n \odot m)(A)$ converges uniformly to $(f \odot m)(A)$. In other words $\lim_n((f_n - f) \odot m)^\bullet(T) = 0$.

Proof. Proposition 7.21. asserts that the hypothesis of theorem 7.10. are satisfied. □

To end this section we state a result that permits integration by pieces.

Proposition 7.24. *Let $\{T_k\}$ be a countable $\mathcal{T}$-partition of T. Suppose a function f is such that $1_{T_k} f$ is integrable for every $k \in \mathbb{N}$. Then f is integrable iff the sequence $\{\int 1_{A_k} f \odot m\}$ is summable whenever $A_k \in \mathcal{T}$ and $A_k \subset T_k$.*

Proof. It is easy to prove that the condition is necessary. Suppose it is satisfied. Put $f_{(n)} = \sum_{k \le n} 1_{T_k} f$. Then $f_{(n)}$ converges everywhere to f. For $A \in \mathcal{T}$ we have:

$$(f_{(n)} \odot m)(A) = \sum_{k \le n} \int (1_{A \cap T_k} f) \odot m$$

Hypothetically the right hand member converges for $n \to \infty$. Following Theorem 7.10. f is integrable. □

8. Tensor integration

In this section we shall first be concerned with the special case where m is a vector measure and G is the vector space $F \widehat{\otimes}_\varepsilon E$ (the completed space of $F \otimes E$ with respect of the norm ε). The symbol $\odot$ has to be replaced by $\otimes$. After which we will replace the ε-norm with the π-norm. We put $\mathcal{L}^1_F(m) = \mathcal{L}^1_F(m^\bullet)$.

A essential tool will be the following notion that enables us to state generalizations of the Orlicz-Pettis's theorem. Let E be a Banach space and H be a subset of E'. Then H is said to have the Orlicz-Pettis's property iff every E-valued sequence x such that the sequence $\{\langle x_n, y\rangle\}$ is summable for every $y \in H$ is summable in E. The classical Orlicz-Pettis's theorem asserts that E' has the Orlicz-Pettis's property. This notion is discussed in [TH], Appendice II. The following proposition is an easy consequence of the definitions.

Proposition 8.1. *Let $(T, \mathcal{T})$ be a measurable space, E a Banach space and H a subset of E'. Let m be a map from $\mathcal{T}$ into E such that the map $A \mapsto \langle m(A), y\rangle$ is a measure for* *all $y \in H$. If H has the Orlicz-Pettis's property with respect to E, then m is a "strong measure", i.e. for all countable $\mathcal{T}$-partition $\{T_n\}$ of T, one has*

$$m(T) = \sum_n m(T_n)$$

for the (norm) convergence in E.

Let $f \in_F$. If $f = \sum \xi_i \mathbf{1}_{A_i}$, ($\{A_i\}$ being a finite partition of T and $\xi_i \in F$), then:

$$\int f \otimes m = \sum \xi_i \otimes m(A_i)$$

Lemma 8.2. *Let $f \in_F$. Then:*

$$\left\| \int f \otimes m \right\|_\varepsilon \leq \|f\|_{\mathcal{L}^1_F(m)}$$

Proof. One has:

$$\begin{aligned}
\|f \otimes m\|_\varepsilon &= \sup\left\{ \sum_i \langle \xi_i, y' \rangle \langle m(A_i), x' \rangle \mid x' \in \mathfrak{B}(E'), y' \in \mathfrak{B}(F') \right\} \\
&= \sup\left\{ \int \langle f(.), y' \rangle\, m_{x'} \mid x' \in \mathfrak{B}(E'), y' \in \mathfrak{B}(F') \right\} \\
&\leq \sup\left\{ \int |\langle f(.), y' \rangle|\, |m_{x'}| \mid x' \in \mathfrak{B}(E'), y' \in \mathfrak{B}(F') \right\} \\
&\leq \sup\left\{ \int \|f(.)\|\, |m_{x'}| \mid x' \in \mathfrak{B}(E') \right\} \\
&= m^\bullet(\|f(.)\|)
\end{aligned}$$

□

Proposition 8.3. *For $F \otimes E$ endowed with the ε-norm, the contextual semi-norm $\underline{w}$ is $m^\bullet$.*

Proof. By lemma 8.2., for every $f \in_F$, we have $\left\| \int f \otimes m \right\|_\varepsilon \leq m^\bullet(\|f(.)\|)$ and therefore $\underline{w}(h) \leq m^\bullet(h)$ for every $h \in_{\mathbf{R}}$.

For the converse inequality, let us pick $y \in F$ such that $\|y\| = 1$ and consider the measure $(F \otimes E)$-valued measure $y \otimes m$ such that $(y \otimes m)(A) = y \otimes m(A)$ for all $A \in \mathcal{T}$.

A easy checking gives: $(y \otimes m)^\bullet = m^\bullet$. On the other hand:

$$\begin{aligned}
(y\otimes m)^\bullet(h) &= \sup\left\{\left\|\int f\; y\otimes m\right\| \;\Big|\; |f| \le h\right\} \\
&= \sup\left\{\sum_i z_i (y\otimes m)(A_i) \;\Big|\; \left|\sum_i z_i \mathbf{1}_{A_i}\right| \le h\right\} \\
&= \sup\left\{\left\|\sum_i (z_i y)\otimes m(A_i)\right\| \;\Big|\; \left|\sum_i z_i \mathbf{1}_{A_i}\right| \le h\right\} \\
&= \sup\left\{\left\|\int (\sum_i z_i \mathbf{1}_{A_i} y)\otimes m\right\| \;\Big|\; \left\|\sum_i z_i \mathbf{1}_{A_i}(.) y\right\| \le h\right\} \\
&\le \underline{w}(h)
\end{aligned}$$

This proves $m^\bullet \le \underline{w}$. □

Consequently the following definition agrees with Definition 5.4..

Definition 8.4. A F-valued function f is said to be Bochner-integrable with respect to the E-valued measure m iff $f \in \mathcal{L}^1_F(m)$.

Definition 8.5. A F-valued function is said to be scalarly integrable with respect to the E-valued measure m iff $\langle f(.), y'\rangle \in \mathcal{L}^1(m_{x'})$ for all $x' \in E'$ and all $y' \in F'$.

In this case for all $x' \in E'$, all $y' \in F'$ and all $A \in \mathcal{T}$, ${}_{\mathrm{w}}\!\int_A f \otimes m$ denotes the element of $(E' \otimes F')^*$ such that

$$\left\langle {}_{\mathrm{w}}\!\int_A f\otimes m, x'\otimes y'\right\rangle = \int_A \langle f(.), y'\rangle\, m_{x'}$$

A F-valued m-measurable function f is said to be ε-Pettis-intégrable iff:

(1) f is m-scalarly integrable

(2) ${}_{\mathrm{w}}\!\int_A f \otimes m \in F \widehat{\otimes}_\varepsilon E$ for all $A \in \mathcal{T}$.

Example 8.6. Every Bochner-integrable function is ε-Pettis-integrable.

Plainly if f is ε-Pettis-integrable, then the map $A \mapsto {}_{\mathrm{w}}\!\int_A f \otimes m$ is a weak measure for the duality $(F \widehat{\otimes}_\varepsilon E, F' \otimes E')$. But we have more:

Theorem 8.7. *If f is ε-Pettis-integrable, then $A \mapsto {}_{\mathrm{w}}\!\int_A f \otimes m$ is a strong measure with values in $F \widehat{\otimes}_\varepsilon E$. This measure will be denoted by $(f \otimes m)_\varepsilon$.*

Proof. Indeed following [TH] (Appendice II, Corollaire II.7), $F' \otimes E'$ has the Orlicz-Pettis-property for $F \widehat{\otimes}_\varepsilon E$. □

Theorem 8.8. *Any F-valued ε-Pettis-integrable function is integrable in the sense of Definition 7.3. and conversely.*

Proof. (1) Suppose f is ε-Pettis-integrable. If is Bochner-integrable, consider a sequence $\{g_n\}$ in $_F$ such that $f = \lim_n g_n$ in $\mathcal{L}^1_F(m)$. By extracting a subsequence we may suppose g_n converges to f m-a.e.. Then f is integrable in the sense of definition 7.3.. Consider now the general case. For every ε-Pettis-integrable function f let us put $p(f) = (f \otimes m)^\bullet_\varepsilon(T)$ Let us take a countable $\mathcal{T}$-partition $\{T_k\}$ such that que $1_{T_k} f \in \mathcal{L}^1_F(m)$ for all k. For every k, there exists a sequence $\{g_{n,k} \mid n \in \mathbb{N}\}$ in $_F(m)$ such that $g_{n,k}$ vanishes outside T_k, $g_{n,k}$ converges m-a.e. to $f1_{T_k}$ and $p(f1_{T_k} - g_{n,k}) \leq \frac{1}{n}2^{-k}$. Let us put $S_k = \cup_{h \leq k} T_h$ and $R_k = T \setminus S_k$.

For all n, let $K(n)$ be such that $p(f1_{R_{K(n)}}) \leq \frac{1}{n}$. Put $g_n = \sum_{k \leq K(n)} g_{n,k}$. Then g_n converges m-a.e. to f and we have:

$$\begin{aligned} p(f - g_n) &\leq p(f1_{R_{K(n)}} - g_n) + p(f1_{R_{K(n)}}) \\ &\leq \sum_{k \leq K(n)} p(f1_{T_k} - g_{n,k}) + p(f1_{R_{K(n)}}) \\ &\leq 2/n \end{aligned}$$

As the sequence $\{g_n\}$ converges m-a.e. and $\lim_n p(f - g_n) = 0$, f is integrable in the sense of Definition 7.3..

(2) Suppose now f is integrable in the sense of Definition 7.3.. Pick $x' \in E'$ and $y' \in F'$. One have $\lim_n \langle f_n(t), y' \rangle = \langle f(t), y' \rangle$ m-p.p.. On the other hand for every $A \in \mathcal{T}$, the sequence $n \mapsto \int_A \langle f_n(.), y' \rangle \, m_{x'}$ converges. Hence by [PB] (théorème 15.10), $\langle f_n(.), y' \rangle \in \mathcal{L}^1(m_{x'})$ and

$$\lim_n \int_A \langle f_n(.), y' \rangle \, m_{x'} = \int_A \langle f(.), y' \rangle \, m_{x'}$$

So f is scalarly integrable.

For $A \in \mathcal{T}$, put $P(A) = \lim_n \int_A f_n \otimes m$. One have $P(A) \in F \widehat{\otimes}_\varepsilon E$. By the Vitali-Hahn-Saks theorem, P is a strong measure. Moreover for all $x' \in E'$ and all $y' \in F'$,

$$\langle P(A), x' \otimes y' \rangle = \lim_n \int_A \langle f_n(.), y' \rangle \, m_{x'} = \int_A \langle f(.), y' \rangle \, m_{x'}$$

Therefore f est ε-Pettis-integrable. □

Example 8.9. If $E = \mathbb{R}$, *i.e.* if m is a scalar measure, then Bochner integrability of a F-valued function in the sense of definition 8.4. coincides with Bochner integrability of a F-valued function in the usual sense for m-measurable functions. The ε-Pettis-integrability is nothing but the usual Pettis-integrability. A glance at theorem 8.8 shows that a F-valued function f is Pettis integrable if there exists a sequence $\{f_n\}$ of simple functions converging m-a.e. to f such that $\int_A f_n m$ converges for every $A \in \mathcal{T}$.

Example 8.10. Taking $F = \mathbb{R}$ we obtain the following definition. If m is a E-valued measure a scalar m-measurable function f is Pettis-integrable iff

(1) $f \in \mathcal{L}^1(m_{x'})$ for every $x' \in E'$

(2) defining $\mathrm{w}\!\int_A fm \in E'^*$ by

$$\forall x' \in E', A \in \mathcal{T} \qquad \left\langle \mathrm{w}\!\int_A fm, x' \right\rangle = \int_A fm_{x'}$$

we have $\mathrm{w}\!\int_A fm \in E$ for all $A \in \mathcal{T}$.

By theorem 8.8. this definition is equivalent to the following: there exists a sequence $\{f_n\}$ of $\mathcal{T}$-simple functions such that f_n converges m-a.e. to f and $\int_A f_n m$ converges for every $A \in \mathcal{T}$. The standard theory of integrability of scalar functions with respect to a vector measure shows the above definition is in fact equivalent to the Bochner-integrability *i.e.* to the property $f \in \mathcal{L}^1(m)$.

We go on to the case $F \otimes E$ is equipped with the π-norm. We suppose that F has the the approximation property, so as $F \widehat{\otimes}_\pi E$ is a subspace of $F \widehat{\otimes}_\varepsilon E$.

Definition 8.11. A F-valued m-measurable function will be said to be π-Pettis-integrable if it is ε-Pettis-integrable and if furthermore $f \otimes m$ is a $(F \widehat{\otimes}_\pi E)$-valued (strong) measure.

For f to be ε-Pettis-intégrable, it is not sufficient $f \otimes m$ to have its values in $F \widehat{\otimes}_\pi E$. It is necessary $f \otimes m$ to be strongly σ addditive. The following theorem gives a very general condition for this condition to be fulfilled.

Theorem 8.12. *If $F' \otimes E'$ has the Orlicz-Pettis's property with respect to $F \widehat{\otimes}_\pi E$, then every ε-Pettis-integrable function f such that $f \otimes m$ has values in $F \widehat{\otimes}_\pi E$ is π-Pettis-integrable.*

In particular if F et E are separable and if F has the metric approximation property, then every ε-Pettis-integrable function f such that $f \otimes m$ is $(F \widehat{\otimes}_\pi E)$-valued is π-Pettis-integrable.

Proof. The first asssertion follows from Proposition 8.1..

For the second assertion call to [PB] (théorème XI.37): the metric approximation implies:

$$\forall X \in F \widehat{\otimes}_\pi E \qquad \|X\|_\pi = \sup\{\langle X, B\rangle \mid B \in E' \otimes F', \|B\|_\varepsilon \leq 1\}$$

By [TH] (théorème II.3 de l'appendice II), this implies that $F' \otimes E'$ has the Orlicz-Pettis's property for $F \widehat{\otimes}_\pi E$. The conclusion is now immediate. □

Proposition 8.13. *If $f \in_{F}$, then*

$$\left\| \int f \otimes m \right\|_\pi \leq \int \|f(.)\| \operatorname{var}(m)$$

Proof. It is a particular case of Proposition 5.2.. □

Corollary 8.14. *The contextual semi-variation $\underline{w}$ attached to the integration with values in $F \widehat{\otimes}_\pi E$ is majorized by* $\operatorname{var}(m)$. *In particular if* $\operatorname{var}(m)$ *is σ-finite then the same is true for $\underline{w}$.*

Theorem 8.15. *Suppose $\underline{w}$ σ-finite. Any F-valued π-Pettis-integrable function is integrable in the sense of Definition 7.3. and conversely.*

Proof. (1) Suppose f is π-Pettis-integrable. If is Bochner-integrable, consider a sequence $\{g_n\}$ in $_F$ such that $f = \lim_n g_n$ in $\mathcal{L}^1_F(\underline{w})$. By extracting a subsequence we may suppose g_n converges to f ma.e.. Then f is integrable in the sense of definition 7.3.. Consider now the general case. For every π-Pettis-integrable function f let us put $p(f) = (f \otimes m)^\bullet_\pi(T)$ Let us take a countable $\mathcal{T}$-partition $\{T_k\}$ such that que $\mathbf{1}_{T_k} f \in \mathcal{L}^1_F(\underline{w})$ for all k. For every k, there exists a sequence $\{g_{n,k} \mid n \in \mathbb{N}\}$ in $_F(m)$ such that $g_{n,k}$ vanishes outside T_k, $g_{n,k}$ converges m-a.e. to $f\mathbf{1}_{T_k}$ and $p(f\mathbf{1}_{T_k} - g_{n,k}) \leq \frac{1}{n} 2^{-k}$. Let us put $S_k = \cup_{h \leq k} T_h$ and $R_k = T \setminus S_k$.

For all n, let $K(n)$ be such that $p(f\mathbf{1}_{R_{K(n)}}) \leq \frac{1}{n}$. Put $g_n = \sum_{k \leq K(n)} g_{n,k}$. Then g_n converges m-a.e. to f and we have:

$$\begin{aligned} p(f - g_n) &\leq p(f\mathbf{1}_{R_{K(n)}} - g_n) + p(f\mathbf{1}_{R_{K(n)}}) \\ &\leq \sum_{k \leq K(n)} p(f\mathbf{1}_{T_k} - g_{n,k}) + p(f\mathbf{1}_{R_{K(n)}}) \\ &\leq 2/n \end{aligned}$$

As the sequence $\{g_n\}$ converges m-a.e. and $\lim_n p(f - g_n) = 0$, f is integrable in the sense of Definition 7.3..

(2) Suppose now f is integrable in the sense of Definition 7.3.. Then it is ε-Pettis-integrable by Theorem 8.8.. Furthermore $f \otimes m$ is a strong measure with values in $F \widehat{\otimes}_\pi E$. Hence f is π-Pettis-integrable. □

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ABOUT AN INTEGRAL OPERATOR PRESERVING THE UNIVALENCE

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Abstract. In this work an integral operator is studied and the author determines conditions for the univalence of this integral operator.

1. Introduction

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the function $f \in A$ which are analytic in U.

Many authors studied the problem of integral operators which preserve the class S. In this sense an important result is due to J. Pfaltzgraff [4].

Theorem A. *[4] If f(z) is univalent in U, α a complex number and $|\alpha| \le \frac{1}{4}$, then the function*

$$G_\alpha(z) = \int_o^z [f'(\xi)]^\alpha d\xi \tag{1}$$

is univalent in U.

Theorem B. *[3] If the function $g \in S$ and α is a complex number, $|\alpha| \le \frac{1}{4n}$, then the function defined by*

$$G_{\alpha,n}(z) = \int_o^z [g'(u^n)]^\alpha du \tag{2}$$

is univalent in U for all positive integer n.

2. Preliminaries

For proving our main result we will need the following theorem and lemma.

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Theorem C. *[1].If the function f is regular in the unit disc U,* $f(z) = z + a_2z^2 + \dots$ *and*

$$\left(1 - |z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3}$$

for all $z \in U$*, then the function f is univalent in U.*

Lemma Schwarz 1. *[2]. If the function g is regular in U,* $g(0) = 0$ *and* $|g(z)| \leq 1$ *for all* $z \in U$*, then the following inequalities hold*

$$|g(z)| \leq |z| \tag{4}$$

for all $z \in U$*, and* $|g'(0)| \leq 1$*, the equalities (in inequality (4) for* $z \neq 0$*) hold only in the case* $g(z) = \epsilon z$*, where* $|\epsilon| = 1$.

3. Principal result

Theorem 1. *Let* γ *be a complex number and the function* $g \in A$, $g(z) = z + a_2z^2 + \dots$. *If*

$$\left|\frac{h''(z)}{g'(z)}\right| \leq \frac{1}{n} \tag{5}$$

for all $z \in U$ *and*

$$|\gamma| \leq \frac{1}{\left(\frac{n}{n+2}\right)^{\frac{n}{2}} \frac{2}{n+2}} \tag{6}$$

then the function

$$G_{\gamma,n}(z) = \int_0^z \left[g'(u^n)\right]^\gamma du \tag{7}$$

is univalent in U for all $n \in N^* - \{1\}$.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[g'(u^n)\right]^\gamma du. \tag{8}$$

The function

$$h(z) = \frac{1}{\gamma} \frac{f''(z)}{f'(z)}, \tag{9}$$

where the constant γ satisfies the inequality (6) is regular in U. From (9) and (8) it follows that

$$h(z) = \frac{\gamma}{|\gamma|} \left[\frac{nz^{n-1}g''(z^n)}{g'(z^n)}\right]. \tag{10}$$

Using (10) and (5) we have

$$|h(z)| \le 1, \tag{11}$$

for all $z \in U$. From (10) we obtain $h(0) = 0$ and applying Schwarz-Lemma we have

$$\frac{1}{|\gamma|}\left|\frac{f''(z)}{f'(z)}\right| \le |z|^{n-1} \le |z| \tag{12}$$

for all $z \in U$, and hence, we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le |\gamma|\left(1-|z|^2\right)|z|^n. \tag{13}$$

Let us consider the function Q:[0, 1] $\to R$, $Q(x) = \left(1-x^2\right)x^n$; $x = |z|, z \in U$, which has a maximum at a point $x = \sqrt{\frac{n}{n+2}}$, and hence

$$Q(x) < \left(\frac{n}{(n+2)^{\frac{n}{2}}}\right)\frac{2}{n+2} \tag{14}$$

for all $x \in (0,1)$. Using this result and (13) we have

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le |\gamma|\left(\frac{n}{(n+2)}\right)^{\frac{n}{2}}\frac{2}{n+2}. \tag{15}$$

From (15) and (6) we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{16}$$

for all $z \in U$. From (16) and (8) and Theorem C it follows that $G_{\gamma,n}$, is in the class S. □

Observation. For $n = 2$, we obtain $|\gamma| \le 4$ and the function $G_{\gamma,2}$ is in the class S.

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ON THE SPLINE APPROXIMATING METHODS FOR SECOND ORDER SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. One proposes an approximation method for the solution of the systems of p second order differential equations by means of spline functions. There is studied the error estimation and the stability of the proposed method.

1. Introduction.

Consider the following system of nonlinear second order differential equations with the initial conditions:

$$\begin{cases} y_i''(x) & = & f_i(x, y_1, \dots, y_p) \\ y_i(x_0) & = & y_{i,0},\ y_i'(x_0) = y_{i,0}', \quad i = \overline{1,p} \end{cases}$$

where $f_i \in C^r([0,1] \times \mathbb{R}^p)$, $i = \overline{1,p}$ and $r, p \in \mathbb{N}$.

The approximate solution of a system of 2 equations of second order was constructed by Th. Fawzy, Z. Ramadan and A.Ayad [1,2]. In this paper we propose a generalization of the method, for system of p equations. The system (1) can be transformed in a system of n equations of the first order, but the order of the method presented is $O(h^{\alpha+r+2m})$. The order of the method presented by G.Micula and Maria Micula[5], for system of n equations of first order, is $O(h^{\alpha+r+m})$.

2. Description of the approximating method.

Let L_i be the

Lipschitz constants satisfied by the functions $f_i^{(q)}$, $i = \overline{1,p}$, $q = \overline{0,r}$:

$$|f_i^{(q)}(x, y_{1,1}, \dots, y_{p,1}) - f_i^{(q)}(x, y_{1,2}, \dots, y_{p,2})| \le L_i \sum_{j=1}^{p} |y_{j,1} - y_{j,2}| \qquad (1)$$

$\forall (x, y_{1,1}, \dots, y_{p,1})$, $(x, y_{1,2}, \dots, y_{p,2})$ in the domain of definition of f_i, $i = \overline{1,p}$.

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Let Δ be a partition of the interval $[0,1]$:

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1,\ h := x_{k+1} - x_k,\ k = \overline{0,n-1}$$

Assume that $f_i \in C^r([0,1] \times \mathbb{R}^p)$ and that the modulus of continuity of the functions $y_i^{(r+2)}$ is $\omega(y_i^{(r+2)}, h)$, and $\omega(h) = \max_{i=\overline{1,p}} \omega(y_i^{(r+2)}, h)$.

The functions $f_i^{(q-1)}$ depending of x, $y_1, \ldots, y_p$ are given from the following algorithm:

Set $f_i^{(0)} = f_i$ and if $f_i^{(q-1)}$ are defined, then:

$$y_i^{(q+2)}(x) = f_i^{(q)}(z) = f_{i_x}^{(q-1)}(z) + f_{i_{y_1}}^{(q-1)}(z) y_1' + \ldots + f_{i_{y_p}}^{(q-1)}(z) y_p' \tag{2}$$

where $z = (x, y_1, \ldots, y_p)$.

We define the spline functions approximating y_i by $s_{i,\Delta}$, $i = \overline{1,p}$, for $x_k \le t_m \le t_{m-1} \le \ldots \le t_1 \le t \le x \le x_{k+1}$, $k = \overline{0,n-1}$ a partition of the interval $[x_k, x_{k+1}]$, in the following way:

$$\begin{aligned} s_{i,\Delta}(x) &:= s_{i,k}^{[m]}(x) := s_{i,k-1}^{[m]}(x_k) + s_{i,k-1}^{'[m]}(x_k)(x - x_k) + \\ &+ \int_{x_k}^{x} \int_{x_k}^{t} f_i[t_1, s_{1,k}^{[m-1]}(t_1), \ldots, s_{p,k}^{[m-1]}(t_1)] dt_1 \, dt \end{aligned} \tag{3}$$

where $s_{i,-1}^{[m]}(x_0) := y_{i,0}$, $s_{i,-1}^{'[m]}(x_0) = y_{i,0}'$ and m is a positiv integer number.

In (4) we use the following m iterations:

$$s_{i,k}^{[0]}(x) := s_{i,k-1}^{[m]}(x_k) + s_{i,k-1}^{'[m]}(x_k)(x - x_k) + \sum_{j=0}^{r} \frac{(x - x_k)^{j+2}}{(j+2)!} M_{i,k}^{(j)} \tag{4}$$

$$\begin{aligned} s_{i,k}^{[j]}(x) &:= s_{i,k-1}^{[m]}(x_k) + s_{i,k-1}^{'[m]}(x_k)(x - x_k) + \\ &+ \int_{x_k}^{x} \int_{x_k}^{t} f_i(t_{m-j+1}, s_{1,k}^{[j-1]}(t_{m-j+1}), \ldots, s_{p,k}^{[j-1]}(t_{m-j+1})) dt_{m-j+1} dt \\ M_{i,k}^{(j)} &:= f_i^{(j)}(x_k, s_{1,k-1}^{[m]}(x_k), \ldots, s_{p,k-1}^{[m]}(x_k)),\ j = \overline{1,m} \end{aligned} \tag{5}$$

It is clear by construction that $s_{i,\Delta} \in C^1[0,1]$, $i = \overline{1,p}$.

3. Error estimation and convergence.

The following notations will be used $y_{i,k}^{(j)} := y_i^{(j)}(x_k)$ for $i = \overline{1,p}$, $j = \overline{0,r+1}$ and $k = \overline{1,n-1}$. The exact solution $y_i := y_i^{[m]}$, $i = \overline{1,p}$ of (1) can be written in the following form:

- By Taylor's expansion, for $y_i^{[0]}$, holds:

$$y_i^{[0]}(x) = \sum_{j=0}^{r+1} \frac{y_{i,k}^{(j)}}{j!}(x - x_k)^j + \frac{y_i^{(r+2)}(\xi_{i,k})}{(r+2)!}(x - x_k)^{r+2} \tag{6}$$

where $\xi_{i,k} \in]x_k, x_{k+1}[$, $i = \overline{1,p}$.

- For $1 \le j \le m$ the exact solution $y_i^{[j]}$ is given by:

$$\begin{aligned} y_i^{[j]}(x) &:= y_{i,k} + y'_{i,k}(x - x_k) + \\ &+ \int_{x_k}^{x}\int_{x_k}^{t} f_i(t_{m-j+1}, y_1^{[j-1]}(t_{m-j+1}), \dots, y_p^{[j-1]}(t_{m-j+1}))dt_{m-j+1}dt \end{aligned} \tag{7}$$

where $i = \overline{1,p}$, $, j = \overline{1,m}$, $k = \overline{0,n-1}$

The error is defined by the usual way, for $i = \overline{1,p}$, $k = \overline{0,n-1}$:

$$\begin{aligned} e_i(x) &:= |y_i(x) - s_{i,\Delta}(x)| \quad , \quad e'_i(x) := |y'_i(x) - s'_{i,\Delta}(x)| \\ e_{i,k} &:= |y_{i,k} - s_{i,\Delta}(x_k)| \quad , \quad e'_{i,k} := |y'_{i,k} - s'_{i,\Delta}(x_k)| \end{aligned} \tag{8}$$

Lemma 3.1. *[1] Let α and β be nonnegative real numbers, $\beta \neq 1$ and $\{A_i\}_{i=0}^{k}$ be a sequence satisfying the conditions:*

$$A_0 \ge 0,\ A_{i+1} \le \alpha + \beta A_i,\ i = 0, 1, \dots, k$$

then the following inequality holds:

$$A_{k+1} \le \beta^{k+1} A_0 + \alpha \frac{\beta^{k+1} - 1}{\beta - 1}$$

Lemma 3.2. *[1] Let α and β be positive real numbers , and $\{A_i\}_{i=1}^{m}$ be a sequence satisfying :*

$$A_1 \ge 0,\ A_i \le \alpha + \beta A_{i+1},\ i = 1, \dots, m-1.$$

then

$$A_1 \le \beta^{m-1} A_m + \alpha \sum_{i=0}^{m-2} \beta^i$$

Definition 3.1. For any $u \in [x_k, x_{k+1}]$, $k = \overline{0,n-1}$, $j = \overline{1,m}$ we define the operator T_{kj} by:

$$T_{kj}(u) := \sum_{i=1}^{p} |y_i^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)|,$$

whose norm is defined by:

$$||T_{kj}|| := \max_{u \in [x_k, x_{k+1}]} \{T_{kj}(u)\}$$

Lemma 3.3. *For any $u \in [x_k, x_{k+1}]$, $k=\overline{0,n-1}$, $j=\overline{1,m}$, the following estimations:*

$$||T_{km}|| \le \{1+pL\sum_{j=0}^{r}\frac{1}{(j+2)!}\}\sum_{i=1}^{p}e_{i,k}+\sum_{i=1}^{p}e'_{i,k}+p\,\omega(h)\frac{h^{r+2}}{(r+2)!} \tag{9}$$

with $L=\max\{L_1, L_2, \dots, L_p\}$, and

$$||T_{k1}|| \le a\sum_{i=1}^{p}e_{i,k}+b\sum_{i=1}^{p}e'_{i,k}+c\,h^{2m+r}\omega(h) \tag{10}$$

hold, where a, b, c are constants independent of h:

$$\begin{aligned} a &= \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^j+2\left(\frac{pL}{2}\right)^m\sum_{j=0}^{r}\frac{1}{(j+2)!}\\ b &= \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^j\\ c &= \frac{p^m}{(r+2)!}\left(\frac{L}{2}\right)^{m-1}\end{aligned}$$

Proof. Using (5) and (7) we get:

$$\begin{aligned}|y_i^{[0]}(u)-s_{i,k}^{[0]}(u)| &\le |y_{i,k}-s_{i,k-1}^{[m]}(x_k)|+|y'_{i,k}-s_{i,k-1}^{'[m]}(x_k)||x-x_k|+\\ &+\sum_{j=0}^{r-1}\frac{|x-x_k|^{j+2}}{(j+2)!}|y_{i,k}^{(j+2)}-M_{i,k}^{(j)}|+\frac{|x-x_k|^{r+2}}{(r+2)!}|y_i^{(r+2)}(\xi_{i,k})-M_{i,k}^{(r)}|\end{aligned} \tag{11}$$

From (9) and (2), we can see that:

$$|y_{i,k}^{(j+2)}-M_{i,k}^{(j)}| \le L_i\{\sum_{j=1}^{p}|y_{j,k}-s_{j,k-1}^{[m]}(x_k)|\} \le L_i\sum_{j=1}^{p}e_{j,k} \tag{12}$$

$$|y_i^{(r+2)}(\xi_{i,k})-M_{i,k}^{(r)}| \le \omega(y_i^{(r+2)},h)+L_i\sum_{j=1}^{p}e_{j,k} \tag{13}$$

where $\omega(y_i^{(r+2)},h)$ is the modulus of continuity of function $y^{(r+2)}$. Using (13) and (14) in (12), we obtain:

$$\begin{aligned}\max_{u\in[x_k,x_{k+1}]}|y_i^{[0]}(u)-s_{i,k}^{[0]}(u)| &\le e_{i,k}+he'_{i,k}+L_i\sum_{j=1}^{p}e_{j,k}\sum_{l=0}^{r}\frac{h^{l+2}}{(l+2)!}+\\ &+\frac{h^{r+2}}{(r+2)!}\omega(h) \le e_{i,k}+e'_{i,k}+L_i\sum_{j=1}^{p}e_{j,k}\sum_{l=0}^{r}\frac{1}{(l+2)!}+\frac{\omega(h)h^{r+2}}{(r+2)!}\end{aligned} \tag{14}$$

Adding in (15) for $i=\overline{1,p}$, we get:

$$||T_{km}|| \le \{1+pL\sum_{j=0}^{r}\frac{1}{(j+2)!}\}\sum_{i=1}^{p}e_{i,k}+\sum_{i=1}^{p}e'_{i,k}+p\frac{h^{r+2}}{(r+2)!}\omega(h) \tag{15}$$

For computing $||T_{kj}||$, we use (6), (8) and (2):

$$|y_i^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)| \le |y_{i,k} - s_{i,k-1}^{[m]}(x_k)| + |y'_{i,k} - s_{i,k-1}^{'[m]}(x_k)||x - x_k| +$$

$$+L_i \int_{x_k}^{x} \int_{x_k}^{t} \{\sum_{i=1}^{p} |y_i^{[m-j-1]}(t_{j+1}) - s_{i,k}^{[m-j-1]}(t_{j+1})|\} dt_{j+1}\, dt$$

$$\max_{u\in[x_k,x_{k+1}]} |y_i^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)| \le e_{i,k} + he'_{i,k} + L_i ||T_{k(j+1)}|| \int_{x_k}^{x} \int_{x_k}^{t} dt_{j+1}\, dt$$

and the result is:

$$||T_{kj}|| \le \sum_{i=1}^{p} e_{i,k} + \sum_{i=1}^{p} e'_{i,k} + pL\frac{h^2}{2}||T_{k(j+1)}|| \qquad (16)$$

Applying Lemma 3.2 we get from (17):

$$||T_{k1}|| \le \left(\frac{pL}{2}\right)^{m-1} h^{2m-2}||T_{km}|| + \sum_{i=1}^{p}(e_{i,k} + e'_{i,k}) \sum_{j=0}^{m-2} \left(\frac{pL}{2}\right)^{j} \qquad (17)$$

and, using (16), it can be shown that:

$$\begin{aligned}
||T_{k1}|| \quad &\le \quad \{\sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^{j} + 2\left(\frac{pL}{2}\right)^{m} \sum_{j=0}^{r} \frac{1}{(j+2)!}\} \sum_{i=1}^{p} e_{i,k} + \\
&+ \quad \sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^{j} \sum_{i=1}^{p} e'_{i,k} + p^m \left(\frac{L}{2}\right)^{m-1} \frac{1}{(r+2)!}\omega(h)h^{2m+r} \\
&\le a \quad \sum_{i=1}^{p} e_{i,k} + b\sum_{i=1}^{p} e'_{i,k} + ch^{2m+r}\omega(h)
\end{aligned}$$

for

$$a = \sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^{j} + 2\left(\frac{pL}{2}\right)^{m} \sum_{j=0}^{r} \frac{1}{(i+2)!}$$

$$b = \sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^{j}, \; c = \frac{p^m}{(r+2)!}\left(\frac{L}{2}\right)^{m-1}$$

□

Lemma 3.4. *For e_i, e'_i defined in (9), there exist constants $\{d_{i1}\}$, $\{d_{i2}\}$, $\{d_{i3}\}$, $\{d_{i4}\}$, $\{d_{i5}\}$, $\{c_{i1}\}$, $\{c_{i2}\}$, $i=\overline{1,p}$ independent of h such that the following inequalities hold:*

$$\begin{aligned} e_i(x) &\leq (1+d_{i1}h)e_{i,k} + hd_{i1}\sum_{\substack{j=1\\ j\neq i}}^{p} e_{j,k} + hd_{i2}e'_{i,k} + \\ &+ hd_{i3}\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} + c_{i1}h^{2m+r+2}\omega(h) \end{aligned} \tag{18}$$

$$\begin{aligned} e'_i(x) &\leq hd_{i4}e_{i,k} + hd_{i4}\sum_{\substack{j=1\\ j\neq i}}^{p} e_{j,k} + (1+hd_{i5})e'_{i,k} + \\ &+ hd_{i5}\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} + c_{i2}h^{2m+r+1}\omega(h) \end{aligned} \tag{19}$$

Proof. Using (8), (6) and (11) we estimate:

$$\begin{aligned} e_i(x) &= \leq |y_{i,k} - s^{[m]}_{i,k-1}(x_k)| + |y'_{i,k} - s'^{[m]}_{i,k-1}(x_k)|(x-x_k) + \\ &+ L_i\int_{x_k}^{x}\int_{x_k}^{t}\{\sum_{j=1}^{p}|y_j^{[m-1]}(t_1) - s_{j,k}^{[m-1]}(t_1)|\}dt_1dt \leq \\ &\leq e_{i,k} + he'_{i,k} + L_i||T_{k1}||\int_{x_k}^{x}\int_{x_k}^{t}dt_1dt \leq \\ &\leq (1+\frac{L_ia}{2})he_{i,k} + \frac{L_ia}{2}h\sum_{\substack{j=1\\ j\neq i}}^{p} e_{j,k} + (1+\frac{L_ib}{2})he'_{i,k} + \\ &+ \frac{L_ib}{2}h\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} + \frac{L_ic}{2}h^{2m+r+2}\omega(h) \end{aligned}$$

So, for $d_{i1} = \dfrac{L_ia}{2}$, , $d_{i2} = 1 + d_{i3}$, $d_{i3} = \dfrac{L_ib}{2}$, $c_{i1} = \dfrac{L_ic}{2}$ we obtain (19). Similarly we prove for e'_i. □

Using the matrix notation:

$$\begin{aligned} E(x) &:= (e_1(x),\dots,e_p(x),e'_1(x),\dots,e'_p(x))^T \\ E_k &:= (e_{1,k},\dots,e_{p,k},e'_{1,k},\dots,e'_{p,k})^T \\ C &:= (c_{11},c_{21},\dots,c_{p1},c_{12},c_{22},\dots,c_{p2})^T \end{aligned}$$

from the Lemma 3.4 we can write:

$$E(x) \le (I + hA)E_k + Ch^{2m+r+1}\omega(h) \tag{20}$$

where I is the unit matrix of order $2p \times 2p$ and

$$A = \begin{pmatrix} d_{11} & \dots & d_{11} & d_{12} & d_{13} & \dots \\ d_{21} & \dots & d_{21} & d_{22} & d_{23} & \dots \\ & & & & \dots & \\ d_{14} & \dots & d_{14} & d_{15} & d_{15} & \dots \\ d_{24} & \dots & d_{24} & d_{25} & d_{25} & \dots \\ & & & & & \end{pmatrix}$$

If for the matrix $M = (m_{ij})$ we defined the norma by:

$$||M|| := \max_i \sum_j |m_{ij}|$$

then on the basis of (21) we can write:

$$||E(x)|| \le (1 + h||A||)||E_k|| + ||C||h^{2m+r+1}\omega(h)$$

The inequality holds for any $x \in [0,1]$. Setting $x = x_{k+1}$ it follows:

$$||E_{(k+1)}|| \le (1 + h||A||)||E_k|| + ||C||h^{2m+r+1}\omega(h).$$

Using Lemma 3.1 and noting that $||E_0|| = 0$, we get:

$$\begin{aligned} ||E(x)|| &\le (1 + h||A||)^{k+1}||E_0|| + ||C||h^{2m+r+1}\omega(h)\frac{(1 + h||A||)^{k-1} - 1}{1 + h||A|| - 1} \le \\ &\le \frac{||C||}{||A||}(e^{||A||} - 1)h^{2m+r}\omega(h) \end{aligned}$$

Now it follows strighforward:

$$e_i^{(j)}(x) \le B_0\omega(h)h^{2m+r}, \text{for } i = \overline{0,p},\ j = 0,1 \tag{21}$$

where $B_0 := \dfrac{||C||}{||A||}(e^{||A||} - 1)$ is a constant independent of h.

We estimate the difference $|y_i^{(q+2)}(x) - s_i^{(q+2)}(x)|$, $q = \overline{0,r}$, $i = \overline{1,p}$

$$\begin{aligned} &|y_i^{(q+2)}(x) - s_i^{(q+2)}(x)| := |\frac{d^{q+2}}{dx^{q+2}}y_i^{[m]}(x) - \frac{d^{q+2}}{dx^{q+2}}s_{i,k}^{[m]}(x)| = \\ = \; &|f_i^{(q)}(t_1, y_1^{[m-1]}(t_1), ..., y_p^{[m-1]}(t_1)) - -f_i^{(q)}(t_1, s_{1,k}^{[m-1]}(t_1), ..., s_{p,k}^{[m-1]}(t_1))| \le \\ \le \; &L_i||T_{k1}|| \le L_i\{a\sum_{j=1}^{p} e_{j,k} + b\sum_{j=1}^{p} e'_{j,k} + ch^{2m+r}\omega(h)\} \le B_{i1}h^{2m+r}\omega(h) \end{aligned} \tag{22}$$

for $B_{i1} = L_i[p(a+b)B_0 + c]$ a constant independent of h.

Thus, we proved the following result:

Theorem 3.1. *Let $(y_1, \dots, y_p)$ be the exact solution of the problem* (1) *and $(s_{1,\Delta} \dots, s_{p,\Delta})$ be the approximate solution for the problem* (1). *If $f \in C^r([0,1] \times \mathbb{R}^p)$, then the following estimations hold for $x \in [0,1]$:*

$$|y_i^{(q)}(x) - s_{i,\Delta}^{(q)}(x)| \leq B_{i2}\omega(h)h^{2m+r} = O(h^{2m+r+\alpha})$$

where $q = \overline{0, r+2}$ and B_{i2}, $i = \overline{1,p}$ are constants independent of h.

4. Stability of the method

. For new initial conditions,
$y_i(x_0) = y^*_{i,0}$, $y'_0(x_0) = y'^*_{i,0}$, we defined the approximate solution by:

$$\begin{aligned} w_{i,\Delta}(x) &:= w_{i,k-1}^{[m]}(x_k) + w_{i,k-1}^{'[m]}(x_k)(x - x_k) + \\ &+ \int_{x_k}^{x}\int_{x_k}^{t} f_i[t_1, w_{1,k}^{[m-1]}(t_1), \dots, w_{p,k}^{[m-1]}(t_1)]dt_1 dt \end{aligned} \tag{23}$$

where $w_{i,-1}^{[m]}(x_0) := y^*_{i,0}$ and $w_{i,-1}^{'[m]}(x_0) = y'^*_{i,0}$, $i = \overline{1,p}$, $k = \overline{0, n-1}$.

In (24) we use m iterations, for $x_k \leq t_m \leq \dots \leq t_1 \leq t \leq x \leq x_{k+1}$, like in (5) and (6).

We use the following notations, for $i = \overline{1,p}$, $k = \overline{0, n-1}$:

$$\begin{aligned} e_i^*(x) &:= |w_i(x) - s_i(x)|,\ e_i^{'*}(x) := |w_i'(x) - s_i'(x)| \\ e_{i,k}^* &:= |w_i(x_k) - s_i(x_k)|,\ e_{i,k}^{'*} := |w_i'(x_k) - s_i'(x_k)| \\ M_{i,k}^{*(j)} &:= f_i^{(j)}(x_k, w_{1,k-1}^{[m]}, \dots, w_{p,k-1}^{[m]}(x_k)),\ j = \overline{1,m} \end{aligned}$$

and we define the operator:

$$T^*_{kj}(u) := \sum_{i=1}^{p} |w_{i,k}^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)|, u \in [x_k, x_{k+1}],\ k = \overline{0, n-1},\ j = \overline{1,m}$$

with the norm:

$$||T^*_{kj}|| = \max_{u \in [x_k, x_{k+1}]} \{T^*_{kj}(u)\}$$

Lemma 4.1. *For any $u \in [x_k, x_{k+1}]$, $k = \overline{0, n-1}$, the estimations*

$$||T^*_{km}|| \leq \{1 + pL\sum_{j=0}^{r}\frac{1}{(j+2)!}\}\sum_{i=1}^{p} e^*_{i,k} + \sum_{i=1}^{p} e^{*'}_{i,k} \tag{24}$$

$$||T^*_{k1}|| \leq a\sum_{i=1}^{p} e^*_{i,k} + b\sum_{i=1}^{p} e^{*'}_{i,k} \tag{25}$$

hold, where

$$a = \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^j + 2\left(\frac{pL}{2}\right)^m \sum_{j=0}^{r}\frac{1}{(i+2)!}$$

$$b = \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^j$$

The proof is similarly with the proof of Lemma 3.3.

Lemma 4.2. *For e^*_i, $e^{*'}_i$, $i = \overline{1,p}$ above defined, the following inequalities hold:*

$$e^*_i(x) \leq (1 + d_{i1}h)e^*_{i,k} + hd_{i1}\sum_{\substack{j=1\\ j\neq i}}^{p} e^*_{j,k} + hd_{i2}e^{*'}_{i,k} + hd_{i3}\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} \tag{26}$$

$$e^{*'}_i(x) \leq hd_{i4}e^*_{i,k} + hd_{i4}\sum_{\substack{j=1\\ j\neq i}}^{p} e^*_{j,k} + (1 + hd_{i5})e^{*'}_{i,k} + hd_{i5}\sum_{\substack{j=1\\ j\neq i}}^{p} e^{*'}_{j,k} \tag{27}$$

where the constants are defined in Lemma 3.4.

The proof is similarly with the proof of Lemma 3.4.

Using the matrix notation

$$E^*(x) := (e^*_1(x), \dots, e^*_p(x), e^{*'}_1(x), \dots, e^{*'}_p(x))^T$$

$$E^*_k := (e^*_{1,k}, \dots, e^*_{p,k}, e^{*'}_{1,k}, \dots, e^{*'}_{p,k})^T$$

then, the estimations $(27-28)$ can be write in the following form:

$$E^*(x) \leq (I + hA)E^*_k$$

where I and A are defined matrix. Applying Lemma 3.1, we get:

$$||E^*(x)|| \leq (1 + h||A||)||E^*_k|| \leq \left(1 + \frac{||A||}{n}\right)^n ||E^*_0|| \leq$$

$$\leq e^{||A||}||E^*_0|| \leq B^*||E^*_0||$$

where $B^* = e^{||A||}$ is a constant independent of h.

Hence:

$$e_i^*(x) \leq B^*||E_0^*|| \quad (28)$$
$$e_i^{'*}(x) \leq B^*||E_0^*||$$

for $i = \overline{1,p}$.

For $|w_i^{(q+2)}(x) - s_i^{(q+2)}(x)|$, $q = \overline{0,r}$ we obtain, like in (23):

$$|w_i^{(q+2)}(x) - s_i^{(q+2)}(x)| \leq B_{i1}^*||E_0^*||$$

where $B_{i1}^* = pL_i(a+b)B^*$ is a constant independent of h.

Thus we proved the following result:

Theorem 4.1. *Let $(s_1, \dots, s_p)$ be the approximate solution of the problem (1) with the initial conditions $y_i(x_0) = y_{i,0}$, $y_i'(x_0) = y_{i,0}'$, $i = \overline{1,p}$ and let $(w_1, \dots, w_p)$ be the approximate spline solution for the same system, but with the initial conditions: $y_i(x_0) = y_{i,0}^*$, $y_i'(x_0) = y_{i,0}^{*'}$, $i = \overline{1,p}$. Then the inequalities:*

$$|w_i^{(q)}(x) - s_i^{(q)}(x)| \leq B_{i2}||E_0^*||$$

hold for all $x \in [0,1]$, $q = \overline{0, r+2}$, where B_{i2}, $i = \overline{1,p}$, are constants independent of h and $||E_0^*|| = \max_i\{|y_{i,0} - y_{i,0}^*|, |y_{i,0}' - y_{i,0}^{'*}|\}$.

5. Numerical example

. Consider the following system of differential equations, for $p = 2$.

$$\begin{cases} y'' = y + z - e^{-x}, & y(0) = 1, \quad y'(0) = 0 \\ z'' = y + z - e^{x}, & z(0) = 1, \quad z'(0) = 0 \end{cases}$$

The method is tested using this example in the interval $[0,1]$ with step 0.1 , where $r = 0$, $m = 1$. The result are tabulated at $x = 1$.

The analytical solution is:

$$y = e^x - x$$
$$z = e^{-x} + x$$

To test the stability of the method, we solve the above example with the new initial conditions:

$$y(0) = 1.000001, \quad y'(0) = 0.000001$$
$$z(0) = 1.000001, \quad z'(0) = 0.000001$$

The results are:

	The convergence		The stability
e_1	$1.48222E-05$	e_1^*	$3.28131E-06$
e_1'	$1.15977E-04$	$e_1^{*\prime}$	$4.48373E-06$
e_1''	$2.41293E-03$	$e_1^{*\prime\prime}$	$6.53131E-06$
e_2	$1.48222E-05$	e_2^*	$3.28133E-06$
e_2'	$1.15977E-04$	$e_2^{*\prime}$	$4.48370E-06$
e_2''	$2.41293E-03$	$e_2^{*\prime\prime}$	$6.53132E-06$

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THE SCHWARZSCHILD-TYPE TWO-BODY PROBLEM: A TOPOLOGICAL VIEW

FERENC SZENKOVITS AND VASILE MIOC

Abstract. The Schwarzschild-type two-body problem (associated to a force function of the form $A/r + B/r^3, A, B > 0$), which models several problems of nonlinear particle dynamics, is being tackled from the standpoint of topology. The corresponding mechanical system is fully characterized, and the first integrals of energy and angular momentum are pointed out. These integrals are used to settle the invariant manifolds and the bifurcation set for the whole allowed interplay among the field parameters, the total energy level, and the angular momentum. The orbits on each manifold are interpreted in terms of physical motion. Besides recovering motions characteristic to classical models, entirely new types of motion are found.

1. Introduction

The theory of orbits in a force field characterized by a force function of the form $A/r + B/r^3$ (with r = distance of a particle to the field source; $A, B > 0$ constants) constitutes an extensively discussed subject. This theory, which models concrete problems belonging to astrophysics, stellar dynamics, celestial mechanics, astrodynamics, cosmogony, etc., was approached by various methods, both qualitative and (especially) quantitative.

Many authors studied quantitatively the motion in such a field (see, e.g., Brumberg, 1972; Chandrasekhar, 1983; Damour and Schaefer, 1986), generally in a relativistic context, showing that the analytic solution of the problem can be obtained in closed form by means of elliptic functions. But the analytic form of these solutions hides the general geometric properties of the model.

As to the rather few qualitative approaches, they dealt mainly with the regularization of motion equations (see, e.g., Saari, 1974; Belenkii, 1981; Szebehely and

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Bond, 1983; Cid *et al.*, 1983). In addition, the quoted authors used only Sundman-type transformations of time.

Stoica and Mioc, 1997 studied qualitatively the problem for any $A \neq 0$, $B \neq 0$, and provided a complete geometric and physical description of the orbits.

Following Smale's topological program (see Smale, 1970, Iacob, 1973 or Abraham and Marsden, 1981), the aim of this paper is to determine the topological type of the energy-momentum invariant manifolds, determined by the first integrals of energy and angular momentum, and the set of bifurcation points, in whose neighbourhood the topological type of the invariant manifolds is changing. Using the geometrical properties of the invariant manifolds, the types of physical motion are briefly characterized for $A, B > 0$.

2. Basic Equations

It is clear that the Schwarzschild-type two-body problem can be reduced to a central force problem (e.g. Arnold 1976). Within this framework, the motion of the particle is confined to a plane. We shall use polar coordinates (r, θ), and follow the treatment presented by Abraham and Marsden (1981, p.656).

The mechanical system with symmetry which describes the problem is (M, K, V, G), where:

$M = (0, \infty) \times S^1$ is the space of the polar coordinates (r, θ), regarded as a Riemannian manifold endowed with the metric

$$\Big\langle (r_1, \theta_1, \dot{r}_1, \dot{\theta}_1), (r_2, \theta_2, \dot{r}_2, \dot{\theta}_2) \Big\rangle = \dot{r}_1 \dot{r}_2 + r_1 r_2 \dot{\theta}_1 \dot{\theta}_2,$$

dots marking time-differentiation;

K is the kinetic energy of the metric above, whose expression on the cotangent bundle T^*M is

$$K(r, \theta, p_r, p_\theta) = (p_r^2 + p_\theta^2/r^2)/2, \tag{1}$$

p_r, p_θ denoting the momenta;

V is the potential energy, given by

$$V(r, \theta) = -A/r - B/r^3; \tag{2}$$

$G = SO(2) \cong S^1$ is the Lie group that acts on M by rotations ($\cong$ denoting isomorphism). Observe that G acts by isometries and leaves V invariant (cf. Abraham and Marsden 1981).

The Hamiltonian of the system is

(3) $$H(r,\theta,p_r,p_\theta) = (p_r^2 + p_\theta^2/r^2)/2 - A/r - B/r^3.$$

The momentum mapping $J : T^*M \longrightarrow \mathbf{R}$ is given by $J(r,\theta,p_r,p_\theta) = p_\theta$, and is invariant under the action of G.

Consider $\mathbf{x} = (r,\theta) \in M$ and the mapping $J_\mathbf{x} : T^*_\mathbf{x}M \longrightarrow \mathbf{R}$. The expression $J_\mathbf{x} : (p_r,p_\theta) \longmapsto p_\theta$ of this mapping shows that $J_\mathbf{x}$ is surjective for all $\mathbf{x} \in M$. In other words,

$$\Lambda = \{\mathbf{x} \in M \mid J_\mathbf{x} : T^*_\mathbf{x}M \longrightarrow \mathbf{R} \text{ is not surjective}\} = \emptyset.$$

Note that $dJ = dp_\theta$, therefore J has no critical points on T^*M.

The problem admits the first integrals of energy and angular momentum, respectively:

(4) $$H(r,\theta,p_r,p_\theta) = K(r,\theta,p_r,p_\theta) + V(r) = h,$$

(5) $$J(r,\theta,p_r,p_\theta) = p_\theta = C,$$

where h and C stand for the integration constants of energy and angular momentum.

3. Effective Potential Energy

Eliminating p_θ between (3)+(4) and (5), one gets

(6) $$p_r^2 = 2(h - V_C),$$

in which

(7) $$V_C(r) = V(r) + C^2/(2r^2) = -A/r + C^2/(2r^2) - B/r^3$$

denotes the so-called *effective potential energy.*

Settled the constant angular momentum C, one sees by (6) that the real motion is possible only in the domains $V_C(r) \leq h$, where h is a fixed total energy level.

The graph of the function $V_C = V_C(r)$ in different cases, for all values C depending on A, B is plotted in Figure 1.

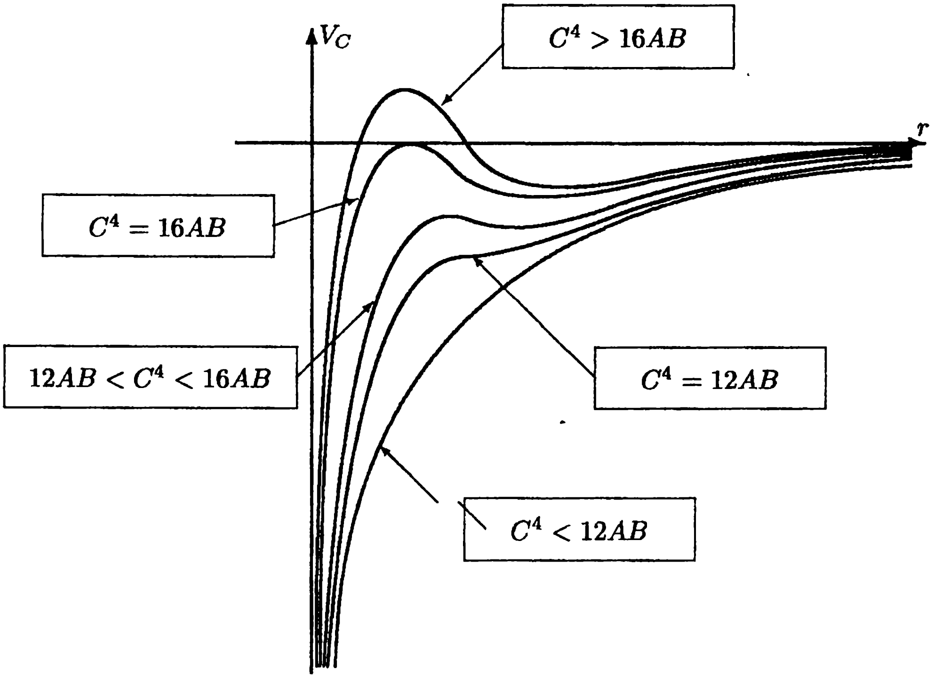

FIGURE 1. The graph of the effective potential energy:

4. Bifurcation Set and Topological Type of Invariant Manifold

To study the motion, we use the invariant manifolds $I_{h,C} = (H \times J)^{-1}(h, C)$, whose defining equations are (4) and (5). Obviously, the topological type of $I_{h,C}$ depends on the condition $V_C(r) \leq h$, and because of the rotational symmetry, each component of $I_{h,C}$ is a product, S^1 being one of the factors. Using the graphs of Figure 1, and observing their significance as regards the allowed domains for r to have real motion, we are able to identify the invariant manifolds diffeomorphic ($\approx$) with $I_{h,C}$ on which the phase curves lie.

To synthesize all possible cases, les us establish and plot the bifurcation set $H \times J$, defined as the set of couples $(h, C) \in \mathbf{R}^2$ for which the energy-momentum mapping $H \times J$ fails to be locally trivial, in other words, those points in whose neighbourhood the topological type of the invariant manifolds is changing (see e.g. Abraham and Marsden 1981).

First we determine the set of critical values $\Sigma'_{H\times J} \subseteq \Sigma_{H\times J}$, defined by the conditions

$$(8) \quad \Sigma'_{H\times J} = \{(h,C) \in \mathbf{R}^2 \mid h = V_C(r), V'_C(r) = 0\} = \bigcup_{i=1}^{2} \{(h,C) \in \mathbf{R}^2 \mid h = (h_{cr})_i\},$$

where

$$(9) \qquad (h_{cr})_i = V_C(r_i), \qquad i = 1, 2;$$

and

$$(10) \qquad r_i = \frac{C^2 + (-1)^{i+1}\sqrt{C^4 - 12AB}}{2A} \qquad i = 1, 2.$$

are the critical points of the effective potential $V_C(r)$, $(V'_C(r) = 0)$.

After some computations we obtain the set of critical values:

$$(11) \quad \Sigma'_{H\times J} = \{(h,C) \in \mathbf{R}^2 \mid 108B^2h^2 + 2C^2(18AB - C^4)h + A^2(16AB - C^4) = 0\}.$$

The graph of this curve is plotted in Figure 2 and has two components, defined for $i = 1, 2$ by:

(12)

$$\{(h,C) \in \mathbf{R}^2 \mid h = (h_{cr})_i\} = \{(h,C) \in \mathbf{R}^2 \mid h = \frac{C^2(C^4 - 18AB) + (-1)^{i+1}(C^4 - 12AB)^{\frac{3}{2}}}{108B^2}\}$$

Note that the complete picture of the set of critical values is symmetric to the $C = 0$ axis, and this symmetry occurs in all the nexts.

The complete bifurcation set is:

$$(13) \qquad \Sigma_{H\times J} = \Sigma'_{H\times J} \cup \{(h,C) \in \mathbf{R}^2 \mid h = 0\}.$$

For different values of the energy and angular momentum constants we found seven cases. The corresponding sets in the (h, C) plane are noted in Figure 2 with (a), (b), ..., (g). In different cases the topological type of the invariant manifolds and the type of orbits in the configuration space is:

(a) If the energy and angular momentum constants are in the domain $\{(h,C) \in \mathbf{R}^2 \mid h \geq 0, h > (h_{cr})_1(C)\}$, then the invariant manifold is diffeomorphic with the reunion of two disjont cylinders ($I_{h,\mu} \approx S^0 \times S^1 \times \mathbf{R}$), and the corresponding orbits in configuration

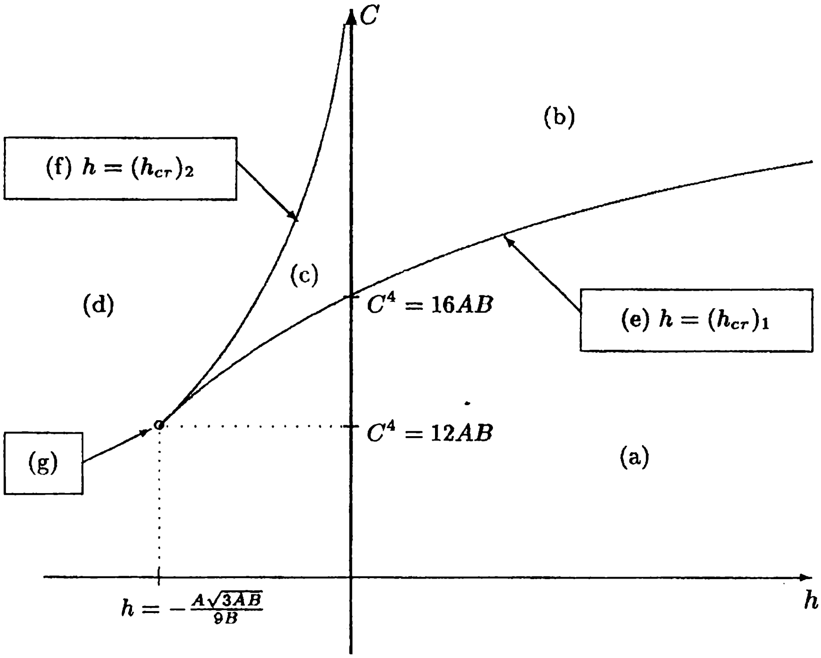

FIGURE 2. The bifurcation set.

space are ejecting from collision and tending to infinity ($0 \to \infty$), or coming from infinity and tending to collisions ($\infty \to 0$).

(b) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid h \geq 0, h < (h_{cr})_1(C)\}$, then $I_{h,C} \approx S^0 \times S^1 \times \mathbf{R}$, but in this case the orbits are comming from infinity and then tending back to infinity ($\infty \to \infty$), or are ejecting from collision and tending back to collision ($0 \to 0$).

(c) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid h < 0, (h_{cr})_2 < h < (h_{cr})_1\}$, then $I_{h,C} \approx (S^1 \times \mathbf{R}) \cup (S^1 \times S^1)$, is the disjont reunion of a cylinder and a torus. The orbits type is ($0 \to 0$), or there are periodic or quasiperiodic orbits (P/QP).

(d) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid (h < 0, h < (h_{cr})_2)$ or $(h < 0, h > (h_{cr})_1)\}$, then $I_{h,C} \approx (S^1 \times \mathbf{R})$ (one cylinder), and the orbits are of the ($0 \to 0$) type.

(e) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid (h = (h_{cr})_1), C^4 > 12AB\}$, then $I_{h,C}$ is diffeomorphic with the reunion of two cylinders which are intersecting in a circle. In this case unstable equilibrium orbits (UE) may exist, or the orbits are of the type ($0 \to$ UE), (UE $\to 0$), ($\infty \to$ UE), (UE $\to \infty$).

(f) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid (h = (h_{cr})_2), C^4 > 12AB\}$ then $I_{h,C} \approx (S^1 \times \mathbf{R}) \cup S^1$ (disjont reunion of a cylinder and a circle), and the orbits are of type $(0 \to 0)$ or (SE), stable equilibrium.

(g) If $C^4 = 12AB, h = (h_{cr})_1 = (h_{cr})_2 = -\frac{A\sqrt{3AB}}{9B}$, then $I_{h,C}$ is homeomorphic (and not diffeomorphic in this case) with $S^1 \times \mathbf{R}$, and the orbits are of type (UE), $(0 \to$ UE) or (UE $\to 0)$.

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