STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

2

1993

CLUJ-NAPOCA

492305

Anul XXXVIII 1993

STUDIA UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

ON SOME GENERALIZED WENDORFF-TYPE INEQUALITIES

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Dedicated to Professor P Szilágyi on his 60th anniversary

Received. November 20, 1993
AMS subject classification 34A40

REZUMAT. - Asupra unor inegalităţi de tip Wendorff generalizate. În lucrare se deduc inegalităţile de tip Wendorff pe baza inegalităţii operatoriale stabilită în [5] pentru operatori monotoni crescători.

1. Introduction. [illegible] and [illegible] proved in [5] an operatorial inequality analogous to those of Gronwall and Bihari for monotonic and continuous operators. If E is a Banach space and K a cone, then $x \geq y$ if $x-y \in K$. Theorem 1 from [5] states: if u verifies the inequality

$$u \leq A\,u + f \tag{1}$$

where f is a fixed element and $A\,E \to E$ is a monotonically increasing operator, and if the equation $y = A\,y + f$ has a unique solution y^*, then $u \leq y^*$

Gronwall's and Bihari's inequalities result immediately from (1). There also was proved a Bernoulli-type inequality analogous to the corresponding equation of the same name. In [3] there was proved a Riccati-type inequality analogous to the equation of the same type In the present paper we use this method to deduce Wendorff-type inequalities for functions of several variables [4].

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2. Main results. First of all we prove

THEOREM 1. *Let* $m, v \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $c \geq 0$. *If the function* $m(x,y)$ *fulfils the inequality*

$$m(x,y) \leq \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\, m(s,t)\, ds\, dt + c\,,\ x \geq x_0\,,\ y \geq y_0 \tag{2}$$

where $v(x,y)$ *is a monotonically increasing function, and if* u^* *is the unique solution of the equation*

$$\frac{\partial u}{\partial x} = \left(\int_{y_0}^{y} v(x,t)\, dt\right) \cdot u(x,y) \tag{3}$$

then $m(x,y) \leq u^*(x,y)$.

Proof Define the function $g(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\, m(s,t)\, ds\, dt + C$, $g(x_0, y_0) = 0$, and consider the operator

$$Am(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\, m(s,t)\, ds\, dt\,,\quad x \geq x_0\,,\ y \geq y_0\,.$$

By (2) it is obvious the $m(x,y) \leq g(x,y)$. It is also clear that

$$\frac{\partial g}{\partial x} = \int_{y_0}^{y} v(x,t)\, m(x,t)\, dt \leq \left(\int_{y_0}^{y} v(x,t)\, dt\right) g(x,y)$$

Since $g(x,y)$ fulfils the inequality

$$\frac{\partial g}{\partial x} \leq \left(\int_{y_0}^{y} v(x,t)\, dt\right) \cdot g(x,y) \tag{4}$$

the comparison theorem [1] and (3), (4) lead to

$$g(x,y) \leq u^*(x,y)\,,$$

hence $m(x,y) \leq u^*(x,y)$ Also, from

$$u^*(x,y) = c\, \exp\left(\int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\, ds\, dt\right) \tag{5}$$

Wendorff's inequality [4] follows clearly

THEOREM 2. *Consider the functions* $m, v, h \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ *If* $m(x,y)$ *fulfils the inequality*

$$m(x,y) \leq h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\, m(s,t)\, ds\, dt\,,\ x \geq x_0\ \ y \geq y_0\,, \tag{6}$$

where v(x,y) is monotonically increasing, and if $u^*(x,y)$ is the unique solutions of the equation

$$\frac{\partial u}{\partial x} = \left(\int_{y_0}^{y} v(x,t)\,dt\right)\cdot u(x,y) + \int_{y_0}^{y} v(x,t)\;h(x,t)\,dt \tag{7}$$

then $m(x,y) \le u^*(x,y) + h(x,y)$

Proof. In this case the function $g(x,y)$ is

$$g(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,m(s,t)\,ds\,dt,\; g(x_0,y) = 0,$$

and we define the operator

$$Am(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,m(s,t)\,ds\,dt,\; x \ge x_0,\; y \ge y_0 \tag{8}$$

From (6) follows immediately $m(x,y) \le g(x,y) + h(x,y)$. Because $g(x,y)$ fulfils the inequality

$$\frac{\partial g}{\partial x} \le \left(\int_{y_0}^{y} v(x,t)\,dt\right)\cdot g(x,y) + \int_{y_0}^{y} v(x,t)\,h(x,t)\,dt \tag{9}$$

then from the comparison theorem [1] and (7), (9) it results $g(x,y) \le u^*(x,y)$, where $u^*(x,y)$ is the unique solution of the equation (7). Since the solution u^* has the form

$$u^*(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,h(s,t)\,\exp\left(\int_{s}^{t}\int_{t}^{y} v(z,r)\,dz\,dr\right)ds\,dt \tag{10}$$

one obtains the generalized Wendorff's inequality [4]

$$m(x,y) \le h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,h(s,t)\,\exp\left(\int_{s}^{x}\int_{t}^{y} v(z,r)\,dz\,dr\right)ds\,dt. \tag{11}$$

Remark. If $h(x,y)$ is increasing, then $m(x,y)$ fulfils the inequality [4]

$$m(x,y) \le h(x,y)\,\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,ds\,dt\right) \tag{12}$$

The inequalities of this kind can be generalized for functions of n variables. In this case the inequalities between vectors are understood to be componentwise [4]. For $x \ge x_0$, $x, x_0 \in \mathbb{R}^n$, let us use the notation

$$\int_{x_0}^{x} v(s)\,ds = \int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2}\cdots\int_{x_{0n}}^{x_n} v(s_1,s_2,\dots,s_n)\,ds_1\,ds_2\dots ds_n,$$

where $x = (x_1, x_2, \dots, x_n)$, $x_0 = (x_{01}, x_{02}, \dots, x_{0n})$

THEOREM 3. *Let* $m, v, h \in C[\mathbb{R}^n, \mathbb{R}_+]$, $x \ge x_0$. *If m(x) fulfils the inequality*

$$m(x) \le h(x) + \int_{x_0}^{x} v(s)\,m(s)\,ds \tag{13}$$

where $v(s)$ is monotonically increasing, and if $u^*(x)$ is the unique solution of the equation

$$\frac{\partial u}{\partial x_1} = \left(\int_{x_{02}}^{x_2} \cdot \int_{x_{0n}}^{x_n} v(x_1, s_2, \ , s_n)\, ds_2 \ \ ds_n\right) u(x) + \\ + \int_{x_{02}}^{x_2} \ \int_{x_{0n}}^{x_n} v(x_1, s_2, \ ., s_n)\, h(x_1, s_2, .., s_n)\, ds_2 \ \ ds_n \tag{14}$$

then $m(x) \le h(x) + u^*(x)$

Proof Define the function $g(x) = \int_{x_0}^{x} v(s)\, m(s)\, ds$, $g(x_{01}, x_2, .., x_n) = 0$, and the operator

$$Am(x) = \int_{x_0}^{x} v(s)\, m(s)\, ds,\ x \ge x_0 \tag{15}$$

From (13) follows obviously $m(x) \le g(x) + h(x)$ Since $g(x)$ fulfils the inequality

$$\frac{\partial g}{\partial x_1} \le \left(\int_{x_{02}}^{x_2} \cdot \int_{x_{0n}}^{x_n} v(x_1, s_2, \ . \ , s_n)\, ds_2 \ \ ds_n\right) g(x) + \\ + \int_{x_{02}}^{x_2} \ \int_{x_{0n}}^{x_n} v(x_1, s_2, \ , s_n)\, h(x_1, s_2, . ., s_n)\, ds_2 \ . \ ds_n, \tag{16}$$

the comparison theorem [1] and (14), (16) lead to $g(x) \le u^*(x)$, where $u^*(x)$ is the unique solution of equation (14) Since u^* has the form [4]:

$$u^*(x) = \int_{x_{01}}^{x_1} \left[\exp\left(\int_s^x v(z)\, dz\right) \int_{x_{02}}^{x_2} .. \int_{x_{0n}}^{x_n} v(x_1, s_2, \ , s_n)\, h(x_1, s_2, .. , s_n)\, ds_2 \ \ ds_n\right] ds \tag{17}$$

or

$$u^*(x) = \int_{x_0}^{x} v(s)\, h(s)\ \exp\left(\int_s^x v(z)\, dz\right) ds \tag{18}$$

one gets the generalized Wendorff's inequality [4]·

$$m(x) \le g(x) + \int_{x_0}^{x} v(s)\, h(s)\ \exp\left(\int_s^x v(z)\, dz\right) ds,\ x \ge x_0 \tag{19}$$

Remark If $h(x)$ is increasing, then $m(x)$ fulfilds the inequality [4]

$$m(x) \le h(x)\ \exp\left(\int_s^x v(z)\, dz\right) \tag{20}$$

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

A FUNCTIONAL-DIFFERENTIAL MODEL FOR PRICE FLUCTUATIONS IN A SINGLE COMMODITY MARKET

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Dedicated to Professor P Szilagyi on his 60^{th} anniversary

Received. November 25, 1993
AMS subject classification: 34K15, 90A12

REZUMAT. - Un model funcţional-diferenţial pentru fluctuaţia preţului într-o piaţă cu o marfă. Fie $x(t)$ preţul de piaţă al unei mărfi la momentul t. În prezenta lucrare se studiază următorul model:

$$x'(t) = F(x(t), x(t-\tau))x(t),\ t \in \mathbb{R}_+,$$
$$x(t) = \varphi(t),\ t \in [-\tau, 0]$$

1. Introduction. There exist many examples of functional - differential model for price fluctuation in a single commodity market (see [3], [4], [5]) For example, Farahant and Grove ([3]) have studied the following model

$$x'(t) = \left[\frac{a}{b + x^n(t)} - \frac{cx^m(t-\tau)}{d + x^m(t-\tau)}\right]x(t), \tag{1}$$

for all $t \in \mathbb{R}_+$,

$$x(t) = \varphi(t),\ \ t \in [-\tau, 0], \tag{2}$$

where $a, b, c, d, \tau, m \in \mathbb{R}_+$ and $n \in [1, +\infty[$

Our purpose here is to study the following model

$$x'(t) = F(x(t), x(t-\tau))x(t),\quad t \in \mathbb{R}_+; \tag{3}$$

$$x(t) = \varphi(t),\ t \in [-\tau, 0]. \tag{4}$$

2. General remarks. We consider the Cauchy problem (3) + (4) In what follow we

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suppose that $F \in (\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $\varphi \in C([-\tau,0], \mathbb{R}_+)$ Let

$X := C([-\tau, +\infty[, \mathbb{R}_+) \cap C^1(\mathbb{R}_+, \mathbb{R}_+)$.

We have

LEMMA 1. *Let $x^* \in X$ be a solution of* (3) + (4) *Then*

(i) $\varphi(0) = 0$ *implies* $x^*(t) = 0$, *for all* $t \in \mathbb{R}_+$

(ii) $\varphi(0) > 0$ *implies* $x^*(t) > 0$, *for all* $t \in \mathbb{R}_+$.

Proof. From (3) + (4) we have that

$$x(t) = \varphi(0) \exp \int_0^t F(x(s), x(s-\tau))\, ds. \tag{5}$$

LEMMA 2. *We suppose that*

(a) *$F(\cdot,v)$ is locally Lipschitz,*

(b) *there exists $M_F > 0$ such that*

$|F(u,v)| \leq M_F$, *for all* $u, v \in \mathbb{R}_+$.

Then the problem (3) + (4) *has in X a unique solution, x^*.*

Proof Let $x^* \in C([-\tau, t_+[, \mathbb{R}_+) \cap C^1([0, t_+[, \mathbb{R}_+)$ be a maximal solution of (3) + (4) From (5) and (b) we have

$x(t) \leq \varphi(0) e^{M_F t}$, for all $t \in [0, t_+[$

From the steps method (see [5], [6], [7]) and the theorem of the maximal solution (see [1]) we have that there exists a unique x^* and $t_+ = +\infty$.

3. A model in the case of naive consumer. Now we consider the following model

$$x'(t) = [f(x(t)) - g(x(t-\tau))]\, x(t), \quad t \in \mathbb{R}_+, \tag{6}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{7}$$

where $\tau > 0$, $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\varphi \in C([-\tau,0), \mathbb{R}_+)$.

We have

THEOREM 1. *We suppose that*

(a) f *is locally Lipschitz,*

(b) $t_1 < t_2$ *implies* $f(t_1) > f(t_2)$, $t_1, t_2 \in \mathbf{R}_+$,

(c) $f(0) > 0$ *and* $\lim_{t \to +\infty} f(t) = 0$,

(d) $t_1 < t_2$ *implies* $g(t_1) < g(t_2)$, $t_1, t_2 \in \mathbf{R}_+$,

(e) $g(0) = 0$ *and* $\lim_{t \to +\infty} g(t) > 0$,

(f) $\varphi(0) > 0$.

Then

(i) *the equation* (6) *has a unique positive equilibrium solution,* r^*;

(ii) *the problem* (6) + (7) *has in* X *a unique solution,* x^*, *and this solution is positive;*

(iii) *there exists* $m, M \in \mathbf{R}_+$, $0 < m < M$, *such that* $m \le x^*(t) \le M$, *for all* $t \in \mathbf{R}_+$;

(iv) *if* x^* *is* r^* *-nonoscillatory, then*

$$\lim_{t \to +\infty} x^*(t) = r^*$$

Proof. (i) Follows from the continuity of f and g and conditions (c) and (e), (b) and (d).

(ii) Follows from Lemma 1 and Lemma 2

(iii) See the proof of the Theorem 1 in [3]

(iv) Let $T > 0$ be such that $x(t) < r^*$, for all $t \ge T$ Then we have that $F(x(t),x(t-\tau)) > 0$ for all $t > T + \tau$. This implies that $x'(t) > 0$ for all $t > T + \tau$ Thus there exists $\lim_{t \to +\infty} x(t)$ We suppose that $\lim_{t \to +\infty} x(t) = 1 < r^*$ Then $F(1,1) < 0$ This is in contradiction with a result of Barbălat (see [2]) from which follows that $\lim_{t \to \infty} x'(t) = 0$

Remark 1 See [3] for $f(u) = \dfrac{a}{b + u^m}$ and $g(u) = \dfrac{cu^m}{d + u^m}$

4. Coincidence points and equilibrium solutions. We consider the equation (6) where f and $g \in C(\mathbb{R}_+,\mathbb{R}_+)$ Let E be the set of equilibrium solutions of (6), and $E_+ = \{r \in E \mid r > 0\}$ We also denote:

$$C(f,g) := \{t \in \mathbb{R}_+ \mid f(t) = g(t)\},$$

$$C_+(f,g) := \{t \in C(f,g) \mid t > 0\}$$

We remark that

$$E_+ = C_+(f,g)$$

From this remark we have the following results on the equilibrium solution of the equation (6).

LEMMA 3. *If there exist $\alpha,\beta \in \mathbb{R}_+$, such that $f(\alpha) > g(\alpha)$ and $f(\beta) < g(\beta)$, then*

$$E_+ \cap \,]\alpha,\beta[\, \neq \varnothing$$

If f is strictly decreasing and g is strictly increasing, then $E_+ = \{r^*\}$

LEMMA 4 *If, $f(0) \neq g(0)$, g is surjective and there exists $t_0 \in \mathbb{R}_+$ such that $f(t_0) < g(t_0)$, then $E_+ \neq \varnothing$*

LEMMA 5. *If $g(\mathbb{R}_+) \supset [a,b]$, $f(\mathbb{R}_+) \subset [a,b]$ and $f(0) \neq g(0)$, then $E_+ \neq \varnothing$.*

LEMMA 6 *We suppose that*

(i) $g(\mathbb{R}_+) = \mathbb{R}_+$ *and* $f(0) \neq g(0)$,

(ii) *there exists $a \in \,]0,1[$ such that*

$$|f(t_1) - f(t_2)| \leq a\,|g(t_1) - g(t_2)|$$

for all $t_1, t_2 \in \mathbb{R}_+$.

Then, $E_+ \neq \varnothing$.

If g is bijective, then, $E_+ = \{r^*\}$

Proof Follows from a general coincidence theorem of Goebel (see [4] and [7])

5. ***Remarks.***

5.1 The following problem arises in the study of the equilibrium solution of the equation (1).

Problem 1. Let $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$. We suppose that $f(\mathbb{R}_+) =]0, M_f]$ and $g(\mathbb{R}_+) = [0, M_g[$. Establish conditions on f and g which imply that $C(f,g) \neq \emptyset$.

5.2 Consider the following problem

$$x'(t) = \left[f(x(t)) - g\left(x\left(\frac{1}{2}t\right)\right)\right]x(t),\ t \in \mathbb{R}_+, \tag{8}$$

$$x(0) = x_0. \tag{9}$$

For the problem (8) + (9) we have

THEOREM 2. *We suppose that*

(a) $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* f *and* g *are locally Lipschitz,*

(b) f *is strictly decreasing,*

(c) $f(0) > 0$ *and* $\lim\limits_{t \to \infty} f(t) = 0$,

(d) g *is strictly increasing,*

(e) $g(0) = 0$ *and* $\lim\limits_{t \to \infty} g(t) > 0$,

(f) $x_0 > 0$.

Then

(i) *the equation* (8) *has a unique positive equilibrium solution,* r^*,

(ii) *the problem* (8) + (9) *has in* $C^1(\mathbb{R}_+, \mathbb{R}_+)$ *a unique solution,* x^*, *and this solution is positive;*

(iii) *there exists* $m, M \in \mathbb{R}_+$, $0 < m < M$, *such that*

$m \le x^*(t) \le M$, *for all* $t \in \mathbb{R}_+$,

(iv) if x^* *is* r^**-nonoscillatory, then* $\lim\limits_{t \to +\infty} x^*(t) = r^*$.

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

ON SOME MODELS OF PRICE FLUCTUATION IN A MARKET ECONOMY

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Dedicated to Professor P. Szilágyi on his 60th anniversary

Received: December 10, 1993
AMS subject classification: 34K15, 90A12

REZUMAT. - Asupra unor modele de fluctuaţii ale preţului într-o economie de piaţă. Consumatorul, într-o economie de piaţă, trebuie să-şi procure marfa la preţul pieţei. Dacă preţul creşte, de obicei cererea scade. Pentru anumite tipuri de mărfuri, un anumit consumator "naiv", nu-şi diminuează cererea deşi preţul creşte. În această lucrare prezentăm un astfel de model de fluctuaţie a preţului, cu întârziere, şi arătăm că există o soluţie pozitivă unică şi mărginită.

In a market economy a consumer must purchase his consumtion of commodity at the market price.

In considering the dynamics of price, production, and consumption of a particular commodity, Bélair and Mackey [1] have studied the model

$$p'(t) = p(t)\, f(p_d, p_s), \tag{1}$$

where $p(t)$ is the function which means the price of commodity at the moment t, and p_d, p_s are the demand price respectively the supply price of this commodity.

A special case of the equation (1) is so called naive consumer model

$$p'(t) = p(t)\left(\frac{a}{b+p^q(t)} - \frac{cp^r(g(t))}{d+p^r(g(t))}\right) \tag{2}$$

because the demand never decreases as price increases.

In the equation (2) we have $a,b,c,d,r \in (0,\infty)$, which are constants, and $q \in [1,\infty)$ is

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a constant.

The function $g \in C([0,\infty),\mathbb{R}_+)$ is the deviating argument which fullfiles the conditions

$$0 \leq g(t) < t, \quad \text{for all } t \in [0,\infty) \tag{3}$$

$t - g(t)$ is bounded for $t \geq 0$, and $\lim_{t\to\infty} (t-g(t)) = \alpha > 0$ (4)

In this paper we prove that there exists the solution p of equation (2), with the conditions

$$p \in C^1([0,\infty),\mathbb{R}_+), \quad p(0) = p_0, \tag{5}$$

and it is unique

Precisely we have

THEOREM 1 *If* $a,b,c,d,r \in (0,\infty)$, $q \in [1,\infty)$ *and the function* $g \in C([0,\infty),\mathbb{R}_+)$ *fullfiles the conditions* (3) *and* (4), *then there exists an unique positive solution of problem* (2)+(5) *and it is bounded for all* $t \in [0,\infty)$.

Proof By the method of steps, it is clear, from the equation (2), that as long as the solution p exists, it satisfies the relation

$$p(t) = p_0 \quad \exp\left(\int_0^t \left(\frac{a}{b+p^q(s)} - \frac{cp^r(g(s))}{d+p^r(g(s))}\right)ds\right) \tag{6}$$

and so p is unique as long as it exists and it is positive as long as it exists

We prove that p is bounded for all $t \in [0,\infty)$, and so in particular, p exists for all $t \in [0,\infty)$.

It is clear that p is bounded from below For prove that p is bounded from above we suppose that this is not the case and so we obtain a contradiction

We suppose that p is not bounded from above Then there exists $T > 0$ and a sequence $(t_n)_{n\geq 1}$, $t_n \to T$ as $n \to \infty$, such that

$$\lim_{n\to\infty} p(t_n) = \infty, \text{ and } p'(t_n) \geq 0$$

The contradiction will come from the consideration of the following two cases:

1° $\lim_{n\to\infty} \inf p(g(t_n)) > 0$, respectively

2°. $\lim_{n\to\infty} \inf p(g(t_n)) = 0$.

1° We suppose that $\lim_{n\to\infty} \inf p(g(t_n)) > 0$. Then there exists $k > 0$ and $n_0 \in \mathbb{N}$, such that $p(g(t_n)) \geq k$ for $n \geq n_0$. This implies that $\dfrac{p^r(g(t_n))}{d+p^r(g(t_n))}$ is bounded away from zero.

How $q \geq 1$ we have also that $\dfrac{p(t_n)}{b+p^q(t_n)}$ is bounded.

It follows from the equation (2), with t replaced by t_n, that $\lim_{n\to\infty} p'(t_n) = -\infty$.

But this is impossible because $p'(t_n) \geq 0$.

2° We suppose that $\lim_{n\to\infty} \inf p(g(t_n)) = 0$. If is necessary, by passing to a subsequence, we may assume that $\lim_{n\to\infty} p(g(t_n)) = 0$

By integrate of equation (2) from $g(t_n)$ to t_n we obtain

$$p(t_n) - p(g(t_n)) = \int_{g(t_n)}^{t_n} \frac{ap(s)}{b+p^q(s)}\,ds - \int_{g(t_n)}^{t_n} \frac{cp^r(g(s))}{d+p^r(g(s))}\,ds \leq$$
$$\leq \int_{g(t_n)}^{t_n} \frac{ap(s)}{b+p^q(s)}\,ds = (t_n - g(t_n))\frac{ap(c_n)}{b+p^q(c_n)},$$

form some $c_n \in [t_n, g(t_n)]$

But this is impossible because $\lim_{n\to\infty}(p(t_n) - p(g(t_n))) = \infty$, $\lim_{n\to\infty}(t_n - g(t_n)) = \alpha > 0$, and the function $F: [0,\infty) \to \mathbb{R}$, given by $F(u) = \dfrac{u}{b+u^q}$, is a bounded function.

Therefore p is a bounded function for all $t \in [0,\infty)$, and so, in particular, p exists for all $t \in [0,\infty)$.

We next claim that $\lim_{t\to\infty} \inf p(t) \neq 0$

For the sake of contradiction, we suppose that this is not the case. Then there exists a sequence $(t_n)_{n\geq 1}$, $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} p(t_n) = 0$ and $p'(t_n) \leq 0$

It is useful to rewrite equation (2) as

$$\frac{p'(t)}{p(t)} = \frac{a}{b+p^q(t)} - \frac{cp^r(g(t))}{d+p^r(g(t))} \tag{7}$$

Then it folllows from equation (7) when t is replaced by t_n that

$$\frac{cp^r(g(t_n))}{d+p^r(g(t_n))} = \frac{a}{b+p^q(t_n)} \to \frac{a}{b} \quad \text{as} \quad n \to \infty,$$

and so there exists $k > 0$ and $n_0 \in \mathbb{N}$ such that $p(g(t_n)) \geq k$ for $n \geq n_0$. Now, by integrating equation (7) from $g(t_n)$ to t_n, we obtain

$$\ln \frac{p(t_n)}{p(g(t_n))} = \int_{g(t_n)}^{t_n} \left(\frac{a}{b+p^q(s)} - \frac{cp^r(g(s))}{d+p^r(g(s))} \right) ds \tag{8}$$

But, this is impossible because

$$\lim_{n\to\infty} \ln \frac{p(t_n)}{p(g(t_n))} = -\infty$$

while the right hand side of relation (8) is bounded.

Thus, the proof is complete.

We have also

THEOREM 2. *The equation* (2) *has an unique positive equilibrium solution.*

Proof The equilibrium solution of equation (2) is that which is independent of t, therefore that for which $p'(t) = 0$, for all $t \in [0,\infty)$. We obtain, with the notation p for the equilibrium solution, equation

$$\frac{a}{b+p^q} - \frac{cp^r}{d+p^r} = 0 \tag{9}$$

From the equation (9) we obtain

$$cp^{q+r} + (bc-a)p^r - ad = 0. \tag{10}$$

Let be the function $f : [0,\infty) \to \mathbb{R}$, defined by

$$f(p) = cp^{q+r} + (bc-a)p^r - ad$$

How $f'(p) = c(q+r)p^{q+r-1} + (bc-a)rp^{r-1}$, the equation $f'(p) = 0$ has the real positive solution only if $bc - a < 0$, and how

$$f(0) = -ad < 0, \quad \lim_{p\to\infty} f(p) = +\infty$$

we get that the equation (10) has an unique positive solution which is the equilibrium solution of the equation (2) The proof is complete

Remark 1 Since the equation (7) can be written in the form

$$\frac{p'(t)}{p(t)} = \frac{a(p-p(t))}{(b+p^q(t))(b+p^q)} + \frac{cd(p^r-p^r(g(t)))}{(d+p^r)(d+p^r(g(t)))}$$

we see that $p(t)$ converges monotonically to p, the equilibrium solution of equation (2)

Remark 2. The Theorem 1 is a generalization of a result of A.M Farahani and E A Grove [2]

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

DIFFERENTIAL INCLUSIONS FOR ELLIPTIC SYSTEMS WITH DISCONTINUOUS NONLINEARITY

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Received *December 10, 1993*
AMS subject classification *35J50, 35R05*

REZUMAT. - Incluziuni diferenţiale pentru sisteme eliptice cu nelinearitate discontinuă. Se studiază probleme la limită pentru sisteme de incluziuni diferenţiale de forma

$$L_i u_i \in f_i - G_i(u), \qquad u=(u_1, \ldots, u_m),$$

unde L_i sunt operatori lineari, de ordinul doi uniform eliptici

ABSTRACT. - Boundary value problems for differential inclusions systems of the form

$$L_i u_i \in f_i - G_i(u), \qquad u=(u_1, \ldots, u_m),$$

are studied, where L_i are uniformly elliptic linear operators of order two

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$ In this paper we study existence conditions for the boundary value problem of the form

$$\begin{cases} L_i u_i \in f_i - G_i(u) & \text{in } \Omega \qquad u=(u_1, \ldots, u_m) \\ u_i|_{\partial\Omega} = 0 & i = 1, \ldots, m \end{cases} \tag{1}$$

where L_i are linear elliptic operators of order two, G_i are multivalued mappings, f_i given functions (functionals) Such differential inclusions appear, for instance, in the study of boundary value problems

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$$\begin{cases} L_i u_i + g_i(u) = f_i & \text{in } \Omega \quad i=1, \dots, m \\ u_i\big|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

when the functions g_i $\mathbb{R}^m \to \mathbb{R}$ have discontinuities In this case we shall replace the functions g_i by multivalued mappings, in which the jumps are filled in.

Let $H^1(\Omega) = \{u \in L^2(\Omega) \mid \frac{\partial u}{\partial x_i} \in L_2(\Omega),\ i=1, \dots, n\}$, $H_0^1(\Omega)$ the space of all the functions of $H^1(\Omega)$ with generalized homogeneous boundary values. In $H_0^1(\Omega)$ we use the scalar product resp norm

$$(u,v)_1 = \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx, \qquad \|u\|_1^2 = \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx \tag{3}$$

$H^{-1}(\Omega)$ will denote the dual space of $H_0^1(\Omega)$

The elliptic operators L_i in (1) are of the form

$$L_i u_i = -\sum_{j,k=1}^n \frac{\partial}{\partial x_j}\left[a_{jk}^i \frac{\partial u_i}{\partial x_k}\right] + a_0^i(x)\, u_i \qquad i=1, \dots, m \tag{4}$$

We assume that a_{jk}^i, $a_0^i \in L^\infty(\Omega)$, $a_0^i(x) \geq 0$ a e. in Ω, and there exists a positive constant γ such that

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq \gamma\|\xi\|^2 \qquad \text{for a e } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n \tag{5}$$

We present here a possible filling in of the jumps of the functions g_i Suppose g_i $\mathbb{R}^m \to \mathbb{R}$, $i=1, \dots, m$ are given functions in $L^\infty_{loc}(\mathbb{R}^m)$, that is, the restriction of g_i to any bounded measurable set $A \subset \mathbb{R}^m$ is in $L^\infty(A)$.

Let $\lambda \in [0,\lambda_0]$ a real variable,

$$\begin{cases} \bar{g}_{i\lambda}(s) = \operatorname*{ess\,sup}_{|t-s|<\lambda} g_i(t) \\ \\ \underline{g}_{i\lambda}(s) = \operatorname*{ess\,inf}_{|t-s|<\lambda} g_i(t) \end{cases} \qquad s,t\in\mathbb{R}^m, \quad \lambda\in[0,\lambda_o] \tag{6}$$

It is clear that $\underline{g}_{i\lambda}(s) \leq \bar{g}_{i\lambda}(s)$, for fixed s, $\underline{g}_{i\lambda}(s)$ is an increasing function in λ, $\bar{g}_{i\lambda}(s)$ is decreasing, both $\underline{g}_{i\lambda}(s)$ and $\bar{g}_{i\lambda}(s)$ are bounded for $\lambda \in [0,\lambda_0]$. Let

$$\underline{g}_i(s) = \lim_{\lambda\searrow 0} \underline{g}_{i\lambda}(s), \qquad \bar{g}_i(s) = \lim_{\lambda\searrow 0} \bar{g}_{i\lambda}(s) \qquad s \in \mathbb{R}^m \tag{7}$$

$$G_i(s) = [\underline{g}_i(s), \bar{g}_i(s)], \quad G(s) = \prod_{i=1}^{m} [\underline{g}_i(s), \bar{g}_i(s)] \tag{8}$$

In this paper we study the following problem:

If $f=(f_1, \ldots, f_m)$ with $f_i \in H^{-1}(\Omega)$ is given, we look for all $u=(u_1, \ldots, u_m)$, $u_i \in H_0^1(\Omega)$ for which there exists at least one $v=(v_1, \ldots, v_m)$, $v_i \in H^{-1}(\Omega) \cap L^\infty(\Omega)$ such that $v(x) \in G(u,x)$ a.e. in Ω and

$$L_i u_i + v_i = f_i \qquad (L_i u_i \in f_i - G_i(u)\ ) \qquad i=1,\ldots,m \tag{9}$$

The equalities are understood in variational sense, that is

$$\int_\Omega \sum_{i=1}^{m} \left\{ \sum_{j,k=1}^{n} a_{jk}^i(x) \frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_k} + a_0^i(x)\, u_i w_i \right\} dx + \int_\Omega \sum_{i=1}^{m} v_i(x) w_i(x)\, dx = \sum_{i=1}^{m} \langle f_i, w_i\rangle \qquad \forall\ w_i \in H_0^1(\Omega). \tag{10}$$

Such problems, for one equation ($m=1$), are studied in many recent papers, i.e. [1-2],[4],[6-8]. We follow some ideas of J. Rauch[7].

THEOREM 1. *If the following conditions are fulfilled:*

1° L_i are uniform elliptic operators of the form (4),

a^i_{jk}, $a^i_0 \in L^\infty(\Omega)$, $\quad a^i_0(x) \geq 0 \quad$ a.e. in Ω

2° $g_i \in L^\infty_{loc}(\mathbb{R}^m) \qquad i=1,\ldots,m$

3° There exist a positive constant r and an angle $\Theta_0 \in \left[0,\frac{\pi}{2}\right)$ *such that*

$$\sum_{i=1}^{m} s_i g_i(s) \geq 0 \quad \text{and} \quad \Theta \leq \Theta_0 \quad \text{for all } s \in \mathbb{R}^m \text{ with } \|s\|_{\mathbb{R}^m} \geq r \tag{11}$$

where $\Theta = \angle(s,g(s))$ *is the angle between s and g(s) in* $\mathbb{R}^m$,

then for all $f=(f_1,\ldots,f_m)$, $f_i \in H^{-1}(\Omega)$ *there exists at least one* $u=(u_1,\ldots,u_m)$, $u_i \in H^1_0$ *and* $v=(v_1,\ldots,v_m)$, $v_i \in H^{-1} \cap L^\infty$ *such that* $v(x) \in G(u(x))$ *a.e. in* Ω *and (10) is fulfilled for all* $w_i \in H^1_0(\Omega)$.

Proof. The space $H^1_0(\Omega)$ is separable, therefore we can find a countable set of finite-dimensional subspaces $V_1 \subset V_2 \subset \ldots \subset V_N \subset V_{N+1} \subset \ldots$ of $H^1_0(\Omega)$ with the property that for all $u \in H^1_0(\Omega)$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and $u_\varepsilon \in V_N$ such that $\|u-u_\varepsilon\| < \varepsilon$.

Let ρ be a mollyfier in $\mathbb{R}^m$, that is: $\rho: \mathbb{R}^m \to \mathbb{R}$, $\rho \in C^\infty_0(\mathbb{R}^m)$, supp $\rho \subset \overline{B(0,1)}$, $\rho(x) \geq 0$ and $\int_{\mathbb{R}^m} \rho(x)dx = 1$. We denote

$$\rho_N(s) = N\rho(Ns), \quad g_{iN}(s) = (\rho_N * g_i)(s) \quad i=1,\ldots,m, \qquad s \in \mathbb{R}^m, \quad N=1,2,\ldots \tag{12}$$

From the hypotheses $g_i \in L^\infty_{loc}(\mathbb{R}^m)$ results, that $g_{iN} \in C^\infty_0(\mathbb{R}^m)$ and the set $\{g_{iN}(s) \mid i=1,\ldots,m,\ N=1,2,\ldots\} \subset \mathbb{R}$ is uniformly bounded with respect to s, when s describes a bounded set.

We consider the following family of perturbed boundary value problems

$$\begin{cases} L_i u_i + g_{iN}(u) = f_i \\ \quad u_i\big|_{\partial\Omega} = 0 \qquad N=1,2,\dots, \qquad u=(u_1,\dots,u_m) \end{cases} \tag{13}$$

and use the Galerkin procedure to determine approximate solutions u_N. We look for $u_i \in V_N$ for which

$$\sum_{i=1}^{m} a_i(u,w) + \int_{\Omega} \sum_{i=1}^{m} g_{iN}(u) w_i \, dx = \sum_{i=1}^{m} \langle f_i, w_i \rangle \quad \forall\, w_i \in V_m \tag{14}$$

where

$$a_i(u,w) = \int_{\Omega} \left[\sum_{j,k=1}^{n} a^i_{jk}(x) \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_k} + a^i_0(x) u_i w_i \right] dx \tag{15}$$

We prove that (14) has at least one solution $u_N=(u_{1N},\dots,u_{mN})$. For this we introduce the mapping $T_N : (V_N)^m \to (V_N')^m$ defined by

$$\langle T_N(u), w \rangle = \sum_{i=1}^{m} a_i(u,w) + \int_{\Omega} \sum_{i=1}^{m} g_{iN}(u) w_i \, dx - \sum_{i=1}^{m} \langle f_i, w_i \rangle \tag{16}$$

From the construction of g_{iN} results that for all $u \in (V_N)^m$ there exists a positive constant C_u such that $|g_{iN}(u(x)) w_i(x)| \le C_u |w_i(x)|$ a.e in Ω $\forall\, w_i \in H^1_0(\Omega)$

Thus the integrals $\int_{\Omega} g_{iN}(u) w_i \, dx$ exist for all $u \in (V_N)^m$ and $w_i \in H^1_0(\Omega)$ (Lebesgue theorem)

We see also that T_N is a continuous operator

We prove that the equation $T_N u = 0$ has at least one solution for all N To see that this is true we apply a corollary of the Brouwer fixed point theorem From (16) for $w = u$ we obtain

$$\langle T_N u, u \rangle = \sum_{i=1}^{m} a_i(u,u) + \int_{\Omega} \sum_{i=1}^{m} g_{iN}(u) u_i \, dx - \sum_{i=1}^{m} \langle f_i, u_i \rangle \tag{17}$$

The operators $L_i u_i$ satisfy the uniform ellipticity conditions (5) and $a^i_0(x) \ge 0$, so we have

$$\sum_{i=1}^{m} a_i(u,u) \geq \gamma \int_\Omega \sum_{j=1}^{m} (\nabla u_j)^2\, dx \tag{18}$$

The hypotheses on g gives that $\sum_{i=1}^{m} g_{iN}(s)s_i$ is uniformly bounded in $\overline{B(0,r)}$ with respect to N and

$$\sum_{i=1}^{m} g_{iN}(s)s_i \geq 0 \quad \text{if} \quad s \in \mathbb{R}^m \setminus B(0,r), \tag{19}$$

where $B(0,r) = \{s \in \mathbb{R}^m \mid \|s\| < r\}$.

Then there exists a positive constant C such that

$$<T_N(u),u> \geq \gamma\|u\|_1^2 - C - \|f\|\cdot\|u\|_1 \qquad \forall\, u \in (V_N)^m \tag{20}$$

Thus we have

$$<T_N u, u> \geq 0 \qquad N=1,2,... \tag{21}$$

if $\|u\|_1 \geq R$, with an $R>0$ conveniently chosen. R doesn't depend on N. V_N is finite-dimensional linear space, therefore all norms on V_N are equivalent. If we change the norm in V_N, then (21) remains valid eventually with a different R. It follows then [5,page 58] that for each N there is a $u_N \in (V_N)^m$ with $T_N(u_N) = 0$ and $\|u_N\|_1 \leq R$.

The space $H_0^1(\Omega)$ is reflexive, $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, therefore the bounded set $\{u_N \mid N=1,2,\ \} \subset H_0^1(\Omega)$ is relatively weakly compact in $H_0^1(\Omega)$ and relatively strongly compact in $L^2(\Omega)$. Thus $\{u_N \mid N=1,2,.\ \}$ contains a subsequence, denoted by $(u_N)_{N\geq1}$ too, so that

$$\begin{aligned} &u_{iN} \rightharpoonup u_i && \text{weakly in } H_0^1(\Omega) \\ &u_{iN} \to u_i && \text{strongly in } L^2(\Omega) \qquad i=1,...,m \\ &u_{iN}(x) \to u_i(x) && \text{a.e in } \Omega \end{aligned}$$

Next we investigate the convergence of the sequence $(g_N(u_N))_{N\geq1}$ For this we use the following criterion on weak compactness in $L^1(\Omega)$. Let $\Omega \subset \mathbb{R}^n$ a bounded domain, $\mathcal{F} \subset L^1(\Omega)$ a bounded part of $L^1(\Omega)$. $\mathcal{F}$ is relatively compact in the topology $\sigma(L^1, L^\infty)$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_A |f(x)|\, dx < \varepsilon \quad \forall f \in \mathcal{F} \text{ and } \forall A \subset \Omega \text{ measurable with } \textit{meas}\, A < \delta$$

[3, page 76].

We try to estimate $|g_{iN}(u_N)|$ by $\int_\Omega |\sum_{i=1}^m u_{iN} g_{iN}(u_N)|\, dx$. Condition (11) gives

$$\int_\Omega |\sum_{i=1}^m u_{iN} g_{iN}(u_N)|\, dx =$$

$$= \int_{\|u_N(x)\|<r} |\sum_{i=1}^m u_{iN} g_{iN}(u_N)|\, dx + \int_{\|u_N(x)\|>r} \sum_{i=1}^m u_{iN} g_{iN}(u_N)\, dx \le \tag{22}$$

$$\le \int_\Omega \sum_{i=1}^m u_{iN} g_{iN}(u_N)\, dx + 2 \int_{\|u_N(x)\|<r} |\sum_{i=1}^m u_{iN} g_{iN}(u_N)|\, dx$$

Since $T_N(u_N) = 0$, we have

$$\sum_{i=1}^m a_i(u_N, u_N) + \int_\Omega \sum_{i=1}^m g_{iN}(u_N) u_{iN}\, dx - \langle f, u_N \rangle = 0 \tag{23}$$

But $\{u_N \mid N=1,2,..\}$ is bounded in $H_0^1(\Omega)$, $a_i(u_N, u_N) \ge \gamma \|u_{iN}\|_1^2 \ge 0$, so we find a positive constant C_1 such that

$$\int_\Omega \sum_{i=1}^m g_{iN}(u_N) u_{iN}\, dx \le C_1$$

This and (22) give

$$\int_\Omega |\sum_{i=1}^m u_{iN} g_{iN}(u_N)|\, dx \le C_1 + \int_{\|u_N(x)\|<r} |\sum_{i=1}^m u_{Ni} g_{Ni}(u_N)|\, dx \le \tag{24}$$

$$\le C_1 + C_2 r \textit{ meas } \Omega \le C_3$$

Using the hypotheses of the theorem and this estimate we prove that the set

$\{g_N(u_N) \mid N=1,2,..\}$ is relatively weakly compact in $L^1(\Omega)$.

Because of condition 3° from theorem 1 we have

$$\sum_{i=1}^{m} s_i g_i(s) \geq C_4 \|s\| \, \|g(s)\| \quad \text{for} \quad \|s\| \geq r,$$

thus

$$\begin{aligned} |g_{iN}(s)| &\leq \|g_N(s)\| \leq \sup_{\|t\|\leq N} \|g_N(t)\| + \frac{\|s\|}{N} \|g_N(s)\| \leq \\ &\leq \sup_{\|t\|\leq N} \|g_N(t)\| + \frac{1}{C_4 N} \sum_{i=1}^{m} |s_i g_{iN}(s)| \end{aligned} \tag{25}$$

For a given $\varepsilon > 0$ we find an $N_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\frac{1}{C_4 N} \int_\Omega \Big| \sum_{i=1}^{m} u_{iN} g_{iN}(u_N) \Big| \, dx < \frac{\varepsilon}{2} \qquad \text{for all} \quad N \geq N_0 \tag{26}$$

and

$$\delta \operatorname*{ess\,sup}_{\|t\|\leq N+1} \|g_N(t)\| < \frac{\varepsilon}{2} . \tag{27}$$

If $A \subset \Omega$ is a measurable set and $meas\ A < \delta$, then by (25), (26) and (27)

$$\begin{aligned} \int_A |g_{iN}(u_N)| \, dx &\leq \int_A \sup_{\|u_N(x)\|\leq N+1} \|g_N(u_N(x))\| \, dx + \frac{1}{C_4 N} \int_A \Big| \sum_{i=1}^{m} u_{iN} g_{iN}(u_N) \Big| \, dx \leq \\ &\leq \delta \operatorname*{ess\,sup}_{\|s\|\leq N+1} \|g_N(s)\| + \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

Thus $\{g_N(u_N)\}$ contains a subsequence, denoted by $\{g_N(u_N)\}$ too, which is weakly convergent in $L^1(\Omega)$ Let v_i the weak-limit of $\{g_{iN}(u_N)\}$. But $T_N(u_N)=0$, therefore

$$\sum_{i=1}^{m} a_i(u_N, w) + \int_\Omega \sum_{i=1}^{m} g_{iN}(u_N) w_i \, dx = \sum_{i=1}^{m} \langle f_i, w_i \rangle \qquad \forall\ w_i \in V_N$$

If $N \to \infty$ we obtain

$$\sum_{i=1}^{m} a_i(u, w) + \int_\Omega \sum_{i=1}^{m} v_i w_i \, dx = \sum_{i=1}^{m} \langle f_i, w_i \rangle \qquad \forall\ w_i \in V_N \tag{28}$$

The number N in (28) is arbitrary, and $\bigcup_{N=1}^{\infty} V_N$ is dense in $H_0^1(\Omega)$, therefore (28) can be extended to all $w_i \in H_0^1(\Omega)$

The correspondence $w_i \mapsto \int_\Omega v_i w_i \, dx$ defines a linear bounded functional on $H_0^1(\Omega)$, so $v_i \in H^{-1}(\Omega)$ and

$$L_i u_i + v_i = f_i$$

Finally we must prove that $v \in G(u)$, or equivalently

$$\underline{g}_i(u(x)) \leq v_i(x) \leq \bar{g}_i(u(x)) \qquad \text{a.e in } \Omega \qquad i=1,\dots,m$$

The shown properties of the sequence $\{u_N\}$ imply that for all $\eta > 0$ there exists a measurable set $A \subset \Omega$, such that $meas\ A < \eta$ and $u_N \to u$ uniformly on $\Omega \setminus A$ Thus for any $\lambda > 0$ we can find an $N_0 \in \mathbf{N}$ such that

$$|u_{iN}(x) - u_i(x)| < \frac{\lambda}{2} \qquad \text{for all } x \in \Omega \setminus A \text{ and } N \geq N_0$$

From the definition of the functions $\underline{g}_{i\lambda}$ and $\bar{g}_{i\lambda}$ results that

$$\underline{g}_{i\lambda}(u(x)) \leq g_{iN}(u_N(x)) \leq \bar{g}_{i\lambda}(u(x)) \qquad x \in \Omega \setminus A, \quad N \geq N_0$$

and

$$\int_{\Omega \setminus A} \underline{g}_{i\lambda}(u(x)) w(x)\, dx \leq \int_{\Omega \setminus A} g_{iN}(u_N(x)) w(x)\, dx \leq \int_{\Omega \setminus A} \bar{g}_{i\lambda}(u(x)) w(x)\, dx$$

$\forall\, w \in L^\infty(\Omega)$ for which $w(x) \geq 0$ a e in Ω But $g_{iN}(u_N) \to v_i$ in $L^1(\Omega)$, thus after passing to limit ($N \to \infty$, $\lambda \searrow 0$) we obtain

$$\int_{\Omega \setminus A} \underline{g}_i(u(x)) w(x)\, dx \leq \int_{\Omega \setminus A} v_i(x) w(x)\, dx \leq \int_{\Omega \setminus A} \bar{g}_i(u(x)) w(x)\, dx$$

Because $w(x) \geq 0$ a.e in Ω (otherwise w is arbitrary from $L^\infty(\Omega)$), results

$$\underline{g}_i(u(x)) \leq v_i(x) \leq \bar{g}_i(u(x)) \qquad \text{a.e. in} \quad \Omega \setminus A \quad .$$

But η can be chosen arbitrarily small and *meas* $A < \eta$, so it follows that

$$\underline{g}_i(u(x)) \leq v_i(x) \leq \bar{g}_i(u(x)) \qquad \text{a.e. in} \quad \Omega \quad .$$

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

POLYNOMIAL NATURAL SPLINE FUNCTIONS OF EVEN DEGREE

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Dedicated to Professor P Szilágyi on his 60^{th} anniversary

Received December 20, 1993
AMS subject classification. 41A15, 65D07

REZUMAT. - Funcţii spline polinomiale naturale de grad par. Lucrarea defineşte şi studiază proprietăţi ale funcţiilor spline polinomiale naturale de grad par. Schimbând condiţiile de interpolare care definesc funcţia spline naturală de interpolare de grad impar, se obţine funcţia spline naturală de grad par ce interpolează derivatele pe mulţimea nodurile funcţiei spline Sunt obţinute câteva proprietăţi extremale remarcabile analoge celor din cazul funcţiilor spline naturale de grad impar

1. Introduction. The rapid development of spline functions is due primarily to their great usefulness in applications Classes of spline functions possess many important properties as well as excelent approximations powers Since they are easy to evaluate and manipulate on computer a lot of applications in the numerical solution of a variety of problems have been found These include for examples data fitting, function approximation, numerical quadrature and differentiation, numerical solution of operator equations, optimal control problems, calculation of eigenvalues and eigenfunctions of operators, numerical methods of probabilities and statistics, and so on For a detailed problematics on spline functions we reffer to the monographs [6,9] and for an exhaustive literature on spline functions and their applications we reffer to [7] Almost all papers underline the fundamental properties of natural spline functions of odd degree, namely the minimum norm property, the best approximation

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property, etc

In this paper, changing the interpolation conditions, we shall define a natural spline function of even degree which is keeping all remarkable properties of odd degree one and we shall develop a theory of natural spline functions of even degree. Our disscusion led us to the conclusion that the space of natural polynomial spline functions of even degree should be useful for approximation purpose

2. Basic definitions and properties. Let $[a,b]$ be a finite closed interval of the real axis, and let

$$\Delta_n = \{x_i\}_1^n, \text{ with } a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n < x_{n+1} = b$$

be a partition of it in n subintervals

$$I_k = [x_k, x_{k+1}[, \quad k = 0, 1, \ldots, n.$$

Let m be a given positive integer.

DEFINITION 1 The function $s : [a,b] \to \mathbb{R}$ is called *the natural spline function of degree* $2m$ if

1^o $s \in C^{2m-1}[a,b]$,

2^o $s|_{I_k} \in \mathcal{P}_{2m}, \quad k = \overline{1, n-1}, \; s|_{I_0} \in \mathcal{P}_m, \quad s|_{I_n} \in \mathcal{P}_m,$

where $\mathcal{P}_k$ is the set of polynomials of degree $\leq k$.

We call the space

$S_{2m}(\Delta_n) = \{ s$ there exists polynomials $s_0, s_1, \ldots, s_n$

such that $s(x) = s_i(x)$ for $x \in I_i$, $i = 0, 1, \ldots, n$;

$D^j s_{i-1}(x_i) = D^j s_i(x_i)$, for $j = 0, 1, \ldots, 2m-1$ }

the space of natural polynomial splines of even degree 2m with the simple knots $x_1, x_2, \ldots, x_n$

The space $S_{2m}(\Delta_n)$ of splines is a subset of $C^{2m-1}[a,b]$ In the most practical applications the natural setting for approximation problems is a closed interval $[a,b]$, but every spline has a natural extension to the whole real line Indeed, if $s \in S_{2m}(\Delta_n)$ then we define

$$s(x) = \begin{cases} s_0(x), & \text{for } x < a, \\ s_n(x), & \text{for } x > b, \end{cases}$$

where s_0 and s_n are the polynomials defining s in the intervals I_0 and I_n, respectively

We now show that $S_{2m}(\Delta_n)$ is a finite dimensional linear space and we give a basis for it

THEOREM 1 *$S_{2m}(\Delta_n)$ is a linear space of dimension $n+1$ Any element $s \in S_{2m}(\Delta_n)$ has the following representation*

$$s(x) = \sum_{i=0}^{m} A_i x^i + \sum_{k=1}^{n} a_k (x-x_k)_+^{2m}, \tag{1}$$

where the real coefficients $(a_k)_1^n$ satisfy the conditions

$$\sum_{k=1}^{n} a_k x_k^i = 0, \quad \textit{for} \quad i = \overline{0, m-1}$$

Proof. The number of all parameters of s are $(2m+1)(n-1) + 2(m+1)$. The continuity conditions (smoothness) in the knots $(x_i)_1^n$ are $2mn$ Thus the free parameters of s are $(2m+1)(n-1) + 2(m+1) - 2mn = n+1$.

From the Definition 1 it is clear that

$$s(x) = \sum_{i=0}^{m} A_i x^i + \sum_{k=1}^{n} a_k (x-x_k)_+^{2m}.$$

But the last condition $s|_{I_n} \in \mathcal{P}_m$ implies $s^{(m+1)}(x) = 0$ for any $x \in I_n$ This means

$$\sum_{k=1}^{n} 2m(2m+1) \quad m\, a_k (x-x_k)^{m-1} = 0, \quad x \in I_n,$$

which is equivalent with

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{m-1} \binom{m-1}{j} x^{m-j-1} (-x_k)^j = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \left(\sum_{k=1}^{n} a_k x_k^j \right) x^{m-j-1} = 0$$

That implies

$$\sum_{k=1}^{n} a_k x_k^j = 0, \quad \text{for} \quad j = 0, 1, \ldots, m-1$$

DEFINITION 2. Let $n \geq m$ be given integer positive numbers and $Y \in \mathbb{R}^{n+1}$, $Y := (y_a, y_1', \ldots, y_n')$ a given vector. The spline function $s \in S_{2m}(\Delta_n)$ is called *the derivative-interpolating spline for the vector Y* if

$$\begin{aligned} &s(x_a) = y_a, \quad x_a \text{ is a given point from } [a,b], \\ &s'(x_i) = y_i', \quad i = \overline{1,n}. \end{aligned} \tag{2}$$

We denote the derivative-interpolating spline for a given Y by s_Y.

THEOREM 2 *Let $n \geq m$ and the vector $Y = (y_a, y_1', \ldots, y_n')$ be given. Then there exists and it is unique a derivative-interpolating function s_Y in $S_{2m}(\Delta_n)$.*

Proof The derivative-interpolating conditions (2) for the $s_Y \in S_{2m}(\Delta_n)$ are.

$$\begin{aligned} &\sum_{i=0}^{m} A_i x_a^i + \sum_{k=1}^{n} a_k (x_a - x_k)_+^{2m} = y_a, \\ &\sum_{i=1}^{m} i A_i x_j^{i-1} + 2m \sum_{k=1}^{n} a_k (x_j - x_k)_+^{2m-1} = y_j', \quad j = \overline{1,n}, \\ &\sum_{k=1}^{n} a_k x_k^j = 0, \quad j = \overline{0, m-1} \end{aligned} \tag{3}$$

This is a linear nonhomogeneous system of $m + n + 1$ equations with the $m + n + 1$ unknowns $A_0, A_1, \ldots, A_m, a_1, a_2, \ldots, a_n$. This system has a unique solution if and only if the corresponding homogeneous system possesses only the trivial solution.

Let $s_0 \in S_{2m}(\Delta_n)$ be the soution of the homogeneous derivative-interpolating problem. Using the generalized integration by parts formula

$$\int_a^b f(x) g^{(m+1)}(x)\,dx = \left[\sum_{j=0}^{m-2} (-1)^j f^{(j)}(x) g^{(m-j)}(x) \right]_a^b + (-1)^{m-1} \int_a^b f^{(m-1)}(x) g''(x)\,dx$$

for $f(x) = s_0^{(m+1)}(x)$ and $g(x) = s_0(x)$ we have

$$\int_a^b \left[s_0^{(m+1)}(x) \right]^2 dx = \left[\sum_{j=0}^{m-2} (-1)^j s_0^{(m+1+j)}(x) s_0^{(m-j)}(x) \right]_a^b + \\ + (-1)^{m-1} \int_a^b s_0^{(2m)}(x) s_0''(x)\, dx$$

But

$$s_0^{(m+1+j)}(a) = s_0^{(m+1+j)}(b) = 0, \quad j = \overline{0, m-2},$$

therefore we have

$$\int_a^b \left[s_0^{(m+1)}(x) \right]^2 dx = (-1)^{m-1} \int_a^b s_0^{(2m)}(x) s_0''(x)\, dx =$$

$$= (-1)^{m-1} \int_{x_1}^{x_n} s_0^{(2m)}(x) s_0''(x)\, dx = (-1)^{m-1} \sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} s_0^{(2m)}(x) s_0''(x)\, dx =$$

$$= (-1)^{m-1} \sum_{k=1}^{n-1} c_k \int_{x_k}^{x_{k+1}} s_0''(x)\, dx = (-1)^{m-1} \sum_{k=1}^{n-1} c_k \left[s_0'(x_{k+1}) - s_0'(x_k) \right] = 0,$$

because $s_0^{(2m)}(x) = c_k = \text{const}$ on $[x_k, x_{k+1}]$ and $s_0'(x_j) = y_j' = 0$. It follows that $s_0^{(m+1)}(x) = 0$, i e $s_0' \in \mathcal{P}_{m-1}$. Taking into account that $s_0'(x_k) = 0$, for $k = \overline{1,n}$, and $n \geq m$ we have $s_0'(x) \equiv 0$, i e. $s_0(x) = \text{const}$ From $s_0(x_\alpha) = 0$ it is clear that $s_0(x) \equiv 0$ and the theorem is proved

COROLLARY 1 *If $f : [a,b] \to \mathbb{R}$ is a given function for that the values $f(x_\alpha)$ and $f'(x_k)$, $k = \overline{1,n}$ are known, there exists and it is unique a natural spline function $s_f \in S_{2m}(\Delta_n)$ which is derivative-interpolating for f,* i.e *s_f satisfies the conditions:*

$$s_f'(x_k) = f'(x_k), \quad k = \overline{1,n},$$

$$s_f(x_\alpha) = f(x_\alpha), \quad \text{for } x_\alpha \text{ fixed from } [a,b]$$

COROLLARY 2 *There exists a unique set of fundamental natural polynomial spline functions $s_k \in S_{2m}(\Delta_n)$, $k = \overline{1,n}$ and $s_\alpha \in S_{2m}(\Delta_n)$ satisfying the following conditions:*

$$s_k'(x_i) = \delta_{ik}, \quad i,k = \overline{1,n}, \quad s_k(x_a) = 0, \quad k = \overline{1,n},$$

$$s_a(x_a) = 1, \quad s_a'(x_k) = 0, \quad k = \overline{1,n}, \quad x_a \text{ fixed from } [a,b]$$

It is clear that the functions s_a, s_k, $k = \overline{1,n}$, form a basis of the linear space $S_{2m}(\Delta_n)$, and for s_f we have the following representation

$$s_f(x) = s_a(x) f(x_a) + \sum_{k=1}^{n} s_k(x) f'(x_k). \tag{4}$$

Remark If $m = n$ it follows that $s_f \in \mathcal{P}_m$.

3. Extremal properties of the natural spline function of even degree. Let introduce the following sets of functions

$$W_2^{m+1}(\Delta_n) := \{f \; [a,b] \to \mathbb{R} \mid f^{(m)} \text{ is abs cont. on each } I_k \text{ and } f^{(m+1)} \in L_2[a,b]\},$$

$$v := \{f \in W_2^{m+1}(\Delta_n) \mid f'(x_k) = y_k', \; k = \overline{1,n}\},$$

and let denote

$$J(f) = \int_a^b \left(f^{(m+1)}(x)\right)^2 dx, \; f \in v$$

THEOREM 1 (Minimal norm property). *If* $s \in S_{2m}(\Delta_n) \cap v$, *then*

$$J(s) = \min \{J(f) \, . \, f \in v\}$$

Remark With the usual notations this theorem asserts that

$$\|s^{(m+1)}\|_2^2 \le \|f^{(m+1)}\|_2^2, \; f \in v, \; \|\cdot\|_2 \text{ being the } L_2\text{-norm}.$$

Proof Taking $f \in v$ and $s \in S_{2m}(\Delta_n) \cap v$ we have.

$$\|f^{(m+1)} - s^{(m+1)}\|_2^2 = \int_a^b \left[f^{(m+1)}(x) - s^{(m+1)}(x)\right]^2 dx =$$

$$= \int_a^b \left[f^{(m+1)}(x)\right]^2 dx - \int_a^b \left[s^{(m+1)}(x)\right]^2 dx -$$

$$-2 \int_a^b s^{(m+1)}(x) \left[f^{(m+1)}(x) - s^{(m+1)}(x)\right] dx$$

It is not difficult to show that the last term is zero Taking in the generalized integration by parts formula of the previous section $f = s^{(m+1)}$ and $g^{(m+1)} = f^{(m+1)} - s^{(m+1)}$ we get

$$\int_a^b s^{(m+1)}(x)\left[f^{(m+1)}(x) - s^{(m+1)}(x)\right]dx =$$

$$= \left[\sum_{j=0}^{m-2} (-1)^j s^{(m+1+j)}(x)\left(f^{(m-j)}(x) - s^{(m-j)}(x)\right)\right]_a^b +$$
$$(-1)^{m-1}\int_a^b s^{(2m)}(x)\left[f''(x) - s''(x)\right]dx$$

From the definition of s follows that

$s^{(m+1+j)}(a) = s^{(m+1+j)}(b) = 0$ for $j = \overline{0, m-2}$ and we have

$$\int_a^b s^{(m+1)}(x)\left[f^{(m+1)}(x) - s^{(m+1)}(x)\right]dx =$$
$$= (-1)^{(m-1)}\int_a^b s^{(2m)}(x)\left[f''(x) - s''(x)\right]dx =$$
$$= (-1)^{m-1}\sum_{k=1}^{n-1} c_k \int_{x_k}^{x_{k+1}}\left[f''(x) - s''(x)\right]dx = (-1)^{m-1}\sum_{k=1}^{n-1} c_k\left[f'(x_k) - s'(x_k)\right] = 0,$$

because $s^{(2m)}(x) = c_k =$ const on $[x_k, x_{k+1}]$ and $f, s \in \mathrm{v}$ So we obtained

$$\left\|f^{(m+1)} - s^{(m+1)}\right\|_2^2 = \left\|f^{(m+1)}\right\|_2^2 - \left\|s^{(m+1)}\right\|_2^2 \geq 0,$$

i e $\left\|s^{(m+1)}\right\|_2^2 \leq \left\|f^{(m+1)}\right\|_2^2$.

COROLLARY 1 *For any* $f \in \mathrm{v}$ *and* $s_f \in S_{2m}(\Delta_n) \cap \mathrm{v}$ *we have*

$$\left\|f^{(m+1)}\right\|_2^2 = \left\|s_f^{(m+1)}\right\|_2^2 + \left\|f^{(m+1)} - s_f^{(m+1)}\right\|_2^2$$

COROLLARY 2 *If* $s \in S_{2m}(\Delta_n) \cap \mathrm{v}$ *and* $\tilde{s} = s + p_m$, *where* $p_m \in \wp_m$, *it follows that*

$$\left\|\tilde{s}^{(m+1)}\right\|_2^2 \leq \left\|f^{(m+1)}\right\|_2^2, \text{ for } f \in \mathrm{v},$$

even in the case that $\tilde{s}$ *does not belong to* v

COROLLARY 3 *For any* $f \in W_2^{m+1}(\Delta_n)$ *we have*

$$\left\|s_f^{(m+1)}\right\|_2^2 \leq \left\|f^{(m+1)}\right\|_2^2$$

COROLLARY 4 *If* $\mathrm{v}_\alpha = \{f \in \mathrm{v} \mid f(x_\alpha) = y_\alpha\}$, *then exists a unique* $s \in \mathrm{v}_\alpha \cap S_{2m}(\Delta_n)$ *such that*

$$\left\|s^{(m+1)}\right\|_2^2 \leq \left\|f^{(m+1)}\right\|_2^2, \quad \forall f \in \mathrm{v}_\alpha$$

holds

THEOREM 2 **(Best approximation property).** *Let* $f \in W_2^{m+1}(\Delta_n)$ *be a given function and* $s_f \in S_{2m}(\Delta_n)$ *the natural derivative-interpolating spline function of even degree. For any* $s \in S_{2m}(\Delta_n)$ *holds*

$$\left\| s_f^{(m+1)} - g^{(m+1)} \right\|_2^2 \leq \left\| s^{(m+1)} - g^{(m+1)} \right\|_2^2 ,$$

where $g \in \mathrm{v}_f = \left\{ h \in W_2^{m+1}(\Delta_n) \mid h'(x_k) = f'(x_k), \quad k = \overline{1,n} \right\}.$

Proof

$$\int_a^b \left[s^{(m+1)}(x) - g^{(m+1)}(x) \right]^2 dx =$$
$$= \int_a^b \left[s^{(m+1)}(x) - s_f^{(m+1)}(x) \right]^2 dx + \int_a^b \left[s_f^{(m+1)}(x) - g^{(m+1)}(x) \right]^2 dx +$$
$$+ 2 \int_a^b \left[s^{(m+1)}(x) - s_f^{(m+1)}(x) \right]^2 \left[s_f^{(m+1)}(x) - g^{(m+1)}(x) \right]^2 dx .$$

In the same manner as in the previous theorem it can be shown that last term of the above equality is zero, and it follows directly that

$$\int_a^b \left[s_f^{(m+1)}(x) - g^{(m+1)}(x) \right]^2 dx \leq \int_a^b \left[s^{(m+1)}(x) - g^{(m+1)}(x) \right]^2 dx$$

for any $g \in \mathrm{v}_f$

Remark In the above relation the equality holds if and only if $s_f - s \in \mathcal{P}_m$

COROLLARY 1 $\left\| s_f^{(m+1)} - f^{(m+1)} \right\|_2^2 \leq \left\| s^{(m+1)} - f^{(m+1)} \right\|_2^2, \ \forall s \in S_{2m}(\Delta_n)$ *with the equality if and only if* $s_f - s \in \mathcal{P}_m$

COROLLARY 2 *In the conditions of the Theorem 2 we have*

$$\left\| s^{(m+1)} - g^{(m+1)} \right\|_2^2 = \left\| s^{(m+1)} - s_f^{(m+1)} \right\|_2^2 + \left\| s_f^{(m+1)} - g^{(m+1)} \right\|_2^2 ,$$

with the equality if and only if $s_f - s \in \mathcal{P}_m$.

COROLLARY 3 $\left\| s^{(m+1)} - s_f^{(m+1)} \right\|_2^2 \leq \left\| s^{(m+1)} - g^{(m+1)} \right\|_2^2, \quad \forall \ s \in S_{2m}(\Delta)$ *and* $g \in \mathrm{v}_f$, *with the equality if and only if* $s_f - s \in \mathcal{P}_m$

It is possible to extend in the same manner also others properties of the natural polynomial splines of the odd degree for the natural splines of even degree with the derivative-interpolation conditions More important, it seems, to be the applications of this kind of spline function, especially to the numerical solution of differential equations, because of their derivative-interpolating conditions. Such kind of applications will be developed in a next paper

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

ON THE REVERSE OF THE KRASNOSELSKII-BROWDER BOUNDARY INEQUALITY

Radu PRECUP*

Dedicated to Professor P Szilágyi on his 60^{th} anniversary

Received October 20, 1993
AMS subject classification: 47H10, 47H05, 47H11

REZUMAT. - Asupra contrarei inegalităţii pe frontieră a lui Krasnoselskii-Browder. Fie G o submulţime deschisă şi mărginită a spaţiului Banach X cu $0 \in G$ şi fie f o aplicaţie de la $\bar{G}$ în spaţiul dual X^*. Inegalitatea pe frontieră a lui Krasnoselskii-Browder: $(x,f(x)) \geq 0$ pentru orice $x \in \partial G$, este pentru anumite tipuri de aplicaţii, suficientă pentru existenţa unei soluţii $x \in \bar{G}$ a ecuaţiei $f(x) = 0$. Prezentul articol se ocupă de contrara acestei inegalităţi, anume: $(x,f(x)) \leq 0$ pentru orice $x \in \partial G$. Arătăm că dacă X este un spaţiu Hilbert infinit-dimensional, $f = I - g$ unde I este identitatea lui X şi $g : \bar{G} \to X$ este complet continuu, atunci inegalitatea $(x,f(x)) \leq 0$ nu are loc pentru toţi $x \in \partial G$. În consecinţă, două teoreme de punct fix demonstrate în [4] nu au obiect fiindcă ipotezele lor nu pot fi satisfăcute. Apoi punem problema dacă un rezultat negativ de tipul celui de mai sus, este valabil şi pentru aplicaţii f de tip monoton, mai generale La această întrebare se dă un răspuns parţial

1. Abstract. Let G be a bounded open subset of a Banach space X with $0 \in G$ and let f be a map from $\bar{G}$ into the dual X^*. The following Krasnoselskii-Browder boundary inequality: $(x,f(x)) \geq 0$ for all $x \in \partial G$ is for some types of maps sufficient for the existence of solutions $x \in \bar{G}$ for equation $f(x) = 0$ This article deals with the reverse of the above inequality, namely

$$(x,f(x)) \leq 0 \text{ for all } x \in \partial G$$

We prove that if X is an infinite-dimensional Hilbert space, $f = I - g$ where I is the identity on X and g $\bar{G} \to X$ is completely continuous, then the inequality $(x,f(x)) \leq 0$ can

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not be true for all $x \in \partial G$ Consequently, two existence theorems proved in [4] have no content since their assumptions are never satisfied We then ask if such a negative result holds true even for more general maps of monotone type. A partial answer is finally given

1. Introduction. Let us start with the definition of a general concept of degree of map

DEFINITION 1 ([1], [2]) Let X and Y be topological spaces Let $\mathcal{O}$ be a class of open subsets of X For each G in $\mathcal{O}$ one considers a family of maps $f:\bar{G} \to Y$; the collection of all such maps for the various G of $\mathcal{O}$ is denoted by $\mathcal{F}$. For each G in $\mathcal{O}$ consider a family of homotopies $\{ f_t \; 0 \le t \le 1 \}$ of maps in $\mathcal{F}$, all having the common domain $\bar{G}$; denote by $\mathcal{H}$ the collection of all such homotopies for the various G in $\mathcal{O}$. Then by a *degree* function on the family $\mathcal{F}$ which is invariant with respect to the homotopies in $\mathcal{H}$ and which is normalized by a given map f_0 from X into Y, one means an integer - valued function $d(f,G,y)$ which is defined for all $G \in \mathcal{O}$, $f \in \mathcal{F}$, $f: \bar{G} \to Y$, $y \in Y \setminus f(\partial G)$ such that the following three conditions are satisfied:

(a) (normalization) If $d(f,G,y) \neq 0$, then $y \in f(G)$ For each G in $\mathcal{O}$, $f_0|_{\bar{G}} \in \mathcal{F}$ and if $y \in f_0(G)$, then $d(f_0|_{\bar{G}}, G, y) = +1$.

(b) (additivity on domain). If $f \in \mathcal{F}$, $f: \bar{G} \to Y$ and $G_1, G_2 \in \mathcal{O}$ are a pair of disjoint open subsets of G such that $y \notin f\left(\bar{G} \setminus (G_1 \cup G_2)\right)$, then $f|_{\bar{G}_1}$ and $f|_{\bar{G}_2}$ both lie in $\mathcal{F}$ and $d(f,G,y) = d\left(f|_{\bar{G}_1}, G_1, y\right) + d\left(f|_{\bar{G}_2}, G_2, y\right)$

(c) (invariance under homotopy). If $\{f_t\,; 0 \le t \le 1\}$ is a homotopy in $\mathcal{H}$ with fixed domain $\bar{G}$ and if $\{y_t\,; 0 \le t \le 1\}$ is a continuous curve in Y such that for all t in $[0,1]$, $y_t \notin f_t(\partial G)$, then $d(f_t, G, y_t)$ is constant on $[0,1]$.

The following proposition is a variant of Proposition 1 and Proposition 3 in [1] and shows how a boundary inequality is useful for existence results

PROPOSITION 1. *Assume a degree function exists as in Definition* 1 *and* Y *is a linear topological space. Let* G *be a set in* $\mathcal{O}$, y *a given element in* $f_0(G)$ *and let* $f\ \bar{G} \to Y$ *lies in* $\mathcal{F}$ *Suppose that* $\mathcal{H}$ *includes the affine homotopy* $f_t = (1-t)f_0|_{\bar{G}} + tf$. *If for each* $x \in \partial G$ *there exists a linear functional* w *on* Y *such that*

$$(w, y) = 0,\ (w, f_0(x)) > 0 \text{ and } (w, f(x)) \geq 0, \tag{1}$$

then $y \in f(\bar{G})$

Proof Suppose $y \notin f(\partial G)$ Since $y \in f_0(G)$, from (a) one has $d(f_0|_{\bar{G}}, G, y) = +1$ To show that $d(f, G, y) = d(f_0|_{\bar{G}}, G, y) = +1$, it suffices to see that $d(f_t, G, y)$ is constant in t, where $\{f_t,\ 0 \leq t \leq 1\}$ is the affine homotopy which joins $f_0|_{\bar{G}}$ and f This follows from (c) if we can verify that $y \notin f_t(\partial G)$ for all t in $[0,1]$. Suppose, however, that for some $x \in \partial G$ and some $t \in [0,1)$, we have $y = f_t(x) = (1-t)f_0(x) + tf(x)$. Then

$$\begin{aligned} 0 = (w, y) = (w, f_t(x)) = (1-t)(w, f_0(x)) + t(w, f(x)) \geq \\ \geq (1-t)(w, f_0(x)) > 0 \end{aligned}$$

which is a contradiction

In the particular case $X = Y$ is a Hilbert space, $\mathcal{O}$ in the class of all bounded open nonempty subsets of X; $\mathcal{F}$ is the family of continuous maps $f.\bar{G} \to X$ with $G \in \mathcal{O}$ and $(I-f)(\bar{G})$ relatively compact in X; $\mathcal{H}$ is the family of continuous homotopies $\{f_t,\ 0 \leq t \leq 1\}$ in $\mathcal{F}$ with a common domain $\bar{G}$ such that there is a compact subset K of X with $(I-f_t)(\bar{G}) \subset K$ for all $t \in [0,1]$, f_0 is the identity of X and $y = 0$, condition (1) is satisfied provided that

$$(x, f(x)) \geq 0 \quad \text{for all} \quad x \in \partial G \tag{2}$$

Condition (2) is just the well-known Krasnoselskii boundary inequality Thus, if we set $f =$

$I - g$, we obtain the fixed point theorem of Krasnoselskii:

PROPOSITION 2 (Krasnoselskii) *Let G be a bounded open subset of a real Hilbert space X with $0 \in G$. Suppose that the completely continuous map g from $\bar{G}$ into X satisfies*

$$(x, g(x)) \leq |x|^2 \quad \textit{for all } x \in \partial G. \tag{3}$$

Then g has at least one fixed point in $\bar{G}$.

An obvious question is what happens if the inequality in (2), equivalently in (3), is reversed This question was asked by Lakshmikantham and Sun in [4] where the following answer was given

PROPOSITION 3 (Lakshmikantham & Sun). *Let X be a real Hilbert space of infinite dimension and G a bounded open set of X with $0 \in G$. Suppose that the completely continuous map g from $\bar{G}$ into X satisfies*

$$(x, g(x)) \geq |x|^2 \quad \textit{for all } x \in \partial G. \tag{4}$$

Then g has at least one fixed point in $\bar{G}$.

An other statement in [4] is the following:

PROPOSITION 4 (Lakshmikantham & Sun). *Let X be a real Hilbert space of infinite dimension and G_1, G_2 two bounded open sets of X such that $0 \in G_1$ and $\bar{G}_1 \subset G_2$. Suppose that the completely continuous map g from $\bar{G}_3$ into X satisfies the following conditions:*

$$(x, g(x)) \geq |x|^2 \quad \textit{and } g(x) \neq x \textit{ for any } x \in \partial G_1 \,; \tag{5}$$

$$(x, g(x)) \leq |x|^2 \textit{ for any } x \in \partial G_2 \,. \tag{6}$$

Then g has at least two fixed points in $\bar{G}_3$.

Proposition 3 is true if X is a space of finite dimension This can be proved by Brouwer's degree ([4], Remark 1) since

$$d(I - g, G, 0) = (-1)^{\dim X} \neq 0$$

Indeed, if g satisfies (4) and $g(x)\neq x$ for any $x\in\partial G$, then g is homotopic to θI for any $\theta > 1$, the homotopy being

$$g_t(x) = (1-t)\theta x + tg(x), \quad 0 \le t \le 1$$

It is easily seen that $g_t(x) \neq x$ for all $x \in \partial G$ and $t \in [0,1]$ Hence,

$$\begin{aligned} d(I-g,G,0) &= d(I-g_1,G,0) = \\ &= d(I-g_0,G,0) = d((1-\theta)I,G,0) = \\ &= d(-I,G,0) = (-1)^{\dim X}. \end{aligned}$$

Let us remark that Proposition 4 is true if X is a space of odd finite dimension. Indeed, under conditions (5), (6) and $g(x) \neq x$ (for all $x \in \partial G_2$), by additivity property of the degree, one has

$$\begin{aligned} d(I-g,G_2\setminus\bar{G}_1,0) &= d(I-g,G_2,0) - \\ &- d(I-g,G_1,0) = 1-(-1)^{\dim X} = 2 \end{aligned}$$

If X is a space of even finite dimension, Proposition 4 is not true as shows the following example:

Example 1. Let $X = \mathbb{R}^{2n}$, $G_1 = \{x\in\mathbb{R}^{2n},\ |x|<1\}$, $G_2 = \{x\in\mathbb{R}^{2n},\ |x|<2\}$, $g:\bar{G}_2\to\mathbb{R}^{2n}$ where

$$g(x)_{2k-1} = 2x_{2k-1}/\left(|x|^2+1\right) + x_{2k}, \quad k = 1,2,..,n$$

$$g(x)_{2k} = 2x_{2k}/\left(|x|^2+1\right) - x_{2k-1}, \quad k = 1,2,..,n.$$

For $|x| = 1$ we have $g(x)_{2k-1} = x_{2k-1}+x_{2k}$, $g(x)_{2k} = x_{2k}-x_{2k-1}$ and so $(x,g(x)) = |x|^2 = 1$ and $g(x) \neq x$ Hence g satisfies (5) For $|x|=2$ we have $(x,g(x)) = \frac{2}{5}|x|^2 = \frac{8}{5} < 4 = |x|^2$, which shows that (6) also holds Nevertheless, the unique fixed point of g is 0 Indeed, if $g(x) = x$, then

$$\left.\begin{aligned} 2x_{2k-1}/\left(|x|^2+1\right)+x_{2k} &= x_{2k-1} \\ 2x_{2k}/\left(|x|^2+1\right)-x_{2k-1} &= x_{2k} \end{aligned}\right\}, \quad k = 1,2, \dots, n \tag{7}$$

If we multiply each of these equalities by x_{2k-1} and x_{2k}, respectively, and add, we obtain

$$2|x|^2/\left(|x|^2+1\right) = |x|^2,$$

which is possible only if $|x| = 1$ or $|x| = 0$. In case $|x| = 1$, from (7) we find $x_{2k-1}+x_{2k} = x_{2k-1}, x_{2k}-x_{2k-1} = x_{2k}$, $k = 1,2, \dots, n$, whence $x=0$, a contradiction Thus, $x=0$ is the unique fixed point of g

3. In spaces of infinite dimension the complete continuity and the reverse of the Krasnoselskii inequality are incompatible.

THEOREM 1. *Let X be a real Hilbert space of infinite dimension, G a bounded open set of X with $0 \in G$ and let g be a completely continuous map from $\bar{G}$ into X. Then, there exists $x \in \partial G$ such that*

$$(x, g(x)) < |x|^2. \tag{8}$$

Proof Suppose otherwise Then

$$(x, g(x)) \geq |x|^2 \text{ for all } x \in \partial G. \tag{9}$$

Since G is open bounded and $0 \in G$ there exist r_1, r_2 such that

$$0 < r_1 \leq |x| \leq r_2 \quad \text{for all } x \in \partial G. \tag{10}$$

On the other hand, the compactness of g implies thet $K = g(\overline{\partial G})$ is compact and

$$|g(x)| \leq R \quad \text{for all } x \in \partial G,$$

where $R > 0$ From (9) and (10), we have

$$r_1|x| \leq |x|^2 \leq (x, g(x)) \leq |x|\,|g(x)| \quad \text{for any } x \in \partial G$$

Hence

$$0 < r_1 \leq |g(x)| \leq R \quad \text{for all} \quad x \in \partial G$$

Again from (9) and (10), we have

$$(x, g(x)) = |x|\,|g(x)| \cos \sphericalangle(x, g(x)) \geq |x|^2 \geq r_1 |x|$$

It follows that

$$\cos \sphericalangle(x, g(x)) \geq r_1/|g(x)| \geq r_1/R \quad (x \in \partial G).$$

Thus

$$\sphericalangle(x, g(x)) \leq \alpha = \arccos(r_1/R) < \pi/2 \quad \text{for all } x \in \partial G$$

Now, for each $y \in K$ define a subset U_y of X by

$$U_y = \{x \in X \setminus \{0\},\ \sphericalangle(x,y) < \pi/2 - \alpha\}.$$

Clearly, the family $\{U_y ; y \in K\}$ is an open cover of K Since K is compact there is a finite subcover of K, say

$$U_{y_1},\ U_{y_2},\ \ ,\ U_{y_m}$$

For any x in ∂G, there exists $i \in \{1, 2, \ ., m\}$ such that $g(x) \in U_{y_i}$

Since $\sphericalangle(x, g(x)) \leq \alpha$ and $\sphericalangle(g(x), y_i) < \pi/2 - \alpha$, we obtain $\sphericalangle(x, y_i) < \pi/2$ and so, $(x, y_i) > 0$ On the other hand, X being of infinite dimension, we may find an element x_0 on ∂G such that $(x_0, y_j) \doteq 0$ for all $j \in \{1, 2, \ \ , m\}$ Therefore we reach a contradiction. Theorem 1 is thus proved

Remark 1 Here is an equivalent statement for Theorem 1

Let X be a real Hilbert space of infinite dimension, G a bounded open set of X with $0 \in G$ and g a completely continuous map from $\bar{G}$ into X. Denote $f = I - g$ Then

$$\sup \{(x, f(x)),\ x \in \partial G\} > 0 \tag{11}$$

Remark 2 Proposition 3 and Proposition 4 have no contents Indeed, by Theorem 1, the assumptions (4) and (5) never hold

4. The reverse of the Krasnoselskii-Browder boundary inequality and maps of monotone type. The question we ask is if (11) holds true even for more general maps of monotone type The answer we give is only a partial one.

Let us first recall some definitions and standard notations of nonlinear functional analysis. If X is a real Banach space and X^* its dual, we denote by (x,w) the pairing between x in X and w in X^* We use the symbol $\to$ for strong convergence and $\rightharpoonup$ for weak convergence If D is a subset of X and f a map from D into X^*, f is said to be *demicontinuous* if it is continuous from the strong topology of X on D to the weak topology of X^*; f is said to be of *class* $(S)_+$ if it is demicontinuous and if for any sequence (x_j) in D with $x_j \rightharpoonup x$ for some $x \in X$ for which $\overline{\lim}\,(x_j - x, f(x_j)) \leq 0$, we have $x_j \to x$; f is said to be *pseudo-monotone* if it is demicontinuous and if for any sequence (x_j) in D with $x_j \rightharpoonup x$ for some $x \in X$ for which $\overline{\lim}\,(x_j - x, f(x_j)) \leq 0$, we have $(x_j - x, f(x_j)) \to 0$, while if $x \in D$, then $f(x_j) \rightharpoonup f(x)$. A multi-valued map T of X into the subsets of X^* is monotone if for any elements $[x,u]$ and $[y,v]$ of its graph, $(x-y,\ u-v) \geq 0$, T is *maximal monotone* if it is maximal in the sense of inclusion of graphs among monotone maps of X into the subsets of X^*

Note that if X is a Hilbert space and g is a completely continuous map from D into X, then the map $f = I - g$ is of class $(S)_+$

The classes of maps we shall deal with include the bounded maps of type $(S)_+$, the bounded pseudo-monotone maps and the sums of maximal monotone maps and bounded maps which are of class $(S)_+$ or pseudo-monotone. For all these classes of maps a degree function is known (see [1], [2]) and with respect to the corresponding degree theory, Krasnoselskii-Browder boundary inequality is a "good" condition in the sense of an existence theorem like Proposition 1 What we can prove for the moment is that, unlike this, the reverse of the

Krasnoselskii-Browder inequality is a "bad" condition for the degree theory

Let X_0 be a finite-dimensional subspace of X, G an open subset of X such that $G \cap X_0 = G_0$ is nonempty and let $f: \bar{G} \to X^*$ be a given map. Then the Galerkin approximant from f is the map $f_0: \bar{G} \to X_0 (=X_0^*)$, $f_0(x) = \varphi^*(f(\varphi(x)))$, where φ is the injection map of X_0 into X and φ^* the corresponding projection of X^* onto X_0^*.

Let Λ be the partially ordered set of finite-dimensional subspaces X_λ of X, ordered by inclusion For each λ, denote by φ_λ the injection map of X_λ into X and by φ_λ^* the corresponding projection of X^* onto X_λ^*

LEMMA 1 (Browder) *Let X be a real reflexive Banach space, G a bounded open subset of X with $0 \in G$ and let $f: \bar{G} \to X^*$ be a bounded map of class $(S)_+$ such that $0 \notin f(\partial G)$ Then, there exists λ_0 in Λ such that for all $\lambda > \lambda_0$, $0 \notin f_\lambda(\partial G_\lambda)$ and Brouwer's degree $d(f_\lambda, G_\lambda, 0)$ is independent of λ (where $G_\lambda = G \cap X_\lambda$ and $f_\lambda = \varphi_\lambda^* f \varphi_\lambda$).*

The common value of $d(f_\lambda, G_\lambda, 0)$ for $\lambda > \lambda_0$ is denoted by $d(f, G, 0)$ and is called Browder's degree of f (see [1]).

The main result of this section is the following theorem

THEOREM 2 *Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$ Suppose that the bounded map f from $\bar{G}$ into X^* is of class $(S)_+$ and satisfies the following condition:*

$$(x, f(x)) \leq 0 \quad \text{for any } x \in \partial G \tag{12}$$

Then f has at least one zero on ∂G

Proof Suppose that the assertion were false Then $0 \notin f(\partial G)$ and by Lemma 1, there would exist λ_0 in Λ such that for all $\lambda > \lambda_0$, $0 \notin f_\lambda(\partial G_\lambda)$ and Brouwer's degree $d(f_\lambda, G_\lambda, 0)$ is independent of λ But, from (12), we have

$$(x, f_\lambda(x)) = (x, f(x)) \le 0 \quad \text{for any } x \in \partial G_\lambda$$

Hence, $d(f_\lambda, G_\lambda, 0) = (-1)^{\dim \lambda}$ and so the degree $d(f_\lambda, G_\lambda, 0)$ could not be independent of λ Thus, the assertion in Theorem 2 is true.

Remark 3 By Theorem 2, the following proposition is true

Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$ and f a bounded map of class $(S)_+$ from $\bar{G}$ into X^*. Then

$$\sup \{ (x, f(x)), x \in \partial G \} \ge 0 \tag{13}$$

and in case that $\sup \{ (x, f(x)), x \in \partial G \} = 0$, there exists $x \in \partial G$ such that $f(x) = 0$

Our question is does the strict inequality hold in (13)? As we have already seen (Remark 1), the answer is positive for maps of the form $f = I - g$ with g completely continuous in Hilbert spaces For the broader class of maps of type $(S)_+$, this is an open problem

COROLLARY 1. *Let X be a real reflexive Banach space of infinite dimension and G a bounded open subset of X with $0 \in G$ Suppose that $f. \bar{G} \to X^*$ is bounded pseudo-monotone Then inequality* (13) *holds*

Proof If $0 \in \overline{f(\partial G)}$, inequality (13) obviously holds Thus we may assume $0 \notin \overline{f(\partial G)}$ Suppose that (13) were false Then there would exist a positive number ε_0 such that

$$(x, f(x)) + \varepsilon_0 |x|^2 \le 0 \quad \text{for any } x \in \partial G \tag{14}$$

For each $0 < \varepsilon \le \varepsilon_0$ the map $f_\varepsilon = f + \varepsilon J$ (J is the duality map of X) is of class $(S)_+$ and bounded By (14), f_ε satisfies condition (12) It follows that there exists $x_\varepsilon \in \partial G$ such that $f_\varepsilon(x_\varepsilon) = 0$, i e

$$f(x_\varepsilon) + \varepsilon J(x_\varepsilon) = 0$$

Letting $\varepsilon \searrow 0$ we find that $0 \in \overline{\mathcal{J}(\partial G)}$, which contradicts our assumption Thus (13) holds.

COROLLARY 2. *Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$. Let T be a maximal monotone map of X into the subsets of X^* with $0 \in T(0)$ and let f be a bounded map of $\bar{G}$ into X^* of class $(S)_+$. Suppose that there exists a sequence (ε_j), $0 < \varepsilon_j$, $\varepsilon_j \to 0$, such that*

$$(x, (T_{\varepsilon_j} + f)(x)) \leq 0 \quad \textit{for all } x \in \partial G, \ j = 1, 2, \ldots, \tag{15}$$

where $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$ is the Yosida approximant of T. Then there exists at least one $x \in \partial G$ such that

$$0 \in f(x) + T(x)$$

Proof The map $T_\varepsilon + f$ ($\varepsilon > 0$) is bounded and of class $(S)_+$ So, by Theorem 1, for each j there exists $x_j \in \partial G$ such that

$$T_{\varepsilon_j}(x_j) + f(x_j) = 0 .$$

Denote $y_j = T_{\varepsilon_j}(x_j) = -f(x_j)$ Since G and f are bounded, we may suppose that we have

$$x_j \rightharpoonup x_0 \ \text{ and } \ y_j \rightharpoonup y_0$$

From $y_j = T_{\varepsilon_j}(x_j)$, we see that

$$y_j \in T(x_j - \varepsilon_j J^{-1}(y_j))$$

Hence, for any $x \in D(T)$ and any $y \in T(x)$, we have

$$(x_j - \varepsilon_j J^{-1}(y_j) - x, y_j - y) \geq 0$$

Thus,

$$(x_j - x, y_j - y) \geq (\varepsilon_j J^{-1}(y_j), y_j - y) \geq$$

$$\geq -(\varepsilon_j J^{-1}(y_j), y) \geq -\varepsilon_j \|y_j\| \, \|y\|$$

Since (y_j) is bounded, we deduce

$$\underline{\lim} \, (x_j - x, y_j - y) \geq 0 \tag{16}$$

If $\overline{\lim}\ (x_j - x_0, f(x_j)) \le 0$, then since f is of class $(S)_+$, we get $x_j \to x_0$ and $y_0 = -f(x_0)$ Next, from (16) we obtain

$$(x_0 - x,\ -f(x_0) - y) \ge 0$$

Since T is maximal monotone and $x \in D(T)$, $y \in T(x)$ were arbitrar, we deduce that $x_0 \in D(T)$ and $-f(x_0) \in T(x_0)$, which finishes the proof It remains only to show that $\overline{\lim}\ (x_j - x_0, f(x_j)) \le 0$ Suppose otherwise, i.e $\overline{\lim}\ (x_j - x_0, f(x_j)) > 0$ Then $\underline{\lim}\ (x_j - x_0, y_j) < 0$ and so, $\underline{\lim}\ (x_j, y_j) < (x_0, y_0)$ On the other hand, by (16)

$$\underline{\lim}\ (x_j, y_j) \ge (x_0 - x, y) + (x, y_0)$$

Therefore

$$(x_0 - x,\ y_0 - y) > 0 \text{ for all } x \in D(T) \text{ and } y \in T(x).$$

Since T is maximal monotone, it follows that $x_0 \in D(T)$ and $y_0 \in T(x_0)$. Thus the above strict inequality must also be true for $x = x_0$ and $y = y_0$, which is absurd The proof is now complete.

Remark 4 Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$, T a maximal monotone map of X into the subsets of X^* with $0 \in T(0)$ and let f be a bounded map of $\overline{G}$ into X^* of class $(S)_+$. Then there exists $\varepsilon_0 > 0$ such that

$$\sup \left\{ (x, (T_\varepsilon + f)(x)),\ x \in \partial G \right\} > 0 \quad \text{for any } 0 < \varepsilon \le \varepsilon_0, \tag{17}$$

or there exists $x \in \partial G$ such that $0 \in f(x) + T(x)$

We conjecture that there is $\varepsilon_0 > 0$ such that the strict inequality (17) always holds

COROLLARY 3 *Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$, T a maximal monotone map of X into the subsets of X^* with $0 \in T(0)$ and let f be a bounded pseudo-monotone map of $\overline{G}$ into X^* Then*

$$\sup\left\{(x,(T_\varepsilon+f)(x)),\ x\in\partial G\right\}\geq 0 \quad \textit{for any } \varepsilon>0 \tag{18}$$

Proof Suppose otherwise Then for some $\varepsilon>0$ there would exist a positive number δ_0 such that

$$(x,(T_\varepsilon+f)(x))+\delta_0\|x\|^2\leq 0 \quad \text{for any } x\in\partial G \tag{19}$$

For each $0<\delta\leq\delta_0$ the map $T_\varepsilon+f+\delta J$ is bounded of class $(S)_+$ and satisfies (12) By Theorem 2 there exists x_δ on ∂G such that

$$T_\varepsilon(x_\delta)+f(x_\delta)+\delta J(x_\delta)=0$$

If we set $x=x_\delta$ in (19) we get $(\delta_0-\delta)\|x_\delta\|^2\leq 0$ $(\delta<\delta_0)$, a contradiction

5. Concluding remarks. Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0\in G$, f a bounded map of $\bar{G}$ into X^* and T a maximal monotone map of X into the subsets of X^* We have established the following results

1) If X is a Hilbert space and $f=I-g$, g completely continuous, then

$$\sup\{(x,f(x)),\ x\in\partial G\}>0$$

2) If f is of class $(S)_+$, then

$$0\notin f(\partial G)\Rightarrow\sup\{(x,f(x)),\ x\in\partial G\}>0$$

3) If f is pseudo-monotone, then

$$\sup\{(x,f(x)),\ x\in\partial G\}\geq 0$$

4) If f is of class $(S)_+$, then

$$0\notin(T+f)(\partial G)\Rightarrow\sup\{(x,(T_\varepsilon+f)(x)),x\in\partial G\}>0 \quad \text{for any } 0<\varepsilon\leq\varepsilon_0$$

5) If f is pseudo-monotone, then

$$\sup\{(x,(T_\varepsilon+f)(x)),\ x\in\partial G\}\geq 0 \quad \text{for any } \varepsilon>0$$

We conjecture that in cases 2) and 4) the strict inequalities on "sup" also hold if $0\in$

$f(\partial G)$ and $0 \in (T+f)(\partial G)$, respectively We recall that the conditions $0 \notin f(\partial G)$ and $0 \notin (T+f)(\partial G)$ are required by the definition of Browder's degree Thus, we have shown that for bounded maps of type $(S)_+$, the reverse of the Krasnoselskii-Browder inequality, namely

$$\sup\{(x, f(x)); x \in \partial G\} \leq 0,$$

implies $0 \in f(\partial G)$, i e it is a bad condition in the degree theory Also, for sums $T+f$ with bounded maps of type $(S)_+$, the reverse of a Krasnoselskii-Browder type condition, i e

$$\sup\left\{(x, (T_{\varepsilon_j}+f)(x)),\ x \in \partial G\right\} \leq 0 \text{ for some sequence } \varepsilon_j \searrow 0,$$

implies $0 \in (T+f)(\partial G)$ Hence it is also a bad condition for the degree theory

We conclude with an application to Leray-Lions maps Let $\Omega \subset \mathbb{R}^n$ be open bounded, $1<p<\infty$, $F\ \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the Caratheodory conditions, i e. $F(\cdot, s, \xi)$ is measurable for all s, ξ and $F(x, \cdot, \cdot)$ is continuous a e $x \in \Omega$ Also assume

$$|F(x,s,\xi)| \leq C(x) + c_1|s|^{p-1} + c_2|\xi|^{p-1}$$

and

$$(\xi - \xi^*, F(x,s,\xi) - F(x,s,\xi^*)) \geq 0,$$

for all ξ, ξ^*, s and a e $x \in \Omega$, where $c_1, c_2 \in \mathbb{R}_+$, $C(x) \geq 0$, $C(x) \in L^{p'}(\Omega)$ $(1/p + 1/p' = 1)$ It is known that the map

$$f\ W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega),\ f(u) = -\operatorname{grad} F(x,u,\operatorname{grad} u)$$

is bounded, continuous and pseudo-monotone

By Corollary 1, for each $R > 0$, we have

$$\sup\left\{(u, f(u)) = \int_\Omega F(x,u,\operatorname{grad} u)\operatorname{grad} u\, dx,\ \|u\|_{W_0^{1,p}} = R\right\} \geq 0$$

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

FIXED POINTS OF RETRACTIBLE MULTIVALUED OPERATORS

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Dedicated to Professor P Szilágyi on his 60th anniversary

Received December 10, 1993
AMS subject classification 47H10, 54H25

REZUMAT. - Puncte fixe pentru operatori multivoci retractibili. Scopul acestei lucrări este de a stabili, folosindu-se tehnica structurilor de punct fix introdusă de I A.Rus în [4], teoreme generale de punct fix pentru operatori multivoci ce nu invariază domeniul de definiţie, din care se va desprinde ca şi consecinţă un rezultat cunoscut.

1. Introduction. The notion "fixed point structure", given by I A Rus in [4] (see also [5] and [6]) is a generalization of some notions as "topological space with fixed point property" (Brouwer, Schauder, Tihonov, etc), "ordered set with fixed point property" (Tarski, Bourbaki-Birkhoff, .), "object with fixed point property" (Lawvere, Lambek, Rus,) Recently, I A Rus extended this technique to multivalued operators and gave new results in the fixed point theory for multivalued operators (see [4])

The object of the present paper is to extend this technique to multivalued operators which doesn't map the domain into itself and to give some new results for such non-self - multivalued operators As consequence, one obtain a well known fixed point theorem for l s c. multivalued operator (see Demling [2])

2. Preliminaries. Let X and Y be two sets We follow terminologies and notations in [4]

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We denote by $M^0(X,Y)$ the set of all multivalued operators $T: X \multimap Y$. If $X = Y$ then $M^0(Y) = M^0(Y,Y)$.

DEFINITION 2.1. Let $T: X \multimap X$ be a multivalued operator (briefly m-operator). Then, by definition an element $x \in X$ is a fixed point of T iff $x \in T(x)$. We denote by F_T the fixed points set of T.

DEFINITION 2.2. Let $T: x \multimap X$ be a m-operator. By definition a subset $A \subset X$ is an invariant subset under T if $T(A) \subset A$. We denote $I(T) = \{A \mid A \subset X,\ T(A) \subset A\}$.

Let X be a nonvoid set and $Y \in P(X)$.

DEFINITION 2.3 ([4]). A triple (X,S,M^0) is a fixed point structure (briely f.p.s) if

(i) $S \subset P(X)$, $S \neq \emptyset$

(ii) $M^0: P(X) \multimap \bigcup_{Y \in P(X)} M^0(Y)$, $Y \multimap M^0(Y) \subset M^0(Y)$ is a mapping such that if $Z \subset Y$, $Z \neq \emptyset$ then $M^0(Z) \supset \{T|_Z \mid T \in M^0(Y) \text{ and } Z \in I(T)\}$

(iii) every $Y \in S$ has the fixed point property with respect to $M^0(Y)$.

Now some examples of fixed point structures.

Example 2.1 ([4]). Let X be a Hausdorff locally convex topological vector space and $M^0(Y) = \{T: Y \to P_{cl,cv}(Y) \mid T \text{ is u.s.c.}\}$. Then $(X,P_{cp,cv}(X),M^0)$ is a f.p.s.

Example 2.2. Let X be a Banach space and $M^0(Y) = \{T: Y \to P_{cl,cv}(Y) \mid T \text{ is l.s.c.}\}$. Then $(X,P_{cp,cv}(X),M^0)$ is a f.p.s.

DEFINITION 2.4 ([4]). Let (X,S,M^0) be a f.p.s., $\theta: Z \to \mathbb{R}_+$ ($S \subset Z \subset P(X)$) and $\mu: P(X) \to P(X)$. The pair (θ,μ) is a compatible pair with (X,S,M^0) if

(i) μ is a closure operator, $S \subset \mu(Z) \subset Z$ and $\theta(\mu(Y)) = \theta(Y)$, for all $Y \in Z$

(ii) $F_\mu \cap Z_\theta \subset S$

Example 2.3 ([4]). Let (X,S,M^0) be as in Example 2.1 or Example 2.2, $Z = P_b(X)$,

$\theta = \alpha_K$ (Kuratowski measure of noncompactness) or $\theta = \alpha_H$ (Hausdorff measure of noncompactness) and $\mu(A) = \overline{co}\, A$ Then the pairs (α_K,μ) and (α_H,μ) are compatible pairs with (X,S,M^0)

Following Halpern-Bergman [3] and Deimling [2] let us introduce

DEFINITION 2 5 ([2]) Let X be a Banach space and $Y \in P_{cl,cv}(X)$ Then $I_Y(x) = \{x + \lambda(y - x) \mid \lambda \geq 0, y \in Y\}$ for $x \in Y$ is the inward set of $x \in Y$ with respect to Y

DEFINITION 2 6 ([2]) Let X be a Banach space and $Y \in P_{cl}(X)$. We let

$$K_Y^*(x) = \left\{ z \in X \mid \lim_{\lambda \to 0_+} \lambda^{-1} D(x + \lambda z, Y) = 0 \right\} \quad \text{for } x \in Y$$

$$K_Y(x) = \left\{ z \in X \mid \lim_{\lambda \to 0_+} \inf \lambda^{-1} D(x + \lambda z, Y) = 0 \right\} \quad \text{for } x \in Y$$

and $Y_Y^*(x) = x + K_Y(x)$, for $x \in Y$

Some basic properties are contained in

PROPOSITION 2 1 ([2], [3]) *Let X be a Banach space and $Y \in P_{cp}(X)$ Then.*

(i) $0 \in K_Y^*(x) \subset K_Y(x)$, for any $x \in Y$

(ii) $K_Y^*(x) = X$, for any $x \in \operatorname{int} Y$

(iii) $K_Y^*(x)$ and $K_Y(x)$ *are closed for any $x \in Y$*

(iv) $\lambda K_Y^*(x) \subset K_Y^*(x)$ and $\lambda K_Y(x) \subset K_Y(x)$, *for all $\lambda \geq 0$ and all $x \in Y$*

(v) *If Y is convex then $K_Y^*(x) = K_Y(x)$ and $K_Y(x)$ is convex, for all $x \in Y$*

(vi) *If Y is convex then $I_Y^*(x) = \overline{I_Y(x)}$, for all $x \in Y$*

(vii) *If Y is convex then $K_Y(\cdot)$ is l s c. on Y*

Now we shall present the notion of retractible multivalued operators (see also [1] and [6])

Let X be a nonvoid set and $Y \in P(X)$

DEFINITION 2 7([1]) A multivalued operator $R\ X \to P(Y)$ is called a retraction of X onto Y if $R|_Y = 1_Y$

DEFINITION 2 8 ([1]) A multivalued $T\ Y \to P(X)$ is retractible onto Y if there is a retraction $R\ X \to P(Y)$ such that $F_{R\circ T} = F_T$.

The next results are given in [1]

PROPOSITION 2 2 *The following two set are equal* $U = \{x \in Y \mid x \in T(x)$ *or* $T(x) \cap R^1(x) = \varnothing\}$ *and* $V = F_T \cup C_Y(F_{R\circ T})$ (*where* $R^1(x) = \{z \in X \mid x \in R(z)\}$)

PROPOSITION 2 3 *The following conditions are equivalent:*

(i) $x \in R(T(Y) \setminus Y) \Rightarrow x \in T(x)$ *or* $T(x) \cap R^1(x) = \varnothing$

(ii) $F_{R\circ T} \subseteq F_T$

(iii) $F_{R\circ T} = F_T$

Example 2 4 ([2]) Let X be a Banach space and $Y \in P(X)$ Consider the metric projection $\Pi_Y\ X \to P(Y)$

$\Pi_Y(x) = \{u \in Y \mid \|x-u\| = D(x,Y)\}$

Evidently $\Pi_Y|_Y = 1_Y$

If $Y \in P_{cp}(X)$ then $\Pi_Y\ X \to P_{cp}(Y)$ is u.s c.

If $Y \in P_{cp,cv}(X)$ then $\Pi_Y(x)$ is convex for all $x \in X$

3. Basic results. The following general results are essential in the fixed point theory of nonself multivalued operators

PROPOSITION 3 1 *Let* (X,S,M^0) *be a f.p.s*, $Y \in S$, $R\ X \to P(Y)$ *a retraction and* $T\ Y \to P(X)$ *a multivalued operator. If*

(i) $R \circ T \in M^0(Y)$

(ii) *T is retractible onto Y by R*

then $F_T \neq \emptyset$

Proof From (ii) we have $F_{R\circ T} = F_T$ From $R \circ T \in M^0(Y)$, $Y \in S$ and (X,S,M^0)-f p s it follows that $F_{R\circ T} \neq \emptyset$ ■

PROPOSITION 3 2 *Let* (X,S,M^0) be a f p s , $Y \in S$, R $X \to P(Y)$ *a retraction of* X *onto* Y

Let T $Y \to P(X)$ *be such that:*

(i) *T admits a continuous selection* t

(ii) $R \circ t \in M^0(Y)$

(iii) *T is retractible onto Y by R.*

Then $F_T \neq \emptyset$

Proof. Let t $Y \to X$ be a continuous selection for T, i e $t(x) \in T(x)$, for all $x \in Y$ From (iii) it follows that $F_{R\circ T} = F_T$ We have $R \circ t \in M^0(Y)$, $Y \in S$ and (X,S,M^0)-f p s , therefore $F_{R\circ t} \neq \emptyset$ The conclusion follows taking into account that $F_{R\circ t} \subseteq F_{R\circ T}$ ■

From Proposition 3 2 we have the following theorem

THEOREM 3 1 *Let X be a Banach space,* $Y \in P_{cp,cv}(X)$ *and* T: $Y \to P_{cl,cv}(X)$ *l.s.c. If* $T(x) \subset I_Y^*(x)$, *for al* $x \in Y$ *then* $F_T \neq \emptyset$

Proof. Let Π_Y $X \to P_{cp,cv}(Y)$ be the metric projection From Example 2.4 $\Pi_Y(\cdot)$ is u.s c. on X By Michael's selection theorem (see [2]) there is continuous operator t $Y \to X$ such that $t(x) \in T(x)$, for all $x \in Y$ Evidently $\Pi_Y \circ t$ $Y \to P_{cp,cv}(Y)$ is u s c Let (X,S,M^0) be the f p s given in Example 2 1 It folows that $\Pi_Y \circ t \in M^0(Y)$ We claim that T is retractible onto Y by Π_Y, i e $F_{\Pi_Y\circ T} \subseteq F_T$

If $x \in (\Pi_Y \circ T)(x)$ then there is an element $y \in T(x)$ such that $x \in \Pi_Y(y)$ We have the following alternative·

a) $y \in Y$ Then $\Pi_Y(y) = y$ So $x = y$ and hence $x \in Tx$, i e

$x \in F_T \neq \emptyset$

b) $y \notin Y$ Since $y \in T(x)$ and $x \in \Pi_Y(y)$ we have that

$$\|y - x\| = \inf_{z \in Y} \|y - z\| = D(y, Y)$$

We shall prove now that for any $h \in]0,1[$

$$D((1-h)x + hy, Y) = \|(1-h)x + hy - x\| \qquad (*)$$

We suppose that there exists $h \in]0,1[$ such that $z \in Y$, $z \neq x$ and $\|(1-h)x + hy - z\| < \|(1-h)x + hy - x\|$

But $\|y - z\| \leq \|(1-h)x + hy - z\| + \|(1-h)(x-y)\| < \|(1-h)x + hy - x\| + \|(1-h)(x-y)\| = \|x - y\|$, which contradicts $x \in \Pi_Y(y)$ It follows that the relation (*) holds

Therefore

$$D(x, Tx) \leq \|x - y\| = \liminf_{h \to 0_+} \frac{D((1-h)x + hy, Y)}{h}$$

Since $y \in T(x) \subset I_Y^*(x) = x + K_Y(x)$ we have that $y - x \in K_Y(x)$ Consequently (see Proposition 2 1) $\liminf_{h \to 0_+} \dfrac{D((1-h)x + hy, Y)}{h} = 0$ Hence $D(x,T(x)) \leq 0$, i e. $x \in T(x)$ Applying Proposition 3 2 one obtain the conclusion ∎

Remark 3 1 Theorem 3 1 appear in [2], where it is established in a different manner

Remark 3 2 A similar result for u s c multivalued operators which satisfies the weakest condition $T(x) \cap I_Y^*(x)$, for all $x \in Y$ is given by *K* Deimling (see [2]) An open problems is to prove such a result using the technique of the fixed point structures for non-self multivalued operators

Remark 3 3 Fixed point theorems for θ-condensing non-self multivalued operators are established in a previous paper A general technique for constructing fixed point theorem in the case of non-self multivalued (θ,φ)-contraction is also given

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

NOTE ON A CRITICAL POINT THEOREM

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Dedicated to Professor P Szilágyi on his 60th anniversary

Received December 22, 1993
AMS subject classification 58E05

REZUMAT. - Notă asupra unei teoreme de punct critic. În lucrare se demonstrează următoarea variantă a unei teoreme privind existenţa a trei puncte critice Fie M o varietate C^2 - Finsler completă fără frontieră şi fie $f:M \to \mathbb{R}$ o funcţie C^1 cu valori reale Dacă f este mărginită inferior, satisface condiţia lui Palais-Smale şi are două minime locale, atunci f posedă cel puţin trei puncte critice distincte

1. Introduction. In the present paper the following variant of the three critical points theorem is proved: Let M be a complete C^2-Finsler manifold, without boundary, and let $f: M \to \mathbb{R}$ be a C^1 real-valued function Assume that f is bounded below, satisfies the Palais-Smale condition and it has two local minima Then f prosses at least three distinct critical points.

Remark that an analogous result is given in M S Berger and M S Berger [4, pp 58-62], R Courant [10, p 223], M Struwe [24, Theorem 1 1 , pp 66-68] for $M = \mathbb{R}^n$ with the coercivity condition $f(x) \to \infty$ when $\|x\| \to \infty$, and in P H Rabinowitz [22, Corollary 3 15] for M a real Banach space with the same regularity hypotheses on f Other versions and applications to various problems in the theory of partial differential equations are presented in H Amann [2], A Castro and A C Lazer [6], K-C Chang [7], [8, Section 5, pp 71-78], [9, pp 128-129], M A Krasnoselski [10], L Nirenberg [17] and P H Rabinowitz [21]

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2. Preliminaries on Finsler manifolds. Let M be a C^1 Banach manifold without boundary (i e $\partial M = \phi$) and led $T(M)$ be the total space of tangent bundle of M A continuous function $\| \ \| \ T(M) \to \mathbf{R}_+$ is a Finsler structure on $T(M)$ if the following conditions are satisfied.

(i) For each $x \in M$, the restriction $\| \ \|_x = \| \ \| \,|_{T_x(M)}$ is an equivalent norm on $T_x(M)$,

(ii) For each $x_0 \in M$, and $k > 1$, there is a trivializing neighbourhood U of x_0 such that

$$\frac{1}{k}\| \ \|_k \leq \| \ \|_{x_0} \leq k\| \ \|_x, \quad (\forall)\, x \in U$$

M is said to be a Finsler manifold if it is regular (as a topological space) and if it has a Finsler structure on $T(M)$

It is known (see R S Palais [18, Theorem 2 11]) that every paracompact C^1 Banach manifold admits Finsler structures and that every C^1 Riemannian manifold is a Finsler manifold (see R S Palais [18, Theorem 2 12])

Suppose that M is connected For $x,y \in M$ define $\Omega(x,y)$ the set of all C^1 - piecewise path σ $[0,1] \to M$ such that $\sigma(0) = x, \sigma(1) = y$. The length of $\sigma \in \Omega(x,y)$ is given by

$$\ell(\sigma) = \int_0^1 \|\sigma(t)\|_{\sigma(t)}\, dt \tag{1}$$

Consider the Finsler metric on M defined as follows

$$d_F(x,y) = \inf\{\ell(\sigma) \ \ \sigma \in \Omega(x,y)\} \tag{2}$$

One can show (see R S Palais [18, Theorem 3 3] or K Daimling [11, Exercise 8, p 376]) that (M,d_F) is a metric space and that the metric topology coincides with the topology of M

To a given Finsler structure on $T(M)$ there correspond a dual structure on the cotangent bundle $T(M)^*$ given by

$$\|\mu\| = \sup\{|\mu(x)| \ \ \|x\|_p = 1\}, \quad \mu \in T(M)^* \tag{3}$$

Let f $M \to \mathbf{R}$ be a C^1 - differentiable mapping A locally Lipschitz continuous vector field $v: M \to T(M)$ such that for each $x \in M$ the following relations are satisfied

(i) $\quad \|v_x\| \leq 2\,\|(df)_x\|$

(ii) $\quad (df)_x(v_x) \geq \|(df)_x\|^2$

where $\|(df)_x\|$ is given by the Finsler structure on $T_x(M)^*$, is called a pseudogradient vector field of f (in short p.g.f. of f).

This important notion was introduced by R. S. Palais [18]. An interesting modification was given by F. E. Browder [5] (see also K. Deimling [11, p. 372]).

It is known (see R. S. Palais [18]) that if M is a C^2 - Finsler manifold and $f: M \to \mathbf{R}$ is a C^1 - differentiable mapping, then $\nabla(f) \neq \phi$, where

$$\nabla(f) = \{v \in X(M) : v \text{ is p. g. f. of } f\} \tag{4}$$

is the set of all pseudo - gradient vector fields of f. Let us note that if M is a Hilbert manifold with the Riemannian structure $\|\ \|$, the norms $\|\ \|_x$ come from inner products by $\|\ \|_x = <\ ,\ >_x^{1/2}$, and we can define a p.g.f. of f by $p \to (\text{grad} f)(p)$, where $(\text{grad} f)(p)$ is given via the well-known Riesz representation theorem by

$$(df)_p(X) = <X, (\text{grad} f)(P)>_p,\ (\forall) X \in T_p(M)$$

(see J. T. Schwartz [23, Chapter IV]).

3. The main results. Let M be a C^2 - Finsler manifold, connected and without boundary. For a C^1 - differentiable real-valued function $f: M \to \mathbf{R}$, define by

$$C(f) = \{p \in M : (df)_p = 0\} \tag{5}$$

the critical set of f and by $B(f) = f(C(f))$ the bifurcation set of f. The elements of $C(f)$ are called the critical points of f and the elements of $B(f)$ represent its critical values. If $p \notin C(f)$, $s \notin B(f)$, then p is a regular point and s is a regular value of the mapping f.

For $s \in \mathbb{R}$ denote by $C_s(f) = C(f) \cap f^{-1}(s)$, the critical point set of f at the level s It is obvious that s is a regular value of f if and only if $C_s(f) = \phi$ We also consider $M_s(f) = f^{-1}((-\infty, s])$

It is well-known that if $s \notin B(f)$ then $f^{-1}(s)$ is ϕ or a differentiable submanifold of M, of codimension 1, and $M_s(f)$ is a differentiable submanifold with boundary of M, of codimension O, and $\partial M_s(f) = f^{-1}(s)$ (see for instance R.Abraham, J E Marsden and T S Ratiu [1, p 197]).

Suppose that the manifold M and the mapping f satisfy the following hypotheses

(a) (Completness) (M, d_ν) is a complete metric space, where d_ν represents the Finsler metric on M defined by (2)

(b) (Boundedness from below) If $B = \inf \{f(x) \ \ x \in M\}$ then $B > -\infty$.

(c) (The Palais-Smale condition) Any sequence $(x_n)_{n \geq 0}$ in M with the properties that $(f(x_n))_{n \geq 0}$ is bounded and $\| (df)_{x_n} \| \to 0$ has a convergent subsequence $(x_{n_k})_{k \geq 0}$, $x_{n_k} \to p$

The above conditions (a) - (c) are sometimes called compactness conditions because if M is a compact manifold they are automatically verified It is clear that the point p, which appears in condition (c) of Palais-Smale, is a critical point of $f, p \in C(f)$

Let $\nu \in \nabla(f)$ be a p g f of f and let $x \in M$ be a fixed point Because ν is locally Lipschitz the following Cauchy problem

$$\begin{cases} \varphi(t) = -\nu_{\varphi(t)} \\ \varphi(0) = x \end{cases} \tag{6}$$

has a unique maximal solution $\varphi^\nu \ \ (\omega_-^\nu(x), \omega_+^\nu(x) \to M$, where $\omega_-^\nu(x) < 0 < \omega_+^\nu(x)$ Denote by $\varphi_t^\nu(X)$ the above solution and by $t \to \varphi_t^\nu(x)$ the corresponding integral curve of (6) Taking into account the hypotheses (a)-(c) it follows that $\omega_+^\nu(x) = +\infty$, i e $\left\{\varphi_t^\nu\right\}_{t \geq 0}$ is a semigroup of diffeomorphisms of M (see R S Palais [18] or K Deimling, [11, Lemma 27 1])

For a vector $v \in X(M)$ let us consider the sets $Z(v) = \{ p \in M \;\; v_p = 0 \}$, $\mathrm{Fix}\;(\varphi^v) = \{ x \in M \;\; \varphi^v_t(x) = x,\ (\forall)\ t \in (\omega^v_-(x), \omega^v_+(x)) \}$ It is easy to see that the following relations hold

$$C(f) = \bigcap_{v \in \nabla(f)} Z(v) \tag{7}$$

$$C(f) = \bigcap_{v \in \nabla(f)} \mathrm{Fix}\;(\varphi^v) \tag{8}$$

If $x \notin C(f)$, then $f(\varphi^v_t(x)) < f(x)$ for $t > 0$ and $f(\varphi^v_t(x)) > f(x)$ **(9)**

for $t < 0$

Our main result is the following

THEOREM *Let M be a C^2 - Finsler manifold, connected and without boundary, and let f $M \rightarrow \mathbb{R}$ be a C^1 - differentiable real-valued mapping. Assume that the hypotheses (a) - (c) are satisfied and there exist two local minima points of f. Then f posses at least three distinct critical points.*

Proof Let $p_1,p_2 \in M$ be the local minima points of f Then $p_1,p_2 \in C(f)$ and $s_i = f(p_i)$, $i = 1,2$ are critical values of f Let us suppose that p_1,p_2 are isolated minima of f and $s_1 \geq s_2$ Then the set $M_{s_1}(f)$ has at least two connected components M_1,M_2 with $p_1 \in M_1$, $p_2 \in M_2$ Because f is continuous it follows that there exists a positive number $\varepsilon_0 > 0$, such that the set $M_{s_1+\varepsilon_0}(f)$ has also at least two connected components M_{1,ε_0}, M_{2,ε_0} with $p_1 \in M_{1,\varepsilon_0}$, $p_2 \in M_{2,\varepsilon_0}$

Let us consider the family of compact subsets of M defined by

$$\mathfrak{R} = \{ K \subseteq M \;\; K \text{ is connected, compact and } p_1,p_2 \in K \}$$

We shall show that the family $\mathfrak{R}$ is invariant with respect to the flow $\{\varphi_t\}_{t \geq 0}$ generated by a p g f of $f, v \in \nabla(f)$ Indeed for any $t \geq 0$ and for any $K \in \mathfrak{R}$ the set $\varphi_t(K)$ is connected and compact since φ_t is a diffeomorphism of the manifold M Moreover, according to the relation (8), it follows that $\varphi_t(p_i) = p_i$, $i = 1,2$, for any $t \geq 0$, i e $p_1, p_2 \in \varphi_t(K)$ for any $t \geq$

0 and for any $K \in \mathfrak{R}$

Define the mini-max of f with respect to the family $\mathfrak{R}$ by

$$\text{mini-max}\,(f, \mathfrak{R}) = \inf_{K \in \mathfrak{R}} \max_{x \in K} f(x)$$

It is obvious that mini-max $(f,\mathfrak{R})$ represent the smallest real number $s \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $K_\varepsilon \in \mathfrak{R}$ with $K_\varepsilon \subseteq M_{s+\varepsilon}(f)$

Because the family $\mathfrak{R}$ is invariant with respect to flow $\{\varphi_t\}_{t\geq 0}$ generated by a p g f $v \in \nabla(f)$, according to the mini-max principle (see R S Palais [18, Theorem 5 18]) it follows that $s =$ mini-max $(f,\mathfrak{R})$ is a critical value of the mapping f It is easy to see that the following inequalities hold $s \geq s_1 \geq s_2$

If $s = s_1$, using the definition of mini-max $(f,\mathfrak{R})$, there exists $K_{\varepsilon_0} \in \mathfrak{R}$ such that $K_{\varepsilon_0} \subsetneq M_{s+\varepsilon_0}(f)$, where ε_0 is the number considered at the beginning of the proof Because K_{ε_0} is connected and $p_1, p_2 \in K_{\varepsilon_0}$, one obtains a contradiction with the choice of ε_0. Therefore $s > s_1 \geq s_2$ and $c_s(f) \neq \phi$ So, there exists a critical point $p_3 \in C_s(f)$ with $p_3 \notin C_{s_i}(f)$, $i = 1,2$, and the assertion is proved

The following two corollaries are obtained from the above main result

COROLLARY 1 *Let M be a* C^2 *- Finsler manifold, connected and without boundary, and let f* $M \to \mathbb{R}$ *be a* C^1 *- differentiable real-valued mapping. Assume that the hypotheses* (a)-(c) *are satisfied and f has a local minimum point which is not a global minimum point Then f posseses at least three distinct critical points*

Proof Let us consider $B = \inf\{f(x) \;\; x \in M\}$ Then B is the global minimum of f (see R S Palais [18, Theorem 5 7] or K-C, Chang [7, Lemma 4], D Andrica [3, Theorem 3 2 9]) and there exists a critical point $p_2 \in C(f)$ such that $f(p_2) = B$ Considering p_1 the local minimum point given in the hypothesis, we can apply the above theorem, and the desired conclusion is obtained

COROLLARY 2 *Let M^m be a m-dimensional C^2-manifold which is closed* (i e M *is compact and without boundary) and connected. If f $M \to \mathbf{R}$ is a C^1- differentiable real-valued mapping with two local minima points, then f posseses at least four distinct critical points.*

Proof Let p_1, p_2 be the local minima points of f given in the hypothesis Because M is compact, f has a maximum point p_4, and it follows $p_1, p_2, p_4 \in M(f)$ (see D Andrica [3, Proposition 2 2 6]) Consider p_3 the critical point obtained by the mini-max principle in Theorem It is clear that the following inequalities are valid $s_4 \geq s_3 > s_2 \geq s_1$, where $s_i = f^{-1}(p_i)$, $i = \overline{1,4}$ If the dimension of M is greater that 2 (i e $m \geq 2$), taking into account the definition of the mini-max of f with respect to the family $\mathfrak{R}$, it results that $s_4 > s_3$, therefore $|C(f)| \geq 4$ If the dimension of M is 1 (i e $m = 1$), using the classification of one-dimensional manifolds (see V.Guillemin, A Pollak [14]) one obtains that M is diffeomorphic with the sphere S^1 and in this case the property is also proved.

Remarks 1) The following example shows that there exist smooth functions f $\mathbb{R}^2 \to \mathbb{R}$ having two global minima points and without other critical points The polynomial function $f(x,y) = (x^2y - x - 1)^2 + (x^2 - 1)^2$ has global minima at the points (1,2) and (-1,0) and no other critical points (this example is simple modification of the polynomial given in A Durffe et al [12])

2) One can show that if M^m is a closed C^2-manifold and $\mathcal{F}(M)$ is the real algebra of C^1 -differentiable real-valued functions defined on M then for every function $f \in \mathcal{F}(M)$ the inequality $|C(f)| \geq 2$ holds (see D Andrica [3, Proposition 2 2 6]) The lower bound of the cardinal number of critical set $C(f)$ is very strong connected to the topology of M For instance if $m = 2$ and the manifold M is not diffeomorphic with the sphere S^2 then $|C(f)| \geq$

3 (see D Andrica [3, Theorem 4 3 6] and M W Hirsch [15, Exercise 13*, pag 29] for a particular case) If $m = 3$ and the fundamental group $\pi_1(M)$ is not free, then $|C(f)| \geq 4$ This inequality follows from a recent result concerning the Lusternik-Schnirelmann category, obtained by Gómez-Larrañaga, J.C González-Acuña, F [13]

Acknowledgements. The author would like to express his gratitude to DEUTSCHER AKADEMISCHER AUSTAUSCHDIENST (DAAD) for the financial support during his staying in Technischen Universität München, where this paper was finished

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

LJUSTERNIK-SCHNIRELMAN THEORY ON CLOSED SUBSETS OF C^1-MANIFOLDS

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Dedicated to Professor P Szilágyi on his 60th anniversary

Received January 5, 1994
AMS subject classification 58E05

REZUMAT. - Teoria Ljusternik-Schnirelman pe submulţimi închise ale unor C^1-varietăţi. Rezultatul principal, Teorema 2 4, generalizează, la C^1-varietăţi, unele rezultate obţinute de Wang şi Szulkin

1. Introduction. Let M be a C^1-Banach manifold which is modeled on the Banach space If (U,φ) is a chart of M at $x_0 \in M$, then denote by $\psi\ U \to T_{x_0}(M)$ the function by $\psi(x) = [U,\varphi,\varphi(x)]_{x_0}$ for each $x \in U$ In what follows we shall denote by TM the tangent bundle of M

DEFINITION 1 1 A Finsler structure on the tangent bundle TM, is a continuous real function $\|\cdot\|\ \ TM \to [0,+\infty[$ such that

i) for every $x \in M$, the restriction $\|\cdot\|_{T_x(M)}$ is an admisible norm on $T_x(M)$,

ii) for every $x_0 \in M$ and $K > 1$, there is a chart (U,φ) of M at x_0 such that the following inequalities

$$\frac{1}{k}\left\|[U,\varphi,h]_x\right\| \le \left\|[U,\varphi,h]_{x_0}\right\| \le k\left\|[U,\varphi,h]_x\right\|$$

hold for all $h \in E$ and $x \in U$

A C^1 - Banach manifold, which is a regular space, whith a fixed Finsler structure for

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TM is called a Finsler manifold

R Palais [4] proved that for every C^1 - Banach paracompact manifold there exists a Finsler structure on the tangent bunde

If x and $y \in M$ are two points in the same component of M, the distance $\rho(x,y)$ is the infimum of $l(\sigma)$, over all C^1 paths joining x to y Then $\rho \; M \times M \to [0,+\infty[$ is a metric on M which is consistent with the topology of M

DEFINITION 1 2 Let M be a C^1 - Banach manifold, $S \subseteq M$ a nonempty subset of M A vector $v_0 \in T_{x_0}(M)$ is called tangent to S at $x_0 \in S$ if there is a chart (U,φ) of M at x_0 such that

$$\lim_{h \uparrow 0} \frac{d(\varphi(x_0) + h\varphi'(x_0)v_0 \quad \varphi(U \cap S))}{h} = 0$$

where d is the distance on E

Let us denote $T_x(S)$ as the set of all tangent vectors at $x \in S$.

D Motreanu [3] proved the following result

THEOREM 1 1 *Let M be a C^1 - Banach manifold, $S \subseteq M$, $x_0 \in S$ and (U,φ) a chart of M at x_0 A vector $v_0 \in T_{x_0}(M)$ is tangent to S at x_0 iff there is a function $u \;]-\alpha,0] \to E$ such that*

i) $\lim_{h \uparrow 0} u(h) = 0$

ii) $\varphi(x_0) + h(v_0 + u(h)) \in \varphi(U \cap S)$, *for all h*

DEFINITION 1 3 Let M be a C^1 - Banach manifold A subset $S \subseteq M$ is locally convex if, for each $x_0 \in S$, there is a chart (U,φ) of M at x_0 such that $\varphi(U \cap S)$ is convex in E

Let M be a C^1 - Finsler manifold, $f \in C^1(M,R)$, $S \subseteq M$, $x \in S$ Consider the following notations

$$\| df(x) \|_S = \sup \{ df(x) \cdot v) \mid v \in T_x(S), \|v\| = 1 \}$$

DEFINITION 1 4 Let M be a C^1 - Finsler manifold S a subset of M and $f \in C^1(M,R)$ A point $x \in S$ is called a critical point of f relative to S if $\| df(x) \|_S = 0$

$KS = \{ x \in S \mid \| df(x) \|_S = 0 \}$ - the set of all critical points of f relative to S,

$KS_c = KS \cap f^{-1}(c)$, for each $c \in R$

DEFINITION 1 5. A function $f \in C^1$ (M,R) is said to satisfy the Palais - Smale condition relative to S at the level $c \in R$, if for each sequence $\{x_n\}_{n \geq 1} \subseteq S$ such that $f(x_n) \to c$ and $\| df(x_n) \|_S \to 0$, there exist a convergent subsequence

The function f is said to satisfy the PS condition relative to S ($(PS)_S$ in short) if f satisfies this condition for all level $c \in R$.

In what follows, we use the Ekeland's variational principle [7]

THEOREM 1 2 *Let (Z,d) be a complete metric space and π $Z \to]-\infty , +\infty]$ a lower semicontinuous, proper ($\pi \neq +\infty$) function which is bounded below. Then for each* $\varepsilon > 0$ *and for each $x \in Z$ such that $\pi(x) \leq \inf\limits_{z \in Z} \pi(z) + \varepsilon^2$, there exists a point $y \in Z$ such that*

1) $\pi(y) \leq \pi(z)$

2) $d(x,y) \leq \varepsilon$

3) $\pi(z) - \pi(y) \geq -\varepsilon d(y,z)$, *for each* $z \in Z$

In what follows, we shall need the definition of the Ljusternik - Schnirelmann category

DEFINITION 1 6 Let X be a topological space and A a subset of X The Ljusternik - Schnirelmann category of A in X is k, if there exists k (but not exist k - 1) closed of X, contractible in X, which cover A Denote $\mathrm{cat}_X(A)=k$ If such a k does not exist then $\mathrm{cat}_X(A)=+\infty$

THEOREM 1 3 *Let X be a topological space and A, B subsets of X The following results hold*

a) $\mathrm{cat}_X(A)=1$ *iff A is contractible to a point in X*

b) *If* $A \subseteq B$ *then* $\mathrm{cat}_X(A) \leq \mathrm{cat}_X(B)$

c) $\mathrm{cat}_X(A \cup B) \leq \mathrm{cat}_X(A) + \mathrm{cat}_X(B)$

d) *If* $\mathrm{cat}_X(B)<+\infty$, *then* $\mathrm{cat}_X(A-B) \geq \mathrm{cat}_X(A) - \mathrm{cat}_X(B)$

e) *If A is a closed subsets of X and* $\alpha\ A\times[0,t_0] \to X$ *is a continuous function such that* $\alpha(a,0) = a, \forall a \in A$, *then* $\mathrm{cat}_X(A) \leq \mathrm{cat}_X(\alpha(A,t_0))$

f) *If M is a* C^1 *- Finsler manifold and* $A \subseteq M$, *then there exists* $U \in \nu_M(A)$ *such that* $\mathrm{cat}_M(\bar{U}) = \mathrm{cat}_M(A)$

2. The Existence of Critical Points. Let M be a C^1 - complete Finsler manifold and let S be a nonempty, closed and locally convex subset of M

DEFINITION 2 1 A pseudogradient vector for a function $f \in C^1(M,R)$, relative to S, at the point $x \in S$, is a vector $v \in T_S(x)$ such that

a) $\|v\| < 2 \, \| df(x) \|_S$

b) $df(x) \cdot v > \| df(x) \|_S^2$

Let $x_0 \in S - KS$ i e $\|df(x_0)\|_S \neq 0$ Using the definition of $\|df(x_0)\|_S$, we get that there exists a tangent vector w_0 to S at x_0 such that $\|w_0\| = 1$ and $df(x_0) \cdot w_0 > \frac{2}{3} \|df(x_0)\|_S$ Denote $v_0 = \frac{3}{2} \| df(x_0) \|_S \, w_0 \quad T_{x_0}(S)$ It is easy to see that v_0 is a pseudogradient vector for f, relative to S, at x_0

Let (U,φ) be a chart of M at x_0 and $\psi = (\varphi'(x_0))^{-1}$

$df(x_0) \cdot v_0 = (f\psi^{-1})\,,(\psi(x_0)) \cdot v_0$

It is easy to check that $\| df(x_0)\|_S^2 < df(x_0) \cdot v_0 \le \| df(x_0)\|_S \, \| v_0 \|$ i e $\| v_0 \| > \| df(x_0)\|_S$ and $df(x_0) \cdot v_0 > \| df(x_0)\|_S^2 > \frac{1}{4} \| v_0 \|$

Using the continuity of the functions $(f \circ \psi^{-1})$, and $U \cap S \ni x \to \|df(x)\|_S$, we get that there exists $r > 0$ such that

$$\|w\| > \|df(x)\|_S, \text{ for all } x \in U \cap S,\ w \in T_{x_0}(M),\ \|w - v_0\| < r$$

and

$$(f\psi^{-1})(y) \cdot w > \frac{1}{4}\|w\|^2, \text{ for all } y \in \varphi(U) \text{ and all } w \in T_{x_0}(M)$$

with $\|w - v_0\| < r$

For a complete metric space (X,ρ) denote by K the set of all subsets of X, closed and bounded On the set K we introduce the Hausdorff metric defined by

$$\text{dist } (A,B) = \max \left\{ \sup_{a \in A} \rho(a,B),\ \sup_{b \in B} \rho(b,A) \right\}$$

Since X is a complete metric space, it follows that (K, dist) is also a complete metric space

We need two results which are proved in Mawhin-Willem [2].

LEMMA 2 1 *Let (X,ρ) be a complete metric space and suppose that X is an ANR. In this case if $\Gamma_j = \{ A \subseteq X \mid \text{cat}_X(A) \ge j,\ A$ compact $\}$, $(\Gamma_j, dist)$ is a complete metric space.*

LEMMA 2 2. *The function $\pi \quad \Gamma_j \to R$, defined by $\pi(A) = \sup_{x \in A} f(x)$ is lower semicontinuous*

THEOREM 2 3 *Let M be a C^1 - Finsler manifold, $S \subseteq M$, nonempty, closed and locally convex Let $f \in C^1(M,R)$ and $x_0 \in S$ a regular point of f relative to S* (i e $\|df(x_0)\|_S \neq 0$) *Suppose that (U,φ) is a chart of M at x_0 such that $\varphi(U \cap S)$ is convex in E. If $r > 0$ is a fixed real number, then there exist $v_1 \in T_{x_0}(S)$ a pseudogradient vector for f, relative to S, at x_0, an open neighborhood V of x_0, $V \subseteq U$, a positive real number $h_0 > 0$ and a* *continuous function $u^*: V \to E$ such that*

a) $\|u^*(x)\| \le r$, *for each* $x \in V$

b) $\varphi(x) - h_0(v_1 + u^*(x)) \in \varphi(u \cap S)$, *for each* $x \in V$

Proof Let $v_0 = [U, \varphi, \tilde{v}_0]_{x_0} \in T_{x_0}(M)$ be a pseudogradient vector for f relative to S at x_0, where $\tilde{v}_0 = \varphi'(x_0) v_0$

Using the Theorem 1 1 obtain that there exists a function u $]-\alpha,0] \to E$ such that

i) $\lim_{h \uparrow -0} u(h) = 0$, and

ii) $\varphi(x_0) + h(v_0 + u(h)) \in \varphi(u \cap S)$, for all $h \in]-\alpha,0]$

If $h_0 > 0$ is fixed, let $\tilde{v}_1 = v_0 + u(-h_0)$ and $V_1 = \varphi'(x_0)^{-1}\tilde{v}_1$ We have $\varphi(x_0) - h_0\,\tilde{v}_1 \in \varphi(U \cap S)$ and since $\varphi(U \cap S)$ is convex in E we get $\varphi(x_0) - h\,\tilde{v}_1 \in \varphi(U \cap S)$, for all h, $0 \le h \le h_0$ Hence $d(\varphi(x_0) - h\,\tilde{v}_1, \varphi(U \cap S))) = 0$, $0 \le h \le h_0$, $v_1 \in T_{x_0}(S)$

If h_0 is small enough we may assume that v_1 is a pseudogradient vector of f, relative to S at x_0

Let V be an open neighborhood of x_0 such that $\|\varphi(x) - \varphi(x_0)\| \le r$, for each $x \in V$

Let u^* $V \to E$ the function given by

$$u^*(x) = \frac{1}{h_0}(\varphi(x_0) - \varphi(x))$$

We have

$$\varphi(x) - h_0(\tilde{v}_1 + u^*(x)) = \varphi(x) - h_0(\tilde{v}_1 + \frac{1}{h_0}(\varphi(x) - \varphi(x_0)) =$$

$$\varphi(x_0) - h_0\tilde{v}_1 \in \varphi(U \cap S),\ \text{for each } x \in V$$

The following theorem is a generalization to C^1 - manifolds of Wang's [8] and Szulkin's [7] rezults

THEOREM 2 4 *Let M be a C^1 - Finsler and S a nonempty, closed and locally convex* *subset of M. Suppose that $f \in C^1(M,R)$ is bounded bellow and satisfies the conditions $(PS)_{S,c}$ for all c_j, $j = \overline{1,k}$ where*

$\Lambda_j = \{ A \subseteq S \mid A$ *compact and* $\mathrm{cat}_M(A) \geq j \}$

$c_j =$ *minimax* $(f, \Lambda_j) = \inf\limits_{A\in\Lambda_j} \sup\limits_{x\in A} f(x)$, *and*

$\Lambda_k \neq 0$ *for some* $k \geq 1$. *Then f has at least k distinct critical points relative to S.*

Proof. Without less of generality we can suppose that M is connected. Since $\Lambda_{j+1} \subseteq \Lambda_j$, for $j = 1, \ldots, k-1$ it follows that

$-\infty < c_1 \leq c_2 \leq \ldots \leq c_k < \infty$

Given j, suppose $c_j = c_{j+1} = \ldots = c_{j+p} = 0$, for some $p \geq 0$. It suffices to show that $\mathrm{cat}_M(KS_c) \geq p + 1$.

Let φ be the collection of all nonempty, closed and bounded subsets of S. On the account of the fact that (S,ρ) is a complete metric space, we obtain that (φ,dist) is a complete metric space too. By Lemma 2.2, we have that the function $\pi : \Lambda_j \to \mathbf{R}$, $\pi(A) = \sup\limits_{x\in A} f(x)$ is lower semicontinuous. Recall that we want to show that $\mathrm{cat}_M(KS_c) \geq p + 1$.

Suppose that $\mathrm{cat}_M(KS_c) \leq p$ and denote

$N_\delta = \{ x \in S \mid \rho(x,KS_c) \leq \delta\}$, for $\delta > 0$.

Let $k \in \left]1, \frac{3}{2}\right[$ be a fixed number. Since f satisfies $(PS)_{S,c}$ it is possible to choose $\delta > 0$ such that $\mathrm{cat}_M(N_{2\delta} KS_c) = \mathrm{cat}_M(KS_c) \leq p$.

Using $(PS)_{S,c}$ we may find an arbitrarily small $\varepsilon > 0$, with the property that

$$\|df(x)\|_S \geq 6\varepsilon \text{, for every } x \in S \cap f^{-1}[c-\varepsilon, c+\varepsilon] - N_\delta(KS_c) \tag{1}$$

Choose an $A_1 \in \Lambda_{j+p}$ such that $\pi(A_1) \leq c + \varepsilon^2$.

Let $A_2 = \overline{A_1 - N_{2\delta}(KS_c)}$. Then $\pi(A_2) \leq c + \varepsilon^2$ and $\mathrm{cat}_M(A_2) \geq \mathrm{cat}_M(A_1) - \mathrm{cat}_M(N_{2\delta}(KS_c)) \geq j + p - p = j$ implying $A_2 \in \Lambda_j$. Using the

Ekeland's variational principle for $\pi \ \Lambda_j \to \mathbb{R}$ we obtain There is an $A \in \Lambda_j$ such that

$$\pi(A_2) \le c + \varepsilon^2 \tag{2}$$

$$\text{dist } (A_1 A_2) \le \varepsilon \text{, and} \tag{3}$$

$$\pi(B) \ge \pi(A) - \varepsilon \text{ dist } (A, B), \quad \forall\, B \in \Lambda_j \tag{4}$$

Since $\varepsilon < \delta$ and dist $(A_1\ A_2) \le \varepsilon$ it follows

$$A \cap N_\delta(KS_c) = \varnothing \tag{5}$$

Our goal is to obtain a cntradiction by constructing a $B \in \Lambda_j$ which will fail to satisfy (4)

Denote $S_1 = A \cap \left\{x \in M \mid f(x) \ge c - \frac{1}{2}\varepsilon\right\}$ Because $A \in \Lambda_j$ we have $\sup\limits_{x\in A} f(x) \ge c > c - \frac{1}{2}\varepsilon$, so that there exists $x \in A$ such that $f(x) \ge x - \frac{1}{2}\varepsilon$, hence $S_1 \ne \varnothing$

Let $x_i \in S_1$ be an arbitrary point Choose a chart (U_i, φ_i) at x_i such that if $\psi_i = \varphi'(x_i)^{-1} \circ \varphi_i$, then

$$\frac{1}{k}\|\cdot\|_x \le \|\cdot\|_{x_i} \le k\|\cdot\|_x, \quad \forall\, x \in U \tag{6}$$

Because $x_i \notin N_\delta(KS_c)$, $\rho(x_i, KS_c) > \delta$. If U_i is sufficiently small then $U_i \subseteq f^{-1}([c-\varepsilon, c+\varepsilon]) - N_\delta(KS_c)$

Hence $\| df(x) \|_S \ge 6\varepsilon$, for each $x \in U_i \cap S$, and therefore $\|df(x_i)\|_S \ne 0$

It follows that there exists a pseudo-gradient vector $v_i = [U_i, \varphi_i, \bar{v}_i]_{x_i}$ for f relative to S at x_i, and there exists $r_i > 0$ such that

$$\|w\| > \|df(x)\|_S \tag{7}$$

$$(f\psi_i^{-1})'(y) \cdot w > \frac{1}{4}\|w\|^2 \tag{8}$$

for all $x \in U_i \cap S$, $y \in \psi_i(U_i)$ $w \in T_{x_i}(M)$, such that $\|w - v_i\|_{x_i} < r_i$

Let V_i be an open neighborhood of x_i, such that $\bar{V}_i \subseteq U_i$ (M is regular space) Let δ_i

> 0 be a real number such that

$$\rho(V_i, M - U_i) \geq \delta_i \text{ and} \tag{9}$$

$$d_{x_i}(\psi(V_i), T_{x_i}(M) - \psi_i(U_i)) \geq \delta_i \tag{10}$$

where d_{x_i} is the distance induced an $T_{x_i}(M)$ by $\|\cdot\|_{x_i}$

Also, we can suppose that $\psi_i(U_i \cap S)$ is a convex set

In this way we obtain an open covering $\{V_i\}$ of S_1 Since S_1 is compact, there exists a finite subcovering V_1, , V_m, to which we may subordinate a continuous partition of unity ξ_1, ,ξ_m, i e ξ_i $M \to [0,1]$ are continuous, supp $\xi_i \subseteq V_i$, $i = 1$, , m and $\sum_{i=1}^{m} \xi_i = 1$ on S_1

Let χ $M \to [0,1]$ be a continuous function such that $\chi|_{S_1} = 1$ and $\chi|_{M-\bigcup_{i=1}^{m} V_i} = 0$, and let $\theta_i = \chi\xi_i$ $M \to [0, 1]$

It follows that

$$\text{supp } \theta_i \text{ supp } \xi_i \subseteq V_i,\ i = 1, \ , m \tag{11}$$

$$\sum_{i=1}^{m} \theta_i = 1 \text{ on } S_1 \tag{12}$$

$$\theta_1 = \quad = \theta_m = 0 \text{ on } M - \bigcup_{i=1}^{m} V_i \tag{13}$$

Denote by $\delta_0 = \min \{\delta_1, \delta_2, \ , \delta_m\}$, $r_0 = \min \{r_1, \ , r_m\}$

We apply now Theorem 2 3 For each $i = 1$, , m there exist the continuous functions u_i $V_i \to T_{x_i}(M)$ such that

$$\|u_i(x)\| \leq r_0, \quad \forall x \in V_i \tag{14}$$

$$\psi_i(x) - h_0(v_i + u_i(x) \in \psi_i(U_i \cap S) \tag{15}$$

where h_0 is a fixed positive number

Using (7) and (8) we obtain

$$\|v_i + u_i(x)\| > \|df(x_i)\|_S, \ \forall x \in A \cap \text{supp } \theta_i \tag{16}$$

$$(f\psi_i^{-1})(y) \cdot (v_i + u_i(x)) > \frac{1}{4}\|v_i + u_i(x)\|^2, \tag{17}$$

$\forall\, x \in A \cap \operatorname{supp} \theta_i,\ \forall\, y \in \psi_i(V_i)$

Fix a real number $t > 0$ such that $t < \min \left\{ \frac{\delta_0}{1+k^2},\ h_0 \left(\min_{i=1,m} \|v_i\| - r_0 \right) \right\}$

Because r_0 is arbitrary small, we can suppose that $r_0 < \min_{i=1,m} \|v_i\|$

$$\text{Let } \alpha_1(t,x) = \begin{cases} \psi_1^{-1}\left(\psi_1(x) - t\,\theta_1(x) \dfrac{v_1 + u_1(x)}{\|v_1 + u_1(x)\|} \right) & \text{if } x \in U_1 \\ x, & \text{otherwise} \end{cases}$$

We will prove that α_1 is well defined and continuous

We have $\left\| t\,\theta_1(x) \dfrac{v_1 + u_1(x)}{\|v_1 + u_1(x)\|} \right\| \le t \le \dfrac{\delta_0}{1+k^2} < \delta_0$, and

$d_{x_1}(\psi_1(x), T_{x_1}(M) - \psi_1(U_1)) \ge \delta_0$, for each $x \in \operatorname{supp} \theta_1$

Therefore $\psi_1(x) - t\,\theta_1(x) \dfrac{v_1 + u_1(x)}{\|v_1 + u_1(x)\|} \in \psi_1(U_1)$, hence α_1 is well defined

Now we claim $\alpha_1(t, A) \subseteq S$ Indeed if $x \in A - \operatorname{supp} \theta_1$ $\alpha_1(t,x) = x \in A$

If $x \in A \cap \operatorname{supp} \theta_1$, then

$$\frac{t\theta_1(x)}{\|v_1 + u_1(x)\|} \le \frac{h_0(\|v_1\| - r_0)}{\|v_1 + u_1(x)\|} \le h_0,$$ and according to (15)

$$\psi_1(x) - t\,\theta_1(x) \frac{v_1 + u_1(x)}{\|v_1 + u_1(x)\|} \in \psi_1(U \cap S)$$

Hence $\alpha_1(t,x) \in U \cap S \subseteq S$

Because α_1 $[0,t] \times M \to M$ is a deformation, according to Theorem 1.4 (*f*) we obtain that $\operatorname{cat}_M(\alpha_1(t,A)) \ge \operatorname{cat}_M(A) \ge j$, and because $\alpha_1(t,A) \subseteq S$, it follows that $\alpha_1(t,A) \in \Lambda_j$

For an arbitrary point $x \in U_1$, let $\sigma_1(s) = \alpha_1(s,x)$, $0 \le s \le t$

Then σ_1 is a C^1 - path joining x to $\alpha_1(t,x)$

Hence $\rho(x, \alpha_1(t,x)) \le \int_0^t \|\dot\sigma_1(s)\|\, ds \le k\,\theta_1(x)\,t$ for every $x \in U_1$

If $x \in M - U_1$, the last inequality is also true Therefore

$$\rho(x, \alpha_1(t,x)) \le k\,\theta_1(x)\,t, \text{ for each } x \in M \tag{18}$$

Now we use the mean value theorem If $x \in U_1$, we have

$$f\alpha_1(t,x) - f(x) = f\psi_1^{-1}\left(\psi_1(x) - t\theta_1(x)\frac{v_1 + u_1(x)}{\|v_1 + u_1(x)\|}\right) - f\psi_1^{-1}(\psi_1(x)) =$$

$$= (f\psi_1^{-1})'\left(\psi_1(x) - \lambda t\theta_1(x)\frac{v_1 + u_1(x)}{\| v_1 + u_1(x)\|}\right)\cdot\left(-t\theta_1(x)\frac{v_1 + u_1(x)}{\| v_1 + u_1(x)\|}\right) \le$$

$$\le \frac{-t\,\theta_1(x)}{\| v_1 + u_1(x)\|}\cdot\frac{1}{4}\,\| v_1 + u_1(x)\|^2 = -\frac{1}{4}\,t\,\theta_1(x)\,\| v_1 + u_1(x)\|$$

But $\| v_1 + u_1(x)\| > \| df(x_1)\|_s$, implying

$$f\alpha_1(t,x) - f(x) \le -\frac{1}{4}\,t\theta_1(x)\cdot\| df(x_1)\|_S \le$$

$$\le -\frac{1}{4}\,t\theta_1(x)\,6\,\varepsilon = -\frac{3}{2}\,t\theta_1(x)\varepsilon\text{, for each } x \in U_1$$

If $x \in U_1$ then $\theta_1(x) = 0$, $\alpha_1(t,x) = x$ and the last inequality holds Therefore, we have

$$f\alpha_1(t,x) - f(x) \le -\frac{3}{2}\,\varepsilon\,t\,\theta_1(x)\text{, for every } x \in M \tag{19}$$

$$\text{Let } \alpha_2(t,x) = \begin{cases} \psi_2^{-1}\left(\psi_2(\alpha_1(t,x) - t\,\theta_3(x)\dfrac{v_2 + u_2(\alpha_1(t,x))}{\| v_2 + u_2(\alpha_1(t,x))\|}\right) \\ \text{if } \alpha_1(t,x) \in U_2 \\ \alpha_1(t,x), \quad \text{otherwise} \end{cases}$$

We shall prove that α_2 is well defined and continuous

Let $x \in \operatorname{supp}\theta_3$ and let be a path joining x to $\alpha_1(t,x)$

Then $\rho(x, M - U_2) \ge \delta_0$ and $\rho(x,\alpha_1(t,x) \le k\,\theta_1(x) \le \dfrac{k\,\delta_0}{1 + k^2} \le \dfrac{\delta_0}{2}$, implying that $\alpha_1(t,x) \in U_2$

If σ leaves U_2, then $l(\sigma) \ge \rho(x, M - U_2) \ge \delta_0$ Therefore $\rho(x,\alpha_1(t,x)) = \inf\{l(\sigma) \mid \sigma \text{ joins } x \text{ to } \alpha_1(t,x),\ \sigma \subseteq U_2\}$

Furthermere, if $\sigma \subseteq U_2$, then

$$l(\sigma) = \int_a^b \|\sigma(t)\|dt \ge \frac{1}{k}\,\|\psi_2(\alpha_1(t,x)) - \psi_2(x)\|_{x_2}$$

Combining these fact we obtain

$$\| \psi_2(\alpha_1(t,x)) - \psi_2(x) \|_{x_2} \le k\rho(x,\alpha_1(t,x)) \le k^2 t$$

Hence

$$\left\| \psi_2(\alpha_1(t,x)) - t\theta_2(x) \frac{v_2 + u_2(\alpha_1(t,x))}{\| v_2 + u_2(\alpha_1(t,x)) \|} - \psi_2(x) \right\|_{x_2} \le$$

$$\le \| \psi_2(\alpha_1(t,x)) - \psi_2(x) \|_{x_2} + \left\| t \cdot \theta_2(x) \frac{v_2 + u_2(\alpha_1(t,x))}{\| v_2 + u_2(\alpha_1(t,x)) \|} \right\|_{x_2} \le$$

$$\le (k^2 + 1) t$$

It follows from (10) that

$\psi_2(\alpha_1(t,x)) - t\theta_2(x) \dfrac{v_2 + u_2(\alpha_1(t,x))}{\| v_2 + u_2(\alpha_1(t,x)) \|} \in \psi_2(U_2)$ showing that α_2 is well defined

Now we prove that $\alpha_2(t,A) \subseteq S$ Indeed if $x \in A \setminus \operatorname{supp} \theta_2$ then $\alpha_2(t,x) = \alpha_1(t,x) \in S$

If $x \in A \cap \operatorname{supp} \theta_2$ then

$\psi_2(\alpha_1(t,x)) - h_0(v_2 + u_2(\alpha_1(t,x))) \in \psi_2(U_2 \cap S)$ and

$\dfrac{t\theta_2(x)}{\| v_2 + u_2(\alpha_1(t,x)) \|} \le \dfrac{h_0(\| v_2 \| - v_0)}{\| v_2 + u_2(\alpha_1(t,x)) \|} \le h_0$, therefore

$$\psi_2(\alpha_1(t,x)) - t\theta_2(x) \frac{v_2 + u_2(\alpha_1(t,x))}{\| v_2 + u_2(\alpha_1(t,x)) \|}) \in \psi_2(U_2 \cap S)$$

According to Theorem 1 4 (f) we obtain that

$\operatorname{cat}_M(\alpha_2(t,A))\ \operatorname{cat}_M(A) \ge j$, so $\alpha_2(t,A) \in A_j$

Proceeding like for α_1 we obtain the following inequalities

$$\rho(x,\alpha_2(t,x)) \le k(\theta_1(x) + \theta_2(x))t \tag{20}$$

$$f\alpha_2(t,x) - f\alpha_1(t,x) \le -\frac{3}{2}\varepsilon t\theta_2(x) \tag{21}$$

So by (19)

$$f\alpha_2(t,x) - f(x) \le -\frac{3}{2}\varepsilon t(\theta_1(x) + \theta_2(x)) \tag{22}$$

Proceeding as above, we define

$$\alpha_m(t,x) = \begin{cases} \psi_m^{-1}(\psi_m(\alpha_{m-1}(t,x) - t\,\theta_m(x) \dfrac{v_m + u_m(\alpha_{m-1}(t,x))}{v_m + u_m(\alpha_{m-1}(t,x))}), \ if\ \alpha_{m-1}(t,x) \in U_m \\ \alpha_{m-1}(t,x), \quad \text{otherwise} \end{cases}$$

and show that

$$\rho(x, \alpha_m(tx) \quad k(\theta_1(x) + \quad + \theta_m(x)) \leq kt \tag{23}$$

$$f\alpha_m(t,x) - f(x) \leq -\frac{3}{2}\varepsilon\, t(\theta_1(x) + \quad + \theta_m(x)) \tag{24}$$

and $\alpha_m(t,A) \in \Lambda_j$

Let $B = \alpha_m(t,A)$ By (23), dist $(A,B) \leq k\,t$

Since $\pi(B) \geq c$ and $f\alpha_m(t,x) \leq f(x)$,

$$\sup_{x\in A} f\alpha_m(t,x) = \sup_{x\in S_1} f\alpha_m(t,x)$$

Recall that $k < \frac{3}{2}$ and $\theta_1(x) + \quad + \theta_m(x) = 1$ on S

Using these facts we obtain

$$-\frac{3}{2}\varepsilon t < -\varepsilon kt < \varepsilon \text{ dist } (A,B) < \pi(B) - \pi(A) = \sup_{x\in S_1} f\alpha_m(t,x) -$$
$$- \sup_{x\in S_1} f(x) \leq \sup_{x\in S} (f\alpha_m(t,x) - f(x)) \leq -\frac{3}{2}\varepsilon\, t,$$

a contradiction

3. Applications to the Geodesic Probem Let M be a finite dimensional Riemannian manifold By $L_1^2(I,M)$ we mean the Hilbert Riemannian manifold of absolutely continuous map from $I = [0,1]$ to M, with square integrable derivative

R S Palais proved in [5] that if M is a C^{k+4} Riemannian manifold, then $L_1^2(I,M)$ is a C^k manifold, and the energy integral $E \quad L_1^2(T,M) \to R$ defined by

$$E(\sigma) = \frac{1}{2}\int_0^1 \| \sigma(t) \|^2 dt,$$

is also of class C^k

If $\sigma \in L_1^2(I,M)$, then the tangent space $T_\sigma L_1^2(I,M)$ consists of all absolutely continuous vector fields X along σ with square integrable covariant derivatives $\nabla_\sigma X$

The Riemannian structure of $L_1^2(I,M)$ is given by

$$<X,Y>_\sigma = \int_0^1 (<X(t), Y(t)>_{\sigma(t)} + <\nabla_\sigma X(t), \nabla_\sigma Y(t)>_{\sigma(t)})\,dt$$

where $X, Y \in T_\sigma L_1^2(I,M)$

If M is a complete Riemannian manifold, then $L_1^2(I,M)$ is also a complete Riemannian manifold

In what follows, suppose that M is a C^5, complete, Riemannian manifold Let $P\ L_1^2(I,M) \rightarrow M \times M$ be the function defined by

$$P(\sigma) = (\sigma(0), \sigma(1))$$

If $N \subseteq M \times M$, then we denote by $\Lambda_N(M) = P^{-1}(N)$, the space of paths in M which start and end in N

T Wang in [10] proved the following theorems·

THEOREM 3 1 *Let N be a localy convex, closed subset of $M \times M$ Then the following assertions are equivalent*

1 $\sigma \in \Lambda_N(M)$ *is a critical point for E relative to* $\Lambda_N(M)$

2 σ *is a geodesic with endpoints in N, orthogonal.*

THEOREM 3 2 *Let N be a closed subset of $M \times M$, such that $P_1(N) \subseteq M$ or $P_2(N) \subseteq M$ is compact Then E satisfies the PS condition relative to* $\Lambda_N(M)$

E Fadell and S Husseini proved the following results

LEMMA 3 3 *Suppose X is a space such that for some field F the cuplong of X over F using singular cohomology is $\geq k$. Then X has a compact subset of category $>k$.*

LEMMA 3 4 *Let M be a finite dimensional ANR, which is not contractible Let M_0 and M_1, subsets of M, which are contractible in M Then* cat $\Lambda_{N_1 \times N_1}(M) = +\infty$ *and contains compact subsets of arbitrary high category*

Usind the Theorem 2 4 and these lemmas, we can immediately get the following results

THEOREM 3 5 *Let M be a C^5, complete, finite dimensional, Riemannian manifold, N a closed, locally convex subset of $M \times M$, such that $P_1(N) \sqsubset M$ or $P_2(N) \sqsubset M$ is compact Then there are at least* cuplong $(\Lambda_N(M)) + 1$ *geodesics with endpoints in N, orthogonal to $P_1(N)$ and $P_2(N)$*

THEOREM 2 6 *Let M be a C^5, complete, finite dimensional, Riemannian manifold, which is not contractible, M_0,* M_1 *locally convex subsets of M which are contractible in M, and M_0 or M_1 is compact Then there are infinitely many geodesics which start in M_0 and end in M_1, orthogonal to M_0 and M_1*

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

HOMOGENEOUS CUBATURE FORMULAS

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Dedicated to Professor P Szilágyi on his 60th anniversary

Received December 18, 1993
AMS subject classification 65D32

REZUMAT. - Formule de cubatură omogene. În lucrare sunt construite câteva formule practice de cubatură, insistându-se asupra formulelor de cubatură omogene

Let D be a domain in $\mathbf{R}^2$, f, $f\ D \to \mathbf{R}$ an integrable function on D and $\lambda_1 f$, , $\lambda_N f$ some given information of f Next, one suppose that $\lambda_i f$ are punctual values of f or of certain of its derivatives

One considers the following problem using the informations $\lambda_1 f$, , $\lambda_N f$, determined a cubature formula

$$I^{xy} f = \iint_D f(x,y)\, dx dy = \sum_{k=1}^{N} A_k \lambda_k f + R_n(f),$$

i e find the coefficients A_k, $k = 1$, , N and the corresponding remainder term $R_n(f)$

The most results are been obtained when D is a regular domain in $\mathbb{R}^2$ (rectangular, simplex, etc) and the information (data) are regularly spaced

An efficient way to construct cubature formulas is based on the extension of the results which are known in the univariate case (for quadrature rules)

At this class of cubature procedure belong the tensorial product and the boolean-sum rules

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The purpose of this paper is to construct some practical cubature formulas with a special emphasis on homogeneous such formulas

1. Let $D \subset \mathbb{R}^2$ be a rectangle ($D = [a,b] \times [c,d]$), $\Delta x.\ a \le x_0 \le x_1 \le \ \ \le x_m \le b$ a partition of the interval $[a,b]$ and $\Delta y.\ c \le y_0 \le y_1 \le \ \ \le y_n \le d$ a partition of $[c,d]$

If $\lambda_i^x f$ and $\lambda_j^y f$ are given informations on f with regard to x respectively y, one considers the quadrature formulas

$$I^x f = \int_a^b f(x,y)\,dx = (Q_1^x f)(\cdot,y) + (R_1^x f)(\cdot,y)$$

and

$$I^y f = \int_c^d f(x,y)\,dy = (Q_1^y f)(x,\cdot) + (R_1^y f)(x,\cdot),$$

where the quadrature rules Q_1^x and Q_1^y are given by

$$(Q_1^x f)(\cdot,y) = \sum_{i=0}^{m} A_i(\lambda_i^x f)(\cdot,y)$$

respectively

$$(Q_1^y f)(x,\cdot) = \sum_{j=0}^{n} B_j(\lambda_j^y f)(x,\cdot),$$

with R_1^x and R_1^y the corresponding remainder operators, i e $R_1^x = I^x - Q_1^x$, $R_1^y = I^y - Q_1^y$.

It is easy to chech the following decomposition of the double integral operator I^{xy}

$$I^{xy} = Q_1^x Q_1^y + (R_1^x I^y + I^x R_1^y - R_1^x R_1^y) \tag{1}$$

and

$$I^{xy} = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) + R_1^x R_1^y \tag{2}$$

The identities (1) and (2) generate so called *product cubature formula*

$$I^{xy} f = Q_1^x Q_1^y f + (R_1^x I^y + I^x R_1^y - R_1^x R_1^y) f \tag{3}$$

respectively the *boolean-sum cubature formula*

$$I^{xy} f = (Q_1^x I^y + Q_1^y I^x - Q_1^x Q_1^y) f + R_1^x R_1^y f \tag{4}$$

Example 1 Let λ_i^x and λ_j^y be the Lagrange's functionals $(\lambda_i^x f = f(x_i, y), \lambda_j^y f = f(x, y_j))$ and Q_1^x respectively Q_1^y the corresponding trapezoidal rules

$$(Q_1^x f)(\cdot, y) = \frac{b-a}{2}[f(a,y)+f(b,y)]$$
$$(Q_1^y f)(x, \cdot) = \frac{d-c}{2}[f(x,c)+f(x,d)]$$

The formulas (3) and (4), in some supplementary differentiability conditions on f, become

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \frac{(b-a)(d-c)}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)] + R_p(f) \tag{5}$$

with

$$R_p(f) = -\frac{(b-a)^3}{12}\int_c^d f^{(2,0)}(\xi_1, y)\,dy - \frac{(d-c)^3}{12}\int_a^b f^{(0,2)}(x,\eta_1)\,dx - \\ -\frac{(b-a)^3(d-c)^3}{144} f^{(2,2)}(\xi_2, \eta_2)$$

respectively

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \frac{b-a}{2}\int_c^d [f(a,y)+f(b,y)]\,dy + \frac{d-c}{2}\int_a^b [f(x,c)+f(x,d)]\,dx - \\ -\frac{(b-a)(d-c)}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)] + R_s(f) \tag{6}$$

with

$$R_s(f) = -\frac{(b-a)^3(d-c)^3}{144} f^{(2,2)}(\xi,\eta),$$

where $\xi, \xi_1, \xi_2 \in [a,b]$ and $\eta, \eta_1, \eta_2 \in [c,d]$

This way, there are obtained the *trapezoidal product cubature formula* (5) and the *trapezoidal boolean-sum cubature* (6)

Let p_1 and q_1 be the approximation order of Q_1^x respectively Q_1^y, i e the sup norm of $R_1^x f$ is $O(|\Delta x|^{p_1})$ and the sup norm of $R_1^y f$ is $O(|\Delta y|^{q_1})$, where $|\Delta x|$ and $|\Delta y|$ are the norms of the partition Δx respectively Δy Next, one supposes that $|\Delta| = |\Delta x| = |\Delta y|$ (or $|\Delta| = \max\{|\Delta x|, |\Delta y|\}$)

From (3) and (4) it follows that the approximation order of the product formula is min

$\{p_1, q_1\}$ while the approximation order of the boolean-sum formula is $p_1 + q_1$

Hence, the boolean-sum cubature rules has the remarkable property regarding its highest approximation order

Otherwise, the boolean-sum formula contains the simple integrals $I^x f$ respectively $I^y f$ But, this simple integrals can be approximated, in a second level of approximation, using new quadrature procedures

From (4), one obtains

$$I^{xy} f = (Q_1^x Q_2^y + Q_2^x Q_1^y - Q_1^x Q_1^y) f + (Q_1^x R_2^y + Q_1^y R_2^x + R_1^x R_1^y) f \tag{7}$$

where Q_2^x and Q_2^y are the quadrature rules used in the second level of approximation

The quadrature rules Q_2^x and Q_2^y can be chosen in many ways First of all, it depends on the given information on the function f

A natural way to choose them is such that the approximation order of the initial boolean-sum formula to be preserved It is obvious that its approximation order cannot be increased

DEFINITION 1 A cubature formula of the form (7) derived from the boolean-sum formula (4) which preserves its approximation order is called *a consistent cubature formula.*

Remark 1 The cubature formula (7) is consistent if the orders p_2 and q_2 of the quadrature procedures Q_2^x respectively Q_2^y, used in the second level of approximation, satisfy the inequalities $p_2 \geq p_1 + q_1 - 1$, $q_2 \geq p_1 + q_1 - 1$

As the approximation order of the boolean-sum cubature cannot be increased, it is preferable to choose the quadrature procedures Q_2^x and Q_1^y such that each of the remainder term of (7) to have the same order of approximation

DEFINITION 2 A cubature formula, of the form (7), of which each term of the remainder has the same order of approximation is called *a homogeneous cubature formula.*

Hence, (7) is a homogeneous cubature formula iff $p_2 = q_2 = p_1 + q_1 - 1$

A homogeneous cubature formula can be derived from the trapezoidal boolean-sum cubature (6) for the standard domain $D_h = [0,h] \times [0,h]$

THEOREM 1 *If* $f^{(4,0)}, f^{(0,4)}, f^{(2,2)} \in C(D_h)$ *then we have the homogeneous formula*

$$\begin{aligned}\iint_{D_h} f(x,y)\,dxdy &= \frac{h^2}{4}[f(0,0)+f(h,0)+f(0,h)+f(h,h)] + \\ &\frac{h^3}{24}[(f^{(1,0)}+f^{(0,1)})(0,0)+(f^{(1,0)}-f^{(0,1)})(0,h) + \\ &(f^{(0,1)}-f^{(1,0)})(h,0)-(f^{(1,0)}+f^{(0,1)})(h,h)]+R(f)\end{aligned} \tag{8}$$

with

$$R(f) = \frac{h^6}{144}\left[\frac{1}{5}f^{(4,0)}(\xi,\eta)+\frac{1}{5}f^{(0,4)}(\xi_2,\eta_2)-f^{(2,2)}(\xi_3,\eta_3)\right]$$

Proof For $D = D_h$, formula (6) becomes

$$\iint_{D_h} f(x,y)\,dxdy = \frac{h}{2}\int_0^h [f(0,y)+f(h,y)]dy + \frac{h}{2}\int_0^h [f(x,0)+f(x,h)]dx - $$
$$-\frac{h^2}{4}[f(0,0)+f(h,0)+f(0,h)+f(h,h)] + R_s(f)$$

where

$$R_s(f) = -\frac{h^6}{144}f^{(2,2)}(\xi,\eta)$$

In order to obtain a homogeneous formula, we can use in the second level, the following quadrature formula

$$\int_0^h g(t)\,dt = \frac{h}{2}[g(0)+g(h)] + \frac{h^2}{12}[g'(0)-g'(h)] + R(g)$$

where

$$R(g) = \frac{h^5}{720}g^{(4)}(\xi)$$

and (8) follows

Remark 2 From the trapezoidal boolean-sum formula (6), it is also obtained a homogeneous cubature if, in the second level of approximation, we use the Simpson's quadrature

$$\int_0^h g(t)\,dt = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right] - \frac{h^5}{2880} f^{(4)}(\xi)$$

One obtains the following *trapezoidal-Simpson formula*

$$\begin{aligned}\iint_{D_h} f(x,y)\,dx\,dy = \frac{h^2}{4}\Big\{ &-\frac{1}{3}[f(0,0) + f(0,h) + f(h,0) + f(h,h)] + \\ &+ \frac{4}{3}\left[f\left(0,\frac{h}{2}\right) + f\left(h,\frac{h}{2}\right) + f\left(\frac{h}{2},0\right) + f\left(\frac{h}{2},h\right)\right]\Big\} + R(f)\end{aligned} \tag{9}$$

where

$$R(f) = -\frac{h^6}{144}\left[\frac{1}{20} f^{(4,0)}(\xi_1,\eta_1) + \frac{1}{20} f^{(0,4)}(\xi_2,\eta_2) + f^{(2,2)}(\xi_3,\eta_3)\right]$$

Remark 3 Formula (8) contains the same nodes as the initial one (6), while in the trapezoidal-Simpson formula appear new nodes

THEOREM 2 *Let f be a integrable function on $D \subset \mathbb{R}^2$ Then from any boolean-sum cubature* (4) *can be derived, in a second level of approximation, a homogeneous cubature formula.*

Proof If in (4), p_1 is the order of approximation of Q_1^x and q_1 is the order of approximation of Q_1^y then both quadrature rules Q_2^x and Q_2^y used in the second level must have the same order of approximation $p_2 = q_2 = p_1 + q_1 - 1$

But, it is know that, for any $p \in \mathbb{N}$, there exists a quadrature rule of order p

Remark 4 If r is the degree of exactness of a quadrature procedure Q then the approximation order of Q is r+2 [4]

2. The product and boolean-sum cubature formulas can be obtained applying the integral operator I^y to both members of the product respectively boolean-sum interpolation formulas correponding to the function f and to the data $\lambda_i^x f,\ i = 0,1,\ ,m$ and $\lambda_j^y f,\ j = 0,1,\ ,n$

Such a practical homogeneous cubature is obtained, for example, using the homogeneous spline interpolation formula generated by linear and cubic spline operators, i e

$$f = (S_1^x S_3^y + S_3^x S_1^y - S_1^x S_1^y) f + (S_1^x R_3^y + S_1^y R_3^x + R_1^x R_1^y) f \tag{10}$$

where S_1 and S_3 are linear respectively cubic spline interpolation operators with R_1 and R_3 the corresponding remainder operators

Taking into account that

$$(S_1 g)(t) = \sum_{i=1}^{n} s_i^1(t) g(t_i) \tag{11}$$

with

$$(R_1 g)(t) = \int_a^b \varphi_1(t,z) f'(z)\, dz$$

and

$$(S_3 g)(t) = \sum_{i=1}^{n} s_i^3(t) g(t_i) \tag{12}$$

with

$$(R_3 g)(t) = \int_a^b \varphi_3(t,z) f''(z)\, dz$$

where s_i^1 and s_i^3 are the corresponding linear respectively cubic cardinal splines, and φ_1 respectively φ_3 are the corresponding Peano's kernels, we have

THEOREM 3 *Let be* $D = [a,b] \times [c,d]$, $\Delta x\ \ a \le x_1 < \ . < x_m \le b$ *and* $\Delta y\ \ c \le y_1 < \ \ < y_n \le d$ *If* $f^{(2,0)}, f^{(0,2)}, f^{(1,1)} \in C(D)$ *then*

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i^1 B_j^2 + A_i^2 B_j^1 - A_i^1 B_j^1) f(x_i, y_j) + R_{mn}(f) \tag{13}$$

where

$$A_i^1 = \int_a^b s_i^1(x)\,dx,\ B_j^1 = \int_c^d \tilde{s}_j^1(y)\,dy$$

$$A_i^2 = \int_a^b s_i^3(x)\,dx,\ B_j^2 = \int_c^d \tilde{s}_j^3(y)\,dy$$

and

$$|R_{mn}(f)| \le (C_{20}M_{20}f + C_{02}M_{02}f + C_{11}M_{11}f)\,|\Delta|^4$$

with $\Delta = |\Delta x| = |\Delta y|$ *and* $C_{20} = \int_a^b |K_3(s)|\,ds$, $C_{02} = \int_c^d |\tilde{K}_3(t)|\,dt$, $C_{11} = \int_a^b |K_1(s)|\,ds \int_c^d |\tilde{K}_1(t)|\,dt$,

where $K_3(s) = \dfrac{(b-s)^3}{6} - \sum_{i=1}^{n} A_i^2 \dfrac{(x_i-s)_+^2}{2}$ *and so on.*

Proof The cubature formua (13) follows from (10), taking into account (11) - (12)

Remark 5 For the standard domain $D_h = [0,h] \times [0,h]$ with $x_0 = y_0 = 0$, $x_1 = y_1 = h/2$ and $x_2 = y_2 = h$, one obtains

$$\int_0^h\int_0^h f(x,y)\,dx\,dy = \frac{h^2}{32}\Big\{f(0,0) + f(0,h) + f(h,0) + f(h,h) + {}$$
$$+ 4\left[f\left(0,\frac{h}{2}\right) + f\left(\frac{h}{2},0\right) + f\left(h,\frac{h}{2}\right) + f\left(\frac{h}{2},h\right)\right] + 12f\left(\frac{h}{2},\frac{h}{2}\right)\Big\} + R_{22}(f)$$

with

$$|R_{22}(f)| \le \frac{1}{32}\left(\frac{19}{24}M_{20}f + \frac{19}{24}M_{02}f + M_{11}f\right)|\Delta|^4$$

where $|\Delta| = h/2$

Remark 6 Taking into account that a quadrature formula obtained from the natural spline interpolation formula is *optimal in sense of Sard* [6], the cubature formula (13) has an optimal character

Such homogeneous cubature formula can be also obtained for a triangular domain Let us consider the standard triangle

$$T_h = \{(x,y) \in \mathbb{R}^2 \mid x \ge 0,\ y \ge 0,\ x + y \le h\}$$

For example, if $f \in B_{11}(0,0)$ then the product P of the Lagrange's operators L_1, L_2, L_3 defined by

$$(L_1 f)(x,y) = \frac{h-x-y}{h-y}f(0,y) + \frac{x}{h-y}f(h-y,y)$$
$$(L_2 f)(x,y) = \frac{h-x-y}{h-x}f(x,0) + \frac{y}{h-x}f(x,h-x)$$

$$(L_3 f)(x,y) = \frac{x}{x+y} f(x+y,0) + \frac{y}{x+y} f(0,x+y),$$

generates the interpolation formula

$$f(x,y) = \frac{h-x-y}{h} f(0,0) + \frac{x}{h} f(h,0) + \frac{y}{h} f(0,h) + (Rf)(x,y) \tag{14}$$

with

$$(Rf)(x,y) = \int_0^h \varphi_{20}(x,y,s) f^{(2,0)}(s,0)\,ds + \int_0^h \varphi_{02}(x,y,t) f^{(0,2)}(0,t)\,dt +$$
$$+ \iint_{T_h} \varphi_{11}(x,y,t,s) f^{(1,1)}(s,t)\,ds\,dt$$

where φ_{20}, φ_{02} and φ_{11} are the corresponding Peano's kernels

THEOREM 4 *If* $f \in B_{11}(0,0)$, $f^{(2,0)}(\cdot,0)$, $f^{(0,2)}(0,\cdot) \in C[0,h]$ *and* $f^{(1,1)} \in C(T_h)$ *then*

$$\iint_{T_h} f(x,y)\,dx\,dy = \frac{h^2}{6}[f(0,0) + f(h,0) + f(0,h)] + R(f)$$

with

$$R(f) = -\frac{h^4}{24}\left[f^{(2,0)}(\xi,0) + f^{(0,2)}(0,\eta) - f^{(1,1)}(\xi,\eta)\right],$$

where $\xi,\eta \in [0,h]$, $(\xi_1,\eta_1) \in T_h$

The proof follows from (14) integrating both hands of this formula

Using the homogeneous interpolation formula [6]

$$f = G_{00} f(0,0) + G_{10} f(h,0) + G_{01} f(0,h) + G_{00}^{10} f^{(1,0)}(0,0) +$$
$$+ G_{10}^{10} f^{(1,0)}(h,0) + G_{01}^{10} f^{(1,0)}(0,h) + G_{00}^{01} f^{(0,1)}(0,0) +$$
$$+ G_{10}^{01} f^{(0,1)}(1,0) + G_{01}^{01}(0,h) + Rf,$$

where

$$G_{00}(x,y) = \frac{(h-x-y)(h^2+hx+hy-2x^2-2y^2)}{h^3}, \quad G_{10}(x,y) = \frac{x^2(3h-2x)}{h^3}$$
$$G_{01}(x,y) = \frac{y[(h-x-y)(2y-h)+(h-x)(h+2x)]}{h^3}, \quad G_{00}^{10}(x,y) = \frac{x(h-x)(h-x-y)}{h^2}$$
$$G_{10}^{10}(x,y) = \frac{x^2(x-h)}{h^2}, \quad G_{01}^{10}(x,y) = \frac{xy(h-x)}{h^2}$$
$$G_{00}^{01}(x,y) = \frac{y(h-y)(h-x-y)}{h^2}, \quad G_{10}^{01}(x,y) = \frac{x^2 y}{h^2},$$

$$G_{01}^{01}(x,y) = \frac{y^2(h-x-y)(2y-h) + xy(h-x)}{h^2}$$

and

$$(Rf)(x,y) = \int_0^h \varphi_{30}(x,y,s) f^{(3,0)}(s,0)\,ds + \int_0^h \varphi_{21}(x,y,s) f^{(2,1)}(s,0)\,ds +$$

$$+ \int_0^h \varphi_{03}(x,y,s) f^{(0,3)}(0,t)\,dt + \iint_{T_h} \varphi_{12}(x,y,s,t) f^{(1,2)}(s,t)\,ds\,dt$$

with φ_{ij} the corresponding Peano's kernels, one obtains a new homogeneous cubature formula

THEOREM 5 *If* $f \in B_{12}(0,0)$, $f^{(3,0)}(\cdot,0), f^{(2,1)}(\cdot,0), f^{(0,3)}(0,\cdot) \in C[0,h]$ *and* $f^{(1,2)} \in C(T_h)$ *then we have*

$$\iint_{T_h} f(x,y)\,dx\,dy = \frac{h^2}{120}[22f(0,0) + 18f(h,0) + 20f(0,h) + 3hf^{(1,0)}(0,0) -$$

$$- 4hf^{(1,0)}(h,0) + 3hf^{(1,0)}(0,h) + 3hf^{(0,1)}(0,0) +$$

$$+ 2hf^{(0,1)}(h,0) - 5hf^{(0,1)}(0,h)] + R(f)$$

where

$$|R(f)| \le \left[C_1\|f^{(3,0)}(\cdot,0)\| + C_2\|f^{(2,1)}(\cdot,0)\| + C_3\|f^{(0,3)}(0,\cdot)\| + C_4\|f^{(1,2)}\|\right]h^5$$

with

$$C_1 = \frac{1}{50}\left(15 - 7\sqrt{4{,}2}\right)$$
$$C_2 = \frac{1}{100}\sqrt[3]{0{,}16}$$
$$C_3 = \frac{1}{720}\left(58 - 25\sqrt{5}\right)$$
$$C_4 = \frac{1}{120}$$

and $\|\cdot\|$ *is the uniform norm.*

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

AN ALMOST EXPLICIT SCHEME FOR A CERTAIN CLASS OF NONLINEAR EVOLUTION EQUATIONS

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Dedicated to Professor P Szilágyi on his 60[th] anniversary

Received December 10, 1993
AMS subject classification 65N35

REZUMAT. - O schemă aproape explicită pentru o clasă de ecuaţii de evoluţie neliniare. Lucrarea prezintă o schemă numerică bazată pe tehnica corecţiei în spaţiul dominant pentru problemele Cauchy provenind din discretizarea spaţială spectrală a problemelor la limită pentru ecuaţii de evoluţie neliniare Se obţine o îmbunătăţire a condiţiei de stabilitate impuse pasului de timp de metoda explicită de bază fără o creştere esenţială a volumului de calcul

Let us consider an equation of the type

$$u_t = Lu + f \quad \text{on} \quad \Omega \times [0,T],\ \Omega \subset R \tag{1}$$

with appropriate initial and boundary value conditions Here we denoted by L a (nonlinear) differential operator in space, like

$$Lu = -u \cdot u_x + \nu u_{xx} \qquad x \in (-1,1) \tag{2}$$

with the given function f and the unknown u assumed to be sufficiently smooth for what follows

When we discretize (1) in space by a spectral method, like Chebyshev collocation method, we define

a) N, the cut-off frequency

b) $\{x_i, i=\overline{0,N}\}$ the set of Gauss-Lobatto collocation points $x_i = \cos(i\pi/N)$, $i=\overline{0,N}$

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c) $\{u_i,\ i=\overline{0,N}\}$ the set of the N+1 sampled values of $u(x)$ $u_i = u(x_i)$, $i=\overline{0,N}$

d) D, the $(N+1)\times(N+1)$ matrix giving the set $\{u_i',\ i=\overline{0,N}\}$ of the N+1 sampled values of the $u_N'(x)$ (where u_N is the Lagrange polynomial approximation of $u(x)$) from the set $\{u_i,\ i=\overline{0,N}\}$

$$u_i' = \sum_{j=0}^{N} D_{ij}\,u_j \qquad i=\overline{0,N}$$

The discretized collocation formulation of (1), (2) reads

$$\frac{du(x_i,t)}{dt} = \sum_{j=0}^{N} \nu (D^2)_{ij}\, u(x_j,t) - u(x_i,t)\sum_{j=0}^{N} D_{ij}\, u(x_j,t) + f(x_i,t) \tag{3}$$

$$u(x_0,t) = u(x_N,t) = 0 \qquad i=\overline{1,N-1}$$

If we denote $v_i(t) = u(x_i,t)$, $g_i(t) = f(x_i,t)$, $i=\overline{1,N-1}$, the problem (3) will be written in compact matrix form as follows

$$v_t = \nu\Delta_2 v - v \odot \Delta_1 v + g \equiv f(t,v) \tag{4}$$

where the $(N-1)\times(N-1)$ matrices Δ_1 and Δ_2 are

$$(\Delta_k)_{ij} = (D^k)_{ij} \qquad i,j=\overline{1,N-1}, \quad k=1,2 \tag{5}$$

and we have (cf [1])

$$\begin{aligned} (\Delta_1)_{ij} &= \frac{(-1)^{i+j}}{x_i - x_j} \qquad i\neq j,\ i,j=\overline{1,N-1} \\ (\Delta_1)_{ii} &= \frac{-x_i}{2(1-x_i^2)} \qquad i=\overline{1,N-1} \end{aligned} \tag{6}$$

It is well known that the spectral radius of the differentiation matrix is $O(N^2)$ [2] and that only some of the numerical eigenmodes correspond to some approximations of the analytical ones, the other numerical eigenmodes being of pure numerical origin, generated by the discretization procedure itself and by round-off effects Hence, the resulting ordinary differential system (4) will be stiff for moderate and large vaues of N, even for linear partial differential equations

Taking into account the fact that the stiffness character as well as an explicit approach lead to severe restriction for the time step, the implicit methods seam to be the most fitted to such problems These implicit techniques allow us to determin the unknown v at the level "n+1" by using an algebraic system of the type

$$v = \gamma + \delta \beta F(t, v) \tag{7}$$

where γ is the known term built on the previous "n" level, δ is the time step while β is a real constant connected with the choosen method This system must be solved at each step, not by simple iteration (which fails to converge unless the time step is again severely restricted) but by expensive Newton iteration

In a preliminary paper [3], in the linear case, we proposed to improve the mentioned implicit procedure by considering an almost fully explicit scheme based on the correction technique in the dominant space [4] Precisely, a pure implicit scheme is considered only for the dominant directions, while the rest of the system is explicitely solved In the nonlinear case, this technique requires the explicit computation of the dominant eigensystem for each time level "n" This can be efficiently performed by an iterative method and homotopies methods [5] will be used to get suitable initial approximations for the dominant spectrum

Let us consider now the nonlinear problem

$$v' = f(t, v), \quad v(0) = v_0 \quad v \in R^{N-1}, \quad t > 0 \tag{8}$$

The correction in the dominant space technique consists of

a) using a conventional *explicit* rational method (the basic method) to advance the solution from $t_n = t_0 + n\delta$ to $t_{n+1} = t_n + \delta$

$$\tilde{v}^{n+1} = B\, v^{(n)} \tag{9}$$

where $v^{(n)}$ is an approximation of the exact solution $v(t_n)$ For example, let us consider the

Adams-Bashforth method

$$\tilde{v}^{(n+1)} = v^{(n)} + \frac{3}{2}\delta f\left(t_n, v^{(n)}\right) - \frac{\delta}{2} f\left(t_{n-1}, v^{(n-1)}\right) \tag{10}$$

b) computing the dominant eigensistem of the Jacobian

$$\tilde{J}_{n+1} = \frac{\partial f}{\partial v}\left(t_{n+1}, \tilde{v}^{(n+1)}\right)$$

Note that if $\lambda^1_{n+1}, \ldots, \lambda^{N-1}_{n+1}$ are the eigenvalues of $\tilde{J}_{n+1}$, we suppose that there exists a constant integer s, $1 \le s \le N-1$ such that $\lambda^1_{n+1}, \ldots, \lambda^s_{n+1}$ are distinct, real and negative and

$$\min_{1 \le i \le s} \left|\lambda^i_{n+1}\right| >> \max_{s+1 \le i \le N-1} \left|\lambda^i_{n+1}\right| \tag{11}$$

Let z^i_{n+1}, $i = \overline{1, N-1}$ be the right (column) eigenvectors of $\tilde{J}_{n+1}$ and y^i_{n+1}, $i = \overline{1, N-1}$ the left (row) eigenvectors of $\tilde{J}_{n+1}$, corresponding to the eigenvalues $\lambda^1_{n+1}, \ldots, \lambda^{N-1}_{n+1}$, normalized according to the following rules

i) $\langle y^i_{n+1}, z^i_{n+1} \rangle = 1$, $i = \overline{1, N-1}$

ii) $\left\| z^i_{n+1} \right\| = 1$, $i = \overline{1, N-1}$ (12)

iii) the first nonvanishing component of z^i_{n+1} is positive

iv) ${}^t z^i_{n+1}$ denotes the t-th component of z^i_{n+1} and ${}^{l(n+1)}z^i_{n+1}$ is a component of z^i_{n+1} with maximum modulus, then

$$\operatorname{sgn}\left\{ {}^{l(n+1)}z^i_{n+1} \right\} = \operatorname{sgn}\left\{ {}^{l(n)}z^i_n \right\} \quad i = \overline{1, N-1},\ n = 0, 1,$$

Of course, we supposed that z^i_n, $i = \overline{1, N-1}$ and y^i_n, $i = \overline{1, N-1}$ are bases on R^{N-1} for each n

The dominant eigensystem is $\left\{\lambda^i_n, z^i_n, y^i_n\right\}$ $i = \overline{1, s}$ and the dominant space is the space spanned by $\left\{z^i_n,\ i = \overline{1, s}\right\}$

c) applying a correction in the dominant space

$$v^{(n+1)} = \tilde{v}^{(n+1)} + \sum_{p=1}^{s} \xi^p_{n+1}\, \tilde{z}^p_{n+1} \tag{13}$$

where ξ^i_{n+1} $i = \overline{1, s}$ are the scalar correction factors and $\tilde{z}^i_{n+1}$ are approximations of

z^{i}_{n+1} $i = \overline{1,s}$, normalized too by the rules (12) Let us suppose that

$$\tilde{z}^{i}_{n+1} = \sum_{j=1}^{N-1} \sigma_{ij}\, z^{j}_{n+1}, \quad \tilde{y}^{i}_{n+1} = \sum_{j=1}^{N-1} \tau_{ij}\, y^{j}_{n+1} \tag{14}$$

where $\sigma_{ii} = 1 + o(\varepsilon)$, $\tau_{ii} = 1 + o(\varepsilon)$ for $i = \overline{1,s}$ and $\sigma_{ij} = o(\varepsilon)$ $\tau_{ij} = o(\varepsilon)$ for $i = \overline{1,s}$, $j = \overline{1,N-1}$, $j \neq i$ and $\varepsilon > 0$

The best correction factors ξ^{p}_{n+1}, $p = \overline{1,s}$ are

$$\xi^{p}_{n+1} = -< \tilde{y}^{p}_{n+1}, \tilde{v}^{n+1} > + \psi^{p}_{n+1}(t_{n+1})$$

where $\psi^{p}_{n+1}(t) = < \tilde{y}^{p}_{n+1}, v(t) >$, the component of the exact solution v on $\tilde{z}^{p}_{n+1}$ The functions $\psi^{p}_{n+1}(t)$ satisfies the Cauchy problems

$$\psi^{p}_{n+1}(t) = < \tilde{y}^{p}_{n+1}, f(t, v(t)) >$$

$$\psi^{p}_{n+1}(t_n) = < \tilde{y}^{p}_{n+1}, v(t_n) >$$

A neighboring Cauchy problem, whose solution $k^{p}_{n+1}(t)$ is an approximation to $\psi^{p}_{n+1}(t)$ is

$$k^{p\prime}_{n+1}(t) = < \tilde{y}^{p}_{n+1}, f\left(t, v^{(n)} + \left(k^{p}_{n+1}(t) - < \tilde{y}^{p}_{n+1}, v^{(n)} >\right)\tilde{z}^{p}_{n+1}\right) > \tag{15}$$

$$k^{p}_{n+1}(t_n) = < \tilde{y}^{p}_{n+1}, v^{(n)} >$$

If we assume that $v^{(n)} = v(t_n)$, then one can show that

$$k^{p}_{n+1}(t_{n+1}) - \psi^{p}_{n+1}(t_{n+1}) = o(\delta^3)$$

If we solve the problem (15) by the Crank-Nicolson method and if we shorten the notation by writing

$$F(t,z) = < \tilde{y}^{p}_{n+1}, f\left(t, v^{(n)} + \left(z - < \tilde{y}^{p}_{n+1}, v^{(n)} >\right) z^{p}_{n+1}\right) > \tag{16}$$

we have the following nonlinear equations for $k^{p}_{n+1} \equiv k^{p}_{n+1}(t_{n+1})$

$$\phi\left(k^{p}_{N+1}\right) \equiv k^{p}_{n+1} - < y^{p}_{n+1}, v^{(N)} > - \frac{\delta}{2}\left[F\left(t_{n+1}, k^{p}_{n+1}\right) + F\left(t_n, < \tilde{y}^{p}_{n+1}, v^{(n)} >\right)\right] = 0 \tag{17}$$

Finally, this system is solved by the following Newton iteration

$$k^{p^{(0)}}_{n+1} = k^{p}_{n}, K^{p^{(l+1)}}_{n+1} = k^{p^{(l)}}_{n+1} - \frac{\phi\left(k^{p^{(l)}}_{n+1}\right)}{1 - \frac{\delta}{2} < \tilde{y}^{p}_{n+1}, f'_{v}\left(t_{n+1}, k^{p^{(l)}}_{n+1}\right)\tilde{z}^{p}_{n+1} >} \tag{18}$$

for $j = 0,1$, where $k_{n+1}^{p(j)} = k_{n+1}^{p(j)}(t_{n+1})$

It is straightforward to show that the convergence of (18) is essentially dependent on the rate of change of the eigensystem and not on the stiffness Note that in order to evaluate all a correction factors, s Cauchy problems of the form (15) must be solved, but these s problems are *uncoupled* and the iterations (17) can be performed separately on each of the s unknowns k_{n+1}^{p} , $p = \overline{1,s}$ This is an important feature because of the possibility of efficient implementation in parallel computer architecture The implicitness of Crank-Nicolson method does not involve us in any matrix inversion, only some matrix by vector products are performed at each time step

Once convergence of (18) has been achieved, the correction factors are given by

$$\xi_{n+1}^{p} = - < \tilde{y}_{n+1}^{p} , \tilde{v}^{(n+1)} > + k_{n+1}^{p} \quad p = \overline{1,s} \tag{19}$$

In order to establish the stability of this scheme, let us consider the linear problem

$$v' = \tilde{J}_{n+1} v \tag{20}$$

In this case the iteration converges in one step and (18) becomes

$$k_{n+1}^{p} = k_{n}^{p} + \frac{\delta < \tilde{y}_{n+1}^{p} , \tilde{J}_{n+1} v^{(n)} >}{1 - \frac{\delta}{2} < \tilde{y}_{n+1}^{p} , \tilde{J}_{n+1} \tilde{z}_{n+1}^{p} >} \tag{21}$$

If $\delta\lambda_{n+1}^{s+1}, \ldots, \delta\lambda_{n+1}^{N-1} \in \mathbf{R}_{B}$ (the region of absolut stabolity of the basic method), we have

$$\begin{aligned} &1 - \frac{\delta}{2} < \tilde{y}_{n+1}^{p} , \tilde{J}_{n+1} \tilde{z}_{n+1}^{p} > = 1 - \frac{\delta}{2} \lambda_{n+1}^{p} - \delta \lambda_{n+1}^{p} o(\varepsilon) \\ &k_{n}^{p} = < y_{n}^{p} , v^{(n)} > + M_{n} o(\varepsilon) \quad \text{where} \quad M_{n} = \| v^{(n)} \| , \ p = \overline{1,s} \end{aligned} \tag{22}$$

and

$$k_{n+1}^{p} = < y_{n}^{p} , v^{(n)} > \frac{1 + \frac{\delta}{2} \lambda_{n+1}^{p}}{1 - \frac{\delta}{2} \lambda_{n+1}^{p} - \delta\lambda_{n+1}^{p} o(\varepsilon)} + \frac{M_{n}\left(1 + \delta\lambda_{n+1}^{1}\right) o(\varepsilon)}{1 - \frac{\delta}{2} \lambda_{n+1}^{p} - \delta\lambda_{n+1}^{p} o(\varepsilon)}$$

It follows that

$$< y_{n+1}^{p}, v^{(n+1)} > = M_{n+1}o(\varepsilon) + k_{n+1}^{p} = \left[M_{n+1} + 2M_n \lambda_{n+1}^{1}/\lambda_{n+1}^{p}\right]o(\varepsilon) +$$

$$+ < y_n^p, v^{(n)} > L, \quad \text{where} \quad |L| < 1 \quad \text{If} \left[M_{n+1} + 2M_n \lambda_{n+1}^{1} / \lambda_{n+1}^{p}\right] o(\varepsilon) \leq$$

$\leq M$ for $n = 0,1,$ and $p = 0,1,$ $,s$, this scheme is dominantly stable i e the restriction that stability imposes on the time step will be essentially those that would arise if the dominant eigenvalues were not present

As an example, let us consider the problem (1)-(2) (the Burgers equation)

$$\begin{aligned} &u_t = 0\,1\,u_{xx} - uu_x + f \quad x \in (-1,1) \\ &u(x,0) = 0 \\ &u(-1,t) = u(1,t) = 0 \end{aligned} \tag{23}$$

where $f(x,t) = (1 - x^2)\cos t + 0\,2\sin t - 2x\sin^2 t + 2x^3\sin^2 t$, with the exact solution $u(x,t) = (1-x^2)\sin t$

After discretization in space by Chebyhev collocation method we have the following Cauchy problem

$$\begin{aligned} &v_t = 0\,1\,\Delta_2 v - v \circ \Delta_1 v + g(t) \equiv F(t,v) \\ &v(0) = 0 \end{aligned} \tag{24}$$

The eigenvalues o $J = F_v$ for the exact solution v, for $N = 8$ and for some t are

t	0	0 3	0 6	0 9	1 2	1 5
λ_1	-21 437	-21 659	-22 040	-22 366	-22 583	-22 677
λ_2	-20 160	-19 879	-19 339	-18 811	-18 419	-18 239
λ_3	-5 497	-5 026	-4 148±	-3 852±	-3 780±	-3 767±
λ_4	-4 053	-4 190	±1 141i	±2 223i	±2 964i	±3 282i
λ_5	-2 219	-2 327	-2 685	-2 597	-2 220±	-2 244±
λ_6	-0 987	-1 108	-1 411	-1 792	±0 630i	±0 857i
λ_7	-0 247	-0 411	-0 829	-1 330	-1 597	-1 661

In this case we have $s = 2$ and λ_n^1, λ_n^2 are the dominant eigenvalues

The computation of the dominant eigensystem for each n was performed in about three steps (ε = 0 001) by a bi-iteration method /6/ started by the n-1 computed eigensystem An indication of the rate of change of the dominant eigenvectors is given by the following numerical values (Tab 1)

Tab 1

t	0 2	0 3	0 4	0 5	0 6	0.7
	0 5720	0 5302	0 4922	0.4580	0.4363	0 4139
	-0 1281	-0 1159	-0 1058	-0 0976	-0 0922	-0 0876
	0 0380	0 0363	0 0361	0 0371	0 0386	0 0408
z^1	-0 0346	-0 0389	-0 0442	-0 0500	-0 0556	-0 0614
	0 0654	0.0749	0 0846	0 0941	0.1026	0 1109
	-0 2020	-0 2171	-0 2310	-0.2436	-0 2536	-0 2630
	0 7802	0.8061	0 8263	0.8418	0.8496	0.8568
	0 8428	0 9040	0 9389	0 9600	0.9702	0.9780
	-0 1627	-0 1615	-0 1542	-0 1440	-0 1319	-0.1196
	0 0116	0 0014	-0 0097	-0 0207	-0 0317	-0 0419
z^2	0 0188	0 0258	0 0323	0.0381	0 0439	0.0486
	-0 0415	-0.0417	-0 0417	-0 0413	-0 0422	-0 0421
	0 1276	0 1049	0 0868	0 0719	0.0635	0.0549
	-0 4947	-0 3785	-0 2902	-0.2212	-0 1804	-0 1425

Tab 2

t	$v_{ex}-v_{comp}\times 10^{3}$	t	$v_{ex}-v_{comp}\times 10^{3}$	t	$v_{ex}-v_{comp}\times 10^{3}$
0 2	0 41	1 1	3 43	4.5	-7 09
0 3	0 81	1 3	3 74	5.0	-6 97
0 4	1 19	1.5	4 29	5.7	-2 66
0 5	1 53	1 7	4 46	6 3	2 67
0 6	1 84	1 9	4.18	7 0	4 94
0 7	2 19	2 5	1 63	7 5	6 23
0 8	2 56	3 0	-2 34	7.9	7 17
0 9	2 90	3 5	-3 92	8 0	7 03
1 0	3 20	4 0	-5 68	8 1	6 77

The correction in the dominant space technique has been used for s = 2, the time step

$\delta = 0.1$, $N = 8$, and the Adams-Bashforth method as basic method The errors for this example after ahout three Newton iterations are as follows (Tab 2)

The conventional Adams-Bashforth method was also used for $\delta = 0.1$ and the instability was observed at about $t = 1$

As a final remark, the correction in the dominant space technique is an almost explicit scheme that seams to be a real way to improve the temporal stability criteria associated with explicit schemes and it could be easily combined with the spectral methods and implemented on parallel computers

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

EFFECTS OF SURFACTANTS ON AN UNDERFORMABLE DROP INITIALLY AT REST

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Dedicated to Professor P Szilágyi on his 60th anniversary

Received November 19, 1993
AMS subject classification 76D07, 76D45, 76T05

REZUMAT. - Efectul surfactanţilor asupra unei picături nedeformabile iniţial în repaus. Curgerea pe suprafaţa unei picături şi mişcarea de translaţie a acesteia, datorate unor gradienţi de tensiune superficială ce apar pe suprafaţa ei, sunt investigate teoretic pentru o picătură nedeformabilă, iniţial în repaus
Repartiţia surfactantului pe suprafaţa picăturii este dată prin legi particulare
Din punct de vedere matematic se rezolvă sistemul Stokes-Oseen printr-o metodă de separare a variabilelor şi se face un studiu asimptotic al forţei (componentelor normală şi tangenţială) ce acţionează asupra picăturii

Abstract. The surface flow and the translational motion of a drop caused by interfacial tension gradients are theoretically investigated in the case of an undeformable drop, initially at rest (or at zero gravity) The interfacial tension gradients are induced by injecting the drop with surfactant The spreading of the surfactant on the interface is described by a particular law A covering degree of the drop by the surfactant it is found out beginning with which the drop undergoes an upward translational motion

Introduction A viscous liquid drop immersed in an immiscible liquid undergoes complicated motions, when interfacial tension gradients arise on its surface The theoretical

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model reported here considers that

- the drop is underformale and initially at rest,

- an interfacial tension gradient is established by injecting a droplet of surfactant in a well-determined point on its surface,

- a real surface flow - Marangoni fow - arises on the drop surface, with a distinct front, which advances continously,

- from all possible motions, induced by the surface tension gradients (translation, rotation, oscillations, waves on its surface, deformation, fission, etc) we shall take into account only the translational motion of the drop,

- the translational velocity varies with the covering degree of the drop by the surfactant;

- no surfactant transfer, inside or outside the drop is considered

1. Governing equations. It will be considered an undeformable drop Ω_1 (density ρ_1) immersed into an immiscible liquid Ω_2 (density ρ_2) If the two liquids, have the same density $\rho = \rho_1 = \rho_2$, the drop is called free and is motionless The two liquids inside and outside the drop (see Fig 1) are Newtonian, incompressible and viscous having the viscosities μ_1 and μ_2 On the physical and chemical aspects of the problem see our previous works [2,8]

On the assumption of undeformability we note the following In the experiments reported in our works [2, 8] the condition is fulfilled that surface tension at the interface between drop and ambient liquid is strong enough to keep it approximately spherical against any deforming effect of viscous forces This condition (see for example Batchelor [1]) reads $\frac{\sigma}{a} \gg \frac{\mu_1 U}{a}$, and expresses that stress due to surface tension should be large, compared with the normal

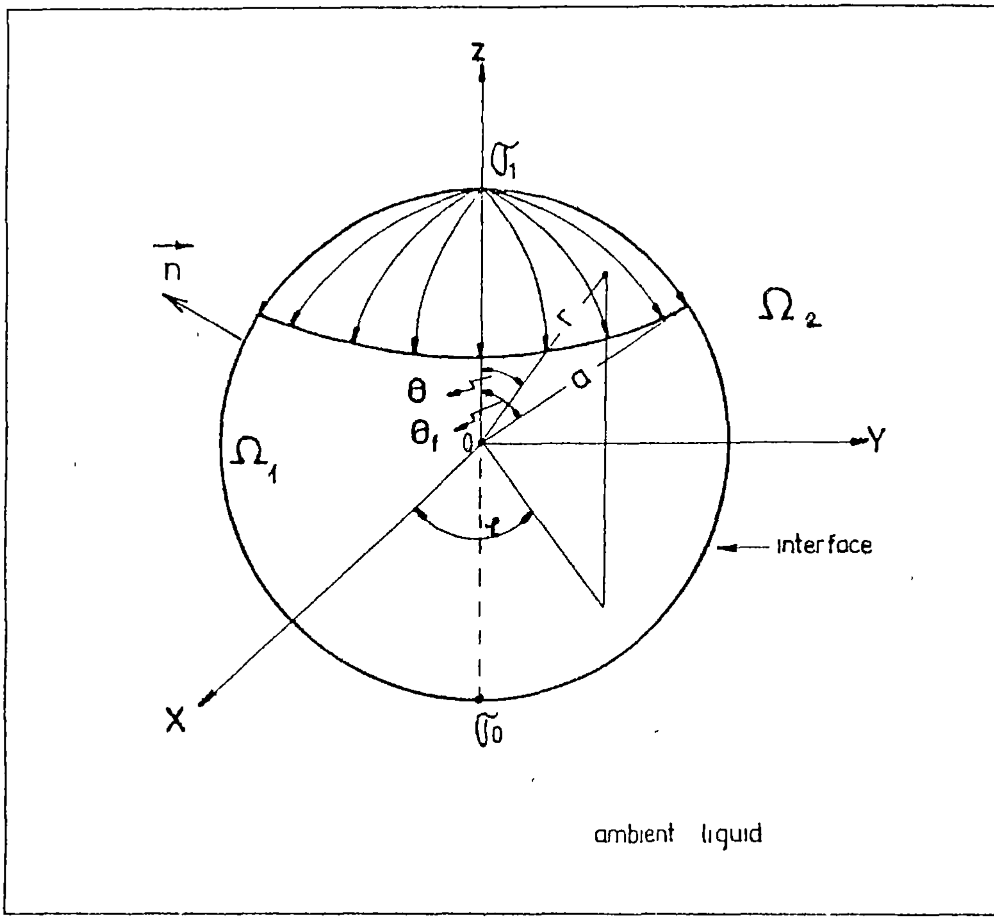

Fig 1

stress due to motion

We must notice that this condition don't contradict the assertion from [2], "the drop behaves like a rigid sphere for small interfacial tension gradient and large viscosity of the drop" First of all the smallness of the term $\frac{\mu_1 U}{a}$ is given essentially by μ_2 from $U(\mu_2 < \mu_1)$. Moreover, we know only by a qualitative point of view that for large interfacial tension gradients and reduced viscosity of the drop, it becomes strongly deformed and phenomena of oscillation or possibly fission may arise

Due to the viscous interface the more viscous fluid from the drop drives the less viscous ambient fluid In [4], the authors state this fact in a suggestive way "high viscosity liquids are the victims of the laziness of the high viscosity liquids because they are easy to pust around"

Because the drop is initially at rest, we don't possess a characteristic velocity U, so we can take that $U = \mu_2/a\rho$, which permits to consider the Reynolds number $Re_2 = 1$ in the system of equations describing the flow of the ambient fluid (exterior flow) We shall call this velocity "viscous" velocity Taking that into account as a characteristic one for the flow inside the drop, we'll obtain

$$Re_1 = \frac{\mu_2}{\mu_1}$$

With the two values of viscosity taken from [8], the Reynolds number corresponding to the drop phase ranges between 1/80 and 1/40

This observations suggested us to couple Oseen's and Stokes' equations, the first one for the ambient liquid and the second for the drop liquid Taking Cartesian axes fixed relative to the drop and (r, θ, φ) spherical polar coordinates, with origin at the centre of the drop, we denote by Ω_1 the interior of the sphere of radius a centered at origin, and by Ω_2 the complementary space of $\bar{\Omega}_1$ in R^3 (see Fig 1) The dimensions of Ω_2 are extremely large compared with the radius a of Ω_1

Using subscripts 1 and 2 related to quantities associated with the drop phase and ambient fluid (liquid) respectively, we denote by $q_i = (q_{r,i}, q_{\theta,i}) = (u_i, v_i)$, $i = 1, 2$ the components of velocity, by $p_{r\theta}$, p_{rr} the tangential and normal components at stress tensor respectively, and by σ the interfacial tension, $\sigma = \sigma(\theta)$

The equations governing the flow considered quasisteady (even steady in Ω_1) and axisymmetric are

$$\mu_1 \nabla^2 q_1 = \nabla p_1 \qquad \text{in } \Omega_1 \tag{1}$$

$$U\frac{\partial q_2}{\partial z} = -\frac{1}{p}\nabla p_2 + \frac{\mu_2}{\rho}\nabla^2 q_2 \qquad \text{in } \Omega_2 \tag{2}$$

$$\nabla q_i = 0 \qquad \text{in } \Omega_2 \cup \Omega_2 \ (i = 1,2) \tag{3}$$

The following boundary conditions are considered

$$|q_2| \to 0, \qquad r \to \infty \tag{4}$$

$$|q_1| \text{ is bounded}, \qquad r = 0 \tag{5}$$

$$q_1 = q_2 \qquad r = a \tag{6}$$

$$(p_{r\theta})_1 = (p_{r\theta})_2 + \frac{1}{a}\frac{\partial\sigma}{\partial\theta} \qquad r = a \tag{7}$$

$$(p_{rr})_1 = (p_{rr})_2 + \frac{2\sigma}{a} \qquad r = a \tag{8}$$

Since the liquid is at rest at infinity we must take condition (4) and because inside the velocity must be bounded - condition (5) The condition (6) expresses the mutual impenetrability of the interface ($r = a$) as well as the continuity of tangential velocity to the surface of the drop This last condition follows from assumption that two immiscible liquids can not slip over each other because of viscosity

In addition to these kinematic conditions there are two boundary dynamic conditions (7) and (8) The first one represents the continuity of tangential stress on crossing the surface of drop at any point We added there the term $\frac{1}{a}\frac{\partial\sigma}{\partial\theta}$ to express the Marangoni spreading of the surfactant Indeed, if we consider that the surface tension of the drop is σ_0 and if in the intersection point of the positive Oz axis (Fig 1) with the drop, the interfacial tension is lowered to $\sigma_1(\sigma_1 < \sigma_0)$ by injecting a small quantity of a surfactant, an interfacial tension difference σ_0-σ_1 appears This interfacial tension difference produces the spreading of the surfactant on the surface We shall note by θ_f the angle characterising the position of the front of the invaded region In this region $0<\theta\leq\theta_f$ and the surface tension varies at $\sigma_1\leq\sigma(\theta)\leq\sigma_0$

The second dynamic condition (8) underlines that at the interface between immiscible viscous fluids in motion, the difference between the normal stress at any point of interface on the convex side and that on the concave side is the quantity which equals the stress due to the surface $2\sigma/a$, the normal being drawn from the concave to the convex side (the outward normal, Fig 1)

As for the pressure we have the following conditions $p_2 - \pi_2 \to 0$, $r \to \infty$ and $p_1 - \pi_1$ is finite everywhere within the drop π_1 and π_2 are respectively hydrostatic pressures within the drop and in ambient fluid When the drop is suspended at rest in an immiscible liquid ($p_1 = \pi_1$, $p_2 = \pi_2$) they satisfy the well known Laplace's equation

$$\pi_1 - \pi_2 = \frac{2\sigma_0}{a}$$

After the start of flow p_1 and p_2 represent from the physical point of view perturbations from π_1 respectively π_2 and they are harmonic functions in Ω_1 respectively Ω_2

Following [7], for example, we introduce stream functions Ψ_1 and Ψ_2 in order to satisfy the equations of continuity (3) by

$$q_{r\,i} = u_i = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi_i}{\partial\theta}, \quad i = 1,2$$

$$q_{\theta\,i} = v_i = \frac{1}{r\sin\theta}\frac{\partial\Psi_i}{\partial r}, \quad i = 1,2$$

The system (1)-(8) will now be written in dimensionless form We introduce as a length scale the radius a, as a velocity scale the characteristic velocity $U = \mu_2/a\rho$ and as interfacial tension scale the value σ_0 With these we have the following dimensionless quantities

$$\bar{r} = \frac{r}{a}, \quad \bar{u}_i = \frac{u_i}{U}, \quad \bar{v}_i = \frac{v_i}{U}, \quad i = 1,2, \quad \bar{p} = \frac{p}{\rho U^2}, \quad \bar{\sigma} = \frac{\sigma}{\sigma_0}$$

Also we have for dimensionless stream functions

$$\bar{\Psi}_i = \frac{\Psi_i}{Ua^2}, \quad i = 1,2$$

In dimensionless form and using $\bar{\Psi}_i$ variables, the equations (1)-(3) become (where the superscript '-' is dropped)

$$E^4\Psi_1 = 0 \qquad \text{in } \Omega_1 \tag{9}$$

$$\left(E^2 - Re_2\frac{\partial}{\partial z}\right)E^2\Psi_2 = 0 \quad \text{in } \Omega_2 \tag{10}$$

where

$$E^2(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(\cdot)}{\partial\theta}\right)$$

Now let us consider in turn boundary conditions (4)-(8) To ensure the asymptotic condition (4) we take

$$\Psi_2 = o(r^2) \quad r \to \infty \tag{11}$$

while (6) gives

$$\frac{\partial\Psi_1}{\partial\theta} = \frac{\partial\Psi_2}{\partial\theta}, \quad r = 1 \tag{12}$$

$$\frac{\partial\Psi_1}{\partial r} = \frac{\partial\Psi_2}{\partial r}, \quad r = 1 \tag{13}$$

The condition (11) shows the free streaming relative to the centre of mass of the drop

It should be noted that the assumption that drop remains spherical in shape as it translates means that

$$u_1 = u_2 = 0, \quad r = 1 \tag{14}$$

may be replaced by

$$\Psi_1 = \Psi_2 = 0, \quad r = 1 \tag{15}$$

The dynamic condition (7) may be rewritten succesively

$$\mu_1 r\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right) + \frac{\mu_1}{r}\frac{\partial u_1}{\partial\theta} = \mu_2 r\frac{\partial}{\partial r}\left(\frac{v_2}{r}\right) + \frac{\mu_2}{r}\frac{\partial u_2}{\partial\theta} + \frac{1}{a}\frac{\partial\sigma}{\partial\theta} \qquad r = a$$

or, by virtute of (14), in dimensionless form, reduces to

$$\frac{1}{Re_1} r \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_1}{\partial r}\right) = r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_2}{\partial r}\right) + Ca \cdot \frac{1-\sigma_1/\sigma_0}{1-\cos\theta_f}\sin^2\theta, \quad r = 1 \tag{16}$$

Here θ_f stands for the angle under which the front of surfactant convers the drop, $0 < \theta_f \leq \pi$, $0 \leq \theta \leq \theta_f$, the function $\sigma(\theta)$ is defined by

$$\sigma(\theta) = \frac{\sigma_0 - \sigma_1}{1-\cos\theta_f}(1-\cos\theta) + \sigma_1, \quad \sigma(0) = \sigma_1, \quad \sigma(\theta_f) = \sigma_0,$$

so $\partial\sigma/\partial\theta = \sin\theta\cdot(\sigma_0 - \sigma_1)/(1-\cos\theta_f)$, and the dimensionless number Ca is a measure for the relative importance of capillary forces to viscous forces $Ca = \sigma_0/U\mu_2$ To unify the notations we have to observe that for proposed "viscous" velocity $Ca = 1/Oh^2$, $Oh = \mu_2/\sqrt{\sigma_0 a\rho}$, being the Ohnesorge number [3] and more $Ca = 1/We$, where We is the Weber number [9]

To be scrupulous, we mention that, as is well known, the surface tension σ usually depends on the scalar fields in the system (e g the electrical field, the temperature field) as well as on the concentration of foreign materials on the surface [6] In the present paper we focus on the variation due to the foreign material given by $\sigma(\theta)$, in fact σ depending not only on θ but on θ_f, σ_0 and σ_1

The normal stress condition (8) gives

$$p_1 - 2\mu_1\frac{\partial u_1}{\partial r} = p_2 - 2\mu_2\frac{\partial u_2}{\partial r} + \frac{2\sigma}{a}, \qquad r = a$$

which by (14) in dimensionless form, reads

$$p_1 + \frac{2}{Re_1 r^2\sin\theta}\cdot\frac{\partial\Psi_1}{\partial r\,\partial\theta} = p_2 + \frac{2}{r^2\sin\theta}\cdot\frac{\partial^2\Psi_2}{\partial r\,\partial\theta} + 2\,Ca\cdot\sigma, \quad r = 1 \tag{17}$$

The conditions (5) and (11) show that suitable forms Ψ_1 and Ψ_2 are ([7], [10], [9])

$$\Psi_1 = \left(Ar^2 + Br^4\right)\sin^2\theta, \qquad r \leq 1 \tag{18}$$

$$\Psi_2 = C(1+\cos\theta)\left\{1 - \exp\left[-\frac{r}{2}(1-\cos\theta)\right]\right\} + \frac{D}{r}\sin^2\theta \quad r \geq 1 \tag{19}$$

Thus there are four constants A, B, C, D to be determined, but five conditions (equations) to be satisfied (15), (13), (16), (17) It must be remembered that, the additional boundary conditions (15), imposed to keep the drop underformable, have replaced the boundary condition (12), and they are not one of the conditions (6)-(8) imposed purely by the kinematics and dynamics of the problem

2. Results and discussions We must observe the fact that in imposing the condition of the tangential stress on the surface of drop, we cannot satisfy the equation (16) exactly (the first term on the right hand side), but it can be satiesfied to $O(1)$ in Re_1 It means that the coupling between exterior flow (the solution of Ossen's equation) and interior flow (the solution of Stokes'equation) is realized only approximativelly A similar observation is valid for the boundary condition $\Psi_2 = 0$ and for the right hand side of (13).

So, on solving the equations given by (15), (13), (16) we obtain

$$A = -\frac{Ca \cdot h(\sigma, \theta_f)}{13/2 + 6/Re_1}, \quad B = -A, \quad C = -2A, \quad D = A \tag{20}$$

where for the sake of brevity, we have noted $\dfrac{1 - \sigma_1/\sigma_0}{1 - \cos\theta_f} = h(\sigma, \theta_f)$

For some values of parameters Re_1, θ_f, Ca etc we give in Fig 2 the streamlines for the flow within the drop (Ψ_1 = const ≤ 0) and in the ambient liquid (Ψ_2 = const ≥ 0)

We observe that because of the aproximativelly imposed boundary conditions (see above), the exterior streamlines present a detachment ($\Psi_2 = 0$ for $r > 1$) from the surface of drop As concern the interior streamlines is observad that they "start" only for a $\theta > 0$, which depends on the constants taken into account This fact is explained by the finite dimension of the surfactant droplet, injected in the north pole of the drop

The expressions of tangential velocities on the surface of drop as limits of interior and exterior flows respectively are

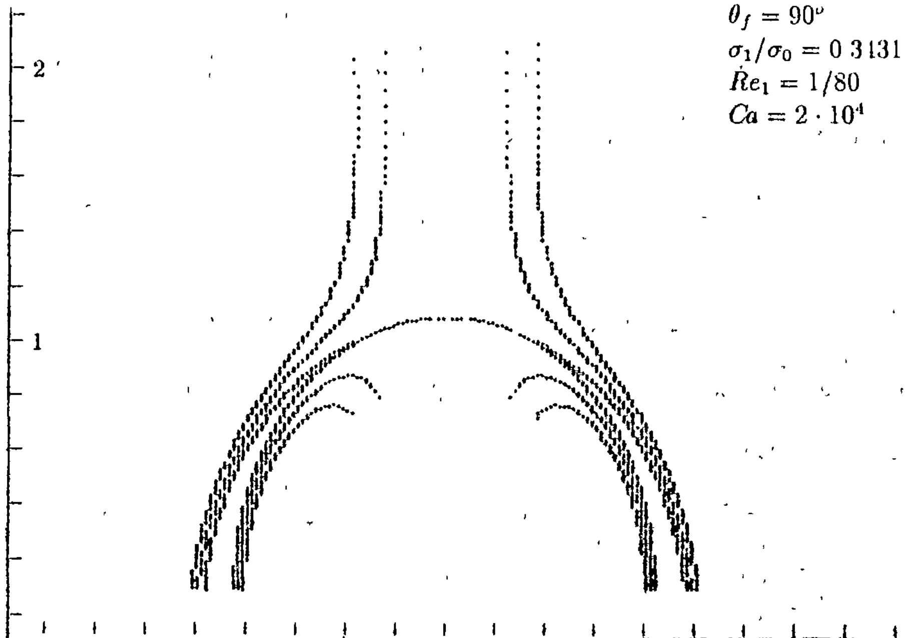

Fig 2

$$v_1 = -2A\sin\theta, \qquad 0 < \theta \le \theta_f \tag{21}$$

$$v_2 = -A\sin\theta\left\{\exp\left[\frac{1}{2}(\cos\theta - 1)\right] + 1\right\}, \quad 0 < \theta \le \theta_f \tag{22}$$

For some values of parameters in Fig 3 are plotted the velocities v_2 on the surface of the drop corresponding to θ_f on x axe The differences between the values of v_1 and v_2 for the same θ are due to the approximativelly imposed boundary conditions

For a given θ_f the velocity of front of surfactant become

$$v_f = -A\sin\theta_f\left\{\exp\left[\frac{1}{2}(\cos\theta_f - 1)\right] + 1\right\}$$

The pressure p_1 within the drop is

$$p_1 = 2\frac{A}{Re_1}r\cos\theta$$

so in the centre of mass of the drop acts only the hydrostatic pressure π_1

With the condition for normal stress (17), not used in the computation of spectrum of flow, and with p_1, we can determine the value of p_2 on the surface of the drop.

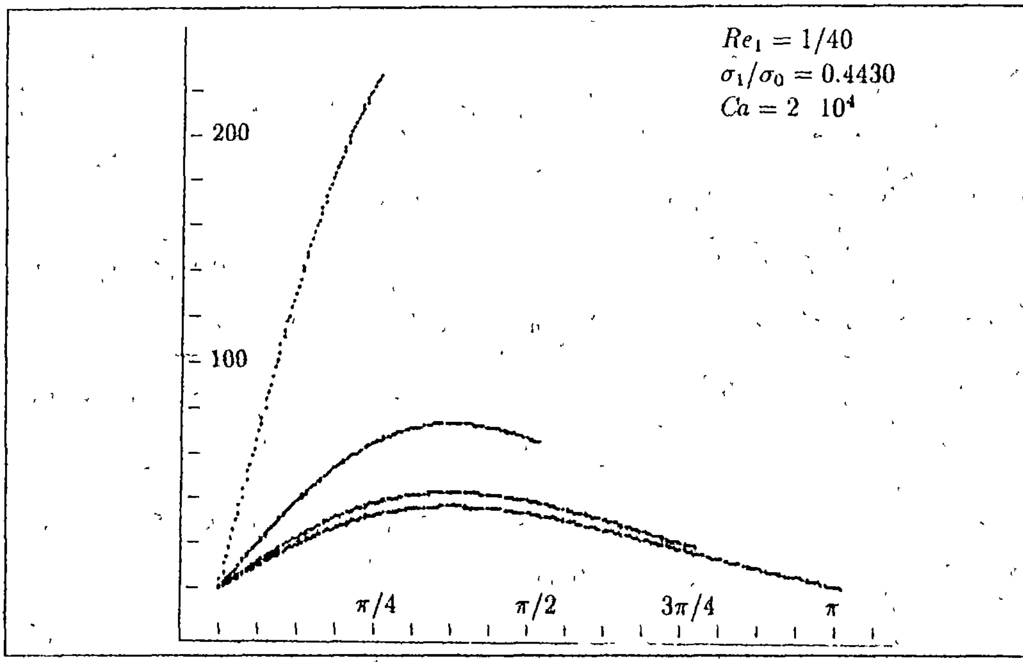

Fig 3

The force acting on the drop may be calculated from the general expression of force [6], which gives in this case

$$F = 2\pi a^2 \int_0^{\theta_f} \left[(p_{rr})_2 \cos\theta - (p_{r\theta})_2 \sin\theta \right] \sin\theta \cdot d\theta ,$$

where $(p_{rr})_2$ and $(p_{r\theta})_2$ are the normal and tangential components of the viscous stress tensor corresponding to the exterior flow We have respectively

$$(p_{rr})_2 = -p_2 + \frac{2}{Re_2} \cdot \frac{\partial u_2}{\partial r}$$

$$(p_{r\theta})_2 = \frac{1}{Re_1} \left(\frac{1}{r} \cdot \frac{\partial u_2}{\partial \theta} + \frac{\partial v_2}{\partial r} - \frac{v_2}{r} \right)$$

The normal component of force F per unit of area, has the following expression

$$F_n = 2\pi Ca\left[\frac{2}{3}(1-\lambda) - \frac{8Re_1}{13Re_1+12}\right]\left(1+\cos\theta_f+\cos^2\theta_f\right) + \\ +2\pi Ca\left(\lambda\cos\theta_f - 1\right)\left(1+\cos\theta_f\right), \quad \lambda = \frac{\sigma_1}{\sigma_0} \tag{23}$$

Using the asymptotic expansion of the function $1/(1 + \varepsilon)$ when $\varepsilon \to 0_+$, for the coefficient A with $\varepsilon = \mathrm{Re}_1$, we may have simply an asymptotic representation for F_n

Fischer, Hsiao and Wendland in [3] obtain an asymptotic representation for the force exerted on a rigid obstacle by the fluid This representation has the form $F = A_0 + A_1\mathrm{Re} + O(\mathrm{Re}^2 \ln \mathrm{Re}^{-1})$ as the Reynolds number $\mathrm{Re}\to 0_+$ and is essentially different to ours by the factor $\ln \mathrm{Re}^{-1}$

From (23) it is observed that the normal (and tangential) component of force acting on the drop, depends direct proportionally on Ca

As a final observation, we have to underline that the representation (23) hides the dependence of F_n on $\mathrm{Re}_2 = 1$

The assumption that the drop is undeformable seem to be too restrictive.

Re / λ	1/20	1/40	1/60	1/80
1/20	165°	169°	171°	173°
1/10	167°	171°	173°	173°
35/102	169°	173°	174°	175°
3/5	173°	175°	176°	177°

Table 1

In fact there are some other effects (see Fig 6 from [2]), so we consider that the force corresponding to $F_n < 0$ is consumed for other type of movements except translation The propulsive (lifting) force, $F_n > 0$, responsible for the upward movement of the drop appears only when the covering of the drop with surfactant is greater than $\pi/2$ However this aspect is only in a qualitative agreement with our previous experimental data [8] From Table 1 results that the smaller the ratio λ is the smaller θ_f for which $F_n > 0$ will be. So, it is clear that, for $\lambda \to 0$ the obtained values of θ_f beginning with a lifting force appears, tends to those obtained experimentally A more clear judgement will be provided considering the shape of drop deformable and, of course, the flow unsteady

Concluding Remarks Perhaps, it would be of some interest to take for characteristic velocity U the experimantal values from our works [2] and [8] That might be the aim of a future work

However, the aspect of our results, the spectra of flows inside and outside of the drop, the existence of the lifting force, as well as the asymptotic representation of force exerted on the drop by ambient liquid due to the variation $\sigma - \sigma_1$, are in good qualitative accordance with experimental results The question of quantitative accordance remain open from both side theoretical and experimental It is very likely that the results presented in this paper would be improved if the differential system (1) - (8) were solved by a numerical method, e g. a spectral method This could also make the topic for a future work

An asymptotic analysis in the spirit of [5] in the assumption of deformability of the drop is almost finished There, all the quantities found in this work, stream functions, pressures, etc will play the role of the first approximations

However, it seems that only by the use of some nonlinear terms (all possible) in vicinities of the surface of the drop inside and outside [5] one could solve some discrepances between theory and experiments

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STUDIA UNIV BABEŞ-BOLYAI, MATHEMATICA, XXXVIII, 2, 1993

ANIVERSĂRI

PROFESOR PAUL SZILÁGYI AT HIS 60th ANNIVERSARY

Professor Paul Szilágyi was born on the 18th june 1933 in Tăşnad, Romania He attented primary and secondary school in Tăşnad and Oradea, then universitary studies (1951-1954) at the Faculty of Mathematics and Physics, Bolyai University of Cluj In 1963 he obtained the title of doctor in mathematical sciences with the thesis "On the solvability of the Dirichlet problem for second order elliptic systems of partial differential equations" under the supervision of Prof. Dumitru V. Ionescu He began his professional activity in 1954 as assistant professor at the Bolyai University (1954-1959), after 1959 working at the Babeş-Bolyai University as assistant professor (1959-1962), lecturer (1962-1980), associate professor (1980-1990), full professor (since 1990) He delivered the basic courses Differential equations, Partial differential equations, The equations of the mathematical physics and many special courses Potential theory, Boundary value problems for elliptic systems, Pseudo-differential operators, Degree theory, Nonlinear elliptic equations and systems, Numerical methods in nonlinear analysis As fellow of the Humboldt Foundation he carried on research activity at the University of Kiel, Germany (1972-73 : 14 months, 1992 2 months) with Prof J Wloka and at the University of Paderborn, Germany (1991-92 4 months) with Prof K Deimling Since 1969 he is fellow-worker of the journal "Zentralblatt für Mathematik", where he published more than 100 reviews In prezent Prof P Szilágyi is vice-dean of the Faculty of Mathematics and Informatics Between 1961-1972 he was secretary of the editorial board of the journal Studia Univ Babeş-Bolyai, now he is member of the editorial board of this journal

In his research work he is intereted in the theory of partial differential equations He studies boundary value problems for linear and nonlinear elliptic equations and systems, variational inequalities with nonlinear elliptic operators and differential inclusions

In some papers he studied the Dirichlet and other boundary value problems for different classes of linear elliptic systems and gave necessary and sufficient conditions for these problems to be of Fredholm, Noether type or normal solvable regardless to geometric form of the domain. In other papers he investigated those second order elliptic linear systems for which the studied boundary value problems may have infinite many solutions or they are not of Noether type He found the canonical form of these systems and gave a geometrical characterization of the domains in which the Dirichlet problem has an infinity of solutions. These domains may have their measure or diameter whatever small.

Later he studied the solvability of usual boundary value problems for nonlinear equations and systems of the form

$$\sum_{|\alpha| \le k} (-1)^{|\alpha|} D^{\alpha} A_{\alpha} (x,u,Du, \ldots ,D^{k}u) + g(x,u) = f$$

resp

$$\sum_{|\alpha| \le k} (-1)^{k} D^{\alpha} A_{\alpha}^{i}(x,u,Du, \ldots ,D^{k}u) + g_{i}(x,u) = f_{i} \quad i = 1,\ldots,m; \; u = (u_1, \ldots ,u_m),$$

where $A_{\alpha}(x,\xi)$ resp $A_{\alpha}^{i}(x,\xi)$ have comparable growth with the polynomial one, and g resp g_i have arbitrary growth with respect to u He obtained sufficient solvability conditions for the studied boundary value problems for bounded and unbounded domains In the case of the above nonlinear operators he gave solvability conditions for variational inequalities too For some nonlinear elliptic systems he extended the notions of upper and lower solution In the study of the nonlinear elliptic equations and systems he used mostly the topological degree theory, the continuations method and the monotone operators

Recently he studies differential equations and boundary value problems with multivalued mappings

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