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ON FACTORIZATION OF GROUP

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REZUMAT. - Asupra factorizării unui grup. În această lucrare sunt stabilite câteva rezultate privind posibilitatea descompunerii unui grup finit în produs de două subgrupuri.

1. Introduction. All groups in this paper are finite. Let G be a group and let M be a subgroup of G (in symbols $M \leq G$). G is factorizable over M if there are $H \leq G$ and $K \leq G$ such that

$$G = HK, \ H \cap K = M.$$

In this case H and K furnish a factorization of G over M and we call K a complement in G of H over M.

Assume that $M \leq H \leq G$. In this paper we present three theorems which give criteria for the existence of a complement in G of H over M. Some special cases are also presented.

We shall begin by reviewing the folloving notions:

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Suppose that G is a group and that S is a (left) G-set (i.e. S is a set on which G acts from the left as a group of permutations). For each element $\alpha \in S$ its G-orbit is the subset $G(\alpha) = \{x\alpha \mid x \in G\}$ of S and its stabilizer in G is the subgroup $G_\alpha = \{x \in G \mid x\alpha = \alpha\}$ of G. It is known that $|G(\alpha)| = |G: G_\alpha|$.

2. Results.

THEOREM 1. *Suppose that $H \leq G$ and that S is a G-set. For an element $\alpha \in S$, the following statements are equivalent:*

(i) *The subgroup H and the stabilizer G_α of α in G furnish a factorization of G over the stabilizer H_α of α in H.*

(ii) *The G-orbit $G(\alpha)$ of α coincides with the H-orbit $H(\alpha)$ of α.*

(iii) $|G : G_\alpha| = |H : H_\alpha|$.

Proof. Assume that $G = HG_\alpha$, $H \cap G_\alpha = H_\alpha$. Then for $x \in G$ we have $x = yz$, $y \in H$, $z \in G_\alpha$. It follows, that $G(\alpha) = \{x\alpha \mid x \in G\} = \{(yz)\alpha \mid y \in H, z \in G_\alpha\} = \{y(z\alpha) \mid y \in H, z \in G_\alpha\} = \{y\alpha \mid y \in H\} = H(\alpha)$. Hence (i) implies (ii).

Suppose that $G(\alpha) = H(\alpha)$. Then it is immediate that $|G : G_\alpha| = |H : H_\alpha|$. Thus (ii) implies (iii).

Suppose that $|G : G_\alpha| = |H : H_\alpha|$. Then $|G| = \frac{|H||G_\alpha|}{|H_\alpha|}$. Since $H \cap G_\alpha = H_\alpha$, it follows, that $|G| = \frac{|H||G_\alpha|}{|H \cap G_\alpha|} = |HG_\alpha|$. Therefore $G = H G_\alpha$, $H \cap G_\alpha = H_\alpha$. Hence (iii) implies (i).

THEOREM 2. *Assume that $M \leq K \leq G$. Then for a subgroup H of G the following statements are equivalent:*

(i) $G = H K$, $H \cap K = M$.

(ii) *There exists a G-set S and $\alpha \in S$ such that the G-orbit G_α of α coincide with the H-orbit H_α of α and $G_\alpha = K$, $H_\alpha = M$.*

Proof. It is easily that the set of left cosets of K in G is a G-set with the operation $(x, yK) \to xyK$. Assume that $\alpha = K$.

If $G = HK$, $H \cap K = M$, then for every $x \in G$, $x = yz$, $y \in H$, $z \in K$. It follows, that $xK = yzK = yK$. Therefore the G-orbit $G(K)$ of K coincides with the H-orbit $H(K)$ of K. If $x \in G$, then $xK = K$ iff $x \in K$. Hence the stabilizer G_K of K coincides with K and $K \cap H = M$ is the stabilizer of K in H. Therefore (i) implies (ii).

The implication (ii) implies (i) is an immediate consequence of Theorem 1.

We note the following particular case of Theorem 1.

THEOREM 3. *Let H be a subgroup of the group G. Then for a subset T of G the following statements are equivalent:*

(i) *The subgroup H and the normalizer $N_G(T)$ of T in G furnish a factorization of G over the normalizer $N_H(T)$ of T in H.*

(ii) *If a subset R of G is conjugate to T in G, then R is conjugate to T with an element of H.*

(iii) $|G : N_G(T)| = |H : N_H(T)|$.

Proof. The set $P(G)$ of the subsets of G is a G-set with the operation $(x,Z) \to x^{-1}Zx$, $x \in G$, $Z \in P(G)$. If $\alpha = T \in P(G)$, then $G_\alpha = N_G(T)$, $H_\alpha = N_H(T)$, $G(\alpha) = \{x^{-1}Tx \mid x \in G\}$, $H(\alpha) = \{y^{-1}Ty \mid y \in H\}$. Hence the statements of Theorem 3 are easily from Theorem 1.

3. Applications. We note that if G is a group, then the condition (ii) of Theorem 3 is satisfied in the following particular cases:

a. H is a normal subgroup of G and T is a Sylow subgroup of H.

b. H is a normal subgroup of G and T is a nilpotent Hall subgroup of H ([1], Th. 5.8., p.285).

c. H is a normal solvable subgroup of G and T is a Hall subgroup of H

([1]. Th. 1.8., p.662).

Hence the following applications of Theorem 3 are immediate:

COROLLARY 1 (The Frattini argument). *If H is a normal subgroup of G and T is a Sylow subgroup of H, then*

$$G = H\,N_G(T),\ H \cap N_G(T) = N_H(T).$$

COROLLARY 2. *If H is a normal subgroup of G and T is a nilpotent Hall subgroup of H, then*

$$G = H\,N_G(T),\ H \cap N_G(T) = N_H(T).$$

COROLLARY 3. *If H is a normal solvable subgroup of G and T is a Hall subgroup of G, then*

$$G = H\,N_G(T),\ H \cap N_G(T) = N_H(T).$$

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CONVEXITY AND INTEGRAL OPERATORS

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REZUMAT. - Convexitate şi operatori integrali. În prima parte a lucrării îmbunătăţim un rezultat al lui V. Zanelli şi dăm o demonstraţie uşoară a sa. Apoi considerăm câţiva operatori integrali şi studiem proprietăţile lor relative la conservarea convexităţii de ordin superior. Obţinem astfel o generalizare a rezultatului din [3].

1. A result of V. Zanelli. In [3] it is proved the following property:

LEMMA 0. *Let* f: $[a,\infty) \to R$ *(with* $a > 0$*) be a positive, decreasing, convex function and*

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

For $a \le y$, $k > 0$, $y + k \le x$, *we have the following inequality:*

$$F(y+k) - F(y) - F(x+k) + F(x) \le k\,[f(y) - f(x)]. \tag{2}$$

The proof is based on a rather complicated geometrical method. We want to eliminate some superfluous hypotheses from the enounce and to give a simple proof of it.

LEMMA 1. *Let* f: $[a,b] \to R$ *be a convex function and* F *be defined by*

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(1). *For $a \le y < x < x + k \le b$ we have the inequality* (2).

Proof. Let us consider the auxiliary function:

$$g(t) = t[f(y) - f(x)] - F(y+t) + F(y) + F(x+t) - F(x),\ t \in [0,k].$$

We have

$$g'(t) = [f(x+t) - f(x)] - [f(y+t) - f(y)] \ge 0$$

because, by the convexity of f, the conditions $x > y$ and $x + t > y + t$ give:

$$\frac{f(x+t) - f(x)}{t} \ge \frac{f(x+t) - f(y)}{x+t-y} \ge \frac{f(y+t) - f(y)}{t}.$$

Obviously $g(0) = 0$ so that $g'(t) \ge 0$ gives $g(k) \ge 0$, that is (2).

It can be remarked that we have renounced at the following hypotheses from Lemma 0: $a > 0$, f is positive and decreasing and $y + k \le x$.

2. Convex functions of higher order. We must remind some definitions.

Let f: $[a,b] \to R$ be an arbitrary function. For arbitrary distinct points x_1, x_2, ..., $x_{n+1} \in [a,b]$ the divided differences of the function f are defined by recurrence:

$$[x_1;f] = f(x_1),\ [x_1, \dots, x_{n+1};f] =$$

$$= ([x_1, \dots, x_{n-1}, x_{n+1};f] - [x_1, \dots, x_n;f])/(x_{n+1} - x_n) \qquad (3)$$

The function f is called convex of order n (or shortly n-convex) if:

$$[x_1,\dots,x_{n+2};f] \geq 0,\ \forall\, x_1,\dots,x_{n+2} \in [a,b] \tag{4}$$

where the points are supposed, as in (3), distinct.

For $n = 1$ we get convexity and for $n = 0$ increasing monotony. It is known (see [2]) that a n-convex function, with $n \geq 1$, is continuous on (a,b), so it is integrable on any subinterval from $[a,b]$.

The main result that we will use is the following:

LEMMA 2. *If the function f is n-convex then:*

$$[x_1,\dots,x_{n+1};f] \leq [y_1,\dots,y_{n+1};f],\ \textit{if } x_i \leq y_i,\ \forall i. \tag{5}$$

Proof. From (3) and (4) we deduce that:

$[x_1,\dots,x_{n-1},x_{n+1};f] \geq [x_1,\dots,x_n;f]$ if $x_{n+1} > x_n$. This gives (5), step by step, because the divided differences are symmetric with respect to the points.

3. Arithmetic integral means. To generalize the result from [3] we consider, for a fixed $k > 0$, some operators.

Let $C[a,b]$ be the set of continuous functions on $[a,b]$. For $f \in C[a,b]$ we denote by $F_k(f)$ the function defined by:

$$F_k(f)(x) = \int_x^{x+k} f(t)\,dt,\ \forall\ x \leq b-k.$$

Then we define:

$$A_k(f)(x) = \frac{1}{k} F_k(f)(x)$$

a sort of arithmetic integral mean and:

$$E_k(f)(x) = A_k(f)(x) - f(x)$$

an "excess" function. We get so the operators F_k , A_k and E_k defined on $C[a,b]$ and with values in $C[a, b-k]$. To study some of their properties, we give simple representation formulas for them.

As:

$$F_k(f)(x) = \int_0^k f(x+t)\,dt$$

making the substitution $t = ks$, we have:

$$A_k(f)(x) = \int_0^1 f(x+ks)\,ds$$

and so

$$E_k(f)(x) = \int_0^1 [f(x+ks) - f(x)]\,ds.$$

Thus $E_k(f) \geq 0$ if f is increasing and Lemma 1 asserts in fact that $E_k(f)$ is increasing if f is convex. We generalize this result as follows.

THEOREM 1. *If the function f is n-convex, then $F_k(f)$ and $A_k(f)$ are also n-convex but $E_k(f)$ is $(n-1)$-convex.*

Proof. If $x_1, \ldots, x_{n+2}$ are distinct points from $[a, b-k]$ we have

$$[x_1, \ldots, x_{n+2}; A_k(f)] = \int_0^1 [x_1 + ks, \ldots, x_{n+2} + ks; f]\,ds \geq 0$$

and

$$[x_1, \dots, x_{n+1}; E_k(f)] = \int_0^1 ([x_1+ks, \dots, x_{n+1}+ks; f] - [x_1, \dots, x_{n+1}; f])\, ds \geq 0$$

by Lemma 2. So the affirmation follows for $A_k(f)$ and $E_k(f)$. As $F_k(f) = kA_k(f)$, it is true also for $F_k(f)$.

We remark that the operator E_k can be defined similarly by:

$$E_k(f)(x) = f(x+k) - A_k(f)(x)$$

having the same properties.

Let us define also the operators F, A, E: $C[a,b] \to C[a,b]$ as follows. For f in $C[a,b]$ we put:

$$F(f)(x) = \int_a^x f(t)\, dt$$

$$A(f)(x) = F(f)(x)/(x-a)$$

and

$$E(f)(x) = f(x) - A(f)(x).$$

Using the substitution $t = a + s(x-a)$, we have:

$$A(f)(x) = \int_0^1 f(a + s(x-a))\, ds$$

and

$$E(f)(x) = \int_0^1 [f(x) - f(sx + (1-s)a)]\, ds.$$

Thus, as above, we can prove

THEOREM 2. *If the function f is n-convex then so is also* $A(f)$, *but* $E(f)$ *is* $(n-1)$*-convex.*

The first result is well known (see [1]) as it is also known that under the same hypotheses, $F(t)$ is $(n+1)$-convex.

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ON SOME INEQUALITIES USEFUL IN THE THEORY OF CERTAIN HIGHER ORDER DIFFERENTIAL AND DIFFERENCE EQUATIONS

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REZUMAT. - Asupra unor inegalităţi din teoria ecuaţiilor diferenţiale şi cu diferenţe de ordin superior. În lucrare sunt stabilite noi inegalităţi integrale şi discrete care pot fi folosite în teoria ecuaţiilor diferenţiale de ordin superior şi a ecuaţiilor cu diferenţe.

Abstract. In this paper we establish some new integral and discrete inequalities which can be used as handy tools in the theory of certain new classes of higher order differential and difference equations.

1. Introduction. The fundamental role played by the integral and discrete inequalities in the development of the theory of differential and difference equations is well known. In the literature there are many papers written on integral and discrete inequalities and their applications in the theory of differential and difference equations, see [1-8, 10-12] and the references cited

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therein. Although stimulating research work related to integral and discrete inequalities used in the theory of differential and difference equations has been undertaken in the literature, it appears that there are certain classes of differential and difference equations for which the existing results on such inequalities do not apply directly. This amounts to finding some useful integral and discrete inequalities in order to achieve a diversity of desired goals. Our main objective here is to establish some new integral inequalities and their discrete analogues, which can be used as handy tools in the study of certain new classes of higher order differential and difference equations. We also present some immediate applications to convey the importance of our results to the literature.

2. Statement of results. In what follows we let $R = (-\infty, \infty)$, $R_+ = [0, \infty)$ and $N_0 = \{0, 1, 2, \ldots\}$. For any function $z(m)$, $m \in N_0$, we define the operator Δ by $\Delta z(m) = z(m+1) - z(m)$ and all $m > n$, $m, n \in N_0$, we use the usual conventions $\sum_{s=m}^{n} z(s) = 0$ and $\prod_{s=m}^{n} z(s) = 1$. We use the following notations for simplification of details of presentation. For $t \in R_+$ and some functions $r_i(t)$, $i = 1, 2, \ldots, n$, we set

$$F[t,r_1,r_2,\dots,r_{n-1},r_n] = r_1(t)\int_0^t r_2(s_2)\int_0^{s_2} r_3(s_3)\int_0^{s_3} r_4(s_4)\dots\int_0^{s_{n-2}} r_{n-1}(s_{n-1})\times$$

$$\times\int_0^{s_{n-1}} r_n(s_n)\,ds_n\,ds_{n-1}\dots ds_4\,ds_3\,ds_2$$

$$+\ r_2(t)\int_0^t r_3(s_3)\int_0^{s_3} r_4(s_4)\int_0^{s_4} r_5(s_5)\dots\int_0^{s_{n-2}} r_{n-1}(s_{n-1})\times$$

$$\times\int_0^{s_{n-1}} r_n(s_n)\,ds_n\,ds_{n-1}\dots ds_5\,ds_4\,ds_3\ +$$

$$+\ r_{n-1}(t)\int_0^t r_n(s_n)\,ds_n + r_n(t)\,.$$

For $m \in N_0$ and some functions $r_i(m)$, $i = 1, 2, \dots, n$, we set

$$H[m,r_1,r_2,\dots,r_{n-1},r_n]$$

$$= r_1(m)\sum_{s_2=0}^{m-1} r_2(s_2)\sum_{s_3=0}^{s_2-1} r_3(s_3)\sum_{s_4=0}^{s_3-1} r_4(s_4)\dots\times\sum_{s_{n-1}=0}^{s_{n-2}-1} r_{n-1}(s_{n-1})\sum_{s_n=0}^{s_{n-1}-1} r_n(s_n)$$

$$+\ r_2(m)\sum_{s_3=0}^{m-1} r_3(s_3)\sum_{s_4=0}^{s_3-1} r_4(s_4)\sum_{s_5=0}^{s_4-1} r_5(s_5)\dots\times\sum_{s_{n-1}=0}^{s_{n-2}-1} r_{n-1}(s_{n-1})\sum_{s_n=0}^{s_{n-1}-1} r_n(s_n)\ +$$

$$+\ r_{n-1}(m)\sum_{s_n=0}^{m-1} r_n(s_n)\ +\ r_n(m).$$

Our main result is given in the following theorem.

THEOREM 1. *Let $y(t)$, $a(t)$, $b(t)$, $p_i(t)$, $i = 1, 2, \ldots, n$ be real-valued nonnegative continuous functions defined for $t \in R_+$.*

(i) *If*

$$y(t) \le a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n y]\, ds, \tag{2.1}$$

for all $t \in R_+$, then

$$y(t) \le a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$

$$\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n b]\, d\tau\right) ds, \tag{2.2}$$

for all $t \in R_+$.

(ii) *Let $G(r)$ be a continuous strictly increasing, convex and submultiplicative function for $r \ge 0$, $G(0) = 0$, $\lim_{r\to\infty} G(r) = \infty$, $\alpha(t)$, $\beta(t)$ be positive continuous functions for $t \in R_+$ and $\alpha(t) + \beta(t) = 1$. If*

$$y(t) \le a(t) + b(t)G^{-1}\left(\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n G(y)]\, ds\right), \tag{2.3}$$

for all $t \in R_+$, where G^{-1} is the inverse of G, then

$$y(t) \le a(t) + b(t)G^{-1}\left(\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n \alpha G(a\alpha^{-1})] \times\right.$$

$$\left.\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n \beta G(b\beta^{-1})]\, d\tau\right) ds\right), \tag{2.4}$$

for all $t \in R_+$.

(iii) *Let $W(y)$ be a real-valued continuous, nondecreasing, subadditive and submultiplicative function defined on interval $I = [y_0, \infty)$ and $W(y) > 0$ on (y_0, ∞), $y_0 \geq 0$ is a real constant, $W(y_0) = 0$. If*

$$y(t) \leq a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(y)]\, ds, \tag{2.5}$$

for all $t \in R_+$, then for $0 \leq t \leq t_1$,

$$y(t) \leq a(t) + b(t)\Omega^{-1}[\Omega(c(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, ds], \tag{2.6}$$

where

$$c(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(a)]\, ds, \tag{2.7}$$

$$\Omega(u) = \int_{u_0}^{u} \frac{ds}{W(s)}, \; u \geq u_0 \text{ with } u_0 > y_0, \tag{2.8}$$

Ω^{-1} is the inverse of Ω and $t_1 \in R_+$ be chosen so that

$$\Omega(c(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, ds \in \mathrm{Dom}\,(\Omega^{-1}),$$

for all $t \in R_+$ lying in the interval $0 \leq t \leq t_1$.

We next establish a more general version of Theorem 1 which may be convenient in some applications.

THEOREM 2. *Let $y(t)$, $a(t)$, $b(t)$, $p_i(t)$, $i = 1, 2, \ldots, n$ be real-valued nonnegative continuous functions defined for $t \in R_+$. Let $f\colon R_+^2 \to R_+$ be a continuous function which satisfies the condition*

$$\text{(A)} \qquad 0 \le f(t,u_1) - f(t,u_2) \le k(t,u_2)(u_1 - u_2),$$

for $t \in R_+$ and $u_1 \ge u_2 \ge 0$, where $k : R_+^2 \to R_+$ is a continuous function.

(iv) *If*

$$y(t) \le a(t) + b(t)\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(fy)]\,ds, \tag{2.9}$$

for all $t \in R_+$, where $(fy)(t) = f(t,y(t))$, then

$$y(t) \le a(t) + b(t)\int_0^t F[s,p_1,p_2,\dots,p_{n-.},p_n(fa)] \times$$

$$\times \exp\left(\int_s^t F[\tau,p_1,p_2,\dots,p_{n-1},p_n(ka)b]\,d\tau\right) ds, \tag{2.10}$$

for all $t \in R_+$, where $(ka)(t) = k(t, a(t))$.

(v) *Let G, G^{-1}, α, β be as in* (ii). *If*

$$y(t) \le a(t) + b(t)G^{-1}\left(\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(f(G(y)))]\,ds\right), \tag{2.11}$$

for all $t \in R_+$, where $(f(G(y)))(t) = f(t,G(y(t)))$, then

$$y(t) \le a(t) + b(t)\,G^{-1}\left(\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(f(\alpha G(a\alpha^{-1})))] \times\right.$$

$$\left.\times \exp\left(\int_s^t F[\tau,p_1,p_2,\dots,p_{n-1},p_n(k(\alpha G(a\alpha^{-1})))(\beta G(b\beta^{-1}))]\,d\tau\right) ds\right), \tag{2.12}$$

for all $t \in R_+$, where $(f(\alpha G(a\alpha^{-1})))(t) = f(t,\alpha(t)\,G(a(t)\alpha^{-1}(t)))$,

$(k(\alpha G(a\alpha^{-1})))(t) = k(t,\alpha(t)G(a(t)\alpha^{-1}(t)))$, $(\beta G(b\beta^{-1}))(t) = \beta(t)\,G(b(t)\beta^{-1}(t))$.

(vi) *Let* W, Ω, Ω^{-1} *be as in* (iii). *If*

$$y(t) \le a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(f(W(y)))]\, ds, \tag{2.13}$$

for $t \in R_+$, *where* $(f(W(y)))(t) = f(t, W(y(t)))$, *then for* $0 \le t \le t_2$,

$$y(t) \le a(t) + b(t)\Omega^{-1}[\Omega(\bar{c}(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(k(W(a)))\, W(b)]\, ds], \tag{2.14}$$

where $(k(W(a)))(t) = k(t, W(a(t)))$, $W(b)(t) = W(b(t))$,

$$\bar{c}(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(f(W(a)))]\, ds, \tag{2.15}$$

and $t_2 \in R_+$ *is chosen so that*

$$\Omega(\bar{c}(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(k(W(a)))\, W(b)]\, ds \in \text{Dom}\,(\Omega^{-1}),$$

for all $t \in R_+$ *lying in the interval* $0 \le t \le t_2$.

The discrete analogues of Theorems 1 and 2 are given in the following theorems.

THEOREM 3. *Let* $y(m)$, $a(m)$, $b(m)$, $p_i(m)$, $i = 1, 2, \ldots, n$ *be real-valued nonnegative functions defined for* $m \in N_0$.

(vii) *If*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n y], \tag{2.16}$$

for all $m \in N_0$, *then*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$

$$\times \sum_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n b]], \tag{2.17}$$

for all $m \in N_0$.

(viii) *Let* G, G^{-1} *be as in* (ii) *and* $\alpha(m)$, $\beta(m)$ *be positive functions defined for* $m \in N_0$ *and* $\alpha(m) + \beta(m) = 1$. *If*

$$y(m) \le a(m) + b(m)G^{-1}\left(\sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n G(y)]\right), \tag{2.18}$$

for all $n \in N_0$, *then*

$$y(m) \le a(m) + b(m)G^{-1}\left(\sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n \alpha G(a\alpha^{-1})] \times \right.$$
$$\left. \times \sum_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n \beta G(b\beta^{-1})]]\right), \tag{2.19}$$

for all $m \in N_0$.

(ix) *Let* W, Ω, Ω^{-1} *be as in* (iii). *If*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(y)], \tag{2.20}$$

for all $m \in N_0$, *then for* $0 \le m \le m_1$, m, $m_1 \in N_0$,

$$y(m) \le a(m) + b(m)\Omega^{-1}[\Omega(d(m) + \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)]], \tag{2.21}$$

where

$$d(m) = \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(a)], \tag{2.22}$$

for $m \in N_0$ *and* $m_1 \in N_0$ *is chosen so that*

$$\Omega(d(m)) + \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)] \in \mathrm{Dom}\,(\Omega^{-1}),$$

for $m \in N_0$ *and* $0 \le m \le m_1$.

THEOREM 4. *Let* $y(m)$, $a(m)$, $b(m)$, $p_i(m)$, $i = 1,2,\dots,n$ *be real-valued nonnegative functions defined for* $m \in N_0$. *Let* $h : N_0 \times R_+ \to R_+$ *be a function which satisfies the condition*

$$\text{(B)} \qquad 0 \le h(m,u_1) - h(m,u_2) \le q(m,u_2)(u_1 - u_2),$$

for $m \in N_0$ *and* $u_1 \ge u_2 \ge 0$, *where* $q(m,r)$ *is a real-valued function defined for* $m \in N_0$, $r \in R_+$.

(x) *If*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s,p_1,p_2,\dots,p_{n-1},p_n(hy)], \tag{2.23}$$

for $m \in N_0$, *where* $(hy)(m) = h(m,y(m))$, *then*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s,p_1,p_2,\dots,p_{n-1},p_n(ha)] \times$$
$$\times \sum_{\tau=s+1}^{m-1} [1 + H[\tau,p_1,p_2,\dots,p_{n-1},p_n(qa)b]], \tag{2.24}$$

for $m \in N_0$, *where* $(qa)(m) = q(m,a(m))$.

(xi) *Let* G, G^{-1}, α, β *be as in* (viii). *If*

$$y(m) \le a(m) + b(m)G^{-1}\left(\sum_{s=0}^{m-1} H[s,p_1,p_2,\dots,p_{n-1},p_n(h(G(y)))]\right), \tag{2.25}$$

for $m \in N_0$, *where* $(h(G(y)))(m) = h(m,G(y(m)))$, *then*

$$y(m) \le a(m) + b(m)G^{-1}\left(\sum_{s=0}^{m-1} H[s,p_1,p_2,\dots,p_{n-1},p_n(h(\alpha G(a\alpha^{-1})))] \times\right.$$

$$\times \prod_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n(q(\alpha G(a\alpha^{-1}))) \times (\beta(G(b\beta^{-1})))]] \Bigg), \tag{2.26}$$

for $m \in N_0$, *where* $(h(\alpha G(a\alpha^{-1})))(m) = h(m, \alpha(m) G(a(m)\alpha^{-1}(m)))$,

$(q(\alpha G(a\alpha^{-1})))(m) =$

$= q(m, \alpha(m) G(a(m)\alpha^{-1}(m))), (\beta(G(b\beta^{-1})))(m) = \beta(m) G(b(m)\beta^{-1}(m))$.

(xii) *Let* W, Ω, Ω^{-1} *be as in* (iii). *If*

$$y(m) \le a(m) + b(m) \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n(h(W(y)))], \tag{2.27}$$

for $m \in N_0$, *where* $(h(W(y)))(m) = h(m, W(y(m)))$, *then for* $0 \le m \le m_2$,

$$y(m) \le a(m) + b(m)\Omega^{-1} [\Omega(\bar{d}(m))$$

$$+ \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n(qW(a)))\, W(b))]], \tag{2.28}$$

where $(q(W(a)))(m) = q(m, W(a(m)))$, $(W(b))(m) = W(b(m))$,

$$\bar{d}(m) = \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n(h(W(a)))],$$

and $m_2 \in N_0$ *be chosen so that*

$$\Omega(\bar{d}(m)) + \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n(q(W(a)))(W(b))] \in \text{Dom}(\Omega^{-1}),$$

for $m \in N_0$ *and* $0 \le m \le m_2$.

3. Proofs of Theorems 1 and 2

(i) Define a function $u(t)$ by

$$u(t) = \int_0^t F[s, p_1, p_2, \dots, p_{n-1}, p_n y]\, ds. \tag{3.1}$$

From (3.1) and using $y(t) \leq a(t) + b(t)\,u(t)$ and the fact that $u(t)$ is monotone nondecreasing for $t \in R_+$, we observe that

$$u'(t) \leq F[t, p_1, p_2, \ldots, p_{n-1}, p_n a] + F[t, p_1, p_2, \ldots, p_{n-1}, p_n b]\,u(t). \quad (3.2)$$

From (3.2) we obtain

$$u(t) \leq \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$

$$\times \exp\left(\int_0^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n b]\, d\tau\right) ds. \quad (3.3)$$

Using (3.3) in $y(t) \leq a(t) + b(t)u(t)$ we get the required inequality in (2.2).

(ii) Rewrite (2.3) as

$$y(t) \leq \alpha(t)\, a(t)\, \alpha^{-1}(t) + \beta(t)\, b(t)\, \beta^{-1}(t) \times$$

$$\times\ G^{-1}\left(\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n G(y)]\, ds\right). \quad (3.4)$$

Since G is convex, submultiplicative and strictly increasing, from (3.4) we have

$$G(y(t)) \leq \alpha(t) G(a(t)\alpha^{-1}(t)) + \beta(t) G(b(t)\beta^{-1}(t)) \times$$

$$\times \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n G(y)]\, ds. \quad (3.5)$$

The estimate given in (2.4) follows by first applying the inequality proved in (i) with $a(t) = \alpha(t)G(a(t)\alpha^{-1}(t))$, $b(t) = \beta(t)G(b(t)\beta^{-1}(t))$ and $y(t) = G(y(t))$ and then applying G^{-1} to both sides of the resulting inequality.

(iii) Define a function $u(t)$ by

$$u(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(Y)]\, ds. \tag{3.6}$$

Using $y(t) \le a(t) + b(t)\, u(t)$ on the right side of (3.6) we observe that

$$u(t) \le \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(a + bu)]\, ds$$

$$\le c(t) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)\, W(u)]\, ds. \tag{3.7}$$

For an arbitrary $T \in R_+$, it follows from (3.7) that

$$u(t) \le c(T) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b) W(u)]\, ds,\ 0 \le t \le T. \tag{3.8}$$

Define a function $v(t)$ by

$$v(t) = \varepsilon + c(T) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b) W(u)]\, ds,\ 0 \le t \le T, \tag{3.9}$$

where $\varepsilon > 0$ is an arbitrary small constant. From (3.9) and using the facts that $u(t) \le v(t)$ and $v(t)$ is monotone nondecreasing for $0 \le t \le T$, we observe that

$$v'(t) \le F[t, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, W(v(t)),\ 0 \le t \le T. \tag{3.10}$$

From (2.8) and (3.10) we have

$$\frac{d}{dt}\Omega(v(t)) \le F[t, p_1, p_2, \ldots, p_{n-1}, p_n W(b)],\ 0 \le t \le T. \tag{3.11}$$

Now integrating both sides of (3.11) from 0 to T we have

$$\Omega(v(T)) \le \Omega(\varepsilon + c(T)) + \int_0^T F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, ds. \tag{3.12}$$

Since T is arbitrary, the inequality (3.12) holds for $t = T$, for all $t \in R_+$ and hence from (3.12) we have

$$v(t) \leq \Omega^{-1}[\,\Omega(\varepsilon + c(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, ds\,]. \quad (3.13)$$

Using (3.13) in $u(t) \leq v(t)$ and the fact that $y(t) \leq a(t) + b(t)\,u(t)$ and letting $\varepsilon \to 0$ we get the desired inequality in (2.6). The subdomain of R_+ for t is obvious. This completes the proof of Theorem 1.

(iv) Define a function $u(t)$ by

$$u(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(fy)]\, ds. \quad (3.14)$$

From (3.14) and using the condition (A) and the facts that $y(t) \leq a(t) + b(t)\,u(t)$ and $u(t)$ is monotone nondecreasing for $t \in R_+$, we observe that

$$u'(t) \leq F[t, p_1, p_2, \ldots, p_{n-1}, p_n(f(a + bu))]$$

$$= F[t, p_1, p_2, \ldots, p_{n-1}, p_n\{(f(a + bu)) - (fa) + (fa)\}]$$

$$\leq F[t, p_1, p_2, \ldots, p_{n-1}, p_n(fa)]$$

$$+ F[t, p_1, p_2, \ldots, p_{n-1}, p_n(ka)b]\, u(t). \quad (3.15)$$

From (3.15) we obtain

$$u(t) \leq \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(fa)] \times$$

$$\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n(ka)b]\, d\tau\right) ds. \quad (3.16)$$

Using (3.16) in $y(t) \leq a(t) + b(t)\,u(t)$ we get the desired inequality in (2.10).

The proofs of (v) and (vi) can be completed by following the same arguments as in the proofs of (ii), (iii) and (iv) given above with suitable modifications. Here we omit the details.

4. Proofs of Theorems 3 and 4. Since the proofs resemble one another, we give the details for (vii) only, the proofs of (viii)-(xii) can be completed by following the proofs of similar results given in [7, ', 11] and closely looking at the proofs of (i)-(iv) and (vii).

(vii) Define a function $z(m)$ by

$$z(m) = \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n y]. \tag{4.1}$$

From (4.1) and using $y(m) \le a(m) + b(m)z(m)$ and the fact that $z(m)$ is monotone nondecreasing for $m \in N_0$, we observe that

$$z(m+1) - z(m) = H[m, p_1, p_2, \dots, p_{n-1}, p_n y]$$

$$\le H[m, p_1, p_2, \dots, p_{n-1}, p_n a] + H[m, p_1, p_2, \dots, p_{n-1}, p_n b]\, z(m). \tag{4.2}$$

The inequality (4.2) implies the estimate (see [7])

$$z(m) \le \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n a] \times$$

$$\times \prod_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n b]]. \tag{4.3}$$

The required inequality in (2.17) now follows by using (4.3) in $y(m) \le a(m) + b(m)z(m)$.

5. Some applications. In this section we present some applications of our results to obtain bounds on the solutions of certain higher order differential and difference equations for which the inequalities available in the existing literature do not apply directly.

Let $p_i(t)$, $0 \leq i \leq n$ be positive continuous functions defined for $t \in R_+$. We define the differential operators L_i, $0 \leq i \leq n$ by

$$L_0 x(t) = \frac{x(t)}{p_0(t)}, \ L_i x(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1} x(t), \ 1 \leq i \leq n.$$

Consider the nonlinear differential equation of the form

$$L_n x(t) = g(t, L_0 x(t), L_1 x(t), \ldots, L_{n-2} x(t), L_{n-1} x(t)), \tag{5.1}$$

with the initial conditions

$$L_{i-1} x(0) = 0, \ i = 1, 2, \ldots, n, \tag{5.2}$$

where $g : R_+ \times R^n \to R$ is a continuous function. For the study of (5.1)-(5.2), see [9] and the references cited therein.

We first convert the problem (5.1)-(5.2) into an equivalent integral equation. Let $y(t) = L_n x(t)$, then we have

$$L_{n-1} x(t) = \int_0^t p_n(s_n)\, y(s_n)\, ds_n, \tag{5.3}$$

$$L_{n-2} x(t) = \int_0^t p_{n-1}(s_{n-1}) \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1}, \tag{5.4}$$

$$L_1 x(t) = \int_0^t p_2(s_2) \int_0^{s_2} p_3(s_3) \int_0^{s_3} p_4(s_4) \ldots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \ldots ds_4\, ds_3\, ds_2, \tag{5.5}$$

$$L_0 x(t) = \int_0^t p_1(s_1) \int_0^{s_1} p_2(s_2) \int_0^{s_2} p_3(s_3) \ldots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \ldots ds_3\, ds_2\, ds_1. \tag{5.6}$$

Consequently the problem (5.1)-(5.2) is equivalent to the following integral equation

$$y(t) = g\Bigg(t, \int_0^t p_1(s_1) \int_0^{s_1} p_2(s_2) \int_0^{s_2} p_3(s_3) \ldots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \ldots ds_3\, ds_2\, ds_1,$$

$$\int_0^t p_2(s_2) \int_0^{s_2} p_3(s_3) \int_0^{s_3} p_4(s_4) \ldots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \ldots ds_4\, ds_3\, ds_2,$$

$$\int_0^t p_{n-1}(s_{n-1}) \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1},$$

$$\int_0^t p_n(s_n)\, y(s_n)\, ds_n\Bigg). \tag{5.7}$$

Suppose that the function g in (5.1) satisfies

$$|g(t, w_0, w_1, \ldots, w_{n-2}, w_{n-1})| \leq a(t) + b(t)[\,|w_0| + |w_1| + \ldots + |w_{n-2}| + |w_{n-1}|\,], \tag{5.8}$$

where $a(t)$ and $b(t)$ are real-valued nonnegative continuous functions defined for $t \in R_+$. From (5.7) and (5.8) we observe that

$$|y(t)| \leq a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n|y|]\, ds. \tag{5.9}$$

Now an application of inequality proved in (i) Theorem 1 yields

$$|y(t)| \leq Q(t), \tag{5.10}$$

where

$$Q(t) = a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times \times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n b]\, d\tau\right) ds.$$

Now using (5.10) in (5.3)-(5.6) we get the bounds on $|L_{n-1}x(t)|$, $|L_{n-2}x(t)|$, ..., $|L_1x(t)|$, $|L_0x(t)|$ in terms of the known quantities. Thus by using the definition of $L_0x(t)$, we get the bound on the solution $x(t)$ of (5.1)-(5.2) in terms of known quantities.

Further, it is be noted that the inequality given in (vii) can be used to obtain upper bound on the solution of the nonlinear difference equation of the form

$$E_n x(m) = g(m, E_0 x(m), E_1 x(m), \dots, E_{n-2} x(m), E_{n-1} x(m)), \qquad (5.11)$$

with the initial conditions

$$E_{i-1} x(0) = 0, \; i = 1, 2, \dots, n, \qquad (5.12)$$

where g is a real-valued function defined on $N_0 \times R^n$, the operators E_j, $0 \le j \le n$ are defined by

$$E_0 x(m) = \frac{x(m)}{p_0(m)}, \; E_j x(m) = \frac{1}{p_j(m)} \Delta E_{j-1} x(m), \; j = 1, 2, \dots, n.$$

$p_j(m)$, $0 \le j \le n$ are positive functions defined for $m \in N_0$. By letting $z(m) = E_n x(m)$ and converting the problem (5.11)-(5.12) into an equivalent form of sum-difference equation and following the same arguments as explained above for the problem (5.1)-(5.2) we get the bound on the solution $x(m)$ of (5.11)-(5.12).

We also note that the inequalities established in (iv) and (x) can be used to obtain bounds on the solutions of the following more general nonlinear higher order differential and difference equations of the forms:

$$L_n x(t) = g(t, f(t, L_0 x(t)), f(t, L_1 x(t)), \dots, f(t, L_{n-2} x(t)), f(t, L_{n-1} x(t))), \qquad (5.13)$$

$$L_{i-1} x(0) = 0, \; i = 1, 2, \dots, n, \qquad (5.14)$$

and

$$E_n x(m) = g(m, h(m, E_0 x(m)), h(m, E_1 x(m)),$$

$$\dots, h(m, E_{n-2}x(m)), h(m, E_{n-1}x(m))), \tag{5.15}$$

$$E_{i-1}x(0) = 0, \; i = 1, 2, \dots, n, \tag{5.16}$$

respectively, under some suitable conditions on the functions involved in (5.13) and (5.15). Since the details of these results are very close to that of given above with suitable modifications, and hence we do not discuss it here.

In concluding this paper we note that there are many possible applications of the inequalities established in this paper to certain classes of higher order differential and difference equations, but those presented here are sufficient to convey the importance of our results to the literature. Various other applications of these inequalities will appear elsewhere.

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SECOND-ORDER DIFFERENTIAL SUBORDINATIONS IN THE HALF-PLANE

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REZUMAT. - Subordonări diferenţiale de ordinul al doilea în semiplan. În lucrare, folosind subordonările diferenţiale, se obţin proprietăţi ale funcţiilor olomorfe în semiplanul complex care satisfac condiţia de normalizare $f(z)-z\to 0$ pentru $z\to\infty$.

Let Δ denote the upper half - plane

$$\Delta = \{z \in \mathbf{C} / \operatorname{Im} z > 0\}$$

and let $A(\Delta)$ denote the class of functions f which are holomorphic in Δ and have the normalization

$$\lim_{\Delta \ni z \to \infty} [f(z) - z] = 0$$

In this paper, using differential subordinations in the half - plane [2], we obtain some properties concerning functions of the class $A(\Delta)$.

DEFINITION 1 [2]. Let $f,g : \Delta \to \mathbf{C}$ be holomorphic functions in Δ. The function f is *subordinate* to the function g in Δ ($f \prec g$) if there is an holomorphic function φ: $\Delta \to \Delta$ such that $\lim\limits_{\Delta \ni z \to \infty} [\varphi(z) - z] = 0$ and

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$f(z) = g(\varphi(z))$, for all $z \in \Delta$.

THEOREM 1. *Let* $f \in A(\Delta)$, $g \in A(\Delta)$ *and* g *is univalent in* Δ. *Then the function* f *is subordinate to the function* g *in* Δ *if and only if* $f(\Delta) \subset g(\Delta)$.

Proof. If $f \prec g$ then using Definition 1 and Schwarz's Lemma for the upper half-plane [3], [4], it results $f(\Delta) \subset g(\Delta)$.

If $f(\Delta) \subset g(\Delta)$ then, using the univalence of the function g, we obtain that $g^{-1}: g(\Delta) \to \Delta$ is an holomorphic function in Δ and we can define the function $\varphi: \Delta \to \Delta$, $\varphi(z) = g^{-1}(f(z))$, $z \in \Delta$. We have

$$|\varphi(z) - z| = |g^{-1}(f(z)) - z| \leq |g^{-1}(f(z)) - f(z)| + |f(z) - z|, \ z \in \Delta$$

and since $\lim\limits_{\Delta \ni z \to \infty} [f(z) - z] = \lim\limits_{\Delta \ni z \to \infty} [g(z) - z] = 0$ it follows that $\lim\limits_{\Delta \ni z \to \infty} [\varphi(z) - z] = 0$.

DEFINITION 2 [2]. We denote by $Q(\Delta)$ the set of functions $q \in A(\Delta)$ which are holomorphic and injective on $\overline{\Delta} - E(q)$, where $E(q) = \{\zeta \in \partial\Delta /$ $\lim\limits_{z \in \zeta} q(z) = \infty\}$, and also $q'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \backslash E(q)$.

DEFINITION 3 [2]. Let Ω be a set in $\mathbf{C}$ and let $q \in Q(\Delta)$. We define the class of *admissible* functions $\psi_\Delta[\Omega, q]$ to be those functions $\psi: \mathbf{C}^3 \times \Delta \to \mathbf{C}$ that satisfy the following admissibility condition:

$$\begin{cases} \psi(r, s, t; z) \notin \Omega, \text{ when } r = q(\zeta), \ s = m \cdot q'(\zeta) \\ \operatorname{Im} \dfrac{t}{s} \geq m \cdot \operatorname{Im} \dfrac{q''(\zeta)}{q'(\zeta)} \text{ and } z \in \Delta \text{ for } \zeta \in \partial\Delta \backslash E(q), \ m \in \mathbf{R}_+ \end{cases} \tag{1}$$

We shall need the following theorem to prove our results:

THEOREM 2[2]. *Let* $\psi \in \psi_\Delta[\Omega, q]$ *and* $p\colon \Delta \to \mathbf{C}$ *be an holomorphic function in* Δ *such that there exists* $a \geq 0$ *with* $p(\Delta_a) \subset q(\Delta)$, *where* $\Delta_a = \{z \in \mathbf{C} / \operatorname{Im} z > a\}$. *If*

$$\psi(p(z), p'(z), p''(z); z) \in \Omega, \text{ for all } z \in \Delta \tag{2}$$

then $p \prec q$.

Remark 1. If $\lim\limits_{\Delta \ni z \to \infty} [p(z) - z] = \lim\limits_{\Delta \ni z \to \infty} [q(z) - z] = 0$ then we obtain that there exists $a \geq 0$ such that $p(\Delta_a) \subset q(\Delta)$. Thus, the condition "$p\colon \Delta \to \mathbf{C}$ be an holomorphic function in Δ such that there exists $a \geq 0$ with $p(\Delta_a) \subset q(\Delta)$" from Theorem 2 can be replaced by $p \in A(\Delta)$.

Let Ω be a set in Δ and let $q(z) = z$, $z \in \Delta$. We will obtain some applications of the Theorem 2 corresponding to this particular Ω and q.

THEOREM 3. *Let* $p \in A(\Delta)$ *and let* $\gamma \in \mathbf{R}$, $\gamma \leq 0$. *If*

$$\operatorname{Im}\left[p(z) + \gamma \cdot \frac{p''(z)}{p'(z)}\right] > 0,\ z \in \Delta \tag{3}$$

then $\operatorname{Im} p(z) > 0$.

Proof. If we let $\psi(r, s, t, z) = r + \gamma \cdot t/s$ then the conclusion will follow from Theorem 2 we show that $\psi \in \psi_\Delta[\Omega, q]$, where $\Omega = \Delta$ and $q(z) = z$. This follows from Definition 3 since

$$\operatorname{Im}\psi(r,s,t;z) = \operatorname{Im}(\zeta+\gamma\cdot t/s) = \operatorname{Im}\zeta+\gamma\cdot\operatorname{Im} t/s \le 0 \text{ for } r=\zeta\in\partial\Delta,$$

$\operatorname{Im} t/s \ge 0$ and $\gamma \le 0$. Hence $\psi \in \psi_\Delta[\Omega,q]$, $p \prec q$ and $\operatorname{Im} p(z) > 0$.

THEOREM 4. *Let* $p \in A(\Delta)$ *and let* $\alpha,\beta \in \mathbb{R}$. *If*

$$\operatorname{Im}\left[\alpha p(z) + \beta\frac{p'(z)}{p(z)}\right] > 0,\ z \in \Delta$$

then $\operatorname{Im} p(z) > 0$.

Proof. If we let $\psi(r,s,t;z) = \alpha r + \beta\cdot s/r$ then we have $\operatorname{Im}\psi(r,s,t;z) =$ $= \alpha\operatorname{Im}\zeta + \beta\cdot m\operatorname{Im} 1/\zeta = 0$ for $r=\zeta\in\partial\Delta$, $s=m\in\mathbb{R}_+$ and $\alpha,\beta\in\mathbb{R}$. Hence $\psi \in \psi_\Delta[\Omega,q]$, $p \prec q$ and $\operatorname{Im} p(z) > 0$.

COROLLARY. *Let* $f: \Delta \to \mathbb{C}$ *be an holomorphic function in* Δ *such that* $-\dfrac{f'}{f}$ *satisfies the conditions of Theorem 4 and* $\alpha \in \mathbb{R}$. *If*

$$\operatorname{Im}\left[(1-\alpha)\frac{f'(z)}{f(z)} + \alpha\frac{f''(z)}{f'(z)}\right] > 0,\ z \in \Delta \tag{5}$$

then $\operatorname{Im}\dfrac{f'(z)}{f(z)} < 0,\ z \in \Delta.$

Remark 2. A function $f \in A(\Delta)$, $f(z) \ne 0$, $z \in \Delta$ is starlike in the half-plane Δ if and only if

$$\operatorname{Im}\frac{f'(z)}{f(z)} < 0,\ z \in \Delta.$$

Using the Corollary, we obtain that a function which satisfies the condition 5 is a starlike function in Δ.

THEOREM 5. *Let* $p \in A(\Delta)$ *and* $\alpha, \beta, \gamma \in \mathbf{R}$, $\gamma \leq 0$. *If*

$$\operatorname{Im}\left[\alpha p(z) + \beta \frac{p'(z)}{p(z)} + \gamma \frac{p''(z)}{p'(z)}\right] > 0,\ z \in \Delta \tag{6}$$

then $\operatorname{Im} p(z) > 0$.

Proof. If we let $\psi(r,s,t;z) = \alpha r + \beta \cdot s/r + \gamma \cdot t/s$ then we have

$\operatorname{Im} \psi(r,s,t;z) = \alpha \operatorname{Im} \zeta + \beta \cdot m \operatorname{Im} 1/\zeta + \gamma \cdot \operatorname{Im} t/s \leq 0$ for $r = \zeta \in \partial\Delta$, $s = m \in \mathbf{R}_+$, $\operatorname{Im} t/s \geq 0$, $\alpha, \beta, \gamma \in \mathbf{R}$, $\gamma \leq 0$. Hence $\psi \in \psi_\Delta[\Omega, q]$, $p \prec q$ and $\operatorname{Im} p(z) > 0$.

Remark 3.

i) If $\gamma = 0$ then Theorem 5 reduces to Theorem 4.

ii) If $\alpha = 1$ and $\beta = 0$ then Theorem 5 reduces to Theorem 3.

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JACK'S, MILLER'S AND MOCANU'S LEMMA FOR HOLOMORPHIC MAPPINGS DEFINED ON DOMAINS WITH DIFFERENTIABLE BOUNDARY OF CLASS C^2

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REZUMAT. - Lema lui Jack-Miller-Mocanu pentru aplicaţii olomorfe pe domenii cu frontieră de clasă C^2. În acest articol vom prezenta varianta n-dimensională a lemei Jack-Miller-Mocanu pentru aplicaţii olomorfe definite pe domenii din $\mathbf{C}^n$ ce au frontieră de clasă C^2. De asemenea vom prezenta şi interpretări geometrice ale rezultatului.

1. Introduction. In several papers [4,5] S.S. Miller and P.T. Mocanu gave the following generalization of the one dimensional Jack's lemma [2] and used it as a basic tool in developing the theory of admissible functions.

LEMMA (Jack-Miller-Mocanu). *Let $f: D \to \mathbf{C}$ be a holomorphic function with $f(0) = 0$ and $f \neq 0$. If $|f(z_0)| = \max_{|z| \le |z_0|} |f(z)|$, $z_0 \in D = \{z \in \mathbf{C} | \; |z| < 1\}$ then there exists a real number $m \ge 1$ such that:*

(i) $\dfrac{z_0 f'(z_0)}{f(z_0)} = m$ *and*

(ii) $\mathrm{Re}\, \dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \ge m.$

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In a previous paper we extend this result to the case of holomorphic mappings defined on the unit ball of $\mathbb{C}^n$. Since in several complex variables the Riemann mapping theorem fails to be true the purpose of this paper is to study an analogous problem to that studied in [2] for holomorphic mappings defined on arbitrary domains. Also we shall give geometric interpretations of result.

We let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n)'$, with the euclidian inner product $<z, w> = \sum_{j=1}^{n} z_j \bar{w}_j$ and the norm $\|z\| = (<z, z>)^{1/2}$.

Vector and matrices marked with the symbol ' and * denote the transposed and the transposed conjugate vector or matrix, respectively.

We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, with the standard operator norm:

$$\|A\| = \sup \{\|Az\| : \|z\| \leq 1\}, \; A \in \mathcal{L}(\mathbb{C}^n).$$

The class of holomorphic mappings $f(z) = (f_1(z), \ldots, f_n(z))'$ from D ($D \subseteq \mathbb{C}^n$ domain) into $\mathbb{C}^n$ is denoted by $\mathcal{H}(D)$.

We denote by $Df(z)$ and $D^2f(z)$ the first and the second Fréchet derivatives of f at z.

We say that $f \in \mathcal{H}(D)$ is locally biholomorphic (locally univalent) at $z \in D$ if f has a local holomorphic inverse at z, or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j, k \le n} \quad \text{is nonsingular.}$$

The open set $D \subseteq \mathbf{C}^n$ is said to have differentiable boundary bD of class C^2, at the point $\dot{z} \in bD$ if there are an open neighborhood U of $\dot{z}$ and a real valued function $\varphi \in C^2(D)$ with the following properties:

$$U \cap D = \{z \in U: \varphi(z) < 0\} \tag{1}$$

$$\frac{\partial \varphi}{\partial z}(z) = \left(\frac{\partial \varphi}{\partial z_1}(z), \ldots, \frac{\partial \varphi}{\partial z_n}(z)\right)' \neq 0 \text{ for } z \in U. \tag{2}$$

bD is of class C^2 if it is of class C^2 at every $z \in bD$.

Notice that (1) and (2) imply

$$U \cap bD = \{z \in U: \varphi(z) = 0\} \text{ and } U - \bar{D} = \{z \in U: \varphi(z) > 0\}. \tag{3}$$

Any function $\varphi \in C^2(U)$ which satisfies (1) and (2) is called a (local) defining function for bD at $\dot{z}$.

For a real valued function $\varphi \in C^2(U)$ ($U \in \mathbf{C}^n$) we define:

$$\frac{\partial^2 \varphi}{\partial z^2}(z) = \left(\frac{\partial^2 \varphi(z)}{\partial z_j \partial z_k}\right)_{1 \le j, k \le n} \tag{4}$$

and

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}}(z) = \left(\frac{\partial^2 \varphi(z)}{\partial z_j \partial \bar{z}_k}\right)_{1 \le j, k \le n} \tag{5}$$

2. Main result.

THEOREM. *Let D be a bounded domain in $\mathbf{C}^n$ with $0 \in D$. Suppose $f \in C(\bar{D}) \cap \mathcal{H}(D \cup \{\dot{z}\})$, $f(0)=0, f \neq 0$ and $Df(\dot{z})$ is nonsingular where $\dot{z} \in bD$ is defined by:*

$$\|f(\dot{z})\| = \max_{z \in \bar{D}} \|f(z)\|.$$

If D has differentiable boundary bD of class C^2 at the point $\dot{z} \in bD$ with the locally defining function φ then there exists a real positive number m such that:

$$\text{(i)} \qquad ((Df(\dot{z}))^*)^{-1}\left(\overline{\frac{\partial\varphi}{\partial z}}(\dot{z})\right) = mf(\dot{z}) \qquad (6)$$

and

$$\text{(ii)} \qquad \frac{w^* \frac{\partial^2\varphi}{\partial z\,\partial \bar{z}} w + \mathrm{Re}\left(w' \frac{\partial^2\varphi}{\partial z^2} w - \left(\frac{\partial\varphi}{\partial z}(\dot{z})\right)' (D^2 f(\dot{z}))^{-1} D^2 f(\dot{z})(w,w)\right)}{\|Df(\dot{z})w\|^2} \geq m \qquad (7)$$

for all $w \in \mathbf{C}^n\setminus\{0\}$ which satisfy $\mathrm{Re} < w, \overline{\frac{\partial\varphi}{\partial z}}(\dot{z}) > = 0$.

Proof. For $z = (z_1, \ldots, z_n)' \in \mathbf{C}^n$, each coordinate z_j can be written as $z_j = a_j + i a_{j+n}$, with $a_j, a_{n+j} \in \mathbf{R}$

The mapping $z \to (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n})' \in \mathbf{R}^{2n}$ establishes an $\mathbf{R}$ linear isomorphism between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$, i.e. we obtain the natural identification between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$.

By using the weak maximum modulus theorem [1] we obtain that $\mathring{z} \in bD$ and $\mathring{z} \in \bar{D}$ is a point of local conditional maximum of the function $\|f(z)\|$ under the condition $z \in bD$.

Since $Df(\mathring{z})$ is nonsingular and D has a differentiable boundary at $\mathring{z} \in bD$ it follows that there exists an open neighborhood U of $\mathring{z}$ and φ a real function such that (1), (2) and (3) hold and also f is injective on U.

Next, we shall use method of Lagrange's multipliers.

Let $F\colon (\mathring{a}_1 - \varepsilon, \mathring{a}_1 + \varepsilon) \times \ldots \times (\mathring{a}_{2n} - \varepsilon, \mathring{a}_{2n} + \varepsilon) \to \mathbf{R}$

$$F(a_1, \ldots, a_{2n}) = \sum_{i=1}^{2n} |f_i(a_1, \ldots, a_{2n})|^2 - \lambda\varphi(a_1, \ldots, a_{2n}) \tag{8}$$

where $\lambda \in \mathbf{R}$ and $\varepsilon > 0$ is sufficiently small so that

$$(\mathring{a}_1 - \varepsilon, \mathring{a}_1 + \varepsilon) \times \ldots \times (\mathring{a}_{2n} - \varepsilon, \mathring{a}_{2n} + \varepsilon) \subset U.$$

Since $(\mathring{a}_1, \ldots, \mathring{a}_{2n})$ is a point of local maximum for the function $\|f(a_1, \ldots, a_{2n})\|^2$ under the condition $\varphi(a_1, \ldots, a_{2n}) = 0$ we obtain:

$$\frac{\partial F}{\partial a_i}(\mathring{a}_1, \ldots, \mathring{a}_{2n}) = 0,\ i \in \{1, \ldots, 2n\} \tag{9}$$

and

$$d^2F(\mathring{a}_1, \ldots, \mathring{a}_{2n})(t, t) \leq 0 \text{ for all } t \in \mathbf{R}^{2n}\setminus\{0\} \tag{10}$$

which satisfy $\sum_{i=1}^{2n} t_i \frac{\partial\varphi}{\partial a_i}(\mathring{a}_1, \ldots, \mathring{a}_{2n}) = 0.$

A simple calculation yields:

$$\frac{\partial F}{\partial a_j}(\mathring{a}_1,\ldots,\mathring{a}_{2n}) = \sum_{l=1}^{n}\left(\frac{\partial f_l}{\partial z_j}(\mathring{z})\bar{f}_l(\mathring{z}) + f_l(\mathring{z})\frac{\partial \bar{f}_l}{\partial \bar{z}_j}(\mathring{z})\right) - \lambda\frac{\partial \varphi}{\partial a_j}(\mathring{z}) = 0, \tag{11}$$

for $j \in \{1, \ldots, n\}$ and

$$\frac{\partial F}{\partial a_{j+n}}(\mathring{a}_1,\ldots,\mathring{a}_{2n}) = i\sum_{l=1}^{n}\left(\frac{\partial f_l}{\partial z_j}(\mathring{z})\bar{f}_l(\mathring{z}) - f_l(\mathring{z})\frac{\partial \bar{f}_l}{\partial \bar{z}_j}(\mathring{z})\right) - \lambda\frac{\partial \varphi}{\partial a_{j+n}}(\mathring{z}) = 0 \tag{12}$$

for $j \in \{1, \ldots, n\}$.

Since

$$\frac{\partial F}{\partial a_j}(\mathring{a}_1,\ldots,\mathring{a}_{2n}) - i\frac{\partial F}{\partial a_{j+n}}(\mathring{a}_1,\ldots,\mathring{a}_{2n}) = 0 \text{ for } j \in \{1,\ldots,n\},$$

we easily obtain:

$$\sum_{l=1}^{n}\frac{\partial f_l}{\partial z_j}(\mathring{z})\bar{f}_l(\mathring{z}) = \lambda\frac{\partial \varphi}{\partial z_j}(\mathring{z}) \text{ for all } j \in \{1,\ldots,n\}. \tag{13}$$

From the relations (13) we get:

$$((Df(\mathring{z}))^{*})f(\mathring{z}) = \lambda\overline{\frac{\partial \varphi}{\partial z}}(\mathring{z}). \tag{14}$$

By using the fact that $Df(\mathring{z})$ is nonsingular we obtain:

$$((Df(\mathring{z}))^{*})^{-1}\left(\overline{\frac{\partial \varphi}{\partial z}}(\mathring{z})\right) = mf(\mathring{z}) \tag{15}$$

where by m we denote the real number $\frac{1}{\lambda}$ (Indeed, if $\lambda = 0$ we obtain $f(\mathring{z}) = 0$ which contradicts the assumption $f \neq 0$).

In order to prove (i) of the Theorem it remains to show that m is positive.

We now let ψ: $f(U) \to \mathbf{R}$ defined by

$$\psi(w) = \varphi((f\,|\,U)^{-1}(w))$$

If t is a small enough real positive number we have that $(1+t)f(\dot z) \in f(U)$ and $(1+t)f(\dot z) \notin f(D)$. Hence $\psi((1+t)f(\dot z)) > 0$.

A simple calculations yields:

$$0 \le \lim_{\substack{t\to 0\\ t>0}} \frac{\psi((1+t)f(\dot z)) - \psi(f(\dot z))}{t} = \sum_{i=1}^{n} \frac{\partial\psi}{\partial w_i}(f(\dot z))f_i(\dot z) =$$

$$= \sum_{i,k=1}^{n} f_i(\dot z)\frac{\partial(f^{-1})k}{\partial w_i}(f(\dot z))\frac{\partial\varphi(\dot z)}{\partial z_k} = \sum_{i=1}^{n} f_i(\dot z)\frac{1}{\lambda}\overline{f_i(\dot z)} = \frac{1}{\lambda}\|f(\dot z)\|^2.$$

Hence $\lambda > 0$ and in consequence $m > 0$ too.

The second differential of F at the point $\dot z$ is negative semidefinite.

Straightforward calculations given us:

$$d^2F(\dot z)(t,t) = \sum_{i,j,k=1}^{n} \frac{\partial^2 f_i(\dot z)}{\partial z_j \partial z_k}\overline{f_i(\dot z)}(t_j + it_{j+n})(t_k + it_{k+n}) +$$

$$+ \sum_{i,j,k=1}^{n} \frac{\partial^2 \overline{f}_i(\dot z)}{\partial \overline{z}_j \partial \overline{z}_k} f_i(\dot z)(t_j - it_{j+n})(t_k - it_{k+n}) +$$

$$+ \sum_{i,j,k=1}^{n} \frac{\partial f_i}{\partial z_j}\frac{\partial \overline{f}_i}{\partial \overline{z}_k}(t_j + it_{j+n})(t_k - it_{k+n}) +$$

$$+ \sum_{i,j,k=1}^{n} \frac{\partial f_i}{\partial z_k}\frac{\partial \overline{f}_i}{\partial \overline{z}_j}(t_k + it_{k+n})(t_j - it_{j+n}) -$$

$$- \frac{1}{m}\sum_{i,j=1}^{2n} \frac{\partial\varphi}{\partial a_j \partial a_k} t_j t_k$$

for $t \in \mathbb{R}^{2n}\setminus\{0\}$ with $\sum_{i=1}^{2n} t_i \frac{\partial\varphi}{\partial a_i}(\dot z) = 0.$

If we note $w_j = t_j + it_{j+n}$, $j \in \{1,\dots,n\}$, $w = (w_1,\dots,w_n)$, and use (4) and (5) then the above inequality becomes:

$$2\,\mathrm{Re}\,((f(\mathring{z}))^{*}D^{2}f(\mathring{z})(w,w)) + 2\,\|Df(\mathring{z})w\|^{2} -$$
$$- \frac{2}{m} w^{*} \frac{\partial^{2}\varphi}{\partial z\,\partial \bar{z}} w - \frac{2}{m}\mathrm{Re}\, w' \frac{\partial^{2}\varphi}{\partial z^{2}} w \le 0. \tag{16}$$

From (15) we get that $(f(\mathring{z}))^{*} = \frac{1}{m}\left(\frac{\partial \varphi}{\partial z}(\mathring{z})\right)'(Df(\mathring{z}))^{-1}$ and substituting into 16) we obtain:

$$\frac{1}{m}\mathrm{Re}\left(\left(\frac{\partial \varphi}{\partial z}(\mathring{z})\right)'(Df(\mathring{z}))^{-1}D^{2}f(\mathring{z})(w,w)\right) + \|Df(\mathring{z})w\|^{2} -$$
$$- \frac{1}{m} w^{*} \frac{\partial^{2}\varphi}{\partial z\,\partial \bar{z}} w - \frac{1}{m}\mathrm{Re}\, w' \frac{\partial^{2}\varphi}{\partial z^{2}} w \le 0$$

which is equivalent with (7).

The condition for $t \in \mathbf{R}^{2n}\setminus\{0\}$ gives the following condition for $w \in \mathbf{C}^{n}\setminus\{0\}$ (obtained by the natural identification between $\mathbf{R}^{2n}$ and $\mathbf{C}^{n}$ mentioned above) $\mathrm{Re}\langle w, \frac{\partial \varphi}{\partial z}(\mathring{z})\rangle = = \sum_{j=1}^{2n} t_j \frac{\partial \varphi}{\partial a_j}(\mathring{z}) = 0$ and this completes the proof.

3. Geometric interpretation of the main result. In the following remarks we shall give some geometric consequences of Theorem.

First we note that if $Df(\mathring{z})$ is nonsingular there exists a neighborhood U of $\mathring{z}$ so that f is injective on U and since f is holomorphic we obtain that f is a biholomorphic mapping between U and $f(U)$. So, if we note by M the intersection between U and bD we obtain that $f(M)$ is a real hypersurface.

Since $((Df(\mathring{z}))^*)^{-1}\left(\overline{\frac{\partial\varphi}{\partial z}(\mathring{z})}\right)$ is an outer normal vector to $f(M)$ at the point $f(\mathring{z})$ the part (i) of the Theorem has the following geometric interpretation.

Remark 1. If f is a function which satisfies the requirements of the Theorem and M is the set defined above then the outher normal vector to $f(M)$ at the point $f(\mathring{z})$ and the position vector $f(\mathring{z})$ are in the same direction.

Let $v = (v_1, \ldots, v_n)'$ be a real tangent vector to $f(M)$ at $f(\mathring{z})$.

It follows that $\operatorname{Re} < ((Df(\mathring{z}))^*)^{-1}\left(\overline{\frac{\partial\varphi}{\partial z}(\mathring{z})}\right), v > \; = 0.$

We define on $F(M)$ an orientation such as the second fundamental form of the real hypersurface $f(M)$ at $f(\mathring{z})$ is

$b(u,u) = \sum_{i,j=1}^{2n} \frac{\partial^2 |f^{-1}(f(\mathring{z}))|^2}{\partial b_i \partial b_j} u_i u_j$ where $u \in \mathbf{R}^{2n} \setminus \{0\}$ is a real tangent vector at $f(M)$ in the point $f(\mathring{z})$.

It is easy to check that the second fundamental form of the real hypersurface $f(M)$ at $f(\mathring{z})$ can be written as:

$$b(v,v) = \frac{v^* \dfrac{\partial^2 \psi(f(\mathring{z}))}{\partial w \, \partial \overline{w}} v + \operatorname{Re} v' \dfrac{\partial^2 \psi(f(\mathring{z}))}{\partial w^2} v}{\left\| ((Df(\mathring{z}))^*)^{-1} \overline{\left(\dfrac{\partial \varphi(\mathring{z})}{\partial z}\right)} \right\|}. \tag{17}$$

We can compute as follows:

$$\frac{\partial^2\psi(f(\mathring{z}))}{\partial w_j \partial w_k} = \sum_i \frac{\partial\varphi}{\partial z_i}(\mathring{z}) \frac{\partial^2 (f^{-1})_i (f(\mathring{z}))}{\partial w_j \partial w_k} +$$

$$+ \sum_{i,k} \frac{\partial^2\varphi(\mathring{z})}{\partial z_i \partial z_k} \frac{\partial (f^{-1})_i (f(\mathring{z}))}{\partial w_k} \frac{\partial (f^{-1})_k (f(\mathring{z}))}{\partial w_j} \tag{18}$$

$$\frac{\partial^2\psi(f(\mathring{z}))}{\partial w_j \partial \bar{w}_k} = \sum_i \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k} \frac{\partial (f^{-1})_i (f(\mathring{z}))}{\partial w_j} \overline{\frac{\partial (f^{-1})_i (f(\mathring{z}))}{\partial w_k}} \tag{19}$$

Next, by using the following connection between the second derivative of a biholomorphic function f and the second derivative of the inverse function f^{-1}:

$D^2 f^{-1}(f(z))(a,b) = -(Df(z))^{-1} D^2 f(z)((Df(z))^{-1}a, (Df(z))^{-1}b)$, $a, b \in \mathbf{C}^n$ and substituting (18) and (19) into (17), we obtain:

$$b(v,v) = \frac{u^* \dfrac{\partial\varphi(\mathring{z})}{\partial z \partial \bar{z}} u + \operatorname{Re}\left(u' \dfrac{\partial^2\varphi}{\partial z^2}(\mathring{z}) u - \left(\dfrac{\partial\varphi}{\partial z}(\mathring{z})\right)' (Df(\mathring{z}))^{-1} D^2 f(\mathring{z})(u,u) \right)}{\left\| ((Df(\mathring{z}))^*)^{-1} \overline{\left(\dfrac{\partial\varphi}{\partial z}(\mathring{z})\right)} \right\|} \tag{20}$$

where u is defined by $u = Df^{-1}(f(\mathring{z}))(v) = (Df(\mathring{z}))^{-1}(v)$.

Since v is a real tangent vector to $f(M)$ at $f(\mathring{z})$ we have:

$$0 = \operatorname{Re} < v, ((Df(\mathring{z}))^*)^{-1} \overline{\frac{\partial\varphi}{\partial z}}(\mathring{z}) > = \operatorname{Re} < (Df(\mathring{z}))^{-1} v, \overline{\frac{\partial\varphi}{\partial z}}(\mathring{z}) > =$$

$$= \operatorname{Re} < u, \overline{\frac{\partial\varphi}{\partial z}}(\mathring{z}) >$$

and by using (ii) of Theorem we obtain:

$$\frac{b(v,v)}{\|v\|^2} \geq \frac{m\|Df(\dot{z})u\|^2}{\|v\|^2\left\|((Df(\dot{z}))^*)^{-1}\left(\frac{\partial\varphi}{\partial z}(\dot{z})\right)\right\|} = \frac{m\|v\|^2}{\|v\|^2\left\|((Df(\dot{z}))^*)^{-1}\left(\frac{\partial\varphi}{\partial z}(\dot{z})\right)\right\|}$$

According to part (i) of the Theorem we get

$$\frac{b(v,v)}{\|v\|^2} \geq \frac{1}{\|f(\dot{z})\|}$$

Since a principal curvature of a real hypersurface $f(M)$ at $f(\dot{z})$ can be writen as $\frac{b(v,v)}{\|v\|^2}$ where v is a principal direction (so v is a real tangent vector) we get the following geometric interpretation of the (ii) of Theorem.

Remark 2. If f is a function which satisfies the requirements of the Theorem and M is the set defined above then all the principal curvatures k_j $(j \in \{1,\dots,2n-1\})$ of $f(M)$ at the point $f(\dot{z})$ satisfy

$$k_j \geq \frac{1}{\|f(\dot{z})\|},\ j \in \{1,\dots,2n-1\}$$

Also the mean curvature of $f(M)$ at $f(\dot{z})$ and the Gaussian curvature of $f(M)$ at $f(\dot{z})$ satisfy the same inequality.

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BERNSTEIN POLYNOMIALS OVER SIMPLICES

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REZUMAT. - Polinoame Bernstein pe simplexuri. În această lucrare autorii se ocupă de studiul unor proprietăţi ale polinoamelor Bernstein definite pe un simplex arbitrar din $\mathbf{R}^s$. Se pun în evidenţă anumite relaţii care au loc între funcţiile convexe în T şi şirurile polinoamelor Bernstein corespunzătoare.

Abstract. In this paper the authors are concerned with a study of the multivariate Bernstein polynomials over an arbitrary simplex in $\mathbf{R}^s$. Some relations between convex functions in T and the sequences of the corresponding Bernstein polynomials are shown.**

Let $T_0, T_1, \ldots, T_s$ be $(s + 1)$ affinely independent points of $\mathbf{R}^s$, $s \geq 1$. The s-dimensional simplex T is defined by

$$T = \operatorname{span}\{T_0, \ldots, T_s\}.$$

Each point $P \in T$ can be uniquely expressed by

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$$P = \sum_{i=0}^{s} u_i T$$

such that $u_i \geq 0$, $i = 0, \ldots, s$, $\sum_{i=0}^{s} u_i = 1$; the $(s+1)$ tuple $u = (u_0, \ldots, u_s)$ is called the barycentric coordinates of P with respect to the simplex T.

Let us define the basic functions

$$B_\alpha^n(P) = \frac{|\alpha|!}{\alpha!} u^\alpha, \tag{1}$$

$\alpha = (\alpha_0, \ldots, \alpha_s) \in Z_+^{s+1}$, $|\alpha| = \sum_{i=0}^{s}$, $\alpha! = \alpha_0! \ldots \alpha_s$, $|\alpha| = n$, $u^\alpha = u_0^{\alpha_0} \ldots u_s^{\alpha_s}$, $\sum_{i=0}^{s} B_\alpha^n(P) = 1$.

The points $x_\alpha = \frac{\alpha}{n}$, $\alpha \in Z_+^{s+1}$ are called nodes of the simplex T, it means that their barycentric coordinates are $\left(\frac{\alpha_0}{n}, \ldots, \frac{\alpha_s}{n}\right)$.

For any function $f(P)$ continuous on T the multivariate Bernstein polynomials defined by

$$B_n(f; P) = \sum_{\substack{|\alpha| = n \\ \alpha \in Z_+^{s+1}}} B_\alpha^n(P) f\left(\frac{\alpha}{n}\right) \tag{2}$$

converge to $f(P)$ uniformly on T as $n \to \infty$. Properties of the multivariate Bernstein polynomials have been also studied in [2], [3], [4], [10], [11], [12], [13], [14].

Now some properties of multivariate Bernstein polynomials are stated.

For a given interior point $P \in T$, $P = (u_0, \ldots, u_s)$ and a number δ, $u_i > \delta$

> 0, $i = 0, \ldots, s$ we define

$$T_{P,\delta} = \{Q = (v_0, \ldots, v_s) \,|\, v_i \geq u_i - \delta,\ i = 0, \ldots, s\}.$$

This is a closed simplex contained by the simplex T and containing P as its focal point. Each edge of $T_{P,\delta}$ is parallel to the corresponding edge of T.

LEMMA 1. *Let* $P = (u_0, \ldots, u_s)$, $P \in \operatorname{int} T$ *and* $0 < \delta < u_i$, $i = 0, \ldots, s$. *Then*

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} \notin T_{P,\delta}}} P_\alpha^n(P) \leq \sum_{i=0}^{s} e^{-\frac{n\delta^2}{4(1-u_i)}}. \tag{3}$$

Proof. By the definition of $T_{P,\delta}$ it is clear that $\frac{\alpha}{n} \notin T_{P,\delta}$ if there exists $k \in \{0, \ldots, s\}$ such that $\frac{\alpha_k}{n} < u_k - \delta$. Then

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} \notin T_{P,\delta}}} P_\alpha^n(P) \leq \sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} < u_i - \delta}} P_\alpha^n(P)$$

Let us define functions $G_i(x)$, $i = 0, \ldots, s$, as follows

$$G_i(x) = \sum_{|\alpha| = n} B_\alpha^n(P)\, e^{x(\alpha_i - u_i n)},\ x \in \mathbf{R}. \tag{4}$$

It is easy to show (using the fact $\sum_{i=0}^{s} u_i = 1$) that

$$G_i(x) = (e^{-xu_i}(1 - u_i) + u_i e^{x(1-u_i)})^n.$$

Let us denote

$$\varphi_i(x) = e^{-xu_i}(1 - u_i) + u_i e^{x(1-u_i)}.$$

And now in the same way as in [9], [6] it can be shown that

$$\varphi_i(x) \le 1 + u_i x^2 (1 - u_i) \le e^{u_i x^2 (1 - u_i)}$$

under the assumption $|x| \le 3/2$.

From it follows

$$G_i(x) \le e^{n u_i x^2 (1 - u_i)}. \tag{5}$$

Let t be an arbitrary positive real number. Then

$$G_i(x) = \sum_{|\alpha| = n} B_\alpha^n(P) e^{x(\alpha_i - u_i n)} \ge \sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t G_i(x)}} B_\alpha^n(P) e^{x(\alpha_i - n u_i)} >$$

$$> \sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t G_i(x)}} B_\alpha^n(P) e^t G_i(x).$$

This gives the following estimate

$$\sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t G_i x)}} B_\alpha^n(P) < e^{-t} \tag{6}$$

Now, using (5) we obtain

$$\sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t e^{n x^2 (1 - u_i)}}} B_\alpha^n(P) < e^{-t}. \tag{7}$$

Let $t = \dfrac{n\delta^2}{4(1 - u_i)}$, $x = -\dfrac{\delta}{2(1 - u_i)}$ then $|x| \le 3/2$ and (7) gives

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha_i}{n} < u_i - \delta}} B_\alpha^n(P) < e^{-\frac{n\delta^2}{4(1 - u_i)}}.$$

And this estimate concludes our proof

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} \notin T_{P,\delta}}} P_\alpha^n(P) \le \sum_{i=0}^{s} e^{-\frac{n\delta^2}{4(1 - u_i)}}. \qquad \blacksquare$$

LEMMA 2. *Let* $P = (u_0, \dots, u_s)$ *be an interior point of* T, $0 < \delta < \frac{u_i}{4s}$, $i = 0, \dots, s$. *Then for* $\frac{\alpha}{n} \in T_{P,\delta}$ *the following inequality*

$$B_\alpha^n(P) \geq K \frac{1}{n^{s/2}} e^{-\frac{3n\delta^2}{4} \sum_{i=0}^{s} \frac{1}{u_i}} \tag{8}$$

holds, where K is an positive constant independent only on s.

Proof. Let us remind Stirling's formula

$$n! = \sqrt{2\pi n}\, n^n e^{-n} H_n,\ H_n = e^{\frac{\theta}{12n}},\ 0 \leq \theta < 1$$

i.e.

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Then

$$B_\alpha^n(P) = \frac{n!}{\alpha!} u^\alpha > \frac{\sqrt{2\pi n}\, n^n e^{-n} \prod_{i=0}^{s} u_i^{\alpha_i}}{\prod_{i=0}^{s} \sqrt{2\pi\alpha_i}\, \alpha_i^{\alpha_i} e^{-\alpha_i} e^{\frac{1}{12\alpha_i}}} =$$

$$= \frac{1}{\prod_{i=0}^{s} e^{\frac{1}{12\alpha_i}}} \frac{1}{(\sqrt{2\pi}\,)^s} \frac{\sqrt{|\alpha|}}{\sqrt{\alpha_0 \dots \alpha_s}} \prod_{i=0}^{s} \left(\frac{|\alpha|}{\alpha_i} u_i\right)^{\alpha_i} \prod_{i=0}^{s} e^{|\alpha|\left(\frac{\alpha_i}{|\alpha|} - u_i\right)}. \tag{9}$$

Denote

$$L_{\alpha_i} = \left(\frac{|\alpha|}{\alpha_i} u_i\right)^{\alpha_i} e^{|\alpha|\left(\frac{\alpha_i}{|\alpha|} - u_i\right)},\ i = 0, \dots, s.$$

As it was proved in [5]

$$\left(\frac{|\alpha|}{\alpha_i} u_i\right)^{\alpha_i} e^{|\alpha|\left(\frac{\alpha_i}{|\alpha|} - u_i\right)} \geq e^{-\frac{3|\alpha|}{4u_i}\left(\frac{\alpha_i}{|\alpha|} - u_i\right)^2} \tag{10}$$

provided that

$$\left| \frac{\alpha_i}{|\alpha|} - u_i \right| < \frac{u_i}{4}, \; i = 0, \ldots, s. \tag{11}$$

It is easy to see that these assumptions are satisfied. From $\frac{\alpha_i}{n} \geq u_i - \delta$ it follows immediately $s\delta \geq \delta \geq u_i - \frac{\alpha_i}{n}$. On the other hand the equalities $\sum_{i=0}^{s} \frac{\alpha_i}{n} = 1$ and $\sum_{i=0}^{s} u_i = 1$ give $s\delta \geq \frac{\alpha_i}{n} - u_i$. Together with the assumptions of lemma we have $\frac{u_i}{4} > s\delta \geq \left| \frac{\alpha_i}{|\alpha|} - u_i \right|$.

Therefore if inequalities (11) are satisfied th(

$$\prod_{i=0}^{s} L\alpha_i \geq \prod_{i=0}^{s} e^{-\frac{3|\alpha|}{4u_i}\delta^2} = e^{-\frac{3|\alpha|}{4u_i}\delta^2 \sum_{i=0}^{s} \frac{1}{u_i}} \tag{12}$$

Further

$$\prod_{i=0}^{s} e^{\frac{1}{12\alpha_i}} = e^{\sum_{i=0}^{s} \frac{1}{12\alpha_i}} < e^{s+1} = (\tag{13}$$

and

$$\left(\frac{|\alpha|}{\prod_{i=0}^{s} \alpha_i} \right)^{\frac{1}{2}} \geq \frac{M}{|\alpha|^{\frac{s}{2}}} \tag{14}$$

for

$$\frac{|\alpha|}{\alpha_i} \geq \frac{1}{\varepsilon + u_i}, \; i = 0, \ldots, s$$

where the constants C and M are independent on α.

Summarizing (9), (10), (13) and (14) we obtain

$$B_\alpha^n(P) \geq K \frac{1}{n^{\frac{s}{2}}} e^{-\frac{3n\delta^2}{4} \sum_{i=0}^{s} \frac{1}{u_i}}. \qquad \blacksquare$$

LEMMA 3. *Let $\Omega \subset T$ be a simplex with edges parallel to those of the given simplex T. Let N_Ω be a number of nodes belonging to Ω. Then there exists a positive number n_0 such that*

$$N_\Omega > \gamma n^s \tag{15}$$

if $n \geq n_0$, where $\gamma > 0$ is a constant.

The proof is simple.

The following theorem can be proved

THEOREM 1. *Let $f \in C(T)$ be convex on T. Then*

$$B_n(f;P) \geq f(P), \quad B_n(f;P) \geq B_{n+1}(f;P)$$

for all $n \geq 1$ and all $P \in T$.

See [3] for the proof.

It is well-known that for univariate Bernstein polynomials so-called converse theorems hold ([5], [7], [8], [15]):

(i) $B_n(f;x) \geq f(x),\ x \in [0,1],\ n \geq 1 \Rightarrow f$ is convex in [0,1].

(ii) $B_n(f;x) \geq B_{n+1}(f;x),\ x \in [0,1],\ n \geq 1 \Rightarrow f$ is convex in [0,1].

But it is impossible to extend directly these converse theorems to the Bernstein polynomials over simplices.

As concerns Bernstein polynomials over triangles this problem was solved in [1]. In [6] there was given a different approach to this problem. Now we are going to prove the following theorem:

THEOREM 2. *Let $f \in C(T)$ and $B_n(f; P) \geq f(P)$ for all $P \in T$ and all natural numbers n. Then the function f does not attain its strict local maximum inside T.*

Proof. Let us suppose that f attains a strict local maximum at the interior point $Q = (u_0, \dots, u_s)$. Without lost of generality it is possible to put $f(Q) = 0$. Then there exists a subsimplex T_{Q,δ_1}, $0 < \delta_1 < u_i$, $i = 0, \dots, s$ containing Q as an interior point such $f(P) \leq 0$ for all $P \in T_{Q,\delta_1}$ and let $t = \min\{u_0, \dots, u_t\}$.

Let us choose δ_2 in such a way that

$$0 < \delta_2 < \min\left(\frac{\delta_1}{4s}, \frac{\delta_1}{\sqrt{3\varepsilon(1-t)}}\right), \quad \varepsilon = \sum_{i=0}^{s} \frac{1}{u_i}.$$

Then $T_{Q,\delta_1} \supset T_{Q,\delta_2}$. Further let $\Omega \subset T_{Q,\delta_2}$ be a subsimplex with edges parallel to the corresponding edges of T_{Q,δ_2} and $f(P) < 0$ for all $P \in \Omega$. Let $(-h)$ be a maximum of f over the subsimplex Ω and let $M = \max_{P \in T} |f(P)|$.

Now let us evaluate $B_n(f; Q)$. It is

$$B_n(f;Q) = \sum_{\frac{\alpha}{n} \notin T_{Q,\delta_1}} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) + \sum_{\frac{\alpha}{n} \in T_{Q,\delta_1} - \Omega} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) + \sum_{\frac{\alpha}{n} \in \Omega} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) \quad (16)$$

Using lemma 1 we obtain for the first sum

$$\left|\sum_{\frac{\alpha}{n}\notin T_{Q,\delta_1}} f\left(\frac{\alpha}{n}\right)B_\alpha^n(Q)\right| \le M \sum_{\frac{\alpha}{n}\notin T_{Q,\delta_1}} B_\alpha^n(Q) \le M(s+1)\, e^{-\frac{n\delta_1^2}{4(1-t)}}. \tag{17}$$

Further as far as the second sum is concerned one can state it is nonpositive.

And now the sum will be estimated: $\Omega \subset T_{\Omega,\delta_2}$ and due to this reason it is

$$\sum_{\frac{\alpha}{n}\in\Omega} B_\alpha^n(Q) \ge K \sum_{\frac{\alpha}{n}\in\Omega} \frac{1}{|\alpha|^{\frac{\delta}{2}}}\, e^{-\frac{3n\delta_2^2}{4}\varepsilon}$$

Now the use of lemma 3 gives

$$\sum_{\frac{\alpha}{n}\in\Omega} B_\alpha^n(Q) \le -hL\frac{n^s}{n^{\frac{s}{2}}}\, e^{-\frac{3n\delta_2^2}{4}\varepsilon}$$

Then

$$B_n(f;Q) \le M(s+1)\, e^{-\frac{n\delta_1^2}{4(1-t)}} - hL\frac{n^s}{n^{\frac{s}{2}}}\, e^{-\frac{3n\delta_2^2}{4}\varepsilon} =$$

$$= e^{-\frac{3n\delta_2^2}{4}\varepsilon}\left(M(s+1)\, e^{-\frac{n\delta_1^2}{4(1-t)}+\frac{3n\delta_2^2}{4}\varepsilon} - hLn^{\frac{s}{2}}\right)$$

Under given assumptions from here it follows that $B_n(f;Q) < 0$ and this contradiction concludes our proof. ■

The following theorem can be proved as the consequence of the theorem 2.

THEOREM 3. *If* $f \in C(T)$ *and the inequality*

$$B_n(f;P) \ge B_{n+1}(f;P)$$

holds for all natural numbers n and all points on T, then the function f does not attain a strict local maximum inside T.

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MONOTONE TECHNIQUE TO THE INITIAL VALUES PROBLEM FOR A DELAY INTEGRAL EQUATION FROM BIOMATHEMATICS

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REZUMAT. - Metoda iterațiilor monotone pentru problema cu valori inițiale relativă la o ecuație integrală din biomatematică. În lucrare este prezentată o matodă constructivă de rezolvare a problemei (1) - (2) în ipotezele (i) - (iv) presupunând că funcția $f(t,x)$ este monotonă în raport cu x. Un aspect nou conținut în acest articol îl constituie adaptarea metodei iterațiilor monotone la cazul operatorilor anti-izotoni, în particular, la cazul când $f(t,x)$ este o funcție necrescătoare în x.

1. Introduction. The following delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds \tag{1}$$

is a model for the spread of certain infectious diseases with a contact rate that varies seasonally. In this equation $x(t)$ is the proportion of infectives in the population at time t, τ is the length of time an individual remains infectious and $f(t, x(t))$ is the proportion of new infectives per unit time.

In [1], [2], [4], [5], [6] sufficient conditions were given for the existence of nontrivial periodic nonnegative and continuous solutions to equation (1) in

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case of a periodic contact rate: $f(t+\omega, x) = f(t,x)$, $f(t,0) = 0$. The tools were Banach fixed point theorem [5], topological fixed point theorems [1], [2], [4], [6], fixed point index theory (the additivity property) [2] and monotone technique [2], [4].

In [3] we dealt with positive and continuous solutions $x(t)$ for equation (1), on a given interval of time $-\tau \le t \le T$, when it is known the proportion $\phi(t)$ of infectives in the population for $-\tau \le t \le 0$, i.e.

$$x(t) = \phi(t), \text{ for } -\tau \le t \le 0. \tag{2}$$

Clearly, we had to assume that ϕ satisfies the following condition:

$$b = \phi(0) = \int_{-\tau}^{0} f(s, \phi(s))\, ds. \tag{3}$$

Under this condition problem (1)-(2) is equivalent with the initial values problem:

$$x'(t) = f(t, x(t)) - f(t-\tau, x(t-\tau)),\ 0 \le t \le T \tag{4}$$

$$x(t) = \phi(t),\ -\tau \le t \le 0.$$

The existence of at least one solution to problem (4) was established in [3] under the following assumptions:

(i) $f(t,x)$ is nonnegative and continuous for $-\tau \le t \le T$ and $x \ge 0$;

(ii) $\phi(t)$ is continuous, $0 < a \le \phi(t)$ for $-\tau \le t \le 0$ and satisfies condition

(3);

(iii) there exists an integrable function $g(t)$ such that

$$f(t,x) \geq g(t) \text{ for } -\tau \leq t \leq T \text{ and } x \geq a \tag{5}$$

and

$$\int_{t-\tau}^{t} g(s)\,ds \geq a \text{ for } 0 \leq t \leq T; \tag{6}$$

(iv) there exists a positive function $h(x)$ such that $1/h(x)$ is locally integrable on $[a, +\infty)$,

$$f(t,x) \leq h(x) \text{ for } 0 \leq t \leq T \text{ and } x \geq a \tag{7}$$

and

$$T < \int_{b}^{\infty} (1/h(x))\,dx. \tag{8}$$

THEOREM 1 [3]. *Suppose that assumptions* (i)-(iv) *are satisfied. Then equation* (1) *has at least one continuous solution* $x(t)$, $x(t) \geq a$, *for* $-\tau \leq t \leq T$, *which satisfies condition* (2).

Moreover, as follows from the proof, each continuous solution $x(t)$ to (1)-(2) satisfying $x(t) \geq a$ for $-\tau \leq t \leq T$, also satisfies

$$x(t) \leq R \text{ for } 0 \leq t \leq T, \tag{9}$$

where R is so that

$$T = \int_{b}^{R} (1/h(x))\,dx. \tag{10}$$

The proof of Theorem 1 was given by using the topological transversality theorem of Granas and can also be done by using Leray-Schauder continuation theorem. A constructive scheme to solve (1)-(2), namely the successive approximations method, was described in [3] only for the particular case where condition (iv) is replased by the more restrictive Lipschitz condition

(iv") there exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

for all $t \in [-\tau, T]$ and $x, y \in [a, +\infty)$.

The aim of this paper is to give a constructive scheme to solve (1)-(2) under assumptions (i)-(iv) provided that $f(t,x)$ is monotone with respect to x. Uniqueness will be also discussed. In case $f(t,x)$ is nondecreasing in x, our results are somewhat similar with those in [2] referring to periodic solutions of (1).

2. Main results. Let E be the Banach space of all continuous functions $x(t)$, $0 \leq t \leq T$ with norm

$$\|x\| = \max_{0 \leq t \leq T} |x(t)|.$$

Consider the closed subset of E:

$$X = \{x \in E;\ x(0) = b \text{ and } x(t) \geq a \text{ for } 0 \leq t \leq T\}$$

and the delay integral operator

$$A\colon E \to X,\ Ax(t) = \int_{t-\tau}^{t} f(s,\tilde{x}(s))\,ds$$

where $\tilde{x}(s) = x(s)$ for $0 < s \leq T$ and $\tilde{x}(s) = \phi(s)$ for $-\tau \leq s \leq 0$. A is completely continuous as an operator from X into X.

THEOREM 2. *Let* (i)-(iv) *be satisfied. Suppose that* $f(t,x)$ *is nondecreasing in* x *for* $a \leq x \leq R$. *Denote*

$$U_0(t) = a \text{ for } 0 \leq t \leq T$$

$$U_n(t) = AU_{n-1}(t) \text{ for } 0 \leq t \leq T \ (n = 1,2,\ldots).$$

Then, $U_n(t) \to x_*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$, $x_*(t)$ *is the minimal solution to* (1)-(2) *in* X *and*

$$a \leq U_1(t) \leq \ldots \leq U_n(t) \leq \ldots \leq x_*(t) \leq R \text{ for } 0 \leq t \leq T.$$

Proof. By Theorem 1 there exists at least one solution in X to (1)-(2). Let $x_1(t)$ be any solution to (1)-(2). We have

$$a = U_0(t) \leq x_1(t) \leq R \text{ for } 0 \leq t \leq T.$$

Consequently, since A is nondecreasing on interval $[a,R]$ of E

$$U_1(t) = AU_0(t) \leq Ax_1(t) = x_1(t).$$

On the other hand, by (iii), we have $a = U_0(t) \leq U_1(t)$. Hence

$$U_0(t) \le U_1(t) \le x_1(t) \text{ for } 0 \le t \le T.$$

Now we inductively find that

$$a \le U_1(t) \le U_2(t) \le \dots \le U_n(t) \le \dots \le x_1(t) \text{ for } 0 \le t \le T.$$

A being completely continuous on X, the sequence $(AU_n)_{n\ge1}$ must contain a subsequence, say $(AU_{n_k})_{k\ge1}$, convergent to some $x_* \in X$. But $AU_{n_k}(t) = U_{n_k+1}(t)$ and taking into account the monotonicity of $(U_n(t))_{n\ge1}$, we obtain that $U_n(t) \to x_*(t)$ uniformly in $t \in [0,T]$ as $n \to \infty$ and

$$U_n(t) \le x_*(t) \le x_1(t) \text{ for } 0 \le t \le T \ (n = 0, 1, \dots).$$

Letting $n \to \infty$ in $AU_n(t) = U_{n+1}(t)$ we get $Ax_*(t) = x_*(t)$, i.e. $x_*(t)$ is a solution to (1)-(2). Finally, by $x_*(t) \le x_1(t)$ where $x_1(t)$ was any solution to (1)-(2), we see that $x_*(t)$ is the minimal solution to (1)-(2) in X.

The following result is concerning with the existence and approximation of the maximal solution in X to (1)-(2).

THEOREM 3. *Let* (i)-(iv) *be satisfied. Suppose that there exists* $R_0 \ge R$ *such that*

$$f(t, R_0) \le R_0/\tau \text{ for } -\tau \le t \le T \tag{11}$$

(i.e. $f(t, \phi(t)) \le R_0/\tau$ *for* $-\tau \le t \le 0$ *and* $f(t, R_0) \le R_0/\tau$ *for* $0 < t \le T$*) and* $f(t,x)$ *is nondecreasing in x for* $a \le x \le R_0$. *Denote* $V_0(t) = R_0$ *for* $0 \le t \le T$,

$$V_n(t) = AV_{n-1}(t) \text{ for } 0 \leq t \leq T \ (n = 1,2,\ldots).$$

Then, $V_n(t) \to x^*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$, $x^*(t)$ *is the maximal solution to* (1)-(2) *in* X *and*

$$x^*(t) \leq \ldots \leq V_n(t) \leq \ldots \leq V_2(t) \leq V_1(t) \leq R_0 \text{ for } 0 \leq t \leq T.$$

Proof. By (11) we have

$$V_1(t) \leq V_0(t) = R_0 \text{ for } 0 \leq t \leq T.$$

Next, the proof is analog to that of Theorem 2.

THEOREM 4. *Let the conditions of Theorem* 2 *be satisfied. Suppose that there exists* $\alpha \in (0,1)$ *such that*

$$f(t,\gamma x) \geq \gamma^\alpha f(t,x) \text{ for all } \gamma \in (0,1),\ t \in [0,T],\ x \in [a,R]. \quad (12)$$

Then, (1)-(2) *has a unique solution in* X.

Proof. Let $x_1(t)$ be any solution in X to (1)-(2). We will show that $x_1(t) = x_*(t)$. Let

$$\gamma_0 = \min_{0 \leq t \leq T} (x_*(t)/x_1(t)).$$

Since $a \leq x_*(t) \leq x_1(t) \leq R$, we have $a/R \leq \gamma_0 \leq 1$. Now, we show $\gamma_0 = 1$. In fact, if $\gamma_0 < 1$, then (12) implies

$$x_*(t) = Ax_*(t) \geq A(\gamma_0 x_1)(t) = \int_{t-\tau}^{t} f(s, \widetilde{\gamma_0 x_1}(s))\, ds$$

$$\geq \gamma_0^\alpha \int_{t-\tau}^{t} f(s, \tilde{x}_1(s))\, ds = \gamma_0^\alpha A x_1(t) = \gamma_0^\alpha x_1(t).$$

Thus $\gamma_0 \geq \gamma_0^\alpha$, which is impossible for $0 < \alpha < 1$. Therefore, $\gamma_0 = 1$ and $x_*(t) = x_1(t)$.

THEOREM 5. *Let the conditions of Theorem* 3 *and Theorem* 4 *be satisfied. Then,* (1)-(2) *has a unique solution $x_*(t)$ in X and for any $x_0(t)$ in E satisfying $a \leq x_0(t) \leq R_0$ for all $t \in [0,T]$, we have $_n(t) \to x_*(t)$ uniformly in $t \in [0,T]$ as $n \to \infty$, where*

$$x_n(t) = Ax_{n-1}(t) \quad (n = 1, 2, \ldots).$$

Proof. We find from

$$a = U_0(t) \leq x_0(t) \leq V_0(t) = R_0$$

that

$$U_n(t) \leq x_n(t) \leq V_n(t) \quad (n = 1, 2, \ldots).$$

On the other hand, by Theorem 2 and Theorem 3, we have that

$$U_n(t) \to x_*(t) \text{ and } V_n(t) \to x_*(t)$$

uniformly in $t \in [0,T]$ as $n \to \infty$. Therefore, $x_n(t) \to x_*(t)$ uniformly in $t \in [0,T]$ as $n \to \infty$.

The following result refers to functions $f(t,x)$ which are nonincreasing in x.

THEOREM 6. *Let* (i)-(iv) *be satisfied. Denote* $R_0 = \max(R, \|U_1\|)$ *and suppose* $f(t,x)$ *is nonincreasing in* x *for* $a \le x \le R_0$. *Also suppose that there exists* $\alpha \in (-1,0)$ *such that*

$$f(t,\gamma x) \le \gamma^{\alpha} f(t,x) \text{ for } \gamma \in (0,1),\ t \in [0,T],\ x \in [a,R_0]. \tag{13}$$

Then, (1)-(2) *has a unique solution* $x_*(t)$ *in* X,

$$a = U_0(t) \le V_1(t) \le \ldots \le U_{2n}(t) \le V_{2n+1}(t) \le \ldots \le x_*(t) \le$$

$$\ldots \le U_{2n+1}(t) \le V_{2n}(t) \le \ldots \le U_1(t) \le V_0(t) = R_0 \quad \text{for } 0 \le t \le T,$$

and $U_n(t) \to x_*(t)$, $V_n(t) \to x_*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$.

Proof. By Theorem 1 there exists as least one solution $x_1(t)$ to (1) - (2) and $a \le x_1(t) \le R$ for $0 \le t \le T$. We have

$$a = U_0(t) \le x_1(t) \le V_0(t) = R_0$$

whence

$$V_1(t) \le x_1(t) \le U_1(t).$$

But, by (iii), $a \le V_1(t)$. Also $U_1(t) \le \|U_1\| \le R_0$. Hence

$$U_0(t) \le V_1(t) \le x_1(t) \le U_1(t) \le V_0(t).$$

It follows

$$U_0(t) \le V_1(t) \le U_2(t) \le x_1(t) \le V_2(t) \le U_1(t) \le V_0(t).$$

Finally

$$a = U_0(t) \le V_1(t) \le \ldots \le U_{2n}(t) \le V_{2n+1}(t) \le \ldots$$

$$\ldots \le x_1(t) \le \ldots \le U_{2n+1}(t) \le V_{2n}(t) \le \ldots \le U_1(t) \le V_0(t) = R_0. \quad (14)$$

A being completely continuous on X, the sequence $(AU_{2n-1}(t))_{n\ge1}$ contains a subsequence convergent to some $y_*(t)$ in X and similarly, $(AV_{2n-1}(t))_{n\ge1}$ contains a subsequence converging to some $y^*(t)$ in X. Now, from (14) we see that

$$U_{2n}(t) \to y_*(t),\ V_{2n+1}(t) \to y_*(t)$$

$$U_{2n+1}(t) \to y^*(t),\ V_{2n}(t) \to y^*(t) \quad (15)$$

uniformly in $t \in [0,T]$ as $n \to \infty$ and

$$y_*(t) \le x_1(t) \le y^*(t). \quad (16)$$

By (15), it follows that

$$y^*(t) = Ay_*(t) \text{ and } y_*(t) = Ay^*(t).$$

Now, we prove that under assumption (13), we have indeed $y_*(t) = y^*(t)$. To do this, let

$$\gamma_0 = \min_{0\le t\le T} (y_*(t) / y^*(t)).$$

Obviously, $0 < a/R_0 \le \gamma_0 \le 1$. We will show that $\gamma_0 = 1$. In fact, if $\gamma_0 < 1$, then (13) implies

$$y^* = Ay_* \le A(\gamma_0 y^*) = \int_{t-\tau}^{t} f(s, \widetilde{\gamma_0 y^*}(s))\, ds \le$$

$$\leq \gamma_0^\alpha \int_{t-\tau}^{t} f(s, \tilde{y}^*(s))\, ds = \gamma_0^\alpha A y^* = \gamma_0^\alpha y_*.$$

Therefore, $\gamma_0^{-\alpha} \leq \gamma_0$ or, equivalently, $\alpha \leq -1$, a contradiction. Thus, $\gamma_0 = 1$ as claimed. Consequently, $y_* = y^*$. The proof is complete.

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ON THE LEGENDRE TRANSFORM AND ITS APPLICATIONS

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REZUMAT. - Transformarea lui Legendre şi aplicaţiile sale. Transformarea lui Legendre este folosită în mecanică la schimbări de variabile în sisteme de ecuaţii diferenţiale. În lucrare se prezintă unele proprietăţi ale transformării în R^n şi se indică aplicaţii în probleme de mecanică generală, mecanică cerească şi electricitate.

1. Introduction. The Legendre transform permits the change of dependent and independent variables. It is very useful in mechanics and thermodynamics. For example, let us consider the inner energy $E = E(S, V)$, which depends on the entropy S and the volume V. Then the total differential of E will be

$$dE = TdS + PdV,$$

with

$$T = E_S(S, V),\ P = E_V(S, V)$$

the absolute temperature and the pressure.

Now T and V will be the new independent variables, which means that

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from $T = E_S(S, V)$ we have to obtain $S = S(T, V)$.

We can find a new function $F = F(T, V)$ given by $F = E - TS$ for which we have

$$dF = -SdT + PdV$$

and hence

$$S = -F_T(V, T),\ P = F_V(V, T).$$

So using the function F we can make the change of variables, of course imposing some conditions on the derivatives of E in order to obtain $S = S(T, V)$. The Legendre transform of E will then be the function $-F$.

The Legendre transform appears in [6], but it seems to have already been known to Euler. A natural generalization was given later by Fenchel [5]. The Fenchel transform has the property that it is defined for arbitrary functions. It is very useful not only in mechanics, but also in optimization. So, this old transform has its place in recent books on mechanics as Arnold [2] or Choquard [4], on differential equations as Amann [1], on convex analysis and optimization as Willem [8], on in more comprehensive ones as that of Zeidler [9]. We mention that this transform can be studied in the more general setting of Banach spaces or dual pairs, as in the books of Barbu and Precupanu [3] or Precupanu

[7], but for the applications given in this paper we consider only the $\boldsymbol{R}^n$ case.

In what follows we expose the definition and the main properties of the Fenchel transform for various classes of functions. Then we emphasize its key role in connecting the Lagrangian and the Hamiltonian setting of some outstanding problems of mechanics and electricity.

2. The conjugate of a function. This section contains general results on the conjugate of a function, as treated for example in [8], [3], [7] or [1].

Let the real function F: $\boldsymbol{R}^n \to]-\infty,\infty]$ be given so that the effective domain of $F, D(F) = \{u \in \boldsymbol{R}^n: F(u) < \infty\}$ is nonvoid.

The *conjugate* (or the *Fenchel transform*) of F is the function F^*: $\boldsymbol{R}^n \to]-\infty,\infty]$ given by

$$F^*(v) = \sup_{u \in D(F)} \{<v, u> - F(u)\}, \tag{1}$$

where $<v,u> = \sum_{k=1}^{n} v_k u_k$ is the inner product on $\boldsymbol{R}^n$.

From the definition we obtain at once the *Fenchel* (*Young*) *inequality*

$$F(u) + F^*(v) \geq <v,u>, \ \forall \ u,v \in \boldsymbol{R}^n. \tag{2}$$

It also follows easily that for two given functions F_i with $D(F_i) \neq \emptyset$, $i = 1, 2$ so that $F_1 \leq F_2$, we have the reversed inequality $F_1^* \geq F_2^*$.

The function F^* is always convex, so we shall remind some related definitions.

A set $C \subset \boldsymbol{R}^n$ is said *convex* if for every two points $x,y \in C$, the line segment

$$[x,y] = \{z \in \boldsymbol{R}^n: z = (1-a)x + ay,\ a \in [0,1]\}$$

lies completely in C.

A function $F: \boldsymbol{R}^n \to]-\infty,\infty]$ is called:

- *convex*, if for every $x,y \in C$ and $t \in]0,1[$,

$$F((1-t)x + ty) \leq (1-t)F(x) + tF(y);$$

- *strictly convex*, if $D(F) \neq \emptyset$ and

$$F((1-t)x + ty) < (1-t)F(x) + tF(y)$$

for every $x,y \in D(F)$, $x \neq y$, $t \in]0,1[$;

- *continuous*, if $u_k \to u$ implies $F(u_k) \to F(u)$;

- *inferior semi-continuous* (i.s.c.), if $u_k \to u$ implies $\underline{\lim}\, F(u_k) \geq F(u)$.

The *epigraph* of the function $F: \boldsymbol{R}^n \to]-\infty,\infty]$ is the set

$$\operatorname{epi} F = \{(u,t) \in \boldsymbol{R}^n: F(u) \leq t\}.$$

It is clear that F is convex if and only if epi F is convex.

If F is a convex function and the graph of F lies above the hyperp

$u \mapsto \langle v, u \rangle$, then $-F^*(v)$ represents the minimal distance from the graph of F to this hyperplane, in the vertical direction. If the hyperplane intersects the graph of F, then $F^*(v)$ represents the maximal distance in the vertical direction between the graph of F and the hyperplane, considering the points for which the graph of F lies under the hyperplane.

A function G: $\mathbf{R}^n \to \mathbf{R}$ is said to be *affine* if it has the form

$$G(u) = \langle v, u \rangle + a, \text{ where } v \in \mathbf{R}^n, a \in \mathbf{R}.$$

For an i.s.c. convex function, the following geometric description holds.

THEOREM 1. *A function* F: $\mathbf{R}^n \to]-\infty,\infty]$ *is i.s.c. and convex if and only if it is the pointwise supremum of the affine functions dominated by* F.

As a consequence of this theorem, F^* is i.s.c. and convex.

Let us denote by $\Gamma_0(\mathbf{R}^n)$ the set of all functions F: $\mathbf{R}^n \to]-\infty,\infty]$ which are convex, i.s.c. and such that $D(F) \neq \emptyset$.

The following theorem holds

THEOREM 2. *If* $F \in \Gamma_0(\mathbf{R}^n)$, *then* $F^* \in \Gamma_0(\mathbf{R}^n)$ *and* $F^{**} = F$, *so the Fenchel transform is an involution of* $\Gamma_0(\mathbf{R}^n)$.

Proof. The function F^* being i.s.c. and convex, we have to prove only that $D(F^*) \neq \emptyset$. From theorem 1 it follows the existence of $(v,a) \in \mathbf{R}^{n+1}$ so that

$$F(u) \geq \langle v, u \rangle - a, \ \forall\, u \in \mathbf{R}^n,$$

so $a \geq \langle v, u \rangle - F(u), \ \forall\, u \in \mathbf{R}^n$. Then $a \geq F^*(v)$ and $v \in D(F^*) \neq \emptyset$.

It is clear that $(v,a) \in \text{epi } F$ if and only if $F(u) \geq \langle v, u \rangle - a, \ \forall\, u \in \mathbf{R}^n$.

So

$$F(u) = \sup_{\substack{(v,a) \in \mathrm{R}^{n+1} \\ (v,\cdot) - a \leq F}} \{\langle v, u \rangle - a\} = \sup_{\substack{v \in D(F^*) \\ a \geq F^*(v)}} \{\langle v, u \rangle - a\} =$$
$$= \sup_{v \in D(F^*)} \{\langle v, u \rangle - F^*(v)\} = F^{**}(\ ), \ \forall\, u \in \mathbf{R}^n,$$

and the equality $F = F^{**}$ is proved. □

We give now some examples for $n = 1$.

Example 1. For $F(u) = |u|^p/p, \ p \in \,]1, \infty[$ we have

$$F^*(v) = |v|^q/q,$$

with q such that $1/p + 1/q = 1$ (q is the conjugate of p). The Fenchel inequality becomes in this case

$$uv \leq |u|^p/p + |u|^q/q,$$

which is the well-known Young's inequality from which some classical inequalities of calculus may be derived.

Example 2. For $F(u) = |u|$, we have

$$F^*(v) = \begin{cases} 0, & |v| \leq 1 \\ +\infty, & |v| > 1. \end{cases}$$

Example 3. For $F(u) = \alpha |u|^p/p, \ \alpha > 0, \ p \in \,]1, \infty[$ we have

$$F^*(v) = \alpha^{-q/p} |v|^q/q,$$

with q the conjugate of p.

Example 4. For $p \in]1,\infty[$ and $c, c' > 0$, if

$$c|u|^p \leq F(u) \leq c'|u|^p,$$

then

$$k|v|^q \leq F^*(v) \leq k'|v|^q,$$

with q the conjugate of p and $k = (c'p)^{-q/p}q^{-1}$, $k' = (cp)^{-q/p}q^{-1}$.

For a function $F\colon \mathbf{R}^n \to]-\infty,\infty]$ such that $D(F) \neq \emptyset$, the *sub-differential* of F at u is the set

$$\partial F(u) = \{v \in \mathbf{R}^n\colon F(w) \geq F(u) + <v, w-u>, \ \forall\, w \in D(F)\}.$$

The function F is said *sub-differentiable* at u if $\partial F(u) \neq \emptyset$.

It is clear that if F is sub-differentiable at u, then $u \in D(F)$; F is subdifferentiable at $u \in D(F)$ iff there exists an affine function which is equal to F at u and is less than F on $\mathbf{R}^n$; the set $\partial F(u)$ is closed and convex in $\mathbf{R}^n$. The function F has a global minimum at u iff $0 \in \partial F(u)$.

THEOREM 3. *If $F \in \Gamma_0(\mathbf{R}^n)$, the following assertions are equivalent*

(a) $v \in \partial F(u)$;

(b) $F(u) + F^*(v) = <v, u>$;

$$\text{(c)} \qquad u \in \partial F^*(v).$$

Proof. (a) $\Leftrightarrow$ (b) follows from the fact that

$$v \in \partial F(u) \Leftrightarrow <v,u> - F(u) \geq <v,w> - F(w),\ \forall\, w \in D(F)$$

$$\Leftrightarrow <v,u> - F(u) \geq \sup_{w \in D(F)} \{<v,w> - F(w)\}$$

$$\Leftrightarrow <v,u> - F(u) = F^*(v).$$

Then, using theorem 2 we have

$$u \in \partial F^*(v) \Leftrightarrow <v,u> = F^*(v) + F^{**}(u) = F^*(v) + F(u),$$

so (b) $\Leftrightarrow$ (c) and the theorem is proved. □

The next result shows the relation between sub-differentiability and convexity.

THEOREM 4. *If* F: $\boldsymbol{R}^n \to]-\infty,\infty]$ *is convex and continuous at* $u \in D(F)$, *then* F *is sub-differentiable at* u.

It the function is convex and differentiable, the sub-differential coincides with the gradient, as the following theorem shows.

THEOREM 5. *Let* F: $\boldsymbol{R}^n \to]-\infty,\infty]$ *be a convex function. If* F *is differentiable at* $u \in \text{int}\, D(F)$, *then*

$$\partial F(u) = \{\nabla F(u)\}.$$

Proof. We show at first that $\nabla F(u) \in \partial F(u)$. The function F being convex, we have

$$F((1-a)u + aw) \le (1-a)F(u) + aF(w)$$

for each $w \in \boldsymbol{R}^n$, $a \in]0,1[$, or

$$[F(u + a(w-u)) - F(u)]/a \le F(w) - F(u).$$

Letting $a \to 0^+$ one has

$$< \nabla F(u), w-u > \le F(w) - F(u),$$

hence $\nabla F(u) \in \partial F(u)$.

We prove now that the unique element in $\partial F(u)$ is $\nabla F(u)$. Let $v \in \partial F(u)$. Then, for each $w \in \boldsymbol{R}^n$

$$F(u) - < v, u > \le F(w) - < v, w >,$$

so the function $F - < v, \cdot >$ has at u a global minimum. From its differentiability at u, we obtain

$$0 = \nabla F(u) - v,$$

hence $v = \nabla F(u)$ and the theorem is proved. □

COROLLARY 6. *The gradient of a convex function* F: $\boldsymbol{R}^n \to]-\infty,\infty]$ *which is differentiable at* $u \in \operatorname{int} D(F)$ *satisfies*

$$F(w) \ge F(u) + < \nabla F(u), u-w >, \textit{ for each } w \in \boldsymbol{R}^n.$$

If $\nabla F(u) = 0$, *then F admits a global minimum at u.*

Proof. The inequality follows from the fact that $\partial F(u) = \{\nabla F(u)\}$. If $\nabla F(u) = 0$, we have $F(w) \geq F(u)$ for each $w \in \mathbf{R}^n$, hence u is a global minimum of F.□

The next theorem gives conditions on F in order to assure the differentiability of F^*.

THEOREM 7. *If $F \in \Gamma_0(\mathbf{R}^n)$ is strictly conv and satisfies a coercivity condition*

$$F(u)/|u| \to \infty \text{ for } |u| \to \infty,$$

then $F^* \in C^1(\mathbf{R}^n, \mathbf{R})$.

Proof. Let $v \in \mathbf{R}^n$ be fixed; we define G_v: Γ $\to \mathbf{R}$,

$$G_v(w) = <v, w> - F(w).$$

The function $-G_v$ is strictly convex and $-G_v(w) \to \infty$, as $|w| \to \infty$, so there is one and only one point $u \in \mathbf{R}^n$ where $-G_v$ attains its infimum. Theorem 3 implies $\partial F^*(v) = \{u\}$.

The function ∂F^*: $\mathbf{R}^n \to \mathbf{R}^n$, $v \mapsto u$ where $\{u\} = \partial F^*(v)$ is continuous. We have from theorem 2 that F^* is i.s.c., hence ∂F^* will have a closed graph. To prove the continuity of ∂F^* it suffices to show that the image of any bounded set is bounded. Let $r > 0$ be given and $|v| \leq r$, $\{u\} = \partial F^*(v)$. Theorem 3 implies

$v \in \partial F(u)$, hence

$$F(0) \geq F(u) - \langle v, u\rangle.$$

Supposing withous loss of generality that $F(0) < +\infty$, from

$$r \geq |v| \geq (F(u) - F(0))/|u|,$$

we obtain using the coercivity condition the existence of $R > 0$ so that $|u| \leq R$ for each v with $|v| \leq r$.

Let us prove that ∂F^* is also differentiable. Let $\{u\} = \partial F^*(v)$ and $\{u_h\} = \partial F^*(v+h)$, $v \in \boldsymbol{R}^n$, $h \in \boldsymbol{R}^n\backslash\{0\}$. Then

$$\langle h, u\rangle \leq F^*(v+h) - F^*(v) \leq \langle h, u_h\rangle$$

and

$$0 \leq [F^*(v+h) - F^*(v) - \langle h, u\rangle]/|h| \leq \langle h, u_h - u\rangle/|h| \leq |u_h - u|.$$

The continuity of ∂F^* implies $|u_h - u| \to 0$ for $|h| \to 0$, so F^* is differentiable at v and $\{\nabla F^*(v)\} = \{u\} = \partial F^*(v)$. It follows that $F^* \in C^1(\boldsymbol{R}^n, \boldsymbol{R})$.□

Let now be given a convex function $F \in C^1(\boldsymbol{R}^n, \boldsymbol{R})$. Using theorems 3 and 5, F^* can be defined implicitly by

$$\begin{cases} F^*(v) = \langle v, u\rangle - F(u) \\ v = \nabla F(u). \end{cases}$$

It the gradient ∇F is locally invertible, these relations define indeed a function of v, considering $u = (\nabla F)^{-1}(v)$. The function F^* is known as the *Legendre transform* of F. If F is strictly convex and $F(u)/|u| \to \infty$ for $|u| \to \infty$, then by theorem 7 the Legendre transform F^* is in the class $C^1(\mathbf{R}^n,\mathbf{R})$.

It is known that for $F \in C^2(\mathbf{R}^n,\mathbf{R})$, F is convex if and only if $D^2F(x)$ is positive semi-definite for every $x \in \mathbf{R}^n$ (i.e. $<D^2F(x)y, y> \geq 0$ for each $y \in \mathbf{R}^n$); if $D^2F(x)$ is positive definite for every $x \in \mathbf{R}^n$ (i.e. $<D^2F(x)y,y> > 0$, for each $y \in \mathbf{R}^n\setminus\{0\}$), then F is strictly convex. For $F \in C^2(\mathbf{R}^n,\mathbf{R})$ with D^2F uniformly positive definite (i.e. there exists $\alpha > 0$ such that $<D^2F(x)y, y> \geq \|y\|^2$ for each $x,y \in \mathbf{R}^n$), then for every $y \in \mathbf{R}^n$, the equation

$$\nabla F(x) = y$$

has a unique solution.

We obtain now the following theorem for C^2 - class functions.

THEOREM 8. *Let $F \in C^2(\mathbf{R}^n,\mathbf{R})$ be given such that D^2F is uniformly positive definite. Then the following statements are true:*

(i) *The transform given by* (1) *has the form*

$$F^*(v) = <v, u> - F(u),$$

u being the solution of $v = \nabla F(u)$;

(ii) *$F^* \in C^2(\mathbf{R}^n,\mathbf{R})$, F^* is stricly convex and $\nabla F^* = (\nabla F)^{-1}$;*

(iii) $F(u) + F^*(v) \geq \langle u, v \rangle$ *for each* $u, v \in \boldsymbol{R}^n$ *and* $F(u) + F^*(v) = \langle u, v \rangle$ *iff* $\nabla F(u) = v$;

(iv) $F^{**} = F$.

Proof. It remains to prove that $F^* \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$. This follows from the equality $\nabla F^* = (\nabla F)^{-1}$ and the theorem of implicit functions.□

3. Euler-Lagrange and Hamiltonian systems. The Legendre transform is of great importance in Mechanics, as it is specified in [2] or [4]. Indeed, it is useful in transforming the implicit Euler-Lagrange systems in the explicit Hamiltonian ones in a very simple way. The following theorem presents this equivalence.

THEOREM 9. *Let* $I \subset \boldsymbol{R}$ *be an open interval and* $D \subset \boldsymbol{R}^n$ *a domain. Consider* $L \in C^2(\boldsymbol{R}^n \times D \times I, \boldsymbol{R})$ *such that for each value of the argument* $(\dot{q}_0, q_0, t_0) \in \boldsymbol{R}^n \times D \times I$, $L_{\dot{q}\dot{q}}(\dot{q}_0, q_0, t_0) \in \mathfrak{L}(\boldsymbol{R}^n)$ *is uniformly positive define. Then the Euler-Lagrange equation*

$$\frac{d}{dt} L_{\dot{q}} = L_q \tag{3}$$

is equivalent to the Hamiltonian system

$$\begin{cases} \dot{p} = -H_q \\ \dot{q} = H_p, \end{cases} \tag{4}$$

where the Hamiltonian $H \in C^2(\boldsymbol{R}^n \times D \times I, \boldsymbol{R})$ *is the Legendre transform of the Lagrangian L with respect to the variable* $\dot{q}$, i.e.

$$H(p,q,t) = \langle p,\dot{q}\rangle - L(\dot{q},q,t), \tag{5}$$

on the right-hand side $\dot{q}$ *being obtained from the equation*

$$p = L_{\dot{q}}, \tag{6}$$

where $H_q := \nabla_q H$ *denotes the gradient with respect to the variable q for fixed t and p.*

Proof. We apply theorem 8 considering L as a function of $\dot{q}$, (and q, t as parameters). Then $H = L^*$ is given by (5), where $\dot{q} = \dot{q}(p,q,t)$ is obtained from the unique solution of (6). We have then $H \in C^2(\boldsymbol{R}^n \times D \times I)$, $H_p = (L_{\dot{q}})^{-1}$ and $L^{**} = L$.

Let now q: $I \to \boldsymbol{R}^n$ be a solution of the Euler-Lagrange equations. From (6) and (3) we obtain $\dot{p} = L_q$. But using (5) we get immediately $L_q = -H_q$, so the first line in (4) is obtained. Because of $H_p = (L_{\dot{q}})^{-1}$ we have from (6) $\dot{q} = H_p$, the second line in (4). It follows that if q is a solution of (3), then (p,q) is a solution of (4).

Conversely, let (p,q) be a solution of (4). Because $L^{**} = L$, we have

$$L(\dot{q},q,t) = <p,\dot{q}> - H(p,q,t),$$

where $\dot{q} = H_p$, i.e. $p = p(\dot{q},q,t) = (H_p)^{-1} = L_{\dot{q}}$. Then $\dot{p} = \frac{d}{dt}(L_{\dot{q}}) = -H_q$ because of (4). But $L_q = -H_q$ and q is a solution of the Euler-Lagrange equations.□

The following theorem states some important properties of a Hamiltonian system.

THEOREM 10. *In the conditions of theorem 9 we have*

a) $$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t};$$

b) *if the system* (4) *is autonomous,* $H(p,q)$ *is a first integral which is by definition the energy*;

c) *if* $\frac{\partial H}{\partial q_i} = 0$, *then* p_i *is a first integral* (q_i *is a cyclic variable*);

d) *if all the* q_i *are cyclic, the system is integrable by quadratures.*

Proof. a) $\frac{dH}{dt} = \frac{\partial H}{\partial t} + <H_p,\dot{p}> + <H_q,\dot{q}> = \frac{\partial H}{\partial t}$, and using the form of H as a Legendre transform, $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$.

b) If the system (4) is autonomous, $\frac{\partial H}{\partial t} = 0$, hence $\frac{dH}{dt} = 0$ and $H(p,q)$ = const for the solutions p and q.

c) Using (4), we have $\dot{p}_i = 0$, hence $p_i = c_i$ gives a first integral.

d) Applying c), we obtain $p_i = c_i$, $i = \overline{1,n}$. From (4) we have $\dot{q}_i = \frac{\partial H}{\partial p_i}(c_1, \ldots, c_n, t)$ and $q_i(t) = q_i(t_0) + \int_{t_0}^{t} \frac{\partial H}{\partial p_i}(c_1, \ldots, c_n, \tau)\, d\tau$.□

4. Applications. In the problems of mechanics, the Lagrangian function has usually the form

$$L(\dot{q}, q, t) = T(\dot{q}, q, t) - U(q, t) = E_{kin} - E_{pot},$$

where

$$T(\dot{q}, q, t) = \frac{1}{2} < A(q,t)\, \dot{q}, \dot{q} >, \tag{7}$$

$A(q,t)$ being a symmetric uniformly positive matrix with entries of C^2-class. Then theorem 9 applies and the Hamiltonian obtained as a Legendre transform will be

$$H(p, q, t) = <p, q> - L(\dot{q}, q, t),$$

where $p = A(q,t)\dot{q}$, hence $\dot{q} = A(q,t)^{-1} p$.

It this case

$$H(p,q,t) = <p, A^{-1}p> - \frac{1}{2}<AA^{-1}p, A^{-1}p> + U(q,t) =$$

$$= \frac{1}{2}<p, A^{-1}p> + U(q,t). \tag{8}$$

So, if the kinetic energy (7) is given by a uniformly positive definite matrix A, then the Hamiltonian is the total energy, expressed in terms of the space and momentum variables. In the case of autonomous systems, we have by theorem 10 b) the energy integral

$$H(p,q) = \text{const.}$$

If $A(q,t) = I_m$, i.e. $T(\dot{q}) = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^2$, it follows $\dot{q}_i = p$ and

$$H(p,q,t) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + U(q,t). \tag{9}$$

Special problems of this type are, for example, those of particles ir Newtonian central field of the harmonic osçillator.

1. A particle in the Newtonian central field. The motion of a punctual mass is described by system of equations

$$\begin{cases} m\ddot{x} = kxr^{-3} \\ m\ddot{y} = kyr^{-3} \\ m\ddot{z} = kzr^{-3} \end{cases},$$

where $k > 0$ is the gravitational constant and $r = (x^2 + y^2 + z^2)^{1/2}$. This system is of the type (3) with $L: \mathbf{R}^3 \times (\mathbf{R}^3 \setminus \{0\}) \to \mathbf{R}$ given by

$$L(\dot{q},q) = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - kr^{-1},$$

where $q = (x,y,z)$. The Hamiltonian will be of the form given in (8),

$$H(p,q) = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + kr^{-1}$$

and the initial system has the Hamiltonian form

$$\begin{cases} \dot{p}_i = kq_i r^{-3} \\ \dot{q}_i = p_i/m \end{cases}, \quad i = 1,2,3.$$

2. The harmonic oscillator. The equation for the harmonic oscillator is

$$m\ddot{q} = -kq,$$

$q \in \boldsymbol{R}$, the constants $m,k > 0$ (which has the known solution $q(t) = C\sin(\omega t + \alpha)$, with the frequency $\omega = \sqrt{k/m}$). The Lagrange function $L = T - U$ is $L: \boldsymbol{R}^2 \to \boldsymbol{R}$,

$$L(\dot{q},q) = (m\dot{q}^2 - kq^2)/2.$$

Hence $L_{\dot{q}\dot{q}} = m > 0$ and theorem 9 applies. From $p = L_{\dot{q}}$ we obtain the momentum $p = m\dot{q}$. The Hamiltonian $H = p\dot{q} - L$ will be $H: \boldsymbol{R}^2 \to \boldsymbol{R}$

$$H(q,p) = \frac{1}{2m}p^2 + \frac{k}{2}q^2,$$

and the Hamilton equations (4)

$$\begin{cases} \dot{p} = -kq \\ \dot{q} = p/m. \end{cases}$$

3. A punctual mass on a torus. The motion of a punctual mass on a torus is governed by a system of the type (3) with $L: \boldsymbol{R}^2 \times (0,2\pi)^2 \to \boldsymbol{R}$ given by

$$L(\dot\theta,\dot\phi,\theta,\phi) = \frac{m}{2}(r^2\dot\theta^2 + (R + r\cos\theta)^2\dot\phi^2) - mgr\sin\theta,$$

with $m > 0$, $R > r > 0$. Denoting $q = (\theta,\phi)$, we have

$$L_{\dot q\dot q} = \begin{pmatrix} mr^2 & 0 \\ 0 & m(R + r\cos q_1)^2 \end{pmatrix}$$

a n d $< D^2Ly,y > = m(r^2y_1^2 + (R + r\cos q_1)^2 y_2^2) \geq m\min\{r^2, (R - r)^2\}(y_1^2 + y_2^2)$,

hence $L_{\dot q\dot q}$ is uniformly positive definite. From

$$p_1 = mr^2\dot q_1$$

$$p_2 = m(R + r\cos q_1)^2\dot q_2,$$

we obtain the Hamiltonian H: $\boldsymbol{R}^2 \times (0, 2\pi)^2 \to \boldsymbol{R}$,

$$H(p_1,p_2,q_1,q_2) = \frac{1}{2mr^2}p_1^2 + \frac{1}{2m}p_2^2/(R + r\cos q_1)^2 + mgr\sin q_1.$$

The system corresponding to (4) is

$$\begin{cases} \dot p_1 = -\dfrac{1}{m}p_2^2(R + r\cos q_1)^{-3}r\sin q_1 - mgr\cos q_1 \\ \dot p_2 = 0 \\ \dot q_1 = \dfrac{1}{mr^2}p_1 \\ \dot q_2 = \dfrac{1}{m}(R + r\cos q_1)^2 p_2. \end{cases}$$

In several problems we have to deal with generalized Lagrangian functions having the form

$$L_1(\dot q, q, t) = T(\dot q, q, t) - <f, \dot q> - U(q,t), \tag{10}$$

with T given by (7) and f a function of q. Applying the Legendre transform we shall get by theorem 9 the Hamiltonian function

$$H_1(p,q,t) = <p,\dot{q}> - L_1(\dot{q},q,t),$$

where $p = A(q,t)\dot{q} - f$, hence $\dot{q} = A(q,t)^{-1}(p+f)$.

Then

$$H_1(p,q,t) = <p,A^{-1}(p+f)> - \frac{1}{2}<AA^{-1}(p+f),A^{-1}(p+f)> +$$
$$+ <f,A^{-1}(p+f)> + U(q,t),$$

hence

$$H_1(p,q,t) = \frac{1}{2}<p,A^{-1}p> + <p,A^{-1}f> + \frac{1}{2}<f,A^{-1}f> + U(q,t). \quad (11)$$

Therefore the transform of a generalized Lagrangian of type (10) is the Hamiltonian (11), the corresponding systems (3) and (4) being equivalent.

For autonomous systems we have in this case an energy integral

$$H_1(p,q) = const$$

given by theorem 10 b).

If $A(q,t) = I_n$, i.e. $T(\dot{q}) = \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^2$, it follows $\dot{q} = p + f$ and

$$H_1(p,q,t) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i f_i + \frac{1}{2}\sum_{i=1}^{n} f_i^2 + U(q,t). \quad (12)$$

The next applications contain problems having generalized Lagrangian functions of type (10).

4. The photogravitational three-body problem. Let us consider the three-dimensional photogravitational three-body problem given by the system of equations

$$\begin{cases} \ddot{x} - 2\dot{y} = \Omega_x \\ \ddot{y} + 2\dot{x} = \Omega_y \\ \ddot{z} = \Omega_z \end{cases}$$

with Ω: $D = \mathbf{R}^3 \setminus \{(\mu,0,0), (\mu - 1,0,0)\} \to \mathbf{R}$ given by

$$\Omega(x,y,z) = (x^2 + y^2)/2 + A_1/r_1 + A_2/r_2,$$

$$A_1 = a_1(1 - \mu),\ A_2 = a_2\mu,$$

$$r_1^2 = (x-\mu)^2 + y^2 + z^2,\ r_2^2 = (x - \mu + 1)^2 + y^2 + z^2,$$

where $\mu \in [0, 1/2]$ and $a_1, a_2 \in\ ]-\infty,1]$.

The system may be written as

$$\frac{d}{dt}L_{\dot{q}} = L_q,$$

where $q = (q_1, q_2, q_3) = (x,y,z)$ and L: $\mathbf{R}^3 \times D \to \mathbf{R}$ is a generalized Lagrangian of the form

$$L(\dot{q},q) = T(\dot{q}) - Z(\dot{q},q).$$

The kinetic energy is given by

$$T(\dot{q}) = (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)/2$$

and the generalized potential Z by

$$Z(\dot{q},q) = \dot{q}_1 q_2 - \dot{q}_2 q_1 - \Omega(q).$$

The Lagrangian is of the type (10) with $A(q,t) = I_3, f(q,t) = (q_2, -q_1, 0)$ and $U(q,t) = -\Omega(q)$.

We have $\dot{q} = A(q,t)^{-1}(p+f)$, hence

$$\dot{q}_1 = p_1 + q_2$$

$$\dot{q}_2 = p_2 - q_1$$

$$\dot{q}_3 = p_3.$$

It follows

$$H(p,q) = (p_1^2 + p_2^2 + p_3^2)/2 + p_1 q_2 - p_2 q_1 + \frac{1}{2}(q_1^2 + q_2^2) - \Omega(q) =$$
$$= (p_1^2 + p_2^2 + p_3^2)/2 + p_1 q_2 - p_2 q_1 - A_1/r_1 - A_2/r_2.$$

The Hamiltonian system (4) is in this case

$$\begin{cases} \dot{p}_1 = p_2 - q_1 + \Omega_{q_1} \\ \dot{p}_2 = -p_1 - q_2 + \Omega_{q_2} \\ \dot{p}_3 = \Omega_{q_3} \\ \dot{q}_1 = p_1 + q_2 \\ \dot{q}_2 = p_2 - q_1 \\ \dot{q}_3 = p_3. \end{cases}$$

5. A charged particle in a magnetic field $B(r) = \text{curl } A(r)$.

The equations are given by

$$m\ddot{q} = \frac{e}{c}(\dot{q} \times \text{curl } A),$$

where $m > 0$ is the mass, e the charge of the particle and $A = (A_1, A_2, A_3)$, with $A_i \in C^1(\mathbf{R}^3)$, $i = 1, 2, 3$.

In this case $q = (q_1, q_2, q_3)$ and L: $\boldsymbol{R}^6 \to \boldsymbol{R}$,

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 + \frac{e}{c} < A(q), \dot{q} >,$$

so we have a generalized Lagrangian of the type (10) with

$$A = mI_3, f = -\frac{e}{c}A, \ U \equiv 0.$$

It follows for $p = (p_1, p_2, p_3)$ that

$$p = m\dot{q} + \frac{e}{c}A,$$

hence

$$\dot{q} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

and from (11) we obtain H: $\boldsymbol{R}^2 \to \boldsymbol{R}$,

$$H(p,q) = \frac{1}{2m}\sum_{i=1}^{3}(p_i - \frac{e}{c}A_i(q))^2 = \frac{1}{2m} < p - \frac{e}{c}A(q), p - \frac{e}{c}A(q) >.$$

The Hamiltonian system is

$$\begin{cases} \dot{p}_i = \dfrac{e}{mc} < p - \dfrac{e}{c}A, A_{q_i} > \\ \dot{q}_i = \dfrac{1}{m}(p_i - \dfrac{e}{c}A_i). \end{cases}$$

6. A charged particle in an electromagnetic field $(E(q,t), B(q,t))$.

In this case, the motion of a particle of mass m and charge e is governed by the equations

$$m\ddot{q} = eE + \frac{e}{c}(\dot{q} \times B),$$

$q = (q_1, q_2, q_3)$ being the coordinates of the particle. These equations admit the Lagrangian L: $\boldsymbol{R}^7 \to \boldsymbol{R}$,

$$L(\dot{q}, q, t) = \frac{m}{2}\dot{q}^2 - e\phi(q,t) + \frac{e}{c}<A(q,t), \dot{q}>,$$

the fields E and B being related to the scalar potential ϕ and the vector potential A by

$$E = -\operatorname{grad}\phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$B = \operatorname{curl} A.$$

The Lagrangian is of the type (10) with

$$A = mI_3, f = -\frac{e}{c}A, \; U = e\phi.$$

It follows that p is given by

$$p = m\dot{q} + \frac{e}{c}A,$$

hence

$$\dot{q} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

and the corresponding Hamiltonian is

$$H(p,q) = \frac{1}{2m}<p - \frac{e}{c}A(q,t), \, p - \frac{e}{c}A(q,t)> + e\phi(q,t).$$

The Hamiltonian system is

$$\begin{cases} \dot{p}_i = -e\phi_{q_i} + \dfrac{e}{mc}<p - \dfrac{e}{c}A, A_{q_i}> \\ \dot{q}_i = \dfrac{1}{m}(p_i - \dfrac{e}{c}A_i) \end{cases}, \; i = 1,2,3.$$

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ON THE PERIOD OF QUASI-CIRCULAR MOTION IN A SPHERICAL POST-NEWTONIAN GRAVITATIONAL FIELD

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REZUMAT. - Asupra perioadei mişcării cvasicirculare într-un câmp gravitaţional post-newtonian sferic. Utilizându-se teoria clasică a perturbaţiilor, se studiază evoluţia perioadei nodale în raport cu perioada kepleriană corespunzătoare în mişcarea cvasicirculară a unei particule de probă într-un câmp gravitaţional post-newtonian sferic (caracterizat de parametri α, β, γ). Se deduc analitic (cu o precizie de ordinul întâi în excentricitate) perturbaţiile relativiste de ordinele întâi şi al doilea ale perioadei nodale. Considerându-se cazul câmpului post-newtonian sferic al lui Einstein ($\beta = \gamma = 1$), se discută evoluţia perioadei nodale pentru trei valori ale parametrului α, atât în cazul general, cât şi în două cazuri particulare. Se discută, de asemenea, influenţa aceluiaşi câmp Einstein asupra mişcării circulare, în trei sisteme de coordonate diferite.

Introduction. One of the oldest methods intended to study the motion in a post-Newtonian (not necessarily relativistic) field used the classic theory of perturbations. According to this method, the force acting on a test particle in such a field is written as a sum of two terms: the Newtonian attraction and a post-Newtonian perturbing force, while the deviations of the orbit from a Keplerian orbit are regarded as perturbations (e.g. [2]).

Such a method was used by different authors (e.g. [3-5]) to determine first

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order relativistic changes of some Keplerian orbital parameters over one anomalistic period. First and second order perturbations in orbital elements over one nodal period were determined in [1, 9, 10] for different relativistic and nonrelativistic post-Newtonian fields.

Few authors dealt with the nodal period behaviour in such a field. An approximate formula for the nodal period as functio. of the orbital elements was given in [5], for the Schwarzschild field, but without expressing the variation of this period. The first and second order changes of the nodal period were obtained in [10, 11] for the Mücket-Treder field, in [1, 7] for the Schwarzschild - de Sitter field, and in [9] for Fock's field.

In this paper we shall treat perturbatively the quasi-circular motion of a test particle in a spherical post-Newtonian gravitational field. We shall determine the first and second order relativistic perturbations of the nodal period.

Notice that the orbits are in fact unperturbed in the considered field, but we shall hereafter use, by abuse of language, a perturbation theory terminology.

2. Starting equations. Let a central body of mass M be the source which generates a spherical post-Newtonian gravitational field, and let $\mu = GM$ be its gravitational parameter (G = gravitational constant). Consider a test particle orbiting M under the action of this field. The relative motion of the test particle can be described in coordinates (t, x) in the form [12]

$$dV/dt = -\mu x/r^3 + a_{PN}. \tag{1}$$

The left-hand side of the above equation is the total acceleration of the test particle. The first term in the right-hand side is nothing but the Newtonian attraction per unit mass (r = radial coordinate), while a_{PN} is the virtual perturbing post-Newtonian acceleration, which has the expression (e.g. [12]; see also [13])

$$a_{PN} = (\mu/c^2)(2(\beta + \gamma - 2\alpha)\mu x/r^4 - (\gamma + \alpha)(V^2/r^3)x +$$

$$+ 3\alpha(x \cdot V)^2 x/r^5 + 2(\gamma + 1 - \alpha)(x \cdot V)V/r^3), \tag{2}$$

where c = speed of light; α = gauge parameter [3]; β, γ are the Eddington-Robertson parameters [14]: β = post-Newtonian parameter describing the amout of nonlinearity of the gravitational field, γ = post-Newtonian parameter describing the space curvature.

Choose a reference frame originated in the mass centre of the body M; and feature the motion of the test particle with respect to this frame throung the Keplerian orbital parameters $\{y \in Y;\ u\}$, all time-dependent, where

$$Y = \{p,\ q = e\cos\omega,\ k = e\sin\omega,\ \Omega,\ i\} \tag{3}$$

and p = semilatus rectum, e = eccentricity, ω = argument of pericentre, Ω = longitude of ascending note, i = inclination, u = argument of latitude.

For our purposes we shall use the definition relation of the nodal period

$$T_\Omega = \int_0^{2\pi} (dt/du)\,du \tag{4}$$

and Newton-Euler equations written with respect to in the form(e.g. [1, 9, 10])

$$\begin{aligned}
dp/du &= 2(Z/\mu)\,r^3 T,\\
dq/du &= (Z/\mu)\,(r^3 kBCW/(pD) + r^2 T(r(q + A)/p + A) + r^2 BS),\\
dk/du &= (Z/\mu)\,(-r^3 qBCW/(pD) + r^2 T(r(k + B)/p + B) - r^2 AS),\\
d\Omega/du &= (Z/\mu) r^3 BW/(pD),\\
di/du &= (Z/\mu) r^3 AW/p,\\
dt/du &= Zr^2(\mu p)^{-1/2},
\end{aligned} \tag{5}$$

where $Z = (1 - r^2 C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, S, T, W = radial, transverse, and binormal components of the perturbing acceleration, respectively.

The change of $y \in Y$ between the initial (u_0) and current (u) positions, which will be used below, is

$$\Delta y = \int_{u_0}^{u} (dy/du)\,du,\ y \in Y, \tag{6}$$

with the integrands given by (5). The integrals are estimated by successive approximations, with $Z \approx 1$.

3. Perturbing acceleration and corresponding equations of motion. The components of the perturbing acceleration $\boldsymbol{a}_{PN}$ have the following expressions [12]

$$S = (\mu/c^2)(\mu/(a^3(1-e^2)^3))(1+e\cos v)^2((2\beta+\gamma-3\alpha)+$$

$$+(\gamma+2)e^2+2(\beta-2\alpha)e\cos v-(2\gamma+2-\alpha)e^2\cos^2 v), \quad (7)$$

$$T = 2(\mu/c^2)(\mu/(a^3(1-e^2)^3))(1+e\cos v)^3(\gamma+1-\alpha)e\sin v,$$

$$W = 0,$$

with a = semimajor axis, v = true anomaly.

Replacing in (7) the well-known formulae

$$p = a(1-e^2), \quad (8)$$

$$u = \omega + v, \quad (9)$$

the definition expression of q and k, and the orbit equation in polar coordinates

$$r = p/(1+e\cos v), \quad (10)$$

then retaining only terms to first order in q and k (because we deal with quasi-circular orbits), the components of the perturbing acceleration reduce to

$$S = (\mu^2/(c^2 p r^2))\,(L_1 + L_2 A q + L_2 B k),$$

$$T = (\mu^2/(c^2 r^3))\,L_3 (Bq - Ak), \qquad (11)$$

$$W = 0,$$

where we abbreviated

$$L_1 = 2\beta + \gamma - 3\alpha,$$

$$L_2 = 2(\beta - 2\alpha), \qquad (12)$$

$$L_3 = 2(\gamma + 1 - \alpha).$$

Focus now our attention to equations (5). It is easy to observe, by the fourth and the sixth equations (5) and by the expression of Z, that $Z = 1$ (because $W = 0$). Substituting (11) in (5), using (10) in the equivalent form

$$r = p/(1 + Aq + Bk), \qquad (13)$$

and performing all necessary calculations, the equations of motion become

$$dp/du = 2(\mu/c^2)\,L_3(Bq - Ak),$$

$$dq/du = \mu/(c^2 p))\,(L_1 B + (L_2 + 2L_3)ABq + (L_2 B^2 - 2L_3 A^2)k),$$

$$dk/du = -(\mu/(c^2 p))\,(L_1 A + (L_2 A^2 - 2L_3 B^2)q + (L_2 + 2L_3)ABk), \qquad (14)$$

$$d\Omega/du = 0,$$

$$di/du = 0,$$

$$dt/du = p(p/\mu)^{1/2}(1 + Aq + Bk)^{-2}.$$

4. Variations of orbital elements. Let us now perform the integrals (6) with the integrands provided by the first five equations (14). We use the successive approximations method, limiting the process to the first order approximation. Accordingly, we consider $y = y_0 = y(u_0)$, $y \in Y$, in the right-hand side of equations (14), and integrate these ones separately. Performing the integrations, and denoting

$$x = \mu/(c^2 p_0) \tag{15}$$

and

$$\begin{aligned} b_1 &= L_3 = 2(\gamma + 1 - \alpha), \\ b_2 &= L_1 = 2\beta + \gamma - 3\alpha, \\ b_3 &= (L_2 + 2L_3)/2 = \beta + 2\gamma + 2 - 4\alpha, \\ b_4 &= (L_2 - 2L_3)/2 = \beta - 2\gamma - 2, \end{aligned} \tag{16}$$

we get the first order (in x) relativistic changes

$$\Delta p = 2xp_0 b_1(-Aq_0 - Bk_0 + A_0 q_0 + B_0 k_0),$$

$$\Delta q = x(-b_2 A + b_3 B^2 q_0 - (b_3 AB - b_4 u)k_0 + $$
$$+ b_2 A_0 - b_3 B_0^2 q_0 + (b_3 A_0 B_0 - b_4 u_0)k_0),$$

$$\Delta k = x(-b_2 B - (b_3 AB + b_4 u)q_0 - b_3 B^2 k_0 + \tag{17}$$
$$+ b_2 B_0 + (b_3 A_0 B_0 + b_4 u_0)q_0 + b_3 B_0^2 k_0),$$

$$\Delta\Omega = 0,$$

$$\Delta_i = 0,$$

where, obviously, $A_0 = A(u_0)$, $B_0 = B(u_0)$.

Observe that, due to the post-Newtonian conservation of the angular momentum, the motion is restricted to a fixed plane (see the last two expressions (17)).

Although this is not the goal of our paper, let us examine briefly what changes ubdergoes the orbit over one nodal period (that is, letting u vary between 0 and 2π). Putting $u_0 = 0$, $u = 2\pi$ in (17), the first three expressions become

$$\Delta p = 0,$$

$$\Delta q = 2\pi x b_4 k_0, \tag{18}$$

$$\Delta k = -2\pi x b_4 q_0.$$

Observing that for quasi-circular orbits $p \approx a$, using the definitions of q and k, and taking into account the last notation (16), relations (18) lead easily to

$$\Delta a = 0,$$

$$\Delta e = 0, \tag{19}$$

$$\Delta\omega = -2\pi x(\beta - 2\gamma - 2).$$

This means that the only first order relativistic effect in the considered field consists of a rotation of the orbit in its plane (apsidat motion). If we particularize the field to the spherical Einstein post-Newtonian gravitational field ($\beta = \gamma = 1$), then, taking into account (15), the last formula (19) reads

$$\Delta\omega = 6\pi\mu/(c^2 p_0), \tag{20}$$

that is, the well-known expression for the relativistic shift of pericentre.

5. Nodal period. Now, let us come back to the main purpose of our paper. As shown in Section 1, we shall determine the first and second order (in x) relativistic perturbations of the nodal period. To do that, we shall resort to the method proposed in [15], extended in [6], and generalized in [8] for some special situations.

According to this method, to second order in a small parameter characterizing the perturbing factor, the nodal period is given by

$$T_\Omega = T_0 + \Delta_1 T + \Delta_2 T, \tag{21}$$

where T_0 (the Keplerian period corresponding to u_0) is determined from (see (4) and the last equation (14))

$$T_0 = p_0(p_0/\mu)^{1/2} \int_0^{2\pi} g^{-2} du, \tag{22}$$

with the abbreviation $g = g(u) = 1 + Aq_0 + Bk_0$.

The general expression of $\Delta_1 T$ and $\Delta_2 T$ were given and explicited in [6, 7] and will not be repeated here. We shall directly particularize them to our perturbing factor (taking into account, for the beginning, the fact that $W = 0$ and the small parameter is just x).

The first order (in x) perturbation of the noda')eriod has (in our case) the form

$$\Delta_1 T = p_0(p_0/\mu)^{1/2}(-2I_q - 2I_k + (3/2)\, I_p/p_0), \tag{23}$$

with [6]

$$\begin{aligned} I_p &= \int_0^{2\pi} g^{-2}\Delta p\, du, \\ I_q &= \int_0^{2\pi} g^{-3}A\Delta q\, du, \\ I_k &= \int_0^{2\pi} g^{-3}B\Delta k\, du. \end{aligned} \tag{24}$$

The second order (in x) perturbation of the nodal period has (in our case) the form

$$\Delta_2 T = 3p_0(p_0/\mu)^{1/2}(I_{qq} + I_{kk} + 2I_{qk} - (I_{pq} + I_{pk})/p_0 + I_{pp}/(8p_0^2)). \tag{25}$$

with [6]

$$I_{pp} = \int_0^{2\pi} g^{-2}(\Delta p)^2 du,$$

$$I_{qq} = \int_0^{2\pi} g^{-4}A^2(\Delta q)^2 du,$$

$$I_{kk} = \int_0^{2\pi} g^{-4}B^2(\Delta k)^2 du, \tag{26}$$

$$I_{pq} = \int_0^{2\pi} g^{-3}A\Delta p\Delta q\, du,$$

$$I_{pk} = \int_0^{2\pi} g^{-3}B\Delta p\Delta k\, du,$$

$$I_{qk} = \int_0^{2\pi} g^{-4}AB\Delta q\Delta k\, du.$$

6. Results. Replacing (17) in (24) and (26), expanding g^{-n}, $n = \overline{2,4}$, to first order in q_0, k_0, and performing the integrations, formulae (24) and (26) become respectively

$$I_p = 4\pi x p_0 b_1 (A_0 q_0 + B_0 k_0),$$

$$I_q = -\pi x b_2 (1 + 3A_0 q_0), \tag{27}$$

$$I_k = -\pi x (b_2 - 2b_4 q_0 + 3b_2 B_0 k_0),$$

and

$$I_{pp} = 0,$$

$$I_{qq} = \pi x^2 b_2(b_2(3/4 + A_0^2) + (6b_2 + b_3(1/2 - 2B_0^2))A_0 q_0 +$$

$$+ 2(b_3 A_0 B_0 + b_4(\pi - u_0))A_0 k_0),$$

$$I_{kk} = \pi x^2 b_2(b_2(3/4 + B_0^2) + 2((b_3 A_0 B_0 - b_4(\pi - u_0))B_0 -$$

$$- 4b_4/3)q_0 + (6b_2 - b_3(3/2 + 2B_0^2))B_0 k_0), \tag{28}$$

$$I_{pq} = -2\pi x^2 p_0 b_1 b_2(2A_0 q_0 + B_0 k_0),$$

$$I_{pk} = -2\pi x^2 p_0 b_1 b_2(A_0 q_0 + 2B_0 k_0),$$

$$I_{qk} = \pi x^2 b_2(b_2/4 + ((b_2 - b_3/4 + b_4/2)A_0 - 2b_4/3)q_0 +$$

$$+ (b_2 - b_3/4 - b_4/2)B_0 k_0).$$

With these expressions, (23) acquires the form

$$\Delta_1 T = 2\pi p_0(p_0/\mu)^{1/2} x(2b_2 - (2b_4 - 3b_1 + b_2)A_0)q_0 +$$

$$+ 3(b_1 + b_2)B_0 k_0), \tag{29}$$

while (25) becomes

$$\Delta_2 T = 3\pi p_0(p_0/\mu)^{1/2} x^2 b_2(3b_2 - (4b_4 - (6b_1 + 8b_2 + b_4)A_0 +$$

$$+ 2b_4(\pi - u_0)B_0)q_0 + ((6b_1 + 8b_2 - b_4)B_0 +$$

$$+ 2b_4(\pi - u_0)A_0)k_0). \tag{30}$$

Finally, replacing (16) in (29) and (30), denoting

$$f_1 = 2(2\beta + \gamma - 3\alpha) + (2(-\beta + 2\gamma + 2) + 3(2\beta + 3\gamma + 2 - 5\alpha)A_0)q_0 +$$

$$+ 3(2\beta + 3\gamma + 2 - 5\alpha)B_0 k_0), \tag{31}$$

$$f_2 = (3/2)(2\beta + \gamma - 3\alpha)(3(2\beta + \gamma - 3\alpha) + (4(-\beta + 2\gamma + 2) +$$

$$+ (17\beta + 18\gamma + 10 - 36\alpha)A_0 - 2(\beta - 2\gamma - 2)(\pi - u_0)B_0)q_0 +$$

$$+ ((15\beta + 22\gamma + 14 - 36\alpha)B_0 + 2(\beta - 2\gamma - 2)(\pi - u_0)A_0)k_0),$$

substituting the resulting expressions in (21), and writing (22) to first order in q_0, k_0 as

$$T_0 = 2\pi p_0(p_0/\mu)^{1/2}, \tag{32}$$

the nodal period (to second order in x) reads

$$T_\Omega = T_0(1 + xf_1 + x^2 f_2). \tag{33}$$

This is the basic formula we searched for and which will be used in the next sections (with f_1, f_2 provided by (31), and with x given by (15)).

7. Two particular cases. We shall consider two particular cases: initial orbital elements corresponding to ascending node, and initially circular orbit. In the first situation we u_0, hence $A_0 = 1$, $B_0 = 0$. So, (31) become

$$f_1 = 2(2\beta + \gamma - 3\alpha) + (4\beta + 13\gamma + 10 - 15\alpha)q_0,$$

$$f_2 = (3/2)(2\beta + \gamma - 3\alpha)(3(2\beta + \gamma - 3\alpha) + (13\beta + 26\gamma + \tag{34}$$

$$+ 18 - 36\alpha)q_0 + 2\pi(\beta - 2\gamma - 2)k_0).$$

If the initial orbit is circular (of radius r_0), then we have $q_0 = 0$, $k_0 = 0$, hence (31) acquire the form

$$f_1 = 2(2\beta + \gamma - 3\alpha),$$

$$f_2 = (9/2)(2\beta + \gamma - 3\alpha)^2. \tag{35}$$

It is easy to see that in this last particular case the perturbation of the nodal period does not depend on the initial position of the test particle.

8. Spherical Einstein post-Newtonian field. Consider that the field in which the test particle moves is the spherical Einstein post-Newtonian gravitational field. In this case $\beta = \gamma = 1$, and formulae (31) read

$$f_1 = 3(2(1 - \alpha) + (2 + (7 - 5\alpha)A_0)q_0 + (7 - 5\alpha)B_0k_0),$$

$$f_2 = (27/2)(1 - \alpha)(3(1 - \alpha) + (4 + 3(5 - 4\alpha)A_0 + 2(\pi - u_0)B_0)q_0 + \tag{36}$$

$$+ ((17 - 12\alpha)B_0 - 2(\pi - u_0)A_0)k_0).$$

The two particular cases (34) and (35) become respectively

$$f_1 = 3(2(1 - \alpha) + (9 - 5\alpha)q_0),$$

$$f_2 = (27/2)(1 - \alpha)(3(1 - \alpha) + (19 - 12\alpha)q_0 - 2\pi k_0), \tag{37}$$

and

$$f_1 = 6(1 - \alpha), \quad f_2 = (81/2)(1 - \alpha)^2. \tag{38}$$

Let us now assign to the gauge parameter α some particular values, which mean some systems of coordinates. The case $\alpha = 0$ means the use of standard post-Newtonian coordinates (spatially isotropic). Expressions (36) become in this case

$$f_1 = 3(2 + (2 + 7A_0)q_0 + 7B_0k_0),$$

$$f_2 = (27/2)(3 + (4 + 15A_0 + 2(\pi - u_0)B_0)q_0 + \qquad (39)$$

$$+ (17B_0 - 2(\pi - u_0)A_0)k_0),$$

while the particular cases (37) and (38) become respectively

$$f_1 = 3(2 + 9q_0), \ f_2 = (27/2)(3 + 19q_0 - 2\pi k_0), \qquad (40)$$

and

$$f_1 = 6, \ f_2 = 81/2. \qquad (41)$$

If we consider $\alpha = 1$, namely the spatial standard coordinate system is used, formulae (36) read

$$f_1 = 6((1 + A_0)q_0 + B_0k_0),$$

$$f_2 = 0, \qquad (42)$$

while (37) and (38) acquire respectively the form

$$f_1 = 12q_0, \ f_2 = 0, \qquad (43)$$

and

$$f_1 = 0, \ f_2 = 0. \qquad (44)$$

Lastly, put $\alpha = 2$. This value of the gauge parameter leads to

$$f_1 = -3(2 - (2 - 3A_0)q_0 + 3B_0k_0),$$

$$f_2 = (27/2)(3 - (4 - 9A_0 + 2(\pi - u_0)B_0)q_0 + \tag{45}$$

$$+ (7B_0 + 2(\pi - u_0)A_0)k_0)$$

for the expressions (36), and to

$$f_1 = -3(2 + q_0), \quad f_2 = (27/2)(3 + 5q_0 + 2\pi k_0), \tag{46}$$

and

$$f_2 = -6, \quad f_2 = 81/2 \tag{47}$$

for the particular cases (37) and (38), respectively.

9. Period behaviour for circular orbits. To end, let us compare the nodal period with the corresponding Keplerian period for circular orbits in the spherical Einstein post-Newtonian gravitational field. Taking into account (38), formula (33) can be written in this situation

$$T_\Omega = T_0(1 + (3/2)(1 - \alpha)x(4 + 27(1 - \alpha)x)). \tag{48}$$

Consider the standard post-Newtonian coordinates ($\alpha = 0$); formula (48) becomes in this case

$$T_\Omega = T_0(1 + (3/2)x(4 + 27x)). \tag{49}$$

Since x is a positive quantity, we have $T_\Omega > T_0$. In other words, for $\alpha = 0$ the post-Newtonian perturbing force acts to decelerate the motion.

For $\alpha = 1$, formula (48) leads immediately to $T_\Omega = T_0$, that is, if we use the spatial standard coordinate system, the motion keeps its Keplerian period.

Lastly, for $\alpha = 2$, expression (48) reads

$$T_\Omega = T_0(1 - (3/2)x(4 - 27x)). \tag{50}$$

This means that there exists a critical value of x, x_c say, such that for $x = x_c = 4/27$ the nodal period and the corresponding Keplerian period coincide. Having in view the expression (15) of x (with p_0 = orbit radius), and recalling the expression of the Schwarzschild radius $R_{Sch} = 2\mu/c^2$, the above coincidence criterion can be formulated as

$$p_0 = (27/8)R_{Sch}. \tag{51}$$

In other words, for an initial radius smaller than $(27/8)R_{Sch}$ the post-Newtonian perturbing force acts to decelerate the motion, and conversely. We may therefore conclude that for concrete astronomical situations the case $\alpha = 2$ entails generally an acceleration of the circular motion as against the Keplerian motion.

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