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THE IRRATIONALITY OF CERTAIN INFINITE PRODUCTS

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ABSTRACT. — The aim of this paper is to prove the irrationality of a class of infinite products. The main theorem generalizes an old theorem of Cantor [3] and a recent result of Sándor [7]. Also, a counter-example for an assertion of Froda [5] and an application of the main result are given.

1. Introduction. An old theorem of Cantor [3] asserts that if (n_k) is a sequence of positive integers with $n_{k+1} > n_k^2$ for all large k, then the value of the product

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{n_k}\right) \tag{1}$$

is irrational.

Recently Sándor [7] gave, among other things, some conditions for which the value of the infinite product

$$\prod_{k=1}^{\infty}\left(1+\frac{m_k}{n_k}\right) \tag{2}$$

is irrational, where (m_k) is a sequence of primes and (n_k) is a sequence of positive integers. Namely, he proved that if (m_k) is a sequence of primes with $\lim\limits_{k\to\infty} m_k = \infty$ and (n_k) is a sequence of positive integers which verify the inequalities

$$n_{h+k} \geqslant m_{h+k} \cdot n_h^{2^k}\ ;\ h \geqslant 1,\ k \geqslant 1 \tag{3}$$

then the value of the product (2) will be irrational.

The purpose of this paper is to give some conditions for which the value of the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right) \tag{4}$$

is irrational, where (a_n) and (b_n) are two sequences of positive integers. Our theorem extends Cantor's theorem which is obtained for $b_n = 1,\ n \geqslant 1$ and Sándor's result.

In the case when the infinite product (4) is divergent the problem of the rationality of his value is needless. Thus we shall assume in what follows that all infinite products which appear are convergent. Another way in which

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we can avoid this trivial case is to make the convention that ∞ is irrational and rational in the same time.

In the proof of the main result we shall use a criterion for irrationality due to Brun [2]. A generalization of Brun's criterion was given by Froda [5] but, as well shall see in the third section of the present paper, Froda's [5] generalization is not true.

As an application of the main result we shall prove that every convergent infinite product of rational numbers greater than [1] has an infinitely many disjoint subproducts (to be defined) with irrational values.

We note in ending that the same method of proof for the main theorem was used in [1] to obtain some criteria for the irrationality of certain series.

2. Main result. The main result of the present paper is the following

THEOREM. *Let (a_n) and (b_n) be two sequences of positive integers such that*

$$a_{n+1} > \frac{b_{n+1}}{b_n} a_n^2 + \frac{b_{n+1}(b_n - 1)}{b_n} a_n + 1 - b_{n+1} \tag{5}$$

holds for all large n. Then the value of the product (4) *is an irrational number.*

For $b_n = 1$ we obtain Cantor's theorem. Also, the above Theorem is more general than Sándor's result. Indeed, if the sequences (m_k) and (n_k) verifie Sándor's conditions then, using (3) for $k = 1$, we shall find that

$$n_{h+1} \geqslant m_{h+1} n_h^2.$$

Because $\lim_{k\to\infty} m_k = \infty$, for all large h we have

$$m_{h+1} n_h^2 > \frac{m_{h+1}}{m_h} n_h^2 + \frac{m_{h+1}(m_h - 1)}{m_h} + 1 - m_{h+1}$$

Thus Sándor's conditions imply the requirements of the main result, i.e. our Theorem is more general than Sándor's result.

Proof of Theorem. We have

$$\prod_{n=1}^{\infty} \left(1 + \frac{b_n}{a_n}\right) = \lim_{n\to\infty} \frac{(a_1 + b_1) \ldots (a_n + b_n)}{a_1 \ldots a_n}. \tag{6}$$

Brun's criterion asserts that a positive real number α which is the limit of an increasing sequence of rationals

$$\alpha = \lim_{r\to\infty} \frac{y_r}{x_r}, \tag{7}$$

where x_r and y_r are positive integers, is irrational when

$$\frac{y_{r+2} - y_{r+1}}{x_{r+2} - x_{r+1}} < \frac{y_{r+1} - y_r}{x_{r+1} - x_r}, \tag{8}$$

for all large r.

Taking into account this theorem we shall prove the inequality (8) for $y_r = (a_1 + b_1) \ldots (a_r + b_r)$ and $x_r = a_1 \ldots a_r$. Because the b_r's are positive integers, we get than (y_r/x_r) is an increasing sequence and thus, keeping in mind Brun's theorem, we shall find that the value of the infinite product (4) is an irrational number.

The inequality (8) in our situation $y_r = (a_1 + b_1) \ldots (a_r + b_r)$ and $x = a_1 \ldots a_r$ is equivalent with the following inequality

$$\frac{y_{r+1}(a_{r+2} + b_{r+2} - 1)}{x_{r+1}(a_{r+2} - 1)} < \frac{y_r(a_{r+1} + b_{r+1} - 1)}{x_r(a_{r+1} - 1)} \tag{9}$$

and thus with

$$\frac{(a_{r+1} + b_{r+1})(a_{r+2} + b_{r+2} - 1)}{a_{r+1}(a_{r+2} - 1)} < \frac{a_{r+1} + b_{r+1} - 1}{a_{r+1} - 1}. \tag{10}$$

From (10) we deduce, by routine calculations, the following equivalent relation

$$b_{r+2}a_{r+1}^2 + a_{r+1}b_{r+1}b_{r+2} + b_{r+1} < b_{r+1}a_{r+2} + b_{r+2}a_{r+1} + b_{r+1}b_{r+2} \tag{11}$$

Hence we have

$$a_{r+2} > \frac{b_{r+2}}{b_{r+1}} a_{r+1}^2 + \frac{b_{r+2}(b_{r+1} - 1)}{b_{r+1}} a_{r+1} + 1 - b_{r+2}. \tag{12}$$

We see that (12) is just (5) with $r + 1$ instead of n.

Therefore, from the assumptions of the Theorem, it follows that (8) holds for every sufficiently large r and thus (via Brun's criterion) the proof of the the main result is complete.

A simple consequence of the main theorem is the following

COROLLARY. *Let k be a positive integer and (a_n) a sequence of positive integers such that*

$$a_{n+1} > a_n^2 + (k - 1)a_n + 1 - k$$

for every sufficiently large n. Then the value of the product $\prod_{n=1}^{\infty}\left(1 + \frac{k}{a_r}\right)$ is an irrational number.

Proof. We take in the above Theorem $b_n = k$.

For $k = 1$ the condition of the above Corollary reduces to $a_{n+1} > a_n^2$, i.e. we obtain Cantor's [3] theorem.

3. A counter-example. A generalization of Brun's irrationality criterion was given by Froda [5]. Froda proved that Brun's criterion is also true when y_r and x_r are positive real numbers such that (8) holds. The same method of proof of our theorem remains valid to show, with the help of Froda's generalization, that our theorem is also true for positive numbers a_n and b_n. However, this is not valid because Froda's generalization is not correct. A counter-example is given in what follows. We note that the fact that Froda's proof for his generalization is not correct was previously known.

Let us define the sequence (c_n) by $c_1 = 2$ and by the recursive relation

$$c_{n+1} = c_n^2 - c_n + 1 \tag{13}$$

We note in passing that this sequence is in many situations a counter-example for some irrationality assertions (see for instance Erdös and Straus [4]).

Let us consider the following two sequences (a_n) and (α_n) given by

$$a_n = \frac{\log 1.5}{2^{1/c_n} - 1}$$

and

$$\alpha_n = \frac{(a_1 + \log 1.5)\dots(a_n + \log 1.5)}{a_1 \dots a_n}.$$

Let us assume that Froda's assertion is true. Then, because (α_n) is increasing, we deduce that

$$\lim_{n\to\infty} \alpha_n = \prod_{h=1}^{\infty}\left(1 + \frac{\log 1.5}{a_n}\right)$$

is irrational whether

$$v_{n+1} < v_n \qquad (14)$$

for every sufficiently large n, where

$$v_n = \frac{(a_1 + \log 1.5)\dots(a_{n+1} + \log 1.5) - (a_1 + \log 1.5)\dots(a_n + \log 1.5)}{a_1 \dots a_{n+1} - a_1 \dots a_n}.$$

Following the same steps as in the proof of our main result and of the Corrollary, we get that (14) is equivalent with the inequality from the statement of the above Corollary with log 1.5 instead of k, i.e. with

$$a_{n+1} > a_n^2 + (\log 1.5 - 1)a_n + 1 - \log 1.5 \qquad (15)$$

We denote $b_n = 1/(2^{1/c_n} - 1)$, then $b_n = a_n/\log 1.5$. With these notations we rewrite (15) as

$$b_{n+1} > b_n^2 \log 1.5 + (\log 1.5 - 1)b_n - 1 + (\log 1.5)^{-1} \qquad (16)$$

Because

$$\prod_{n=1}^{\infty}\left(1 + \frac{\log 1.5}{a_n}\right) = \prod_{n=1}^{\infty}\left(1 + \frac{1}{b_n}\right) = \prod_{n=1}^{\infty} 2^{1/c_n} = 2^{\sum_{n=1}^{\infty} 1/c_n}$$

is convergent (see Gleason, Greenwood and Kelly [6, pp. 429–430]) we obtain that b_n tends with n to infinity. Hence

$$(\log 1.5 - 1)b_n - 1 + (\log 1.5)^{-1} < 0 \qquad (17)$$

for every sufficiently large n.

On the other hand we have

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n^2} = \lim_{n\to\infty} \left(\frac{2^{1/c_n} - 1}{1/c_n}\right)^2 \frac{c_{n+1}}{c_n^2} \frac{1/c_{n+1}}{2^{1/c_{n+1}} - 1} =$$

$$= (\log 2) \lim_{n\to\infty} \frac{c_{n+1}}{c_n^2} = \log 2 > \log 1.5$$

because $\lim_{x\to 0} \frac{2^x - 1}{x} = \log 2$ and c_n tends with n to infinity by (13). It results that

$$b_{n+1} > b_n^2 \log 1.5 \tag{18}$$

for every sufficiently large n.

From (17) and (18) it follows That (16) holds for all large n. Therefore, if Froda's assertion is true, the number

$$\lim_{n\to\infty} \alpha_n = \prod_{n=1}^{\infty} \left(1 + \frac{\log 1.5}{a_r}\right) = \prod_{n=1}^{\infty} 2^{1/c_n}$$

is an irrational number. Passing to series we find that

$$\prod_{n=1}^{\infty} 2^{1/c_n} = 2^{\sum_{n=1}^{\infty} 1/c_n}$$

is irrational. But $\sum_{n=1}^{\infty} 1/c_n = 1$ (see [6, pp. 429–430)], so we get that 2 is an irrational number, which is of course a contradiction.

Hence Froda's assertion is not true in general.

4. An application. We say that $\prod_{k=1}^{\infty} v_k$ is a *subproduct* of a given product $\prod_{k=1}^{\infty} u_k$ if (v_k) is a subsequence of the sequence (u_k). Two subproducts $\prod_{k=1}^{\infty} v_k$ and $\prod_{k=1}^{\infty} w_k$ of the same product $\prod_{k=1}^{\infty} u_k$ are *disjoint* if (w_k) is a subsequence of the sequence (u_k) from which has been taken the subsequence (v_k).

As an application of the main result we shall prove the following unexpectedly

PROPOSITION. *Every convergent infinite product of rational numbers greater than* 1 *has an infinitely many disjoint subproducts with irrational values.*

This proposition is similar with a result from [1] where we proved that every convergente infinite serie of positive rationals has an infinitely many disjoint subseries with irrational sums (the notions of subseries and disjoint subseries are defined similary).

Proof of Proposition. Let $P = \prod_{n=1}^{\infty} c_n$ be a convergent infinite product with $c_n = 1 + b_n/a_n$, where b_n and a_n are positive integers, $n = 1, 2, \ldots$. Because the product P is convergent, the sequence (b_n/a_n) tends to zero when n tends to infinity. Because a_n and b_n are positive integers we get that the sequence (a_n) tends with n to infinity.

Hence there are an infinitely many disjoint subsequences $(a_{n(k)}^{(p)})_k$ and $(b_{n(k)}^{(p)})_k$, $p = 1, 2, \ldots$ of the sequences (a_n) and (b_n) respectively, such that

$$a_{n(k+1)} > \frac{b_{n(k+1)}}{b_{n(k)}} a_{n(k)}^2 - \frac{b_{n(k+1)}(b_{n(k)} - 1)}{b_{n(k)}} a_{n(k)} + 1 - b_{n(k+1)}$$

for all large k.

Now, using the main result, we find that the subproducts of P generated by the subsequences $(a_{n(k)}^{(p)})_k$ and $(b_{n(k)}^{(p)})_k$ have irrational values.

The proof is complete.

Finally, we propose the following

QUESTION. *Is there a convergent infinite product of rationals greater than 1 such that all its subproducts have irrational values?*

An affirmative answer to this question would provide a negative answer to the problem of replacing in the above Proposition the word *„irrational"* by *„rational"*. We note that the corresponding question for the series has an affirmative answer [1]. Thus the above problem has a negative answer for series and this negative result may be explained by the fact that the set of irrationals is uncountable while the set of rationals is „only" denumerable.

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GENERALIZATIONS OF AN ASYMPTOTIC FORMULA OF RAMANUJAN

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REZUMAT. — Generalizări ale formulei Ramanujan asimptotice. În lucrare se arată că dacă f este o funcţie numerică total multiplicativă mărginită, atunci pentru $k > 0$ are loc evaluarea asimptotică (3.4). Ca un caz particular se obţine relaţia (3.5) iar în cazul $k = 1$ se obţine rezultatul lui Ramanujan (3.6).

1. Introduction. Let $\sigma_k(n)$ denote, as usual, the sum of the k-th powers of all positive divisors of n and let $\sigma_1(n) \equiv \sigma(n)$ denote the sum of all positive divisors of n.

In 1916 Srinivasa Ramanujan ([5], eq. 19) stated without proof the following asymptotic formula

$$\sum_{n \leqslant x} \sigma^2(n) = \frac{5}{6} \zeta(3) x^3 + 0(x^2 \log^2 x), \tag{1.1}$$

where $\zeta(s)$ is the Riemann Zeta function. Several years later B. M. Wilson ([7], § 7.) mentioned that using analytical methods another formula of Ramanujan, namely

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\,\sigma_b(n)}{n^s} = \frac{\zeta(s)\ \zeta(s-a)\ \zeta(s-b)\ \zeta(s-a-b)}{\zeta(2s-a-b)} \tag{1.2}$$

([5], eq. 15) leads to an asymptotic formula for the more general sum $\sum_{n \leqslant x} \sigma_a(n) \cdot \sigma_b(n)$, which reduces to (1.1) in case $a = b = 1$.

The aim of this paper is to establish an asymptotic formula for the sum $\sum_{x \leqslant n} \sigma_k^2(n)$, $k > 0$ (it is the case $a = b = k > 0$) using a simple elementary method based on two convolutional identities (corollaries 2.1. and 2.3.). In fact we will deduce a slightly more general result (theorem 3.2.) and obtain as a consequence the asymptotic formula for $\sum_{n \leqslant x} \sigma_k^2(n)$ (corollary 3.3.).

2. Preliminaries. Throughout this paper x is assumed real and $\geqslant 2$, and $n \geqslant 1$ denotes an integer. Let $*$ denote the Dirichlet convolution of arithmetical functions defined by

$$(f * g)(n) = \sum_{d \mid n} f(d) \cdot g\left(\frac{n}{d}\right)$$

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We recall that an arithmetical function is multiplicative if $f(mn) = f(m)f(n)$ when m and n are coprime and completely multiplicative if $f(mn) = f(m)(n)$ for every integers $m, n \geqslant 1$.

It is well-known that the Dirichlet convolution of two multiplicative functions is also multiplicative and the multiplicative functions form a commutative group under the Dirichlet convolution.

Let μ denote the Möbius function and let $E_k(n) = n^k$, $E_0(n) \equiv U(n) = 1$ for every n. We have $\mu * U = I$, where $I(n) = 1$ or 0 according as $n = 1$ or $n > 1$ is the Dirac function (the unit element of the group) and $\sigma_k = U * E_k$. It is also well-known that the functions μ, $\mu^2 = \mu \cdot \mu$ the characteristic function of the square-free numbers, σ_k are multiplicative and E_k, U, I are completely multiplicative.

LEMMA 2.1. *If f and g are completely multiplicative functions, then*

$$(f * g) \circ E_2 = f^2 * g^2 * \mu^2 fg, \tag{2.1}$$

where o denotes the ordinary composition of functions.

Proof. Both sides of (2.1) are mutiplicative, being the Dirichlet convolution of multiplicative functions (the composition by E_2 preserves the multiplicativity). So it is enough to verify the above identity for $n = p^i$, a prime power. Noting $f(p) = a$ and $g(p) = b$ we have

$$(f^2 * g^2 * \mu^2 fg)(p^i) = (f^2 * g^2)(p^i) + (f^2 * g^2)(p^{i-1}) f(p) g(p) =$$
$$= (f^2(p^i) + f^2(p^{i-1}) g^2(p) + \dots + f^2(p) g^2(p^{i-1}) + g^2(p^i)) + (f^2(p^{i-1}) +$$
$$+ f^2(p^{i-2}) g^2(p) + \dots + f^2(p) g^2(p^{i-2}) + g^2(p^{i-1})) f(p) g(p) =$$
$$= (a^{2i} + a^{2i-2} b^2 + \dots + a^2 b^{2i-2} + b^{2i}) + (a^{2i-2} + a^{2i-4} b^2 + \dots + a^2 b^{2i-4} +$$
$$+ b^{2i-2}) ab = a^{2i} + a^{2i-1} b + a^{2i-2} b^2 + \dots + a^2 b^{2i-2} + ab^{2i-1} + b^{2i} =$$
$$= f(p^{2i}) + f(p^{2i-1}) g(p) + \dots + f(p) g(p^{2i-1}) + g(p^{2i}) = (f * g)(p^{2i})$$

and the proof is complete.

COROLLARY 2.1. ($g = E_k$) *If f is completely multipicative*

$$(f * E_k) \circ E_2 = f^2 * E_{2k} * \mu^2 f E_k \tag{2.2}$$

COROLLARY 2.2. ($g = E_k$, $f = U$)

$$\sigma_k \circ E_2 = \sigma_{2k} * \mu^2 E_k, \text{ that is}$$

$$\sigma_k(n^2) = \sum_{d\delta = n} \sigma_{k2}(d) \mu^2(\delta) \delta^k \text{ for all } n \geqslant 1.$$

LEMMA 2.2. *If f and g are completely multiplicative functions, then*

$$(f * g)^2 = f^2 * g^2 * fg * \mu^2 fg \tag{2.3}$$

Proof. The function f is completely multiplictive so its inverse under the Dirichlet convolution is $f^{-1} * = \mu f$ (see for example [1], theorem 2.) and

$f(k_1 * k_2) = fk_1 * fk_2$ for arbitrary functions k_1 and k_2 (see [1], theorem 5). So the above identity is equivalent to

$$(f * g)^2 * \mu f^2 = g^2 * fg * \mu^2 fg \text{ or}$$

$$(f * g)^2 * \mu f^2 = g(f * g * \mu^2 f).$$

Let denote $h = f * g$, where $g = h * f^{-1} * = h * \mu f$ and we have to prove

$$h^2 * \mu f^2 = g(h * \mu^2 f)$$

Because of multiplicativity it is enough to verify this identitiy for $n = p^i$,

$$(h^2 * \mu f^2)(p^i) = h^2(p^i) + h^2(p^{i-1})\mu(p)f^2(p) = h^2(p^i) - h^2(p^{i-1})f^2(p) =$$

$$= (h(p^i) - h(p^{i-1})f(p))(h(p^i) + h(p^{i-1})f(p)) = g(p^i)(h * \mu^2 f)(p^i) = (g(h * \mu^2 f))(p^i)$$

which proves the lemma.

Remark 2.1. Using that $U * \mu^2 = 2^\nu$, where $\nu(n)$ denotes the number of distinct prime factors of n, formula (2.3) leads to

$$(f * g)^2 = f^2 * g^2 * (U * \mu^2)fg \quad \text{or}$$

$$(f * g)^2 = f^2 * g^2 * 2^\nu fg,$$

which is analogous to the elementary formula $(a + b)^2 = a^2 + b^2 + 2ab$

Remark 2.2. For particular functions f and g (2.3) gives interesting formas. We mention a single example: if $f = g = U$, $f * g = U * U = \tau$ the divisor function and we have

$$\tau^2 = \tau * U * \mu^2 \text{ or } \tau^2 * \mu = \tau * \mu^2. \qquad [8]$$

For a further generalization of (2.3) see [4], theorem 2.

An immediately consequence of lemmas 2.1. and 2.2 is

LEMMA 2.3. *If f and g are completely multiplicative functions, then*

$$(f * g)^2 = (f * g) \circ E_2 * fg \tag{2.4}$$

COROLLARY 2.3. $(g = E_k)$ *If f is completely multiplicative*

$$(f * E_k)^2 = (f * E_k) \circ E_2 * fE_k \tag{2.5}$$

COROLLARY 2.4. $(g = E_k,\ f = U)$

$$\sigma_k^2 = \sigma_k \circ E_2 * E_k, \text{ that is}$$

$$\sigma_k^2(b) = \sum_{d\delta = n} \sigma_k(d^2)\delta^k.$$

Define $D(f, s)$ by the Dirichlet series

$$D(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},\ s > 1.$$

We have $D(U, s) = \zeta(s)$, $s > 1$. f is bounded and completely multipicative the series $D(f, s)$ is absolutely convergent for $s > 1$ and

$$D(f, s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}, \qquad (2.6)$$

where the product extends over the primes p (cf. [2], ch. 7. theorem 5).

LEMMA 2.4. *If f is bounded and completely multiplicative, then*

$$D(\mu^2 f, k) = \frac{D(f, k)}{D(f^2, 2k)}, \quad k > 1 \qquad (2.7)$$

Proof. The series $D(\mu^2 f, k)$ is absolutely convergent for $k > 1$ and the general term is a multiplicative function of n so it can be expanded into an infinite product of Euler type (cf. [2] ch. 7., theorem 5):

$$D(\mu^2 f, k) = \prod_p \left(\sum_{i=0}^{\infty} \frac{\mu^2(p^i) f(p^i)}{p^{ki}}\right) = \prod_p \left(1 + \frac{f(p)}{p^k}\right) =$$

$$= \prod_p \left(1 - \frac{f^2(p)}{p^{2k}}\right) \Big/ \prod_p \left(1 - \frac{f(p)}{p^k}\right) = \frac{D(f, k)}{D(f^2, 2k)} \text{ by (2.6)}$$

We will use the following familiar estimates:

LEMMA 2.5.

$$\sum_{n \leq x} n^k = \frac{x^{k+1}}{k+1} + 0(x^k), \quad k \geq 0 \qquad (2.8)$$

$$\sum_{n \leq x} \frac{1}{n} = 0 \ (\log x) \qquad (2.9)$$

$$\sum_{n \leq x} \frac{1}{n^k} = 0(x^{1-k}), \quad 0 < k < 1 \qquad (2.10)$$

$$\sum_{n > x} \frac{1}{n^k} = 0\left(\frac{1}{x^{k-1}}\right), \quad k > 1 \qquad (2.11)$$

LEMMA 2.6. (cf. [3], lemma 2.3) *If f is a bounded arithmetical function, then for $k > 0$*

$$\sum_{n \leq x} (f * E_k)(n) = \frac{D(f, k+1)}{k+1} x^{k+1} + 0(A_k(x)), \qquad (2.12)$$

where $A_k(x) = x^k$, $x \log x$ or x according as $k > 1$, $k = 1$ or $k < 1$.

Proof. By (2.8) we have

$$\sum_{n\leqslant x}(f * E_k)(n) = \sum_{n=d\delta\leqslant x} f(d)\delta^k = \sum_{d\leqslant x} f(d) \sum_{\delta\leqslant \frac{x}{d}} \delta^k =$$

$$=\sum_{d\leqslant x} f(d)\left\{\frac{1}{k+1}\left(\frac{x}{d}\right)^{k+1} + 0\left(\left(\frac{x}{d}\right)^k\right)\right\} = \frac{x^{k+1}}{k+1}\sum_{d\leqslant x}\frac{f(d)}{d^{k+1}} + 0\left(x^k \sum_{d\leqslant x}\frac{1}{d^k}\right) =$$

$$= \frac{x^{k+1}}{k+1}\sum_{d=1}^{\infty}\frac{f(d)}{d^k} - \frac{x^{k+1}}{k+1}\sum_{d>x}\frac{f(d)}{d^{k+1}} + 0\left(x^k\sum_{d\leqslant x}\frac{1}{d^k}\right) =$$

$$= \frac{x^{k+1}}{k+1} D(f, k+1) + 0\left(x^{k+1}\sum_{d<x}\frac{1}{d^{k+1}}\right) + 0\left(x^k\sum_{d\leqslant x}\frac{1}{d^k}\right).$$

The first 0-term is $0\left(x^{k+1}\frac{1}{x^k}\right) = 0(x)$ by (2.11) and the second 0-term is for $k > 1$ $0(x^k)$; for $k = 1$ $0(\log x)$ by (2.9) and for $k < 1$ it is $0(x^k \cdot x^{1-k}) = 0(x)$ using (2.10) and the proof is complete.

3. Main rezult.

THEOREM 3.1. *If f is completely multiplicative and bounded, then for $k > 0$*

$$\sum_{n\leqslant x}(f * E_k)(n^2) = \frac{D(f^2, 2k+1)\, D(f, k+1)}{(2k+1)\, D(f^2, 2k+2)}\, x^{2k+1} + 0(B_k(x)), \tag{3.1}$$

where $B_k(x) = x^{2k}$, $x^2 \log x$, $x^{\frac{3}{2}} \log x$, or x^{k+1} according as $k > 1$, $k = 1$, $k = \frac{1}{2}$ or $k < 1$ and $k \neq \frac{1}{2}\cdot$

Proof. By corollary 2.1. and lemma 2.6. we have

$$\sum_{n\leqslant x}(f * E_k)(n^2) = \sum_{n\leqslant x}(f^2 * E_{2k} * \mu^2 f E_k)(n) =$$

$$= \sum_{d\delta=n\leqslant x}\mu^2(d)f(d)d^k(f^2 * E_{2k})(\delta) = \sum_{d\leqslant x}\mu^2(d)f(d)d^k\sum_{\delta\leqslant \frac{x}{d}}(f^2 * E_{2k})(\delta) =$$

$$= \sum_{d\leqslant x}\mu^2(d)f(d)d^k\left\{\frac{D(f^2, 2k+1)}{2k+1}\left(\frac{x}{d}\right)^{2k+1} + 0\left(A_{2k}\left(\frac{x}{d}\right)\right)\right\} =$$

$$= \frac{D(f^2, 2k+1)}{2k+1}\, x^{2k+1}\sum_{d\leqslant x}\frac{\mu^2(d)f(d)}{d^{k+1}} + 0\left(\sum_{\leqslant x} d^k A_{2k}\left(\frac{x}{d}\right)\right) =$$

$$= \frac{D(f^2, 2k+1)}{2k+1}\, x^{2k+1}\sum_{d=1}^{\infty}\frac{\mu^2(d)\,f(d)}{d^{k+1}} + 0\left(x^{2k+1}\sum_{d>}\frac{1}{d^{k+1}}\right) +$$

$$+ 0\left(\sum_{d\leqslant x} d^k A_{2k}\left(\frac{x}{d}\right)\right), \text{ where} \sum_{d=1}^{\infty}\frac{\mu^2(d)\,f(d)}{d^{k+1}} = D(\mu^2 f, k+1) =$$

$$= \frac{D(f, k+1)}{D(f^2, 2k+2)} \text{ by lemma 2.4.}$$

Using (2.11) the first 0-term is $0\left(x^{2k+1}\dfrac{1}{x^k}\right) = 0(x^{k+1})$ and the second 0-term is for $k > 1$ $0\left(\sum_{d\leqslant x} d^k\left(\dfrac{x}{d}\right)^{2k}\right) = 0(x^{2k})$; for $k = 1$ $0\left(\sum_{d\leqslant x} d\left(\dfrac{x}{d}\right)^{2}\right) = 0\left(x^2\sum_{d\leqslant x}\dfrac{1}{d}\right) =$ $= 0\,(x^2 \log x)$ by (2.9); in case $k = \dfrac{1}{2}$ it is $0\left(\sum_{d\leqslant x} d^{\frac{1}{2}}\dfrac{x}{d}\log\dfrac{x}{d}\right) =$ $0\left(x\log x\sum_{d\leqslant x} d^{-\frac{1}{2}}\right) = 0\left(x\log x\cdot x^{\frac{1}{2}}\right) = 0\left(x^{\frac{3}{2}}\log x\right)$ using (2.10) and for $k<1$, $k\neq\dfrac{1}{2}$ the second remain term is $0\left(\sum_{d\leqslant x} d^k\dfrac{x}{d}\right) = 0\left(x\sum_{d\leqslant x}\dfrac{1}{d^{1-k}}\right) = 0(x^{k+1})$ or $0\left(\sum_{d\leqslant x} d^k\left(\dfrac{x}{d}\right)^{2k}\right) = 0\left(x^{2k}\sum_{d\leqslant x}\dfrac{1}{d^k}\right) = 0(x^{k+1})$ according as $2k<1$ or $2k>1$ by (2.10).

COROLLARY 3.1. *($f = U$) For $k > 0$*

$$\sum_{n\leqslant x}\sigma_k(n^2) = \frac{\zeta(2k+1)\,\zeta(k+1)}{(2k+1)\,\zeta(2k+2)}\,x^{2k+1} + 0(B_k(x)), \tag{3.2}$$

where $B_k(x)$ is given in theorem 3.1.

COROLLARY 3.2. ($f = U$, $k = 1$)

$$\sum_{n\leqslant x}\sigma(n^2) = \alpha x^3 + 0(x^2\log x), \tag{3.3}$$

where $\alpha = \dfrac{\zeta(3)\,\zeta(2)}{3\zeta(4)} = \dfrac{5\zeta(3)}{\pi^2}$.

Now we prove our principal result.

THEOREM 3.2. *If f is completely multiplicative and bounded, then for $k > 0$*

$$\sum_{n\leqslant x}(f * E_k)^2(n) = \frac{D(f^2, 2k+1)\,D^2(f, k+1)}{(2k+1)\,D(f^2, 2k+2)}\,x^{2k+1} + 0(C_k(x)), \tag{3.4}$$

where $C_k(x) = x^{2k}$, $x^2\log^2 x$, $x^{\frac{3}{2}}\log^2 x$ or $x^{k+1}\log x$ according as $k > 1$, $k = 1$, $k = \dfrac{1}{2}$ or $k < 1$ and $k \neq \dfrac{1}{2}$.

Proof. We use corollary 2.3. and the above theorem.

$$\sum_{n\leqslant x}(f * E_k)^2(n) = \sum_{n\leqslant x}((f * E_k)\circ E_2 * fE_k)(n) =$$

$$= \sum_{d\delta=n\leqslant x} f(d)d^k(f * E_k)(\delta^2) = \sum_{d\leqslant x} f(d)\,d^k\sum_{\delta\leqslant\frac{x}{d}}(f * E_k)(\delta^2) =$$

$$= \sum_{d \leq x} f(d)\, d^k \left\{ \frac{D(f^2, 2k+1)\, D(f, k+1)}{(2k+1)\, D(f^2, 2k+2)} \left(\frac{x}{d}\right)^{2k+1} + 0\left(B_k\left(\frac{x}{d}\right)\right)\right\} =$$

$$= \frac{D(f^2, 2k+1)\, D(f, k+1)}{(2k+1)\, D(f^2, 2k+2)}\, x^{2k+1} \sum_{d \leq x} \frac{f(d)}{d^{k+1}} + 0\left(\sum_{d \leq x} d^k B_k\left(\frac{x}{d}\right)\right),$$

where $\sum_{d \leq x} \frac{f(d)}{d^{k+1}} = \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + 0\left(\sum_{d > x} \frac{1}{d^{k+1}}\right) = D(f, k+1) + 0\left(\frac{1}{x^k}\right)$

and the remain term is : case $k > 1$ $0\left(\sum_{d \leq x} d^k \left(\frac{x}{d}\right)^{2k}\right) = 0(x^{2k})$;

case $k = 1$ $0\left(\sum_{d \leq x} d\left(\frac{x}{d}\right)^2 \log \frac{x}{d}\right) = 0\left(x^2 \log x \sum_{d \leq x} \frac{1}{d}\right) = 0(x^2 \log^2 x)$;

case $k = \frac{1}{2}$ $0\left(\sum_{d \leq x} d^{\frac{1}{2}}\left(\frac{x}{d}\right)^{\frac{3}{2}} \log \frac{x}{d}\right) = 0\left(x^{\frac{3}{2}} \log x \sum_{d \leq x} \frac{1}{d}\right) = 0\left(x^{\frac{3}{2}} \log^2 x\right)$;

and in case $k < 1$, $k \neq \frac{1}{2}$ it is $0\left(\sum_{d \leq x} d^k \left(\frac{x}{d}\right)^{k+1}\right) = 0(x^{k+1} \log x)$

which proves the theorem.

COROLLARY 3.3. ($f = U$) *For* $k > 0$

$$\sum_{k \leq 1} \sigma_k^2(n) = \frac{\zeta(2k+1)\, \zeta^2(k+1)}{(2k+1)\, \zeta(2k+2)}\, x^{2k+1} + 0(C_k(x)), \tag{3.5}$$

where $C_k(x)$ is given in theorem 3.2.

COROLLARY 3.4. ($f = U$, $k = 1$; Ramanujan)

$$\sum_{n \leq x} \sigma^2(n) = \beta x^3 + 0(x^2 \log^2 x), \tag{3.6}$$

where $\beta = \frac{\zeta(3)\, \zeta^2(2)}{3\zeta(4)} = \frac{5}{6}\, \zeta(3)$.

Remark 3.1. In 1970 R. A. Smith [6] improved the error term of Ramanujan's formula (3.6) into

$$0(x^2 \log^{\frac{5}{3}} x),$$

using analytical methods.

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NORMAL Π — COMPLEMENTS IN FINITE GROUPS

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REZUMAT. — **Π-complement normal în grupuri finite.** În cele patru teoreme din lucrare se dau condiţii necesare şi suficiente pentru existenţa π-complementului normal în grupuri finite.

1. Preliminaires. Let G denote a finite group, π a set of prime numbers and π' the complementary set of π. A normal π—complement in G is a normal Hall π'-subgroup of G.

The purpose of the present paper is to give some necessary and sufficient conditions for the existence of normal π-complements in finite groups.

The notations and the terminology used are largely standard.

If $g \in G$, then g has a unique decomposition in the form $g=g_\pi \cdot g_{\pi'} = g_{\pi'} \cdot g_{\pi'}$, where g_π is a π-element and $g_{\pi'}$, is a π'-element of G. We call g_π, $g_{\pi'}$ respectively, the π-factor, π'-factor of g. If π consists only of one prime p, we write g_p for g_π. Each $g \in G$ is the mutually commuting product of its p-factors $g_p \neq 1$ for the different primes.

Two elements $g, h \in G$ will be said to be π-conjugate in G, if their π-factors are conjugate in G in the ordinary sense. Since the π-factor of an element g is a power of g, conjugate elements are also π-conjugate. The π-conjugacy is an equivalence relation in G. We can thus speak of the π-conjugate classes in G. We use $K_{G,\pi}(g)$ for the π-conjugate class of the element g in G.

It is clear that:

— $K_{G,\pi}(g) = K_{G,\pi}(g_\pi)$ for every $g \in G$.

— $K_{G,\pi}(1)$ is the set of π'-elements of G.

— If G has a normal normal π-complement K, then $K = K_{G,\pi}(1)$ (Hence the normal π-complement is uniquely determined by the set π).

LEMMA 1. (see [4], lemma (20.4), p. 106). *For every π-element $g \in G$, if $C_G(g)$ is the centralizer of g in G, then*

$$|K_{G,\pi}(g)| = |G : C_G(g)| \cdot |K_{C_G(g),\pi}(1)| \tag{1}$$

LEMMA 2. *If H is a Hall π-subgroup of G, then the following conditions are equivalent:*

(a_1) *If two elements of H are conjugate in G, they are conjugate in H.*

(a_2) *For every $h \in H$, $K_{G,\pi}(h) \cap H = K_{H,\pi}(h)$.*

Proof. We observe first that, since H is a π-subgroup, $h = h_\pi$ for every for every $h \in H$. Hence the π-conjugate classes of H ocincides with the conjugate classes of H.

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Assume (a_1). It is obvious that $K_{H,\pi}(h) \subseteq K_{G,\pi}(h) \cap H$ for any $h \in H$. Let $k \in K_{G,\pi}(h) \cap H$. It results that $h(=h_\pi)$ and $k(=k_\pi)$ are conjugate in G. Hence from (a_1) h and k are conjugate in H. Thus $k \in K_{H,\pi}(h)$.

Conversely, assume (a_2). If $h, k \in H$ are conjugate in G, then $K_{G,\pi}(h) = K_{G,\pi}(k)$. It follows from (a_2) that $K_{H,\pi}(h) = K_{H,\pi}(k)$.
Hence h and k are conjugate in H.

2. Necessary and sufficient conditions for the existence of the normal π-complements

THEOREM 1. (A reformulation of a theorem of R. Brauer). *The following conditions are necessary and sufficient for the existence of normal π-complement in G:*

(A_1) *There exist a Hall π-subgroup H of G.*

(A_2) *For every $h \in H$, $K_{G,\pi}(h) \cap H = K_{H,\pi}(h)$*

(A_3) *If E is an elementary π-subgroup of G, then E is conjugate in G to a subgroup of H.*

Proof. The statements are immediate consequences of the Brauer's theorem (see [1], Th. 3) and of the LEMMA 2.

Remark 1. ([2], Th. 2) The condition (A_3) in THEOREM 1 can be replaced by the following:

(A_3') *If $1 \neq h \in H$ and P is a Sylow p-subgrup of $C_G(h)$, for some $p \in \pi$ not dividing the order of h, then the elementary π-subgroup $\langle h\rangle \times P$ is conjugate in G to a subgroup of H.*

Remark 2. ([3], Th. 1). The condition (A_3) in THEOREM 1. can be replaced by the statement:

(A_3'') *If $h \in H$ satisfier the condition $C_G(h) \neq G$, then $C_H(h)$ is a Hall π-subgroup of $C_G(h)$ and $C_G(h)$ has normal π-complement.*

THEOREM 2. *The following three conditions are equivalent*

(i) *The finite group G has normal π-complement.*

(ii) (B_1) *There exist a Hall π-subgroup H of G.*
(B_2) *$C_H(h)$ is a Hall π-subgroup of $C_G(h)$ and $|K_{C_G(h),\pi}(1)| = |C_G(h) : C_H(h)|$ for every $h \in H$.*

(iii) (C_1) *There exist a Hall π-subgroup H of G.*
(C_2) *Any π-element of G is conjugate in G to an element of H.*
(C_3) *$|K_{C_G(h),\pi}(1)| = |C_G(h)|_{\pi'}$ for every $h \in H$. ($|C_G(h)|_{\pi'}$ is the largest integer dividing $|C_G(h)|$ al of whose prime factors are in π')*

Proof. (i) implies (ii). Suppose that G has a normal π-complement. The condition (B_1) follows by THEOREM 1 and (B_2) is an immediate consequence of REMARK 2.

(ii) implies (iii). Assume (B_1) and (B_2). It is obious that (C_1) and (C_3) follows from (B_1) and (B_2).

In order to prove (C_2) we use industion on the number of p-factors of π-elements of G. Let g be a π-element of G and $g = g_p h = h g_p$, where $g_p \neq 1$

is the p-factor of g, for some $p \in \pi$. If $h = 1$, then (B_1) and Sylow's theorem shows that g is conjugate in G to an element of H. If $h \neq 1$, then by induction h is conjugate in G to an an element of H. Replacing g by a conjugate, we may suppose that $h \in H$. It follows from (B_2) that $C_H(h)$ contain a Sylow p-subgroup P of $C_G(h)$. Since $g_p \in C_G(h)$, it results that g_p is conjugate in $C_H(h)$ to an element of $P \subseteq C_H(h) \subseteq H$. Hence $g = hg_p$ is conjugate in G to an element of H. Thus (C_2) holds.

(iii) implies (i). Suppose that (C_1), (C_2) and (C_3) hold. We apply THEOREM 1. It is clear that (A_1) holds. It remains to prove (A_2) and (A_3).

Let $K_{H,\pi}(h_i)$ $(i = 1, 2, \ldots, n)$ be the different π-conjugate classes of H. Then

$$H = \bigcup_{i=1}^{n} K_{H,\pi}(h_i) \text{ (disjoint).} \tag{2}$$

Hence

$$|H| = \sum_{i=1}^{n} |K_{H,\pi}(h_i)| \tag{3}$$

By (C_2) any π-element of G are conjugate to an element of H. Hence, for every $g \in G$, there exist an $h_i \in H$ such that $K_{G,\pi}(g) = K_{G,\pi}(h_i)$. It follows that $K_{G,\pi}(h_i)$ $(i = 1, 2, \ldots, n)$ are the all π-conjugate classes of G, i.e.

$$G = \bigcup_{i=1}^{n} K_{G,\pi}(h_i) \tag{4}$$

The equility (1) implies from (C_3) that

$$|K_{G,\pi}(h_i)| = |G : C_G(h_i)| \cdot |C_G(h_i)|_{\pi'}. \tag{5}$$

Since H is a Hall π-subgroup of G, it results that $C_H(h_i)$ is a π—subgroup of $C_G(h_i)$. Hence

$$|C_G(h_i)|_{\pi'} \leqslant |C_G(h_i) : C_H(h_i)|.$$

This inequality implies from (5) that

$$|K_{G,\pi}(h_i)| \leqslant |G : C_G(h_i)| \cdot |C_G(h_i) : C_H(h_i)|. \tag{6}$$

It results that

$$|K_{G,\pi}(h_i)| \leqslant |G : H| \cdot |H : C_H(h_i)| = |G : H| \cdot |K_{H,\pi}(h_i)|$$

Hence by (3) we have

$$\sum_{i=1}^{n} |K_{G,\pi}(h_i)| \leqslant |G : H| \left(\sum_{i=1}^{n} |K_{H,\pi}(h_i)| \right) = |G : H| \cdot |H| = |G|$$

This implies from (4) that

$$G = \bigcup_{i=1}^{n} K_{G,\pi}(h_i) \text{ (disjoint)} \tag{7}$$

It follows that $K_{G,\pi}(h) \cap H = K_{H,\pi}(h)$ for every $h \in H$. Hence (A_2) holds.

It remains to prove (A_3).

We first note, that from (6) and (7) it follows that

$$|K_{G,\pi}(h_i)| = |G : C_G(h_i)| \cdot |C_G(h_i) : C_H(h_i)|$$

Hence from (5) we obtain that

$$|C_G(h_i)|_{\pi'} = |C_G(h_i) : C_H(h_i)| \tag{8}$$

This shows that $C_H(h)$ is a Hall π-subgroup of $C_G(h)$ for every $h \in H$.

Let E be an elementary π-subgroup of G, i.e. E is the direct product of a π-element h and a *p-subgrup* P_0 for some $p \in \pi : E = \langle h \rangle \times P_0$. It follows from (C_2) that h is conjugate in G to an element of H. Replacing E by a conjugate, we may assume that $h \in H$. Since $C_H(h)$ is a Hall π-subgroup of $C_G(h)$, it results that $C_H(h)$ contain a Sylow p-subgroup P of $C_G(h)$. Since P_0 is a p-subgroup of $C_G(h)$, the Sylow's theorem shows that P_0 is conjugate in $C_H(h)$ to a subgroup of $P \subseteq H$. This proves that E is conjugate in G to a subgroup of H. Hence (A_3) holds.

COROLLARY. *If the finite group G has a nilpotent Hall π-subgroup H. then the following conditions are equivalent*

(i) *G possesses normal π-complement*

(ii) $|K_{C_G(h),\pi}(1)| = |C_G(h)|_{\pi'}$ *for every* $h \in H$

Proof. The statements follows from THEOREM 2 and from a Theorem of Wielandt ([6]. The 5.8, p. 285)

As an application of Corollary, we obtain.

THEOREM 3. *Tge following conditions are lquivalent:*

(i) *G is a p-nilpotent group*

(ii) *If P is a Sylow p-subgroup of G, then for every $h \in P$ the number of p'-elements of $C_G(h)$ is* $|C_G(h)|_{p'}$.

Remark 3. It is known (see [5], p. 137) the following conjecture: If n divide the order of a finite group G and the number of solutions of $x^n = 1$ in G is exactly n, then these solutions form a normal subgroup of G.

It $n = |G|_{\pi'}$ then it is obvions that $K_{G,\pi}(1)$ is the set of solutions of $x^n = 1$. We hawe the following reformulation of THEOREM 2.

THEOREM 4. *If $n = |G|_{\pi'}$, then the following three conditions are equivalent:*

(i) *The solutions of $x^n = 1$ in G form a normal subgroup of G.*

(ii) (B'_1) *There exist a Hall π — subgroup H of G*
(B'_2) *For every $h \in H$, $C_H(h)$ is a Hall π — subgroup of $C_G(h)$ and if $|n_h| = |C_G(h)|_{\pi'}$, then the number of solutions of $x^{n_h} = 1$ in $C_G(h)$ is n_h.*

(iii) $\begin{cases} (C'_1) \text{ There exist a Hall } \pi \text{ — subgroup } H \text{ of } G \\ (C'_2) \text{ Any } \pi \text{ — element of } G \text{ is conjugate in } G \text{ to an element of } H \\ (C'_3) \text{ For every } h \in H \text{, the number of solutions of } x^{n_h} = 1 \text{ in } C_G(h) \text{ is } \end{cases}$

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THE DEFINITION OF DISTANCE AND DIAMETER IN FUZZY SET THEORY

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REZUMAT. — Definirea distanţei şi diametrului în teoria mulţimilor fuzzy. În lucrare se defineşte distanţa între două mulţimi nuanţate (fuzzy) şi diametrul unei mulţimi nuanţate. În cazul clasic aceste definiţii se reduc la cele cunoscute.

1. Introduction. In any metric space (S, d) it is possible to define the *distance* between two subsets X and Y of S by setting $\delta(X, Y) = 0$ if $X = \emptyset$ or $Y = \emptyset$ and

$$\delta(X, Y) = \inf \{d(x, y)/x \in X,\ y \in Y\} \text{ otherwise.} \tag{1}$$

The distance between a point x and a set X is defined by setting $\delta(x, X) = \delta(\{x\}, X)$.

This allows, for instance, to characterize the non-empty closed sets as the sets X for which $x \in X$ if and only if $\delta(x, X) = 0$.

Another foundamental concept is that of *diameter* $\Delta(X)$ of a set. One defines it by setting $\Delta(X) = 0$ if $X = \emptyset$ and

$$\Delta(X) = \sup \{d(x, y)/x \in X,\ y \in X\} \text{ otherwise.} \tag{2}$$

In this paper our aim is to define analogue concepts for the fuzzy sets. So we define the *distance* between two fuzzy sets and, hence, between a fuzzy point and a fuzzy set.

We call *closed* a fuzzy set containing all the fuzzy points that have distance from it equals to zero, and we show that the complements of closed sets determine a fuzzy topology, the fuzzy topology of lower semi-continuous functions.

Also, we define the *diameter* of a fuzzy set. This will allow to characterize the fuzzy points as the fuzzy sets with diameter equals to zero.

2. Prerequisites and definitions. Let X a set and $\mathbf{R}$ the set of real numbers. We say *fuzzy subset* of X or, more simply, *fuzzy set* [8] a function $f: X \to [0, 1]$ where $[0, 1]$ denotes the set $\{\alpha \in R/0 \leqslant \alpha \leqslant 1\}$.

We denote by $F(X)$ the class of the fuzzy subsets of X. If $f, g \in F(X)$ then we set $f \leqslant g$ iff $f(x) \leqslant g(x)$ for any $x \in X$. Moreover $-f$, *the complement of* f is the fuzzy subset of X defined by setting $(-f)(x) = 1 - f(x)$ for any $x \in X$. If $(f_i)_{i \in I}$ is a family of fuzzy subsets of X then $\bigvee_{i \in I} f_i$ and $\bigwedge_{i \in I} f_i$ are the fuzzy subsets of X defined by setting

$(\bigvee_{i \in I} f_i)(x) = \sup_{i \in I}\{f_i(x)\}$ and

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$(\bigwedge_{i\in I} f_i)(x) = \inf_{i\in I}\{f_i(x)\}$ for any $x \in X$.

We denote by f_0 and f_1 the fuzzy sets for which $f_0(x) = 0$ and $f_1(x) = 1$ for any $x \in X$.

Moreover, if $\alpha \in [0, 1]$, we call *α-cut* of a fuzzy set f the subset $C_f^\alpha = \{x \in X/f(x) \geqslant \alpha\}$.

A fuzzy set f is called *crisp* if $f(x) \in \{0, 1\}$ for any $x \in X$. The fuzzy sets crisp can be interpreted as characteristic functions of subsets of X and, hence, they can be identified with these subsets.

For any $a \in X$ and $\alpha \in (0, 1] = \{x \in \mathbf{R}/0 < x \leqslant 1\}$ the fuzzy set f_a^α, defined by setting $f_a^\alpha(x) = 0$ if $x \neq a$ and $f_a^\alpha(x) = \alpha$ if $x = a$, is called *fuzzy point* ([7], [3], [4]).

We say that the fuzzy point f_a^α *belongs* to the fuzzy set f, $f_a^\alpha \in f$, if $f_a^\alpha \leqslant f$ that is if $f(a) \geqslant \alpha$.

We can now define the concept of fuzzy topological space (see references). To this aim we give the following definitions.

DEFINITION 1. *A class τ of fuzzy subsets of X constitutes a fuzzy topology if the following conditions are verified:*

a) $f_0, f_1 \in \tau$

b) *if* $f, g \in \tau$ *then* $f \wedge g \in \tau$

c) $\bigvee_{i\in I} f_i \in \tau$ *for any family* $(f_i)_{i\in I}$ *of elements in* τ.

The pair (X, τ) is named *fuzzy topological space*; the elements of τ are named *open*, the complements of these elements are named *closed*.

The following definition is dual of Definition 1.

DEFINITION 2. *A class $C \subset F(x)$ is a system of closed fuzzy subsets of X if the following conditions are verified:*

a) $f_0, f_1 \in C$

b) *if* $f, g \in C$ *then* $f \vee g \in C$

c) $\bigwedge_{i\in I} f_i \in C$ *for any family* $(f_i)_{i\in I}$ *of elements of* C.

Obviously, the class of complements of a system of closed fuzzy set is a fuzzy topology and the class of complements of a fuzzy topology constitutes a system of closed fuzzy sets.

3. Distance between two fuzzy sets. Let (S, d) be a metric space. We define a *distance* between two fuzzy subsets f, g of S in the following way:

$$d(f, g) = \int_0^1 \delta(C_f^\alpha, C_g^\alpha)\,d\alpha \qquad (3)$$

Note that if $\beta \geqslant \alpha$ then $C_f^\beta = \{x \in S/f(x) \geqslant \beta\} \subseteq C_f^\alpha = \{x \in S/f(x) \geqslant \alpha\}$ and, hence, $\delta(C_f^\beta, C_g^\beta) \geqslant \delta(C_f^\alpha, C_g^\alpha)$. This proves that $\delta(C_f^\alpha, C_g^\alpha)$ is an increasing function of α and, hence, that the distance between two fuzzy sets is defined for any $f, g \in F(S)$, even if it is finite or infinite. An example of a pair of fuzzy sets with infinite distance is the following.

Let (S, d) be the set of real numbers with the usual distance, and consider ${}^1_0$ and f, where f is the fuzzy set for which $f(x) = x/x + 1$; then $d(f_0^1, f)$ is $f$equal to ∞.

If in f and g there are two crisp points, that is if there exist x, y in S for which f_x^1 and f_y^1 belong respectively to f and g, then, being any contribution $\delta(C_f^\alpha, C_g^\alpha) \leqslant d(x, y)$, the integral in (3) assumes a finite value.

If f and g are the characteristic functions of two subsets X and Y of S then $C_f^\alpha = X$ and $C_g^\alpha = Y$ for every $\alpha > 0$, hence $d(f, g) = \int_0^1 \delta(X, Y)\, d\alpha = 1 \cdot \delta(X, Y) = \delta(X, Y)$. Then (3) generalizes the classical definition of distance between two subsets of a metric space.

Obviously the distance between a fuzzy point f_x^α and a fuzzy set g is $\int_0^\alpha \delta(x, C_g^\beta)\, d\beta$. Moreover the distance between two fuzzy points f_b^β and f_c^γ is equal to $\int_0^{\beta \wedge \gamma} \delta(\{b\}, \{c\})\, d\alpha$ and therefore

$$d(f_b^\beta, f_c^\gamma) = [\gamma \wedge \beta] \cdot d(b, c). \tag{4}$$

This proves that, for the fuzzy points crisp, the distance defined by (3) coincides with the usual one between points.

It is interesting to examine the case that f and g assume values in a finite subset $\{\gamma_0, \ldots, \gamma_n\}$ of $[0, 1]$. Then, if $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_n = 1$ we have

$$d(f, g) = \sum_{i=1}^{n} \delta(C_f^{\gamma_i}, C_g^{\gamma_i}) \cdot (\gamma_i - \gamma_{i-1})\,; \tag{5}$$

if, for $i = 1, \ldots, n$, $\gamma_i - \gamma_{i-1} = 1/n$

$$d(f, g) = 1/n \cdot \left(\sum_{i-1}^{n} \delta(C_f^{\gamma_i}, (C_g^{\gamma_i}) \right) \cdot \tag{6}$$

In general, we can also utilize Formulas (5) and (6) to compute a suitable approximation of the distance between two fuzzy subsets.

We can give a definition of closure for fuzzy sets:

DEFINITION 3. *A fuzzy set f is metrically closed if either $f = f_0$ or, for every fuzzy point f_x^α, $f_x^\alpha \in f$ iff $d(f_x^\alpha, f) = 0$. We denote by C the class of the metrically closed fuzzy sets.*

PROPOSITION 1. *The set C is a system of closed fuzzy subsets of X. Equivalently, the set τ of the relative complements defines a fuzzy topology.*

Proof. It is obvious that f_0 and f_1 are elements of C. Let $f \in C$ and $g \in C$, and let f_x^α a fuzzy point. If $f_x^\alpha \in f \vee g$ it is obvious that $d(f_x^\alpha, f \vee g) = 0$. Conversely, suppose that $d(f_x^\alpha, f \vee g) = 0$, then $\delta(x, C_{f \vee g}^\beta) = 0$ for every

$\beta < \alpha$. Suppose, by absurd that $f_x^\alpha \notin f \vee g$, then $f(x) < \alpha$ and $g(x) < \alpha$, i.e. $f_x^\alpha \notin f$ and $f_x^\alpha \notin g$. This implies that $d(f_x^\alpha, f) > 0$ and $d(f_x^\alpha, g) > 0$ and therefore that $\delta(x, C_f^\gamma) > 0$ and $\delta(x, C_g^\gamma) > 0$ for a suitable $\gamma < \alpha$. It follows that $\delta(x, C_f \cup C_g) > 0$ and, since $C_{f \vee g}^\gamma \subseteq C_f^\gamma \cup C_g^\gamma$, $\delta(x, C_{f\vee g}^\gamma) \geqslant \delta(x, C_f^\gamma \cup C_g^\gamma) > 0$, an absurd. This prove that $f_x^\alpha \in f \vee g$ and therefore that $f \vee g \in C$.

Let $(f_i)_{i \in I}$ be a family of elements of C and set $f = \bigwedge_{i \in I} f_i$: we have to prove that $f \in C$. If $f_x^\alpha \in f$ it is obvious that $d(f_x^\alpha, f) = 0$. Assume that $d(f_x^\alpha, f) = 0$ then $\delta(x\, C_f^\beta) = 0$ for every $\beta < \alpha$. If, by absurd, $\alpha > f(x)$, then $\alpha > f(x_j)$, and therefore $f_x^\alpha \notin f_j$, for a suitable $j \in I$. Thus $\int_0^\alpha \delta(x, C_{f_j}^\beta)\, d\beta > 0$ and there exists $\gamma < \alpha$ such that $\delta(x, C_{f_j}) > 0$. Since $C_f^\gamma \subseteq C_{f_j}^\gamma$, we have also that $\delta(x, C_f^\gamma) \geqslant (x, C_{f_j}^\gamma) > 0$, an absurd. Thus we have proved that $\alpha \leqslant f(x)$ and therefore that $f_x^\alpha \in f$. This complete the proof.

Now we show that the above defined fuzzy topology τ coincides with the *natural fuzzy topology* defined in [2].

PROPOSITION 2. *C is the class of the upper semicontinuous functions from S to* $[0, 1]$. *It follows that τ is the class of the lower semicontinuous functions.*

Proof. Let $f \in C$, then, to prove that f is upper semicontinuous, it suffices to prove that $\{x \in S / f(x) < \alpha\}$ is open for every $\alpha \in [0, 1]$. Equivalently, we can prove that C_f^α is closed. Let $x \in S$ and $\delta(x, C_f^\alpha) = 0$, then $\delta(x, C_f^\beta) = 0$ for every $\beta < \alpha$ and therefore $d(f_x^\alpha, f) = \int_0^\alpha \delta(x, C_f^\beta)\, d\beta = 0$. Thus $f_x^\alpha \in f$ and $x \in C_f^\alpha$. This proves that C_f^α is closed.

Conversely, suppose f upper semicontinuous or, equivalently, that C_f^α is closed for every $\alpha \in [0, 1]$. Moreover, suppose that $d(f_x^\alpha, f) = \int_0^\alpha \delta(x, C_f^\beta)\, d\beta = 0$, then $\delta(x, C_f^\beta) = 0$ for every $\beta < \alpha$. This implies that $x \in C_f^\beta$ and therefore that $f(x) \geqslant \beta$ for every $\beta < \alpha$. In conclusion $f(x) \geqslant \alpha$ and $f_x^\alpha \in f$. This proves that $f \in C$.

4. Diameter of a fuzzy set. Let f be a fuzzy subset of S, then we set

$$\Delta(f) = \sup \{d(x, y) / x \text{ and } y \text{ are fuzzy points of } f\}. \qquad (7)$$

The number $\Delta(f)$ may be either finite or infinite, we call it the *diameter of the fuzzy set f.*

If $\Delta(f) < \infty$ then f is called *bounded.*

Being $\delta(f_x^\alpha, f_y^\beta) = d(x, y) \cdot (\alpha \wedge \beta)$, it is obvious that

$$\Delta(f) = \sup \{d(x, y) \cdot [f(x) \wedge f(y)]/x, y \in S\}. \tag{8}$$

PROPOSITION 3. *If f is crisp then the definition of diameter is the classical one. Moreover if $f \leqslant g$ then $\Delta(f) \leqslant \Delta(g)$.*

Proof. If f is the characteristic function of the set X then $\Delta(f) = \sup \{d(x, y) \cdot [f(x) \wedge f(y)]/f(x) \neq 0\ f(y) \neq 0\} = \sup \{d(x, y)/x \in X,\ y \in X\}$.

Suppose that $f \leqslant g$, then $d(x, y) \cdot [f(x) \wedge f(y)] \leqslant d(x, y) \cdot [g(x) \wedge g(y)]$ and $\Delta(f) \leqslant \Delta(g)$.

PROPOSITION 4. *The diameter of a fuzzy set $f \neq f_0$ is equal to zero iff f is a fuzzy point.*

Proof. It is obvious that the diameter of a fuzzy point is zero. Conversely, suppose that f is a fuzzy set for which $\Delta(f) = 0$. Then, by (8), $d(x, y) \cdot [f(x) \wedge f(y)] = 0$ for every $x, y \in S$. By hypothesis, there exists $a \in S$ for which $f(a) \neq 0$ and, if $y \neq a$, since $d(a, y) \neq 0$ then $f(a) \wedge f(y) = 0$. This proves that $f(y) = 0$ for every $y \neq a$ and therefore that f is a fuzzy point.

PROPOSITION 5. *For any $f \in F(S)$* and $\alpha \in (0, 1]$

$$\Delta(C_\alpha^f) \leqslant \Delta(f)/\alpha \tag{9}$$

Then every α-cut of a bounded fuzzy set is bounded while the converse falls,

Proof. If $x, y \in C_f^\alpha$, i.e. $f(x) \geqslant \alpha$, $f(y) \geqslant \alpha$, then $d(x, y) \cdot [f(x) \wedge f(y)] \geqslant d(x, y) \cdot \alpha$. This proves that $\Delta(f) \geqslant \alpha \cdot d(x, y)$ or, equivalently, $d(x, y) \leqslant \Delta(f)/\alpha$.

To prove that there exists a fuzzy set f such that $\Delta(f) = \infty$ and $\Delta(C_f^\alpha) < \infty$ for any $\alpha \in [0, 1]$, let S be the positive real numbers set and define $f: S \rightarrow [0, 1]$ by setting $f(x) = 1/(\sqrt{x} + 1)$. Now $\Delta(f) \geqslant d(0, x) \cdot (f(0) \wedge f(x)) = x/(\sqrt{x} + 1)$ for any $x \in S$. Then $\Delta(f) = \infty$ while it is obvious that every cut of f is bounded.

Proposition 5 shows that our definition of bounded fuzzy set is different from Kaufmann's definition [6].

In metric space theory one proves that a subset is bounded if and only if it is contained in a suitable circle. In order to obtain a similar result for fuzzy subsets we give the following definition.

DEFINITION 4. *We call f-circle with center f_c^γ and radius r, the fuzzy set $C(f_c^\gamma, r)$ such that, for any fuzzy point f_b^β, $f_b^\beta \in C(f_c^\gamma, r)$ iff $d(f_b^\beta, f_c^\gamma) \leqslant r$ and $\beta \leqslant \gamma$.*

PROPOSITION 6. *The f-circle $C(f_c^\gamma, r)$ is the fuzzy set defined by*

$$f(z) = \begin{cases} \gamma \text{ if } d(z, c) \leqslant r/\gamma \\ r/d(z, c) \qquad \textit{otherwise.} \end{cases} \tag{10}$$

Moreover the diameter of $C(f_c^\gamma, r)$ *is not greater than* $2r$.

Proof. By definition $f = \bigvee \{f / \beta_x^\beta \leqslant \gamma,\ x \in S$ *and* $d(f_x^\beta, f_c^\gamma) \leqslant r\}$, then $f(z) = \bigvee \{f_z^\beta / \beta \leqslant \gamma$ *and* $d(f_z^\beta, f_c^\gamma) \leqslant r\} = \bigvee \{\beta / \beta \leqslant \gamma$ *and* $\beta \cdot d(x, c) \leqslant r\}$. This proves (10).

To show that $\Delta(f) \leqslant 2r$ observe that, for every pair of fuzzy points $f_b^\beta, f_{b'}^{\beta'}$ with $\beta \leqslant \gamma$ and $\beta' \leqslant \gamma$, the following triangular inequality holds:

$$d(f_b^\beta, f_{b'}^{\beta'}) \leqslant d(f_b^\beta, f_c^\gamma) + d(f_c^\gamma, f_{b'}^{\beta'}). \tag{11}$$

In fact, by $d(b, b') \leqslant d(b, c) + d(c, b')$, we have

$$(\beta \wedge \beta') \cdot d(b, b') \leqslant (\beta \wedge \beta') \cdot d(b, c) + (\beta \wedge \beta') \cdot d(c, b') \leqslant \beta \cdot d(b, c) +$$

$$+ \beta' \cdot d(c, b') = (\beta \wedge \gamma) \cdot d(b, c) + (\beta' \wedge g) \cdot d(c, b') = d(f_b^\beta, f_c^\gamma) + d(f_c^\gamma, f_{b'}^{\beta'}).$$

But $(\beta \wedge \beta') \cdot d(b, b') = d(f_b^\beta, f_{b'}^{\beta'})$ and then (11) is proved.

From this it follows that $\triangle(f) \leqslant 2r$.

PROPOSITION 7. *Let f be a bounded fuzzy set,* $\gamma = \sup \{f(x)\}$ *and $c \in S$ a point such that* $f(c) > 0$. *Then f is contained in the f-circle* $C(f_c^\gamma, \Delta(f)/f(c))$. *It follows that a fuzzy set f is bounded if and only if it is contained in an f-circle.*

Proof. Let $r = \Delta(f)/f(c)$ and denote by g the f-circle $C(f_c^\alpha, r)$. If $d(z, c) \leqslant r/\gamma$ then $g(z) = \gamma = \sup \{f(x)\}$ and therefore $g(z) \geqslant f(z)$. If $d(z, c) > r/\gamma$ then $g(z) = r/d(z, c)$. Since $f(c) \cdot f(z) \leqslant f(c) \wedge f(z)$, we have also that $d(c, z) \cdot f(c) \cdot f(z) \leqslant d(c, z) \cdot (f(c) \wedge f(z)) \leqslant \Delta(f)$. This proves that $f(z) \leqslant r/d(z, c) = g(z)$.

Finally, observe that, if (S, d) is the euclidean plane, then the diameter of an f-circle $C(f_c^\gamma, r)$ is just $2r$. Indeed, let z and z' two points collinear with c such that $d(z, c) = d(z', c) = r/\gamma$. Then $d(z, z') = 2r/\gamma$ and $d(f_z^\gamma, f_{z'}^\gamma) = \gamma \cdot d(z, z') = 2r$. Since f_z^γ and $f_{z'}^\gamma$, are fuzzy points of the f-circle $C(f_c^\gamma, r)$, this proves that the relative diameter is $2r$.

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PARTICULAR n — α-CLOSE-TO-CONVEX FUNCTIONS

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REZUMAT. — Funcţii particulare n — α-aproape convexe. În lucrare se dau unele rezultate referitoare la funcţii particulare din clasele de funcţii $AC_n(\delta)$, $AC_n\left(\frac{1}{2}\right)$, $C_n\left(\frac{1}{2}\right)$ introduse de H. S. Al-Amiri în [1].

1. Introduction. Let A be the class of functions $f(z)$, analytic in the unit disc U with $f(0) = f'(0) - 1 = 0$. Like in [2] we denote by $K_{n,\alpha}(\delta)$ the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] > \delta,\quad z \in U$$

where $\alpha \geqslant 0$, $\delta < 1$ and $D^nf(z) = \dfrac{z}{(1-z)^{n-1}} * f(z) = \dfrac{z(z^{n-1}f(z))^{(n)}}{n!}$, where $(*)$ stands for the Hadamard product. Note that the classes $K_{n,\alpha}(\delta)$ and $Z_n(\delta) \equiv \equiv K_{n,0}(\delta)$ are studied in [2] and the classes $K_{n,\alpha}\left(\frac{1}{2}\right)$ and $Z_n\left(\frac{1}{2}\right)$ were introduced by H. S. Al — Amir [1] and S. Ruscheweyh [6] respectively.

Like in [3] we denote by $AC_n(\delta)$ (the class of *n-closeto-convex functions of order* δ) the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)} > \delta,\ z \in U,\ \text{where } g \in Z_{n+1}(\delta)$$

and we denote by $C_{n,\alpha}(\delta)$ (the class of *n — α-close-toconvex functions of order* δ) the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}f(z)} + \alpha\frac{D^{n+2}g(z)}{D^{n+2}g(z)}\right] > \delta,\ z \in U \text{ where}$$

$g \in Z_{n+2}(\delta)$. The classes $AC_n\left(\frac{1}{2}\right)$ and $C_{n,\alpha}\left(\frac{1}{2}\right)$ were introduced by H. S. Al — Amiri [1] and some properties of $AC_n(\delta)$ and $C_{n,\alpha}(\delta)$ given in [3] by using sharp subordination results (see [4], [5]).

In this paper we will present some results concerning particular functions of this classes.

2. Preliminaries. Let f and g be regular in U. We say that f is subordinate to g, written $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

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We will need the following lemmas to prove our main results and we denote by $b_\gamma(z) = \sum_{j=1}^{\infty} \frac{\gamma + 1}{\gamma + j} z^j$, $\operatorname{Re}\gamma > -1$.

LEMMA A. [3, Theorem 2]. *Let* $\gamma > -1$ *and*

$$\delta_0 = \max\left\{\frac{n-\gamma}{n+1}, \frac{2n-\gamma}{2(n+1)}\right\} \leqslant \delta < 1.$$

If $f \in Z_n(\delta)$ *then* $f * b_\gamma \in Z_n(\tilde{\delta}(n, \gamma, \delta))$, *where*

$$\tilde{\delta}(n, \gamma, \delta) = \frac{1}{n+1}\left(\frac{\gamma+1}{F\left(1, 2(n+1)(1-\delta), \gamma+2, \frac{1}{2}\right)} - \gamma + n\right)$$

and this result is sharp.

LEMMA B. [4]. *Let* $h, q \in H(U)$ *be univalent in* U *and suppose* $q \in H(U)$. *If* $\psi : C^3 \to C$ *satisfies:*

a) ψ *is analytic in a domain* $D \subset C^3$

b) $(q(0), 0, 0) \in D$ *and* $\psi(q(0), 0, 0) \in h(U)$

c) $\psi(r, s, t) \notin D$ *when* $(r, s, t) \in D$, $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re}(1 + t/s) \geqslant m \operatorname{Re}(1 + \zeta q''(\zeta)/q'(\zeta))$, *where* $|\zeta| = 1$, $m \geqslant 1$, *then for all* $p \in H(U)$ *so that* $(p(z), zp'(z), z^2p''(z)) \in D$, $z \in U$ *we have:*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \;D\; p(z) \prec q(z).$$

3. Main results.

LEMMA 1. *Let* $\gamma > -1$ *and* $\delta_0 = \max\left\{\frac{n-\gamma}{n+1}, \frac{2n-\gamma}{2(n+1)}\right\} \leqslant \delta \leqslant \frac{2n-\gamma+1}{2(n+1)}$ *then* (1)

$$\delta \leqslant \tilde{\delta}(n, \gamma, \delta) = \frac{1}{n+1}\left(\frac{\gamma+1}{F\left(1, 2(n+1)(1-\delta), \gamma+2; \frac{1}{2}\right)} - \gamma + n\right).$$

Proof. The above inequality is equivalent to

$$F\left(1, 2(n+1)(1-\delta), \gamma+2; \frac{1}{2}\right) \cdot ((n+1)\delta + \gamma - n) \leqslant \gamma + 1. \quad (2)$$

Because $F\left(1, 2(n+1)(1-\delta), \gamma+2; \frac{1}{2}\right) = 1 + \sum_{k+1}^{\infty} \frac{a_k}{2^k}$ where

$a_k = \frac{b}{c}\frac{b+1}{c+1} \cdot \ldots \cdot \frac{b+k-1}{c+k-1}$ and $b = 2(n+1)(1-\delta)$, $c = \gamma + 2$ we can easily show that if $\delta_0 \leqslant \delta \leqslant \frac{2n-\gamma+1}{2(n+1)}$ then $a_k \leqslant \left(\frac{b}{c-1}\right)^k$ for all $k \in N$, hence (2) holds.

THEOREM 1. *Let* $\gamma > -1$ *and*

$$\max\left\{\frac{n-\gamma+1}{n+2}, \frac{2n-\gamma+2}{2(n+2)}\right\} \leqslant \delta \leqslant \frac{2n-\gamma+3}{2(n+2)}.$$

If $f \in AC_n(\delta)$ *related to* $g \in Z_{n+1}(\delta)$ *then* $f * b_\gamma \in AC_n(\delta)$ *related to* $g * b_\gamma \in Z_{n+1}(\delta)$.

Proof. Because $g \in Z_{n+1}(\delta)$, by using Lemma A we have $G \equiv g * b_\gamma \in Z_{n+1}(\bar{\delta}(n+1, \gamma, \delta))$ and from (1) we deduce that $G \in Z_{n+1}(\delta)$. Let $F(z) \equiv f(z) * b_\gamma(z)$ and $D^{n+1}F(z)/D^{n+1}G(z) = p(z)$, $p(0) = 1$. From the well-known formulas [6]

$$z(D^k f(z))' = (k+1)\, D^{k+1}f(z) - kD^k f(z) \tag{3}$$

$$z(D^k F(z))' = (\gamma+1)D^k f(z) - \gamma D^k f(z),\ \operatorname{Re}\gamma > -1,\ k \in N$$

we obtain

$$\frac{D^{n+1}f(z)}{D^{n+1}g(z)} = p(z) + \alpha(z)\, zp'(z), \text{ where } \alpha(z) = \frac{1}{1+\gamma}\frac{D^{n+1}G(z)}{D^{n+1}g(z)}.$$

Using again (3) and because $G \in Z_{n+1}(\delta)$ we obtain

$$\operatorname{Re}\alpha(z) = \frac{1}{1+\gamma}\left[(n+2)\ \operatorname{Re}\frac{D^{n+2}G(z)}{D^{n+1}G(z)} + \gamma - n - 1\right] > \frac{1}{F\left(1, 2(n+2)(1-\delta), \gamma+2; \frac{1}{2}\right)}$$

hence $\operatorname{Re}\alpha(z) > 0$, $z \in U$. Because $f \in AC_n(\delta)$ related to $g \in Z_{n+1}(\delta)$, then

$$p(z) + \alpha(z)\ zp'(z) \prec h_\delta(z) = \frac{1-(1-2\delta)z}{1+z}.$$

Without loss of generality we can assume that p and h satisfy the conditions of the theorem on the closed disc $\bar{U}$; if not we can replace $p(z)$ by $p_r(z) = p(rz)$ and $h_\delta(z)$ by $h_{\delta, r}(z) = h_\delta(rz)$, $0 < r < 1$, and these new functions satisfy the conditions of the theorem on $\bar{U}$. We would then prove $p_\delta(z)_{\delta, r} \prec h_\delta(z)$ for all $0 < r < 1$ and by letting $r \to 1^-$ we have $p(z) \prec h_\delta(z)$.

Let $\psi(r, s) = r + \alpha(z)s$ which is analytic in C^2 and $\psi(h_\delta(0), 0) = h_\delta(0) \in h_\delta(U)$. A simple calculus shows that

$$\operatorname{Re}\frac{\psi_0 - h_\delta(\zeta_0)}{\zeta_0 h_\delta(\zeta_0)} = m_0\ \operatorname{Re}\alpha(z) > 0,\ z \in U, \text{ where}$$

$$\psi_0 = h_\delta(\zeta_0) + \alpha(z)\, m_0 \zeta_0 h'_\delta(\zeta_0),\ m_0 \geqslant 1,\ |\zeta_0| = 1.$$

Using the fact that $\zeta_0 h'_\delta(\zeta_0)$ is an outward normal to the boundary of the convex domain $h_\delta(U)$ we conclude that $\psi_0 \notin h_\delta(U)$ and using Lemma B we have $p(z) \prec h_\delta(z)$ i.e.

$$F \in AC_n(\delta) \text{related to } G \in Z_{n+1}(\delta).$$

Taking $n = 0$ in Theorem 1 we obtain:

COROLLARY 1. *Let* $\gamma > -1$, $\max\left\{\frac{1-\gamma}{2}, \frac{2-\gamma}{4}\right\} \leqslant \delta \leqslant \frac{3-\gamma}{4}$ *and* $f, g \in$ *Then*

$$\mathrm{Re}\frac{f'(z)}{g'(z)} > \delta,\ z \in U \text{ where } \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > 2\delta - 1,\ z \in U \text{ implies}$$

$$\mathrm{Re}\frac{F'(z)}{G'(z)} > \delta,\ z \in U \text{ where } \mathrm{Re}\left(1 + \frac{zG''(z)}{G'(z)}\right) > 2\delta - 1,\ z \in U \text{ and}$$

$$F = f * b_\gamma,\ G = g * b_\gamma.$$

Remark. Taking $\delta = \frac{1}{2}$ in this corollary we obtain that if $\gamma \in [n, n+1$ and $f \in AC_n\left(\frac{1}{2}\right)$ related to $g \in Z_{n+1}\left(\frac{1}{2}\right)$ then $f * b_\gamma \in AC_n\left(\frac{1}{2}\right)$ related to g $\in Z_{n+1}\left(\frac{1}{2}\right)$. This last result holds for all $\gamma \in C$ with $\mathrm{Re}\,\gamma \geqslant \frac{n}{2}$, $n \in N$ Theorem 2].

COROLLARY 2. *If* $-1 < \gamma \leqslant 0$ *then* $f \in Z_n\left(\frac{n-\gamma}{n+1}\right)$ *implies that* $f * b_\gamma$ $\in Z_n\left(\tilde{\delta}\left(n, \gamma, \frac{n-\gamma}{n+1}\right)\right)$, *where*

$$\tilde{\delta}\left(n, \gamma, \frac{n-\gamma}{n+1}\right) = \frac{1}{(n+1)\sqrt{\pi}} \frac{\Gamma(\gamma + 3/2)}{\Gamma(\gamma + 1)} + \frac{n-\gamma}{n+1} \text{ and this result is sharp.}$$

Proof. If $-1 < \gamma \leqslant 0$ then $\max\left\{\frac{n-\gamma}{n+1}, \frac{2n-\gamma}{2(n+1)}\right\} = \frac{n-\gamma}{n+1}$ and using Lemma A for $\delta = \frac{n-\gamma}{n+1}$ we obtain our result.

Taking $\gamma = 0$ in Lemma A we obtain:

COROLLARY 3. *Let* $\frac{n}{n+1} \leqslant \delta < 1$ *and* $f \in Z_n(\delta)$; *then* $f * b_0 \in Z_n(\tilde{\delta}(n, 0, \delta))$, *where*

$$\tilde{\delta}(n, 0, \delta) = \begin{cases} \frac{1}{n+1}\left[\frac{1 - 2(n+1)(1-\delta)}{2 - 2^{2(n+1)(1-\delta)}} + n\right], & \text{for } \delta \neq \frac{2n+1}{2(n+1)} \\ \frac{1}{n+1}\left[\frac{1}{2\ln 2} + n\right], & \text{for } \delta = \frac{2n+1}{2(n+1)} \end{cases}$$

and this result is sharp.

Taking $n = 0$ in the above corollary we obtain:

COROLLARY 4. *Let* $0 \leqslant \delta < 1$ *and* $f \in A$ *with* $\mathrm{Re}\frac{zf'(z)}{f(z)} > \delta$, $z \in U$. $\mathrm{Re}\frac{zF'(z)}{F(z)} > \tilde{\delta}$, $z \in U$ *where*

$$\tilde{\delta} = \begin{cases} \dfrac{2\delta - 1}{2 - 2^{2(1-\delta)}}, & for \ \delta \neq \dfrac{1}{2} \\ \dfrac{1}{2 \ln 2}, & for \ \delta = \dfrac{1}{2} \end{cases}$$

and $F = f * b_0$ *and this result is sharp.*

COROLLARY 5. *If* $\gamma > 0$ *then* $f \in Z_n\left(\dfrac{2n-\gamma}{2(n+1)}\right)$ *implies that* $f * b_\gamma \in Z_n\left(\dfrac{2n-\gamma+1}{2(n+1)}\right)$ *and this result is sharp.*

Proof. If $\gamma > 0$ then $\max\left\{\dfrac{n-\gamma}{n+1}, \dfrac{2n-\gamma}{2(n+1)}\right\} = \dfrac{2n-\gamma}{2(n+1)}$ and using Lemma A for $\delta = \dfrac{2n-\gamma}{2(n+1)}$ we obtain the above result.

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LE CALCUL DE L'INFLUENCE DES PAROIS SUR UN ÉCOULEMENT COMPRESSIBLE ROTATOIRE

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ABSTRACT. — Calculation of the Influence of the Walls upon a Compressible Rotating Flow. The problem of the influence of the walls on a fluid flow, produced by a rotational displacement of a thin profile in the fluid mass, is envisaged. Using the Bessel functions and the equations obtained in [2] the authors provide a special technique which allows the computing of the above mentioned influence in the case of a circular wall (of a ,,channel'' type) or of a straight unlimited wall. The flow is considered plane, potential and compressible, the fluid being inviscid.

On sait que le mouvement irrotationnel d'un fluide idéal dû au déplacement dans la masse fluide d'un corps solide rigide S, de dimensions finies, rapporté au repère inertial fixe $0x, y_1 z_1$, est régissé par l'équation fondamentale [2]

$$\Delta\varphi - \frac{1}{c^2}\, v \cdot \operatorname{grad} w - \frac{1}{c^2}\,\frac{\delta w}{\delta t} = 0$$

où l'on a posé

$$W = \frac{\delta\varphi}{\delta t} + \frac{1}{2} v^2$$

Ici on a designé par φ le potentiel des vitesses, par $v = |\vec{v}|$ le module de la vitesse absolute $\vec{v}$, par $\delta/\delta t$ la dérivée partielle par rapport à t dans le système de coordonnées x_1, y_1, z_1, t, et par c la vitesse de propagation du son.

On a supposé aussi que le fluide, au repos à l'infini est assujetti à une loi de compressibilité barotropique $\rho = \rho(p)$ — réliant la densité ρ à la pression p — et que les forces massiques soient négligeables.

Si on écrit l'équation fondamentale dans les variables x, y, z, t liées au corps S en mouvement alors en désignant par $v_r(v_1^{(r)}, v_2^{(r)}, v_3^{(r)})$ la vitesse rélative au répère $Oxyz$, par $v_e(v_1^{(e)}, v_2^{(e)}, v_3^{(e)})$ la vitesse d'entrainement avec le solide S, par $\omega\ (\omega_1, \omega_2, \omega_3)$ la vitesse de rotation, on obtient avec la convention de sommation par rapport aux indices muets [2]

$$\left(\delta_{ij} - \frac{1}{c^2} v_i^{(r)} v_j^{(r)}\right) \frac{\partial^2\varphi}{\partial x_i \partial x_j} + \frac{1}{c^2}\, \varepsilon_{ijk}\, v_i^{(e)} \omega_j \frac{\partial\varphi}{\partial x_k} =$$

$$= \frac{1}{c^2}\,\frac{\partial^2\varphi}{\partial t^2} + \frac{2}{c^2} v_i^{(r)} \frac{\partial^2\varphi}{\partial x_i\, \partial t} - \frac{1}{c^2}\,\frac{\partial v_i^{(e)}}{\partial t}\,\frac{\partial\varphi}{\partial x_i}$$

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où δ_{ij} sont les composantes du tenseur de Kronecker, ε_{ijk} celles du tenseur de Ricci et $\partial/\partial t$ est l'opérateur de dérivation partielle par rapport à t dans le système de coordonnées x, y, z, t.

Si l'écoulement rélatif au répère $Oxyz$ est stationnaire et si, de plus, le solide S est animé d'une rototranslation uniforme alors, dans le cas particulier du mouvement plan-parallèle, l'équation prend la forme simplifiée [2]

$$\left[1 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial x} + \omega y\right)^2\right]\frac{\partial^2\varphi}{\partial x^2} + \left[1 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial y} - \omega x\right)^2\right]\frac{\partial^2\varphi}{\partial y^2} -$$

$$-\frac{2}{c^2}\left(\frac{\partial\varphi}{\partial x} + \omega y\right)\left(\frac{\partial\varphi}{\partial y} - \omega x\right)\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\omega^2}{c^2}\left(x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y}\right) = 0$$

où $\omega = \omega_3$ est la vitesse angulaire de rotation autour de l'axe $O_1z_1 \equiv Oz$ du solide S qui est maintenant un cylindre à génératrices parallèles à cet axe, les points O et O_1 coincidant.

A cette équation on doit ajouter la condition aux limites

$$\left.\frac{\partial\varphi}{\partial n}\right|_c = -\frac{\omega}{2}\frac{d}{ds}(x^2 + y^2)\Big|_c$$

où C est la frontière de la plaque D d'intersection du solide S avec le plan Oxy du mouvement, le vecteur $\vec{n}$ représente le vecteur unitaire de la normale à C, orientée positivement vers l'exterieur (donc vers le fluide en mouvement) et s désigne l'abscisse curviligne croissante dans le sens direct de parcours de C.

D'autre part si on introduit la fonction de courant ψ du mouvement relatif, celle-ci est reliée au potentiel des vitesses absolues φ, par le système

$$\frac{\partial\varphi}{\partial x} + \omega y = \frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\varphi}{\partial y} - \omega x = -\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x}$$

où u s'exprime d'une façon non linéaire au moyen des dérivées premières de la fonction φ.

Si on élimine la fonction φ de ce système on aboutit à l'equation aux dérivées partielles que vérifie la fonction de courant ψ du mouvement relatif

$$\left(1 - \frac{u_r^2}{c^2}\right)\frac{\partial^2\psi}{\partial x^2} - \frac{2u_rv_r}{c^2}\frac{\partial^2\psi}{\partial x\partial y} + \left(1 - \frac{v_r^2}{c^2}\right)\frac{\partial^2\psi}{\partial y^2} =$$

$$= 2\omega\left(1 - \frac{u_r^2 + v_r^2}{c^2}\right) + \frac{\omega^2}{c^2}\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right)$$

où $u_r = v_r^{(1)}$ et $v_r = v_r^{(2)}$

équation à laquelle on ajoute la condition aux limites

$$\psi|_c = 0$$

Si le mouvement fluide plan est produit par la rotation autour du poir O d'un profil mince P et peu courbé et si l'incidence par rapport à la vites d'entraînement de rotation est assez petite, l'équation fondamentale en φ pe être linearisée Précisément dans un système de coordonnées polaires – av le pôle en O et l'axe polaire Ox — l'intrados et l'extrados du profil P soie définis respectivement par $r = r_1(\theta)$, $r_2 = r_2(\theta)$, $\theta' < \theta < \theta''$ de sorte que l' ait $r_2(\theta) \geqslant r_1(\theta)$, $r_2(\theta) \approx R$, $r_1(\theta) \approx R$, où R est une constante positive. L rotation de ce profil mince produit un mouvement fluide absolu de perturb tion, comportant des vitesses assez petites, donc on peut négliger les carr et les produits de $\frac{\partial\varphi}{\partial x}$ et $\frac{\partial\varphi}{\partial y}$ ce qui conduit finalement à l'équation simpli

$$\left(1 - \frac{\omega^2 y^2}{c_0^2}\right)\frac{\partial^2\varphi}{\partial x^2} + \frac{2\omega^2 xy}{c_0^2}\frac{\partial^2\varphi}{\partial x\partial y} + \left(1 - \frac{\omega^2 x^2}{c_0^2}\right)\frac{\partial^2\varphi}{\partial y^2} +$$

$$+ \frac{\omega^2}{c_0^2}\left(x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y}\right) = 0$$

Si on transcrit cette équation en coordonnées polaires elle devient

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\left(1 - \frac{r^2\omega^2}{c_0^2}\right)\frac{\partial^2\varphi}{\partial \theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = 0$$

Alors en cherchant ses solutions, sous la forme $\varphi = \Phi(r)\,\psi(\theta)$ on about aux équations differentielles ordinaires suivantes

$$\Phi''(r) + \frac{1}{r}\Phi'(r) + K^2\left(\frac{\omega^2}{c_0^2} - \frac{1}{r^2}\right)\Phi(r) = 0$$

et

$$\psi''(\theta) + K^2\psi(\theta) = 0$$

En posant

$$r = \frac{c_0 X}{k\omega}\ (k \neq 0)$$

on trouve

$$\Phi_k = c_k^1 J_k(X) + c_k^2 Y_k(X) \text{ et}$$

$$\psi_k = \cos(k\theta + k^1)$$

où $J_k(X)$ et $Y_k(X)$ sont les functions de Bessel de première et seconde espè d'indice réel k, c_k^1, c_k^2 et k' étant des constantes.

On trouve ainsi des solutions $\varphi_k = \Phi_k\psi_k$ de l'équation considérée faisa intervenir les fontions de Bessel. Pour avoir la solution du problème mécaniq envisagé il faut essayer d'extraire de ces solutions φ_k celle qui satisfait la co dition simplifiée de glissement sur l'aile mince c'est-à-dire à

$$\frac{\partial\varphi_k}{\partial r}(R, \theta) = -\omega r_j^1(\theta),\ j = 1,2$$

qui vont sur les deux côtés de l'arc circulaire de rayon R représentant squelette du profil pour lequel $\theta' \leqslant \theta \leqslant \theta''$.

Pour résourdre effectivement ce problème on observe d'abord que pour valeurs arbitraires des constantes c_k^1 et c_k^2 le comportement à l'infini — qui exprime le repos du fluide aux grandes distances — est assuré grâce aux suivantes représentations pour les fonctions de Bessel

$$J_k(X) \approx \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(X - \frac{k\pi}{2} - \frac{\pi}{4}\right) \text{ pour } X \to \infty$$

$$Y_k(X) \approx \left(\frac{2}{\pi X}\right)^{1/2} \sin\left(X - \frac{k\pi}{2} - \frac{\pi}{4}\right) \text{ pour } X \to \infty$$

En ce qui concerne la condition de glissement sur l'extrados et l'intrados du profil P en cherchant la solution du problème sous la forme

$$\varphi = \sum_{k=1}^{\infty} [c_k^1 J_k(X) + c_k^2 Y_k(X)] \cos k\theta$$

et en considérant le développement Fourier en sinus de $r_j(\theta)$ c'est-à-dire

$$r_j^1(\theta) = \sum_{k=1}^{\infty} - k a_k^{(j)} \cos k\theta, \qquad j = 1, 2$$

le système algèbrique suivant dans les inconnues c_k^1 et c_k^2 (pour k naturel arbitraire) assurerait la solution complète du problème

$$\left| c_k^1 \frac{\partial J_k(x)}{\partial x} + c_k^2 \frac{\partial Y_k(x)}{\partial x} \right|_{r = R^{(j)}} = c_0 a_k^{(j)}, \quad j = 1, 2$$

Ici on a désigné par $R^{(1)}$ et $R^{(2)}$ la distance minimale, respectivement maximale, entre les points de l'intrados et de l'extrados du profil P et le point fixe O.

Supposons maintenant que l'écoulement fluide produit par la rotation du profil P ait lieu dans un tuyau circulaire fixe dont la séction, dans le plan d'écoulement est donnée par la circonférence $x_1^2 + y_1^2 = R_1^2$. Dans les points de cette circonférence nous aurons la condition de glissement suivante qui remplace, dans ce cas, la condition de comportement à l'infini

$$\vec{v}_a \vec{n} \Big|_{r = R_1} = \frac{\partial \varphi}{\partial r}(R_1, \theta) = 0$$

En cherchant de nouveau la solution du problème sous la forme

$$\varphi = \sum_{k=1}^{\infty} [c_k^1 J_k(X) + c_k^2 Y_k(X)] \cos k\theta$$

la condition d'au-dessus s'exprime par une rélation de dépendence, pour quelque k naturel, entre c_k^1 et c_k^2 Si on accepte que R_1 est suffisament **grand** pour que le developpements assymptotiques des $J_k(X)$ et $Y_k(X)$ aient lieu, cet rélation revient à

$$\operatorname{tg}\left(\frac{k\omega R_1}{c_0}-\frac{k\pi}{2}-\frac{\pi}{4}\right)\approx\frac{c_k^2\left(\frac{2c_0}{\pi k\omega R_1}\right)^{1/2}-c_k^1\cdot\frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{c_0}{k\omega R_1}\right)^{3/2}}{c_k^1\left(\frac{2c_0}{\pi k\omega R_1}\right)^{1/2}+c_k^2\cdot\frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{c_0}{k\omega R_1}\right)^{3/2}}.$$

En fin en assimilant l'extrados et l'intrados du profil P avec le bord supérieur, respectivement inférieur, de l'arc $\frac{r_1(\theta)+r_2(\theta)}{2}$, la condition de glissement sur ce profil revient à

$$\left[c_k^1\frac{\partial J_k(X)}{\partial x}+c_k^2\frac{\partial Y_k(X)}{\partial x}\right|_{r=R}=c_0a_k$$

où a_k sont les coefficients du développement Fourier en sinus de la fonction donnée $\frac{r_1(\theta)+r_2(\theta)}{2}$ c'est-à-dire

$$\frac{r_1'(\theta)+r_2'(\theta)}{2}=-\sum_{k=1}^{\infty}-ka_k\cos k\theta$$

La dernière condition, tout ensemble avec celle d'ou-dessus, détermineront univoquement les coefficients c_k^1 et c_k^1 de la solution du problème.

Considerons maintenent le cas quand l'écoulement fluide produit par rotation du profil P ait lieu en présence d'une paroi rectiligne illimitée (supposée parallèle à l'axe fixe Ox_1) d'équation $y_1=-y_0$. En remarquant que l'équation de la paroi peut s'écrire encore, par rapport au repère mobile xy sous la forme

$$x\sin\alpha+y\cos\alpha+y_0=0\quad\text{ou bien}$$

$r=-\frac{y_0}{\sin(\theta+\alpha)}$ $(\alpha=\widehat{x_1Ox})$, la condition de glissement sur lui devient

$$0=\vec{v}_a\,\vec{n}\Big|_\Gamma=\frac{\partial\varphi}{\partial y_1}\Big|_\Gamma=\frac{\partial\varphi}{\partial x}\sin\alpha+\frac{\partial\varphi}{\partial y}\cos\alpha\Big|_\Gamma=$$

$$=\left(\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)\sin\alpha+\left(\sin\theta\frac{\partial\varphi}{\partial r}+\cos\theta\frac{\partial\varphi}{\partial t}\right)\cos\alpha\Big|_\Gamma$$

c'est-à-dire

$$\frac{\partial\varphi}{\partial r}\sin(\theta+\alpha)+\frac{1}{r}\cos(\theta+\alpha)\Big|_{r=-\frac{y_0}{\sin(\theta+\alpha)}}=0$$

Cherchant de nouveau la solution du problème sous la forme

$$\varphi=\sum_{k=1}^{\infty}[c_k^1J_k(X)+c_k^2Y_k(X)]\cos k\theta$$

cette cdition cononduit à son tour à une rélation de dépendance entre les constantes c_k^1 et c_k^2, plus précisément il faut satisfaire

$$\sum_{k=1}^{\infty} \frac{\omega}{c_0} \left[c_k^1 \frac{\partial J_k(X)}{\partial X} + c_k^2 \frac{\partial Y_k(X)}{\partial X} \right]_{r=-\frac{Y_0}{\sin(\theta+\alpha)}} \cos k\theta =$$

$$= - \sum_{k=1}^{\infty} \frac{Y_0}{\sin(\theta+\alpha)} \left[c_k^1 J_k(x) + c_k^2 Y_k(x) \right]_{r=-\frac{y_0}{\sin(\theta+\alpha)}} \sin k\theta$$

Mais, en utilisant les formules de récurrence pour les dérivées des fonctions de Bessel on obtient

$$\sum_{k=1}^{\infty} \frac{\omega}{2c_0} \{ c_k^1 [J_{k-1}(x) - J_{k+1}(x)] + c_k^2 [Y_{k-1}(x) - Y_{k+1}(x)] \} \cdot \cos k\theta =$$

$$= - \sum_{k=1}^{\infty} \frac{\cos(\theta+\alpha)}{y_0} \left[c_k^1 J_k(x) + c_k^2 Y_k(x) \right] \sin k\theta$$

Ou bien

$$\frac{\omega}{2c_0} \left[c_1^1 J_0(x) \cos\theta + c_1^1 \frac{\cos(\theta+\alpha)}{y_0} J_1(x) \sin\theta \right] +$$

$$+ \frac{\omega}{2c_0} \left[c_1^2 Y_0(x) \cos\theta + c_1^2 \frac{\cos(\theta+\alpha)}{y_0} Y_1(x) \sin\theta \right] +$$

$$+ \sum_{k=2}^{\infty} \left\{ J_k(x) \left[(c_{k+1}^1 - c_{k-1}^1) \frac{\omega}{2c_0} \cos k\theta + c_k^1 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta \right] + \right.$$

$$\left. + Y_k(x) \left[(c_{k+1}^2 - c_{k-1}^2) \frac{\omega}{2c_0} \cos k\theta + c_k^2 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta \right]_{r=-\frac{y_0}{\sin(\theta+\alpha)}} \right\}$$

Choisisant alors $c_2^1 = c_1^2 = 0$ et approximant $Y_k(x)$ par $J_k(x) \operatorname{tg}\left(x - \frac{k\pi}{2} - \frac{\pi}{4}\right)$ — forme inspirée par le comportement assymptotique des $J_k(x)$ et $Y_k(x)$ — on aboutit à l'accomplissement tout au long de la paroi $r = \frac{y_0}{\sin(\theta+\alpha)}$, de la rélation

$$(c_{k+1}^1 - c_{k-1}^1) \frac{\omega}{2c_0} \cos k\theta + c_k^1 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta +$$

$$+ \operatorname{tg}\left(- \frac{k\omega}{c_0} \frac{y_0}{\sin(\theta+\alpha)} - \frac{k\pi}{2} - \frac{\pi}{4} \right) \left[(c_{k+1}^2 - c_{k-1}^2) \frac{\omega}{2c_0} \cos k\theta + c_k^2 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta \right] = 0$$

(pour $k = 2, 3, \ldots$)

Mais parce qu'au voisinage de $\theta = -\frac{\pi}{2}$ la condition de glissement sur la paroi est pratiquement satisfaite (conséquence du caractère rotatoire de l'écoulement) et pour $\theta \notin V\left(-\frac{\theta}{2}\right)$ étant justifiées les approximations $\cos(\theta + \alpha)$ $\approx \cos\alpha$ et $\sin(\theta + \alpha) \approx \sin\alpha$ on est conduit, dans ce cas, à la rélation s vante entre coefficients

$$(c^1_{k+1} - c^1_{k-1}) + \mathrm{tg}\left(-\frac{k\omega y_0}{c_0 \sin\alpha} - \frac{k\pi}{2} - \frac{\pi}{4}\right)(c^2_{k+1} - c^2_{k-1}) = 0$$

Evidemment à cette dernière condition pour les coefficients c^j_k on d ajouter la rélation écrite au—dessus (entre les mêmes coefficients et qui expri le glissement du fluide sur le contour du profil), ce qui détermine complèteme le problème.

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ABELIAN GROUPS WITH PSEUDOCOMPLEMENTED LATTICE OF SUBGROUPS

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ABSTRACT. — In this paper we prove that for the lattice of the subgroups of an abelian group pseudocomplementation and distributivity are equivalent. We also characterize abelian groups which have a Stone lattice or a Heyting algebra of subgroups.

Let L be a lattice with zero and $0 \neq a \in L$. If $C_a = \{x \in L / a \wedge x = 0\}$, the greatest element of C_a (if it exists) is called the pseudocomplement of a in L. (Note that the "pseudocomplement" is differently used for an unspecified maximal element of C_a). If every element in L has a pseudocomplement, L is called a pseudocomplemented lattice.

We first recall the following known facts:

(A) Every distributive compactly generated lattice is pseudocomplemented.

(B) If A is an abelian group, the lattice $L(A)$ of all the subgroups of A is compactly generated.

1. LEMMA. *Let P be an inductive poset. The following conditions are equivalent:* (i) *P has a unique maximal element*; (ii) *P has a greatest element.*

Indeed, if m is the unique maximal element of P and $a \in P$ then $P_a = \{x \in P / a \leqslant x\}$ is inductive and has (by Zorn's lemma) maximal elements which are also maximal in P. So $a \leqslant m$. The converse is obvious.

2. COROLLARY. *Let L be an upper continuous lattice. The following conditions are equivalent:* (i) *C_a has a unique maximal element*; (ii) *C_a has a greatest element.*

Indeed, in an upper continuous lattice, C_a is inductive.

The key result for our paper, from [5] is the following:

(C) Let $B \neq 0$ be a subgroup of an abelian group A. There is a unique B-high subgroup if and only if A/B is a torsion group and for each prime p either $B[p] = A[p]$ or $B[p] = 0$ holds.

3. COROLLARY. *Let $B \neq 0$ be a subgroup of an abelian group A. The following conditions are equivalent:* (i) *B has a pseudocomplement in $L(A)$*; (ii) *there is only one B-high subgroup in A*; (iii) *A/B is a torsion group and for every prime p either $B[p] = A[p]$ or $B[p] = 0$ holds.*

4. PROPOSITION. *For an abelian group A the following conditions are equivalent:* (a) *every nontrivial quotient group of A is a torsion group*; (b) *A is either a torsion group or a torsion-free group of rank* 1.

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Proof. Clearly no mixed group has the property (a): if $0 \neq T(A) \neq A$ then $A/T(A)$ is torsion-free. Obviously every torsion group has the property (a). Now, if A is torsion-free of rank $r_0(A) \geqslant 2$ and $0 \neq a \in A$ then $r_0(A/\langle a \rangle) \geqslant 1$ so that $A/\langle a \rangle$ is not torsion. Finally, if A is torsion-free of rank 1, A has the property (a), as every rational group has it.

5. PROPOSITION. *For a torsion group A the following conditions are equivalent:* (c) *for each subgroup B of A and each p prime either $B[p] = A[p]$ or $B[p] = 0$ holds*; (d) *A is a direct sum of cocyclic groups corresponding to different primes.*

Proof. We can obviously reduce our problem to p-groups. But $B[p] = 0$ iff if and only if $B = 0$ so that only the case $B[p] = A[p]$ needs care. If A is a p-group such that $B[p] = A[p]$ holds for each subgroup $B \neq 0$ of A then $A[p] = S(A)$ (the socle) is contained in every nonzero subgroup B of A. In this case, having a smallest nonzero subgroup, A is cocyclic. The converse is obvious.

6. COROLLARY. *If A is an abelian group, the lattice $L(A)$ is pseudocomplemented if and only if A is either a direct sum of cocyclic groups corresponding to different primes or a torsion-free group of rank* 1.

Proof. Using 3, 4 and 5 we only need to observe that (c) is trivially true for torsion-free groups.

7. THEOREM. *For an abelian group A the following conditions are equivalent:* (i) *$L(A)$ is a distributive lattice*; (ii) *$L(A)$ is a pseudocomplemented lattice*; (iii) *A is a locally cyclic group*; (iv) $r_0(A) + \max\limits_p r_p(A) \leqslant 1$; (v) *$A$ is either a direct sum of cocyclic groups corresponding to different primes or a torsion-free group of rank* 1.

Proof. One can use [3, p 86, ex. 5] and [2, p 301, T. 78.2]. The rest is done by the previous corollary.

A pseudocomplemented distributive lattice is called a Stone lattice if $a^* \vee a^{**} = 1$, where a^* denotes the pseudocomplement of a in L. If B is a subgroup of A such that A/B is a torsion group, let π be the set of all the primes such that $B[p] = 0$ holds and $B[p] = A[p]$ holds for $p \notin \pi$. Using proposition 2 and 3 from [5] we have $B^* = \bigoplus\limits_{p \in \pi} (T(A))_p$ and $B^{**} = \bigoplus\limits_{p \notin \pi} (T(A))_p$ so that $B^* + B^{**} = T(A)$. Hence only the torsion groups from 7 have Stone lattices of subgroups:

8. PROPOSITION. *For an abelian group A the following conditions are equivalent:* (i) *$L(A)$ is a Stone lattice*; (ii) *A is a direct sum of cocyclic groups corresponding to different primes.*

A lattice with zero is called a Heyting algebra (or a relative pseudocomplemented lattice) if for every $a, b \in L$ the subset $\{x \in L / a \wedge x \leqslant b\}$ has a greatest element denoted $a * b$.

We finally mention the following characterization [1]: (D) A bounded lattice L is a Heyting algebra if and only if L is distributive and for each $b \in L$ the sublattice $1/b = \{x \in L / b \leqslant x\}$ is pseudocomplemented.

The pseudocomplementation and the distributivity of the lattice of all the subgroups of an abelian group being equivalent it immediately follows that

$L(A)$ is a Heyting algebra if and only if $L(A)$ is distributive (any sublattice of a distributive lattice is distributive too).

Remark. The characterization of the class of all abelian groups which have the lattice $L(A)$ a Boole algebra is an easy consequence of 8 (cf. 2, p. 302, Cor. 78.5).

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CRITICAL RADII AND MAXIMUM MASSES OF RELATIVISTIC STEPENARS

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ABSTRACT. — For the relativistic stepenars, the critical radii and the maximum masses are computed. When the index of the stepenar varies in the range $0 \leqslant \alpha \leqslant 10$, the critical radius and the maximum mass vary, respectively, in the intervals $1.125 \leqslant R_{min}/R_g \leqslant 4.658$; $0.955 \leqslant M_{axm}/M_{\odot} \leqslant 8.046$, depending on the values assumed for the non-dimensional central pressure. The obtained results are given in tables and plotted on graphs.

1. Introduction. In the newtonian theory of stellar structure, the class of stellar models with the distribution of the density as a power law, having the form

$$\rho = \rho_c(1 - r/R)^{\alpha}, \quad \alpha \geqslant 0, \tag{1}$$

where the notations are the usual ones, was introduced by Huseynov and Kasumov (1972). They named these models stepenars or pseudo-polytropes.

The relativistic stellar models with the distribution of the mass-energy density having the form (1) were firstly studied in our papers (Ureche, 1983 a, 1983 b). These models have been named relativistic stepenars.

2. Main Properties of Relativistic Stepenars. If we introduce the non-dimensional variables (see Ureche, 1980 a), the distribution of the density (1) takes the form

$$\psi = (1 - \eta/\eta_s)^{\alpha}, \quad \alpha \geqslant 0, \tag{2}$$

where η_s is the non-dimensional radius of the star. With the change of variable $\eta = \eta_s y$ and using the non-dimensional form of the equations of relativistic stellar equilibrium from the last cited paper, we obtain the main properties of the relativistic stepenars, namely:

— The non-dimensional mass distribution is given by

$$m(y) = \frac{\eta_s^3}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} f(y), \tag{3}$$

where

$$f(y) = 2 - (1 - y)^{\alpha+1}[(\alpha + 1)(\alpha + 2)y^2 + 2(\alpha \mp 1)y + 2], \tag{4}$$

the total mass of the relativistic stellar configuration having the expression

$$m_s \equiv m(1) = \frac{2\eta_s^3}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} \tag{5}$$

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— The degree of the concentration of the matter (-energy) towards the centre of the relativistic stellar configuration is given by the ratio between the central density ρ_c and the mean density $\bar{\rho}$, that is

$$\rho_c/\bar{\rho} = (\alpha + 1)(\alpha + 2)(\alpha + 3)/6 \tag{6}$$

— The scale factors from the change of variables are given by

$$a = R\sqrt{\frac{4}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} \cdot \frac{R}{R_g}}, \tag{7}$$

$$M^* = 2M\sqrt{\frac{4}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} \cdot \frac{R^3}{R_g^3}}, \tag{8}$$

where

$$R_g = 2GM/c^2 \tag{9}$$

is the gravitational (Schwarzschild) radius of the relativistic configuration.

— The radius and the mass of the configuration have respectively the expressions

$$R = \frac{(\alpha + 1)(\alpha + 2)(\alpha + 3)}{4\eta_s^2} R_g, \tag{10}$$

$$M = \frac{c^3}{\pi^{1/2}G^{3/2}(\alpha + 1)(\alpha + 2)(\alpha + 3)} \cdot \frac{\eta_s^3}{\rho_c^{1/2}}. \tag{11}$$

3. Critical Radii of Relativistic Stepenars. For the distribution of the mass of stellar model, the exact solution (3) was obtained, while the distribution of the pressure results from the numerical solution of the differential problem

$$\frac{dp}{dy} = -\eta_s^2 \frac{[p + (1 - y)\alpha]\,[f(y) + (\alpha + 1)(\alpha + 2)(\alpha + 3)y^3p]}{(\alpha + 1)(\alpha + 2)(\alpha + 3)y^2 - 2\eta_s^2 yf(y)} \tag{12}$$

$$p(1) = 0, \quad \eta_s^2 < (\alpha + 1)(\alpha + 2)(\alpha + 3)/4,$$

where the function $f(y)$ is given by the expression (4).

In a previous paper (Ureche, Oproiu, Imbroane, 1985) a numerical analysis of the differential equation (12) was performed. So, for different values of the parameters α (the index of the stepenar) and η_s^2, the tables of the function $p(y)$ were obtained. Here we shall concentrate our attention on those models in which the non-dimensional central pressure takes the values $p_c = 1/3$, $p_c = 1$ and $p_c = \infty$. In Table 1, for the different values of the index of the stepenar α, the values of the parameter η_s^2, corresponding to $p_c = 1/3$, $p_c = 1$ and $p_c = \infty$, are given.

The values of η_s^2 given in Table 1 are the maximum values of this parameter, for $p_c = 1/3$ (classical constraint of GRT), $p_c = 1$ (causal law) and $p_c = \infty$ (absolute limit, which does not depend on the equation of state). Let

Table 1

α	η_s^2		
	$p_c = 1/3$	$p_c = 1$	$p_c = \infty$
0	5/6	9/8	4/3
1	3.2491	4.0289	4.495
2	7.186	8.654	9.46
3	12.57	14.91	16.25
4	19.48	22.89	24.86
5	27.88	32.56	35.27
6	37.75	43.92	47.48
7	49.12	56.97	61.49
8	61.96	71.70	77.31
9	76.29	88.13	95.76
10	92.10	106.25	114.36

η_s^{*2} be one of the values given in Table 1. From (10) for the corresponding minimum radius of the configuration we obtain

$$R_{\min} = \frac{(\alpha + 1)(\alpha + 2)(\alpha + 3)}{4\eta_s^{*2}} R_g. \quad (13)$$

So, using the values in Table 1 with the expression (13) we have computed the minimum radii of relativistic stepenars, that is the critical radii at which the gravitational collapse is inevitable. The obtained results, expressed in terms of the gravitational radius of the configuration, are listed in Table 2.

For the three values of p_c, the quantity $R_{\min}/R_g$ is plotted in Figure 1 as function of α. From Table 2 and Figure 1 one can observe that the critical radii increase with the index of stepenar α. The equation (6) points out the fact that the degree of the concentration of matter towards the centre of configuration also increase with α. Therefore the critical radii increase with the

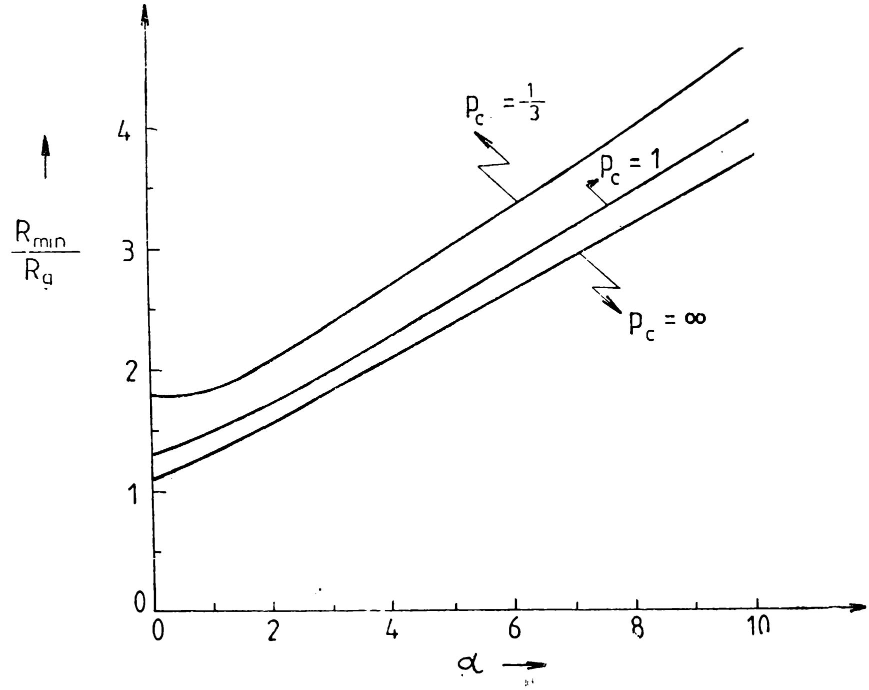

Fig. 1

Table 2

Critical radii of the relativistic stepenars

α	$R_{\min}/R_g$		
	$p_c = 1/3$	$p_c = 1$	$p_c = \infty$
0	1.800	1.333	1.125
1	1.847	1.489	1.335
2	2.087	1.733	1.586
3	2.387	2.012	1.846
4	2.695	2.294	2.112
5	3.013	2.580	2.382
6	3.338	2.869	2.654
7	3.664	3.160	2.927
8	3.995	3.452	3.201
9	4.326	3.744	3.446
10	4.658	4.038	3.751

Table 3

Maximum masses of the relativistic stepenars

α	$M_{\max}/M_{\odot}$		
	$p_c = 1/3$	$p_c = 1$	$p_c = \infty$
0	3.976	6.236	8.046
1	3.826	5.283	6.226
2	3.184	4.207	4.809
3	2.604	3.364	3.828
4	2.170	2.764	3.123
5	1.836	2.317	2.612
6	1.574	1.976	2.221
7	1.369	1.710	1.917
8	1.203	1.497	1.676
9	1.067	1.325	1.501
10	0.955	1.183	1.321

increasing of the degree of the concentration of matter (-energy) towards the centre of the relativistic star. An interesting problem would be the study of the asymptotical behaviour of the quantity $R_{\min}/R_g$ for $\alpha \rightarrow \infty$.

4. Maximum Masses of Relativistic Stepenars. From the equations (6) and (11) for the maximum mass of a relativistic stepenar we obtained the expression:

$$M_{\max} = \frac{6^{1/2} c^3}{\pi^{1/2} G^{3/2} (\alpha + 1)^{3/2} (\alpha + 2)^{3/2} (\alpha + 3)^{3/2}} \cdot \frac{\eta_s^{*3}}{\bar{\rho}^{1/2}} \tag{14}$$

With this expression, using the values of the parameter η_s^* from Table 1, we have computed the maximum masses of the relativistic stepenars. For this purpose we took $\bar{\rho} = \rho_{\rm nuc} = 2 \cdot 10^{17}$ kg/m^3 (Brecher, Caporaso, 1977). The computations were performed for the same three values: $p_c = 1/3$, $p_c = 1$ and $p_c = \infty$. So, we obtained the maximum masses of relativistic stepenars. These ones are the limiting masses for the considered models. Over these masses the gravitational collapse is inevitable. The obtained results, expressed in solar masses, are given in Table 3.

The ratio $M_{\max}/M_{\odot}$ is plotted in Figure 2, as function of α, for the considered values of p_c. From Table 3 and Figure 2 one can observe that the maximum (critical) masses, called Oppenheimer-Volkoff limiting masses (Zeldovich, Novikov, 1971) decrease with the increasing of the degree of the concentration of matter (-energy) towards the centre of the relativistic star. An interesting problem would also be the study of the asymptotical behaviour of the quantity $M_{\max}/M_{\odot}$ for $\alpha \rightarrow \infty$.

We note that the results obtained here for $\alpha = 0$ (homogeneous model) and $\alpha = 1$ (linear model) are in agreement with those given in the previous papers (Ureche, 1980 a, b; 1982).

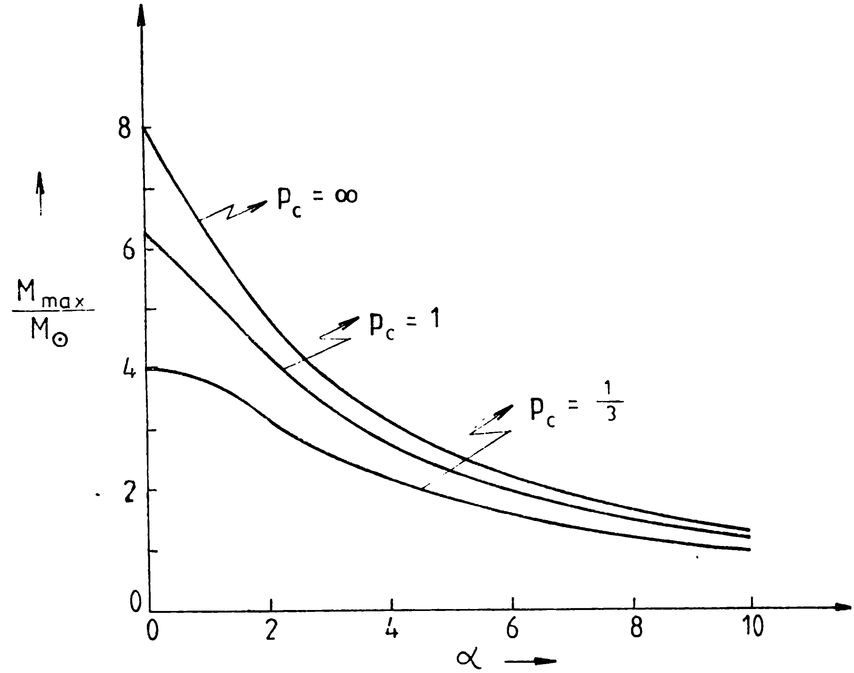

Figure 2

We conclude that the obtained results are equivalent with the following criteria of stability for the relativistic stellar configuration with the power law density distribution (relativistic stepenars)

$$R > R_{min}, \qquad M < M_{max}. \tag{15}$$

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CARACTERISATION DES FONCTIONS CONVEXES A L'AIDE DES OPERATEURS CONVOLUTIFS POSITIFS

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RÉSUMÉ. — Nous allons faire référence a quelque résultats particuliers concernant la caracterisation de fonctions convexes a l'aide de certains opérateurs linéaires et positifs.

DEFINITION — *Soit* $f: [a, b] \to \mathbf{R}$. *Si* $\forall x_1, x_2 \in [a, b]$ *et* $\forall \alpha_1 > 0,\ \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1$, *on a*

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

alors on dit que fonction f est convexe.

Nous allons noter par $[x_1, x_2, x_3; f]$ la différence divisée du deuxiéme ordre de la fontion f.

LEMMA. *Soit* $f \in C[a, b]$. *Une condition nécessaire et suffisante pour que f soit convexe est que :*

$$[x_1, x_2, x_3; f] \geqslant 0 \quad \forall x_1, x_2, x_3 \in [a, b]$$

La démonstration de ce lemma on peut voir, par exemple, dans [6].

Considération les opérateurs de Bernstein

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right),\ f \in C[0, 1],\ x \in [0, 1]$$

On a

THÉORÈME. *Soit* $f \in C[0, 1]$. *Une condition nécessaire et suffisante pour que f soit convexe est :*

$$B_n\left(f; \frac{i}{n}\right) \geqslant f\left(\frac{i}{n}\right),\ i = 0, 1, \ldots, n;\ n = 1, 2, \ldots \tag{1}$$

DEMONSTRATION. Ce théoréme est donné en [1]. Nous allons indiquer une démonsstration simple du théorème.

On a [4]

$$B_n(f; x) - f(x) = \frac{x(1 - x)}{n} \sum^{n-1} p_{n-1, k}(x) \left[x, \frac{k}{n}, \frac{k+1}{n}; f\right], \quad x \in [0, 1] \tag{2}$$

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où

$$p_{n,k}(x) = \binom{k}{n} x^k (1-x)^{n-k}$$

Si f est convexe, d'après le lemme ci-dessus et le fait que $p_{n-1,k}(x) \geqslant 0$, $k = 0, 1, \ldots, n$; $n = 1, 2, \ldots$ il résulte la relation (1) du théorème.

Supposons maintenant que (1) soit vérifiée et montrons que f est convexe. Supposons le contraine, donc

$$\exists x_1, x_2, x_3 \in [0, 1] \text{ tels que } [x_1, x_2, x_3; f] = c < 0.$$

La différence entre

$$\sum_{k=0}^{n-1} p_{n-1,k}(x) \left[x, \frac{k}{n}, \frac{k+1}{n}; f\right] \text{ et } [x_1, x_2, x_3; f]$$

peut être redue aussi petite que l'on veut pour un n suffismment grand dans un point appartenant à [0, 1], correspondant à x_1, x_2, x_3. Il résulte que la somme respective peut être négative ce qui est en contradiction avec (1), donc f est convexe.

Pour les opérateurs de Stancu [5]

$$P_n^{[\alpha]}(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \cdot \frac{x(x+\alpha)\ldots(x+\overline{k-1}\alpha)(1-x)(1-x+\alpha)\ldots(1-x+\overline{n-k-1}\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\alpha)} \quad (3)$$

$$\alpha \in \mathbf{R},\ f \in C[0, 1]$$

on a une formule semblable à (2) et on peut démontrer de la même façon le théorème suivant :

THÉORÈME. *Soit $f \in C[0, 1]$ et $P_n^{[\alpha]}$, $\alpha > 0$ les opérateurs* (3) *associe à cette fonction. Une condition nécessaire et suffisante pour que f soit convexe est que :*

$$P_n^{[\alpha]}\left(f; \frac{i}{n}\right) \geqslant f\left(\frac{i}{n}\right), \quad i = 0, 1, \ldots, n;\ n = 1, 2, \ldots$$

Pour les opérateurs de Bernstein on a aussi :

THÉORÈME. *Une condition nécessaire et suffisante pour qu'une fonction continue f soit convexe sur* [0, 1] *est que la suite $\{B_n(f; x)\}$, $n = 1, 2, \ldots$; $x \in [0, 1]$ soit non croissante.*

La démonstration de ce théorème est donné en [2].

Pour la démonstration on utilise la relation suivante, satisfaite par les polinomes de Bernstein.

$$B_n(f; x) - B_{n+1}(f; x) = \frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[x, \frac{k}{n}, \frac{k+1}{n+1}; f\right] \quad (4)$$

Pour les opérateurs $P_n^{[\alpha]}$ on a aussi :

THÉORÈME. *Une condition nécessaire et suffisante pour qu'une fonction continue f soit convexe sur* [0, 1] *est que la suite* $\{P_n^{[\alpha]}(f; x)\}$, $n = 1, 2, \ldots$; $x \in [0, 1]$ *soit non croissante.*

Remarque. On a les mêmes résultats pour les opérateurs $B_n^{[s]}$

$$B_n^{[s]}(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[s(x)]^k\,[1 - s(x)]^{n-k}$$

où $s(x)$ satisfaire des conditions que assurent leur convergence uniforme vers f.

Considérons maintenant les opérateurs convolutifs positifs de type binomial de la forme :

$$L_n(f; x) = A_n(x)\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} P_k(u(x))P_{n-k}(v(x)) \tag{5}$$

$$A_n(x) = [P_n\,(u(x) + v(x))]^{-1} < \infty.$$

Hypothese :

Supposons que les fonctions $a_k : [0, 1] \to \mathbf{R}$

$$b_{n-k} : [0, 1] \to \mathbf{R}$$

$$a_k(x) = \frac{P_n(u(x) + v(x))}{P_{k+1}(u(x) + v(x))} \cdot \frac{P_{k+1}(u(x))}{P_k(u(x))}$$

$$b_{n-k}(x) = \frac{P_n(u(x) + v(x))}{P_{n+1}(u(x) + v(x))} \cdot \frac{P_{n-k+1}(v(x))}{P_{n-k}(v(x))}$$

$$k = 0, 1, \ldots, n;\; n = 1, 2, \ldots$$

satisfassent la relation $1 = a_k(x) + b_{n-k}(x)$

THÉORÈME. a) *Dans l'hypothèse ci-dessus pour les opérateurs convolutifs* (4) *on a :*

$$L_n(f; X) - L_{n+1}(f; x) = \frac{A_{n+1}(x)}{n(n+1)}\sum_{k=0}^{n-1}\binom{n+1}{k} P_{k+1}(u(x)).$$

$$P_{n-k}(v(x))\left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n};\, f\right]$$

$$n = 1, 2, \ldots$$

b) Si $f: [0, 1] \to \mathbf{R}$ *est convexe alors la suite* $\{L_n\}$, $n = 1, 2, \ldots$ *est croissante.*

DEMONSTRATION a) On a :

$$L_{n+1}f; x) = A_{n+1}(x)\sum_{k=1}^{n}\binom{n+1}{k} f\left(\frac{k}{u+1}\right) P_k(u(x))P_{n-k+1}(v(x)) +$$

$$+ A_{n+1}(x)f(0)P_{n+1}(v(x)) + A_{n+1}(x)f(1)P_{n+1}(u(x))$$

En utilisant la relation $1 = a_k(x) + b_{n-k}(x)$ on peut écrire :

$$L_n(f; x) = A_n(x)\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) a_k(x)P_k(u(x))P_{n-k}(v(x)) +$$

$$+ A_n(x)\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) P_k(u(x))P_{n-k}(v(x))b_{n-k}(x) =$$

$$= A_n(x)\sum_{k=1}^{n+1}\binom{n}{k-1} f\left(\frac{k-1}{n}\right) a_{k-1}(x)P_{k-1}(u(x))P_{n-k+1}(v(x)) +$$

$$+ A_n(x)\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) P_k(u(x))P_{n-k}(v(x))b_{n-k}(x)$$

En tenant compte des valeurs fonctions a_k et b_{n-k} données dans l'hypothèse obtient

$$L_{n+1}(f; x) - L_n(f; x) = A_{n+1}(x)\sum_{k=1}^{n}\left[\binom{n+1}{k} f\left(\frac{k}{n+1}\right) - \binom{n}{k-1} f\left(\frac{k-1}{n}\right) - \right.$$

$$\left. - \binom{n}{k} f\left(\frac{k}{n}\right)\right] \cdot P_k(u(x))P_{n-k+1}(v(x))$$

De là il résulte la relation donnée.

b) La dèmonstration rèsulte de a)

EXEMPLES

1° Si $L_n = B_n$ (polynôme Bernstein) alors $u(x) = x$, $v(x) = 1-$ $P_k(u(x)) = x^k$, $A_n(x) = 1$ et l'hypothèse ci-dessus on a $a_k(x) = x$, $b_{n-k}(x)$ $= 1 - x$ et donc évidemmnet $1 = a_k(x) + b_{n-k}(x)$.

La relation du théorème devient alors (4).

2° Si $L_n = P_n^{[\alpha]}$ l'opérateur Stancu alors

$$P_k(u(x)) = x(x + \alpha) \ldots (x - (k-1)\alpha)$$

$$P_k(u(x) + v(x)) = (1 + \alpha)(1 + 2\alpha) \ldots (1 + (n-1)\alpha)$$

et on obtient :

$$a_k(x) = \frac{1}{1 + n\alpha}(x + k\alpha), \quad b_{n-k}(x) = \frac{1}{1 + n\alpha}(1 - x + (n-k)\alpha)$$

pour lesquels $1 = a_k(x) + b_{n-k}(x)$.

La relation du théorème devient alors une relation semblable à (4) p les opérateurs $P_n^{[\alpha]}$, [5].

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LOOP-EXIT SCHEMES AND GRAMMARS; PROPERTIES, FLOWCHARTABLIES

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ABSTRACT. — Some properties for the Loop-Exit grammars and an algorithm for construction of one flowchart for one Loop-Exit scheme are presented in the paper.

1. Introduction. The flowcharts is a traditional tool for the algorithm description. Recently, the Loop-Exit schemes [2, 6] have been are used for the algorithm description, too. Some programming languages such as BLISS [7], Ada [5 and some Pascal implementation [8], used for flowcontrol statements of Loop-Exit type. In this paper, some properties for the Loop-Exit grammars [3, 4 are presented. Also, an elegant algorithm for construction of one flowchart for one Loop-Exit scheme is described.

2. The definition of a Loop-Exit Scheme. Let $\Sigma = \mathbf{AM} \cup \mathbf{TM}$ be a terminal alphabet where **AM** and **TM** are the sets of *assignment and test mark* respectively let

RES = {"+", " –", ";", NULL, IF, THEN, ELSE, ENDIF, LOOP, ENDLOOP, EXIT} be a set of some *reserved symbols* and let $\mathbf{LM} = \{i_1, i_2, \ldots, i_l\}$ be a set of *loop-marked symbols*. Usually, when there is not confusion, we assume that $\mathbf{LM} = \{1, 2, \ldots, l\}$. Suppose that $\mathbf{RES} \cap (\Sigma \cup \mathbf{LM}) = \emptyset$.

Definition 1. A *Loop-Exit-Free Scheme* (LEFS) *over* Σ *is recursively defined as follows:*

a) "NULL; "*is a* LEFS. *For each* $a \in \mathbf{AM}$, "a;" *is a* LEFS.

b) *If* $t \in TM$, α *and* β *are* LEFS *and* i, j, k $\in$ **LM**, *then the followings are* LEFS:

b1) „$\alpha\beta$"

b2) „IF t THEN α[EXIT$_i$;] [ELSE β[EXIT$_j$;]]ENDIF;"

where [δ] *means that* δ *is optional.*

b3) „LOOP$_k$ α ENDLOOP$_k$;"

c) *Each* LEFS *is obtained from a and b rules which satisfies:*

c1) *each two* LOOPs *must have two distinct loop-mark symbols from* **LM**

c2) *for each* LOOP$_k$ α ENDLOOP$_k$; α *has at least an* EXIT$_k$ *in it;*

c3) *for each* EXIT$_k$, *if* LOOP$_k$ α ENDLOOP$_k$; *is in* LEFS, *then*

$\alpha = \alpha'$ EXIT$_k$; α''

Definition 2. *A Loop-Exit Scheme* (LES) *is a* LEFS *such that:*

c3') *for each* EXIT$_k$ *there is* LOOP$_k$ α' EXIT$_k$; α'' ENDLOOP$_k$ *into* LEFS.

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Let S be a LES. All the symbols IF of S will be indexed by **1, 2, ...** For each IF_i, the corresponding symbols THEN, ELSE — if this exists — and ENDIF will be indexed by the same index i. In the LES from the examples below we have marked this indexing.

Let $\mathbf{N} = \{I_j \mid \text{if } \mathrm{IF}_j \text{ is into } S\} \cup \{L_k, B_k \mid \text{if } \mathrm{LOOP}_k \text{ is into } S\}$ be a set of nonterminal symbols where $\mathbf{N} \cap (\Sigma \cup \mathbf{LM}) = \emptyset$.

Definition 3. *Let* α *be a* LEFS. *Through* $\mathfrak{J}(\alpha)$ *we denote the skeleton word over* α, *obtained by the following rules:*

a) *If* $\alpha = \varepsilon$, *then* $\mathfrak{J}(\alpha) = \varepsilon$;

b) *If* α_1 *and* α_2 *are* LEFS, *we have:*

b1) *If* $a \in \mathbf{AM}$ *and* $\alpha = \alpha_1\, a\, ;\, \alpha_2$ *then* $\mathfrak{J}(\alpha) = \mathfrak{J}(\alpha_1) a\, \mathfrak{J}(\alpha_2)$.

b2) *If* $\alpha = \alpha_1\, \mathrm{NULL}\, ;\, \alpha_2$ *then* $\mathfrak{J}(\alpha) = \mathfrak{J}(\alpha_1)\, \mathfrak{J}(\alpha_2)$.

b3) *If* $\alpha = \alpha_1 \mathrm{IF}_j\, \beta\, \mathrm{ENDIF}_j\, ;\, \alpha_2$ then $\mathfrak{J}(\alpha) = \mathfrak{J}(\alpha_1)\, \mathrm{I}_j\, \mathfrak{J}(\alpha_2)$.

b4) If $\alpha = \alpha_1 \mathrm{LOOP}_k\, \beta\, \mathrm{ENDLOOP}_k\, ;\, \alpha_2$ then $\mathfrak{J}(\alpha) = \mathfrak{J}(\alpha_1)\, L_k\, \mathfrak{J}(\alpha_2)$.

Definition 4. *Let* $X_1\, \alpha\, X_2 \beta Y_2 \delta Y_1$ *be a* LEFS, *where* $X_i = \mathrm{IF}_{j_i}$ *a* THEN_{j_i} *or* $X_i = \mathrm{IF}_{j_i}$ *a* $\mathrm{THEN}_{j_i}\ \gamma\ \mathrm{ELSE}_{j_i}$ *or* $X_i = \mathrm{LOOP}_{k_i}$, $i = \overline{1, 2}$ *and according to* X_i *we have* $Y_i = \mathrm{ENDIF}_{j_i}$; *or* $Y_i = \mathrm{ENDLOOP}_j$; $i = \overline{1, 2}$ *respectively.* Through $\mathfrak{D}(X_1 \alpha X_2)$ *we denote the directly word from* X_1 *to* X_2, *obtained by the ollowing rules:*

a) *If* α *is a* LEFS *then:*

a1) *If* $X_1 = \mathrm{IF}_{j_1}\, a\, \mathrm{THEN}_{j_1}$ *then* $\mathfrak{D}(X_1 \alpha X_2) = a + \mathfrak{J}(\alpha)$.

a2) *If* $X_1 = \mathit{IF}_{j_1}\ a\ \mathrm{THEN}_{j_1}\, \gamma\, \mathrm{ELSE}_{j_1}$ *then* $\mathfrak{D}(X_1 \alpha X_2) = a - \mathfrak{J}(\alpha)$.

a3) *If* $X_1 = \mathrm{LOOP}_{k_1}$ *then* $\mathfrak{D}(X_1\, \alpha\, X_2) = \mathfrak{J}(\alpha)$

b) *Otherwise:*

b1) *If* $\alpha = \alpha_1\ \mathrm{IF}_n\, b\ \mathrm{THEN}_n\ \alpha_2$ and $\delta = \delta_2 \mathrm{ENDIF}_n\, ;\ \delta_1$ then
$\mathfrak{D}(X_1\ \alpha X_2) = \mathfrak{D}(X_1 \alpha_1\ \mathrm{IF}_n)\ \mathfrak{D}(\mathrm{IF}_n\ \alpha_2 X_2)$.

b2) *If* $\alpha = \alpha_1\ \mathrm{IF}_n\ b\ \mathrm{THEN}_n\ \gamma_1\ \mathrm{ELSE}_n\ \alpha_2$ *and* $\delta = \delta_2\ \mathrm{ENDIF}_n\, ;\ \delta_1$ *then*
$\mathfrak{D}(X_1\, \alpha X_2) = \mathfrak{D}(X_1 \alpha_1 \mathrm{IF}_n)\ \mathfrak{D}(\mathrm{IF}_n \alpha_2 X_2)$.

b3) *If* $\alpha = \alpha_1\ \mathrm{LOOP}_m\ \alpha_2$ *and* $\delta = \delta_2\ \mathrm{ENDLOOP}_m\, ;\ \delta_1$ *then*
$\mathfrak{D}(X_1\, \alpha\, X_2) = \mathfrak{D}(X_1\ \alpha_1\ \mathrm{LOOP}_m)\, B_m\, \mathfrak{D}(\mathrm{LOOP}_m\ \alpha_2 X_2)$.

Definition 5. *Let* S *be a* LES. *The language* $L(S)$ *associated to* S *is generated from the following* CFG:

$$G_S = (\{\nabla\} \cup \{I_j, L_k, B_k, . j \geqslant 0,\ k \geqslant 0\},\ \Sigma \cup \{+, -\},\ \mathfrak{P},\ \nabla).$$

where „∇" *is a new symbol,* I_j *is a nonterminal for* IF_j — *if this exists* — L_k *and* B_k *are two nonterminals for* LOOP_k — *if this exists* — *and the set* $\mathfrak{P}$ *of the productions is constructed by the following rules:*

a) $\nabla \to \mathfrak{J}(S)$.

b) *For each* $\mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \alpha\ \mathrm{ENDIF}_j$; *the productions:*

b1) $I_j \to b-$

b2) $I_j \to b + \mathfrak{J}(\alpha)$ *if does not exist* EXIT_k *such that* $\alpha = \alpha'\ \mathrm{EXIT}_k$; *are in* $\mathfrak{P}$;

c) *For each* $\mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \alpha\ \mathrm{ELSE}_j\ \beta\ \mathrm{ENDIF}_j$; *the productions:*

c1) $I_j \to b + \mathfrak{I}(\alpha)$ *if* $\alpha \neq \alpha'\ \mathrm{EXIT}_k$;

c2) $I_j \to b - \mathfrak{I}(\beta)$ *if* $\beta \neq \beta'\ \mathrm{EXIT}_k$; *are in* $\mathfrak{L}$;

d) *For each* $\mathrm{LOOP}_k\ \alpha_1\alpha_2\ \delta\ \mathrm{ENDLOOP}_k$; *the productions:*

d1) $L_k \to \mathfrak{I}(\alpha_1\ \alpha_2\ \delta)L_k$ *and* $B_k \to \mathfrak{I}(\alpha_1\alpha_2\ \delta)B_k \mid \varepsilon$

d2) $L_k \to \mathfrak{D}(\mathrm{LOOP}_k\ \alpha_1\ \mathrm{IF}_j)\ b + \mathfrak{I}(\beta)$, if

$\alpha_2 = \mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \beta\ \mathrm{EXIT}_k;\ \mathrm{ENDIF}_j$; *or*

$\alpha_2 = \mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \beta\ \mathrm{EXIT}_k;\ \mathrm{ELSE}_j\ \gamma\ \mathrm{ENDIF}_j$;

d3) $L_k \to \mathfrak{D}(\mathrm{LOOP}_k\ \alpha_1\ \mathrm{IF}_j)\ b - \mathfrak{I}(\beta)$, if

$\alpha_2 = \mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \gamma\ \mathrm{ELSE}_j\ \beta\ \mathrm{EXIT}_k;\ \mathrm{ENDIF}_j$; *are in* $\mathfrak{L}$.

Definition 6. *Let S be a* LES. *The static word associated to S is obtained f S by erasing all reserved symbols.*

Example 1.

```
LOOP1
 IF1 a1 THEN1
       LOOP2
         a2;
         IF2 a3 THEN2 EXIT2;
               ELSE2
                IF3 a4 THEN3 NULL; ELSE3 EXIT1; ENDIF3;
         ENDIF2;
       ENDLOOP2;
     ELSE1
       IF4 a5 THEN4 NULL; ELSE4 EXIT1; ENDIF4;
 ENDIF1;
ENDLOOP1;
```

The static word is "$a_1\ a_2\ a_3\ a_4\ a_5$".

Example 2. Let us consider LES from the example 1. The associated mmar has the follows productions:

$\nabla \to L_1$

$L_1 \to I_1L_1 \mid a_1 + B_2a_2a_3 - a_4 - \mid a_1 - a_5 -$ $\qquad B_1 \to I_1B_1 \mid \varepsilon$

$I_1 \to a_1 + L_2 \mid a_1 - I_4$

$L_2 \to a_2I_2L_2 \mid a_2a_3 +$ $\qquad B_2 \to a_2I_2B_2 \mid \varepsilon$

$I_2 \to a_3 - I_3$

$I_3 \to a_4 +$

$I_4 \to a_5 +$.

3. Orderly properties in the Loop-Exit grammars. *Let S be a LES, let* $G_S = (\{\nabla\} \cup \mathbf{N},\ \Sigma \cup \{+, -\},\ \mathfrak{P},\ \nabla)$ be the attached grammar to S and let $a_1 a_2 \dots a_n$ be the static word attached to S.

Consider $\alpha \in (\{\nabla\} \cup \mathbf{N} \cup \Sigma \cup \{+, -\})^{+}$. Similar with [1] we definev:

Definition 7. If $\alpha \neq +\alpha'$ *and* $\alpha \neq -\alpha'$ *then*

$$\text{FIRST}\ (\alpha) = \{a \in \Sigma \mid \alpha \Rightarrow a\beta\}.$$

Suppose that $\alpha = \alpha' +$ *or* $\alpha = \alpha' -$ *if and only* $\alpha = \alpha'' a +$ *or* $\alpha = \alpha'' a -$ *with* $a \in \mathbf{TM}$. *Then*

$$\text{LAST}\ (\alpha) = \{a \mid a \in \mathbf{AM},\ \alpha \overset{*}{\Rightarrow} \beta a\} \cup \{a + \mid a \in \mathbf{TM},\ \alpha \overset{*}{\Rightarrow} \beta a +\} \cup \{a - \mid a \in \mathbf{TM},\ \alpha \overset{*}{\Rightarrow} \beta a -\}.$$

Using the definitions 1—7 we can directly prove the following lemmas:

LEMMA 1. *For each* $A \in \mathbf{N}$, *A being a useful and accesible symbol, there is a symbol* a_i *from the static word so that:*

a) $\text{FIRST}\ (A) = \{a_i\}$;

b) *For each* $A \to \alpha \in \mathfrak{P}$, $\text{FIRST}\ (\alpha) = \{a_i\}$.

LEMMA 2. *If* $a_i \in AM$ *is a symbol from the static word then:*

a) *It does not exist* $w \in L(G_S)$ *so that* $w = xa_i + y$ *or* $w = xa_i - y$

b) *For each* $A \to \alpha a_i \beta \in \mathfrak{P}$, *if* $\varepsilon \neq \beta \neq A$ *then* $\text{FIRST}\ (\beta) = \{a_{i+1}\}$.

LEMMA 3. *If* $a_i \in \mathbf{TM}$ *is a symbol from the static word, then*

a) *Each production from* $\mathfrak{P}$ *which contains* a_i *is either* $A \to \alpha a_i + \beta$ *or* $A \to \alpha a_i - \beta$ *and there are productions of both forms.*

b) *If it exists* $A \to \alpha a_i + \beta$, *so that* $\varepsilon \neq \beta \neq A$ *then it exists* a_j *a symbol from the static word so that* $i < j$ *and for each* $A \to \alpha a_i + \beta \in \mathfrak{P}$. $\varepsilon \neq \beta \neq A$ *it results that* $\text{FIRST}\ (\beta) = \{a_j\}$.

c) *If it exists* $A \to \alpha a_i - \beta$ *so that* $\varepsilon \neq \beta \neq A$ *then it exists* a_k *a symbol from the static word so that* $i < k$ *and for each* $A \to \alpha a_i - \beta$, $\varepsilon \neq \beta \neq A$ *it results that* $\text{FIRST}\ (\beta) = \{a_k\}$.

d) *If the symbol* a_j *verifies b) and* a_k *verifies c) then it results that* $j < k$.

LEMMA 4. *The following properties hold:*

a) $\text{FIRST}\ (a_i) = \{a_i\}$ *for each* $a_i \in \Sigma$

b) $\text{LAST}\ (a_i X) = \{a_i X\}$ *where* $X \in \{\varepsilon, +, -\}$

c) *If* $Y = a_i X$ *where* $X \in \{\varepsilon, +, -\}$ *then*

$$\text{LAST}\ (\alpha Y) = \text{LAST}(Y)$$

d) $\text{LAST}\ (A) = \bigcup \{\text{LAST}(\alpha) \mid A \to \alpha \in \mathfrak{P}\}$

e) *For each two productions* $A \to \alpha$ *and* $A \to \beta$ *we have:*

$$\text{LAST}\ (\alpha) \cap \text{LAST}(\beta) = \emptyset$$

Example 3. Let us consider LES from the example 1, having the associated grammar in the example 2. After eliminating the inaccesible and useless symbols, only the productions $B_1 \to I_1L_1 | \varepsilon$ must be erased. The FIRST and LAS relations are:

FIRST $(a_i) = \{a_i\}$, $i = \overline{1, 5}$, LAST $(a_2) = \{a_2\}$,

LAST $(a_i +) = \{a_i +\}$ and LAST $(a_i -) = \{a_i -\}$, for $i \neq 2$.

For nonterminals we have:

	Δ	L_1	I_1	L_2	B_2	I_2	I_3	I_4
FIRST	a_1	a_1	a_1	a_2	a_2	a_3	a_4	a_5
LAST	a_4- a_5-	a_4- a_5-	a_3+ a_5+	a_3+	a_4+	a_4+	a_4+	a_5+

4. An algorithm for conversion a LES into a flowchart. Now we give method for conversion to flowchart from LES without inaccessible LEFS [4

ALGORITHM 1.

Input. A LES A without inaccesible LEFS.

Output. An equivalent flowchart $(\mathfrak{X}, \mathfrak{U})$ with S.

Step 1. Using the definition 6 we'll construct the associated grammar G. Using the algorithms from [1] we eliminate the inaccesible and useless symbol

Step 2. Using the lemmas 1—4 for each symbol Y from G_S, the FIRST (Y) and LAST (Y) relations are found.

Step 3. If $a_1a_2 \ldots a_n$ is the static word associated to S, then the set of vertices $\mathfrak{X}$ is obtained as follows:

— for each symbol a_i from the static word, if $a_i \in$ **AM** then A_i : "$\boxed{a_i}$" is a vertex else, (if $a_i \in$ **TM**) A_i: "$\langle a_i \rangle$" is a vertex;

— for each $a_i X \in$ LAST (∇) we have one stop vertex „Δ";

— the start vertex "∇" is added to $\mathfrak{X}$.

Step 4.

Let $\mathfrak{U} := \{(\nabla, \text{FIRST } (\nabla)\} \cup \{(A_i, \Delta)$ marked $X | a_i X \in \text{LAST } (\nabla), X \in \{\varepsilon, +, -\}\}$

Step 5. For each production $A \to \alpha\beta$ from G_S, $\alpha \neq \varepsilon \neq \beta$, add $\{(A_i, A_k)$ marked $X | \{a_k\} = \text{FIRST } (\beta), a_i X \in \text{LAST } (\alpha), X \in \{\varepsilon, +, -\}\}$ to the set $\mathfrak{U}$.

Example 4. Let us consider LES from the example 1, having the grammar in the example 2 and FIRST and LAST relations from the example 3. After applying the step 4, we have:

$\mathfrak{U} = \{(\nabla, A_1)$ unmarked (marked with ε), (A_4, Δ) and (A_5, Δ), both marked "$-$"$\}$.

When we apply the step 5 to $L_1 \to I_1L_1$ with $\alpha = I_1$ and $\beta = L_1$, we obtain the edges (A_3, A_1) and (A_5, A_1), both marked "$+$". When we apply the step 5 to $L_2 \to a_2I_2L_2$ with $\alpha = a_2$ and $\beta = I_2L_2$, we obtain the edge (A_2A_3) unmarked.

After applying the algorithm 1, we obtain the flowchart from fig. 1.

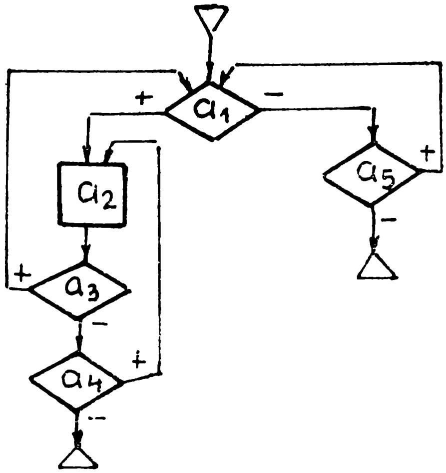

Fig. 1

THEOREM 1. *For each* LES *without inaccessible* LEFS [4] *using the algorithm* 1, *a flowchart* (X, U) *equivalent with* LES *is obtained.*

Proof this theorem was presented in [3].

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CONTINUOUS SELECTIONS OI MULTI-VALUED MAPS WITH NONCONVEX RICHT-HAND SID. AND THE PICARD PROBLEM FOR THE MULTI-VALUED YPERBOLIC EQUATION $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$

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ABSTRACT. — The Picard problem is considered for the multivolued hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, where F is a continuous multi-valued map defined on $A \subset R^{n+2}$ with compact values, but nonconvex in R^n. An existence theorem of a continuous selection is proved for $F(x, y, z)$, with $z \in K$, where K is a compact, convex, set of absolutely continuous functions, submitted to certain conditions. An operator is then defined by means of this selection, for which one applies the Schander Fixed Point Theorem — the fixed point being just the solution of the Picard probleme.

1. Introduction. In this paper we are concerned with the Picard proble for the multi-valued hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, where F is a mult valued map, defined in a suitable subset of $\mathbf{R}^{n+2}$, with values that are nonempt compact but not necessarily convex subsets of $\mathbf{R}^n$. The Picard problem is defin by analogy with the Picard problem for quasilinear hyperbolic equations [1 in [2], [3], where F is a multi-valued map defined on a subset of $\mathbf{R}^{n+2}$ and valu in the set of compact convex nonempty subsets of $\mathbf{R}^n$, satisfying conditions Caractheódory type. Using the Fixed Point Theorem of Kakutani-Ky Fan o proves that the problem above has at least a solution.

In this note one proves an existence theorem of a continuous selection i each of the maps $(x, y) \to F(x, y, z(x, y))$ relative to a given family of continuo maps $(x, y) \to z(x, y)$, as in [4] — [8], and using the Schauder Fixed Po Theorem one obtains the existence of a solution of the Picard problem.

2. Continuous approximate selections. Let be the multivalued map $F: A$- comp X, where $A \subset \mathbf{R}^{n+2}$, $A = D \times B$, $D = [0, a] \times [0, b] \subset \mathbf{R}^2$, $B \subset \mathbf{R}^n$ the closed ball centered in origin with radius $c = M_1 + \text{Mab}$, M_1 given by (3 M given by (4), $X \subset \mathbf{R}^n$ is the closed ball centered in origin with radius M, being a compact metric space with the metric d induced on X by the norm defin on $\mathbf{R}^n$.

Let H be the Hausdorff-Pompeiu metric [9] on comp X induced by d. Th comp X is a compact metric space with respect to the metric H.

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Let $C(D\,;\,\mathbf{R}^n)$ be the Banach space of continuous functions defined on D and valued in $\mathbf{R}^n$ and $L^1(D\,;\,\mathbf{R}^n)$ the Banach space of equivalence classes of Lebesgue integrable functions defined on D and valued in $\mathbf{R}^n$.

Let the following hypotheses be satisfied :

(H_0) The curve $\gamma : x = \psi(y),\ 0 \leqslant y \leqslant b$, is defined by the function $\psi \in C^1([0, b]\,;\,\mathbf{R})$, satisfying the conditions

$$\psi(0) = 0, 0 \leqslant \psi(y) \leqslant a,\ o \leqslant y \leqslant b, \tag{1}$$

(H_1) The functions $P \in AC([0, a]\,;\,\mathbf{R}^n)$, $Q \in AC([0, b]\,;\,\mathbf{R}^n)$, where $AC([\alpha_1, \alpha_2]\,;\,\mathbf{R}^n)$ is the space of obsolutely continuous functions $f:[\alpha_1, \alpha_2] \to \mathbf{R}^n$, endowed with the norm

$$|f| = \sup_{t \leqslant [\alpha_1, \alpha_2]} ||f(t)|| + \int_{\alpha_1}^{\alpha_2} ||f'(t)||dt,$$

satisfy the condition $P(0) = Q(0)$,

(H_2) The function $\alpha : D \to \mathbf{R}^n$ defined by

$$\alpha(x, y) = P(x) + Q(y) - P(\psi(y)),\ (x, y) \in D, \tag{2}$$

is bounded and therefore, the is $M_1 > 0$ such that

$$||\alpha(x, y)|| \leqslant M_1,\ (x, y) \in D. \tag{3}$$

It follows that α is absolutely continuous ;

$$\alpha \in C^*(D\,;\,\mathbf{R}^n),\ [10],\ \S\S\ 565-568.$$

Let K be the set of absolutely continuous functions $z : D \to \mathbf{R}^n$, $z \in C^*(D\,;\,\mathbf{R}^n)$, [10], satisfying the conditions (3), (4), (5), where

$$\left\|\frac{\partial^2 z(x, y)}{\partial x\,\partial y}\right\| \leqslant M, \text{ a.e. } (x, y) \in D, \tag{4}$$

and

$$\begin{cases} z(x, 0) = P(x),\ 0 \leqslant x \leqslant a. \\ z(\psi(y), y) = Q(y),\ 0 \leqslant y \leqslant b. \end{cases} \tag{5}$$

Then, the following two propositions hold :

Proposition 1. *K is a nonempty convex and compact subset of the Banach space* $C(D\,;\,\mathbf{R}^n)$.

Proof. The relation $z \in K$ implies $z \in C(D\,;\,\mathbf{R}^n)$. We observe that $\frac{\partial^2 z}{\partial x \partial y}$ exists a.e. $(x, y) \in D$, as $z \in C^*(D\,;\,\mathbf{R}^n)$, [10].

Let $M(x, y)$ be any point of D. Consider the parallel to x-axis, that intersects the curve γ in the point $N(\psi(y), y)$. Let $M_0(x, 0)$ and $N_0(\psi(y. 0)$ be the ortogonal projections of M and N on the x-axis. Denote $D_0(x, y)$ the rectangle MNN_0M_0, given by

$$D_0(x, y) = \{\psi(y) \leqslant u \leqslant x, 0 \leqslant v \leqslant y\}.$$

Integrating $\frac{\partial^2 z(x,y)}{\partial x \partial y}$ over $D_0(x,y)$, one obtains

$$\iint\limits_{D_0(x,y)} \frac{\partial^2 z(u,v)}{\partial u \partial v} du dv = \int_0^y dv \int_{\psi(y)}^x \frac{\partial^2 z(u,v)}{\partial u \partial v} du = \int_0^y dv \frac{\partial z}{\partial v}(u,v)\Big|_{u=\psi(y)}^{u=x} =$$

$$= \int_0^y \frac{\partial z}{\partial v}(x,v)\, dv - \int_0^y \frac{\partial z}{\partial v}(\psi(y), v) dv = z(x,y) - z(x, 0(\psi(y), y) +$$

$$+ z(\psi(y), 0) = z(x,y) - P(x) - Q(y) + P(\psi(y)), \; (x,y) \in D.$$

Using (2), it follows

$$z(x,y) = P(x) + Q(y) - P(\psi(y)) + \iint\limits_{D_0(x,y)} \frac{\partial^2 z(u,v)}{\partial u\, \partial v} du dv =$$

$$= \alpha(x,y) + \iint\limits_{D_0(x,y)} \frac{\partial^2 z(u,v)}{\partial u\, \partial v} du dv, \; (x,y) \in D, \qquad (6)$$

or

$$z(x,y) = P(x) + Q(y) - P(\psi(y)) + \int_0^y dv \int_{\psi(y)}^x \frac{\partial^2 z(u,v)}{\partial u\, \partial v} du, \; (x,y) \in D. \qquad (6')$$

The compactness of K follows using (6) or (6') and the Arzelà-Ascoli Theorem. The convexity of K is obvious.

Remark. The relation $z \in K$ implies $(x, y, z(x,y)) \in A$ for each $(x, y \in D$. Because each $z \in K$ generate a multi-valued map $(x,y) \to F(x, y, z(x,y))$ from D to comp X, we shall denote this map by $G(z)$,

$$G(z)(x,y) = F(x, y, z(x,y)), \; (x, y) \in D. \qquad (7)$$

Proposition 2. *Let* $F : A \to \text{comp } X$ *be a multi-valued continuous map. Then, for each* $\varepsilon > 0$, *there exists a continuous function* $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ *such that for each* $z \in K$ *we have*

$$d(g(z)(x,y), G(z)(x,y)) < \varepsilon, \; a.e. \; (x,y) \in D. \qquad (8)$$

Proof. The proof is analogous to that given in [4]—[8] and is based on the construction of the function g by means of the continuous partition of the unity. Let $\varepsilon > 0$ be given. In view of the fact that F is continuous on A and A is compact, F is uniformly continuous on A and there is $\Delta > 0$ such that

$$H(F(x,y,z), F(\xi, \eta, \bar{z})) < \varepsilon,$$

for any two points (x, y, z), $(\xi, \eta, \bar{z})$ in A with $||(x, y) - (\xi, \eta)|| < \Delta$, $||z - \bar{z}|| < \Delta$.

Let $\{\mathcal{U}_k\}_{1\leqslant k\leqslant N}$ be a finite open cover of K such that diam $\mathcal{U}_k < \Delta$ for any $k = \overline{1, N}$. Let $\{p_k\}_{1\leqslant k\leqslant N}$ be the continuous partition of unity subordinate to $\{\mathcal{U}_k\}_{1\infty k\infty N}$; select for each k a point $z_k \in \mathcal{U}_k$ and let $\{v_k\}_{1\leqslant k\leqslant N}$ be a sequence of Lebesgue measurable functions $v_k : D \to \mathbf{R}^n$ such that, for every k, $v_k(x, y) \in G(z_k)(x, y)$ a.e. $(x, y) \in D$. Such function v_k exists because each $G(z_k)$ is continuous and measurable in D, [11]; $v_k \in \mathfrak{L}^1(D; \mathbf{R}^n)$ for every k. We can take $N = N_1N_2$. Denote $\mathcal{U}_k = \mathcal{U}_{ij}$, $v_k = v_{ij}$, $z_k = z_{ij} \in \mathcal{U}_{ij}$, $p_k(z) = p_{ij}(z)$ and suppose

$$p_{ij}(z) = q_i(z)r_j(z),\ i = \overline{1, N_1},\ j = \overline{1, N_2}.$$

The functions $p_{ij} : K \to \mathbf{R}$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$ satisfy the properties:

a) $0 \leqslant p_{ji}(z) \leqslant 1$, for $z \in K$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$,

b) $p_{ij}(z) = 0$ if $z \notin \mathcal{U}_{ij}$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$,

c) $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} p_{ij}(z) = \sum_{i=1}^{N_1}\sum_{i=1}^{N_2} q_i(z)r_j(z) = 1$, for $z \in K$.

For each $z \in K$ define the continuous functions $\lambda_{ij} : K \to \mathbf{R}$

$$\lambda_{ij}(z) = x_i(z)y_j(z),\ i = \overline{1, N_1},\ j = \overline{1, N_2},$$

where

$$\begin{cases} x_0(z) = 0 \\ x_i(z) = x_{i-1}(z) + aq_i(z)\sum_{j=1}^{N_2} r_j(z),\ i = \overline{1, N_1}, \end{cases}$$

and

$$\begin{cases} y_0(z) = 0 \\ y_j(z) = y_{j-1}(z) + br_j(z)\sum_{i=1}^{N_1} q_i(z),\ j = \overline{1, N_2}. \end{cases}$$

For each $z \in K$ define the rectangles

$$D_{ij}(z) = [x_{i-1}(z), x_i(z)) \times [y_{j-1}(z), y(z)),\ i = \overline{1, N_1},\ j = \overline{1, N_2},$$

which constitute a partition of D excepting lines $x = a$ and $y = b$.

We construct the desired function $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$,

$$\begin{cases} g(z)(x, y) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \chi\,[D_{ij}(z)](x, y)\, v_{ij}(x, y),\ 0 \leqslant x < a,\ 0 \leqslant y < b, \\ g(z)(a, y) = v_{l, N_2}(a, y),\ l = \min\{i \geqslant 1;\ x(z) = a\}, \\ g(z)(x, b) = v_{N_1, p}(x, b),\ p = \min\{j \geqslant 1;\ y(z) = b\}. \end{cases} \tag{9}$$

where $\chi[D_{ij}(z)]$ is the characteristic function of $D_{ij}(z)$.

Obviously, g maps K into $\mathfrak{L}^1(D; \mathbf{R}^n)$. Moreover, for a given $z \in K$ and any fixed $(x, y) \in D$, there exists a unique (i, j) such that $(x, y) \in D_{ij}(z)$ and this implies $z \in \mathfrak{U}_{ij}$. Thus, $g(z)\,(x, y) = v_{ij}(x, y)$ and $\|z(x, y) - z_{ij}(x, y)\| < \Delta$ so that

$$d(g(z)\,(x, y), G(z)(x, y) \leqslant d\,(v_{ij}(x, y), G(z_{ij})(x, y)) + H\,(G(z_{ij})\,(x, y); G(z)\,(x, y)) < d(v_{ij}(x, y), G(z_{ij})\,(x, y)) + \varepsilon. \tag{10}$$

It follows that, for each $z \in K$, $d(g(z)\,(x, y), G(z)\,(x, y)) < \varepsilon$ a.e. $(x, y) \in D$, therefore (8) holds. We show that g is continuous on K. Then, for any points z, w in K and any $(x, y) \in D$, $0 \in x < a$, $0 \leqslant y < b$,

$$\|g(z)\,(x, y) - g(w)\,(x, y)\| \leqslant \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \chi\,[D_{ij}(z)\,\Delta D_{ij}(w)]\,(x, y)\,\|v_{ij}(x, y)\| \tag{11}$$

where $D_{ij}(z)\,\Delta D_{ij}(w) = (D_{ij}(z) - D_{ij}(w)) \cup (D_{ij}(w) - D_{ji}(z))$.

Since K is compact, $\{\lambda_{ij}\}_{i=\overline{1, N_1},\, j=\overline{1, N_2}}$ is a uniformly equicontinuous family of real valued functions. Thus, for every $\eta > 0$, there exists a $\delta > 0$ such that, for any $z \in K$, $w \in K$ satisfying $\|z(x, y) - w(x, y)\| < \delta$ at every $(x, y) \in D$,

$$|\lambda_{ij}(z) - \lambda_{ij}(w)| < \varepsilon/2MN,$$

and hence $\mu(D_{ij}(z)\,\Delta D_{ij}(w)) < \eta/MN$,

so that (11) implies

$$|g(z) - g(w)|\,|_{\mathfrak{L}^1} = \iint_D \|g(z)\,(x, y) - g(w)\,(x, y)\|\,dxdy < \eta. \tag{12}$$

Therefore $g : K \to \mathfrak{L}^1(D\,; \mathbf{R}^n)$ is uniformly continuous.

3. Continuous selections. On analogy of [4]—[8] one proves the following existence theorem of a continuous selection for multi-valued map $G(z)$.

Theorem 1. *If $F : A \to \operatorname{comp} X$ is a multi-valued continuous map, then there exists a continuous function $g : K \to \mathfrak{L}^1(D\,; \mathbf{R}\,)$ such that, for any $z \in K$, $g(z)(x,y) \in G(z)(x, y)$, a.e. $(x, y) \in D$, that is $g(z)$ is a continuous selection for $G(z)$, given by* (7).

Proof. Define a sequence of continuous functions $g^n : K \to \mathfrak{L}^1(D\,; \mathbf{R}^n)$, $n \in N$, submitted to the following conditions:

1) $d(g^n(z)\,(x, y), G(z)\,(x, y)) < \dfrac{1}{2^{n+1}}$, a.e. $(x, y) \in D$,

2) $\mu\left(\left\{(x, y) \in D \mid \|g^n(z)\,(x, y) - g^{n-1}(z)\,(x, y)\| \geqslant \dfrac{1}{2^{n-1}}\right\}\right) < \dfrac{1}{2^n}$.

The condition 2) shows that for each $z \in K$, the sequence $\{g^n(z)\}_{n \in \mathbf{N}}$ converges, in the norm of $\mathfrak{L}^1(D\,; \mathbf{R}^n)$, to an element $g(z)$ and the convergence is uniform on K, because the condition 2) is satisfied uniformly for any $z \in K$. Using the Lebesgue Dominated Convergence Theorem it follows that $g(z) \in \mathfrak{L}^1(D\,; \mathbf{R}^n)$ for each $z \in K$. By continuity of the functions g^n, $n \in \mathbf{N}$, it follows that $g : K \to \mathfrak{L}^1(D\,; \mathbf{R}^n)$ is continuous. Therefore, for any $z \in K$, there exists a measurable function $g(z) : D \to X$ such that the sequence $\{g^n(z)\}_{n \in \mathbf{N}}$ converges to $g(z)$ a.e. in measure, and a subsequence of $\{g^n(z)\}_{n \in \mathbf{N}}$ that converges to $g(z)$ a.e. on D. Then, from the condition 1), follows that for each $z \in K$ we have $g(z(x, y)) \in G(z)(x, y)$, a.e. $(x, y) \in D$, because $G(z)(x, y)$ is closed for any $z \in K$.

The sequence $\{g^n(z)\}_{n \leq N}$ is obtained by induction. From Proposition 2 it follows that there exists a continuous function $g^0 : K \to \mathfrak{L}^1(D ; \mathbf{R}^n)$ such that for any $z \in K$

$$d(g^0(z)(x, y), G(z)(x, y)) < \frac{1}{2}, \text{ a.e. } (x, y) \in D. \tag{13}$$

Also, from Proposition 2 and the continuity of F on $A = D \times B$ there exists $\Delta_1 > 0$ such that

$$H(F(x, y, z), F(\xi, \eta, \tilde{z})) < \frac{1}{4} \tag{14}$$

for each (x, y, z), $(\xi, \eta, \tilde{z})$ in A with $||(x, y) - (\xi, \eta)|| < \Delta_1$, $||z - \tilde{z}|| < \Delta_1$ and

$$\mu(\{(x, y) \in D \mid ||g^0(z)(x, y) - g^0(\tilde{z})(x, y)|| > 0\}) < \frac{1}{2} \tag{15}$$

for each $z \in K$, $\tilde{z} \in K$ satisfying $||z(x, y) - \tilde{z}(x, y)|| < \Delta_1$ for any $(x, y) \in D$.

By analogy with the Proposition 2, let $\{\mathfrak{U}_k^1\}_{1 \leq k \leq N(1)}$ be an open finite cover of K, such that diam $\mathfrak{U}_k^1 < \Delta_1$, for any k; let $\{p_k^1\}_{1 \leq k \leq N(1)}$ be the continuous partition of unity subordinate to $\{\mathfrak{U}_k^1\}_{1 \leq k \leq N(1)}$; we select for each k a point $z_k^1 \in \mathfrak{U}_k^1$ and a Lebesgue measurable function $v_k^1 : D \to \mathbf{R}^n$ such that

$$v_k^1(x, y) \in G(z_k^1)(x, y), \text{ a.e. } (x, y) \in D,$$

and

$$||v_k^1(x, y) - g^0(z_k^1)(x, y)|| = d(g^0(z_k^1)(x, y), G(z_k^1)(x, y)). \tag{16}$$

It follows from the continuity of each $G(z_k^1)$, that are measurable on D, [12]. By analogy with the Proposition 2, consider $N(1) = N_1(1)\, N_2(1)$ and denote $\mathfrak{U}_k^1 = \mathfrak{U}_{ij}^1$, $v_k^1 = v_{ij}^1$, $z_k^1 = z_{ij}^1 \in \mathfrak{U}_{ij}^1$ and $p_k^1(z) = p_{ij}^1(z) = q_i^1(z)\, r_j^1(z)$, $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$.

The continuous partition of unity, $\{p(z_{ij}^1)\}$, $p_{ij}^1 : K \to \mathbf{R}$ satisfies:

a) $0 \leq p_{ij}^1(z) \leq 1$ for all $z \in K$, $i = 1, N_1(1)$, $j = 1, N_2(1)$,

b) $p_{ij}^1(z) = 0$ if $z \notin \mathfrak{U}_{ij}^1$,

c) $\sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} p_{ij}^1(z) = \sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} q_i^1(z) r_j^1(z) = 1$ for all $z \in K$.

For each $z \in K$, define the continuous functions $\lambda_{ij}^1 : K \to \mathbf{R}$

$$\lambda_{ij}^1(z) = x_i^1(z)\, y_j^1(z), \; i = \overline{1, N_1(1)} \; j = \overline{1, N_2(1)}$$

with

$$\begin{cases} x_0^1(z) = 0 \\ x_i^1(z) = x_{i-1}^1(z) + aq_i^1(z) \sum\limits_{j-1}^{N_2(1)} r_j^1(z), \; i = \overline{1, N_1(1)}, \end{cases}$$

and

$$\begin{cases} y_0^1(z) = 0 \\ y_i^1(z) = y_{j=1}^1(z) + br_j^1(z) \sum\limits_{i=1}^{N_1(1)} q_i^1(z), \; j = \overline{1, N_2(1)}. \end{cases}$$

For each $z \in K$ consider the rectangles

$$D_{ij}^1(z) = [x_{i-1}^1(z), x_i^1(z)) \times [y_{j-1}^1(z), y_j^1(z)), \; i = \overline{1, N_1(1)}, \; j = \overline{1, N_2(1)},$$

establishing a partition of D, except for the lines $x = a$ and $y = b$.),

Define the function $g^1 : K \to \mathfrak{L}^1(D ; \mathbf{R}^n)$

$$\begin{cases} g^1(z)\,(x, y) = \sum\limits_{i=1}^{N_1(1)} \sum\limits_{j=1}^{N_2(1)} \chi[D_{ij}^1(z)](x, y) v_{ij}^1(x, y), \; 0 \leqslant x < a, \; 0 \leqslant y < b, \\ g^1(z)(a, y) = v_{l,\, N_2(1)}^1(a, y), \quad l = \min\{i \geqslant 1 \,;\, x_i^1(z) = a\}, \\ g^1(z)\,(x, b) = v_{N_1(1),\, p}(x, b), \quad p = \min\{j \geqslant 1 \,;\, y_i^1(z) = b\}. \end{cases} \tag{17}$$

The function g^1 is continuous (see the proof of continuity of g in the Proposition 2). To verify the conditions 1), 2) suppose that $z \in K$ is given, and $(x, y) \in D$ fixed $(0 \leqslant x < a, \; 0 \leqslant y < b)$.

Then $(x, y) \in D_{ij}^1(z)$ for a unique pair of indices (i, j), therefore $p_{ij}^1(z) < \vec{0}$.

Then

$$g^1(z)(x, y) = v_{ij}^1(x, y) \tag{18}$$

and $\| z(x, y) - z_{ij}^1(x, y) \| < \Delta_1$ such that

$$\begin{aligned} d(g^1(z)\,(x, y), G(z)\,(x, y)) &= d(v_{ij}^1(x, y), G(z)\,(x, y)) \leqslant \\ &\leqslant d(v_{ij}^1(x, y), G(z_{ij}^1)(x, y)) + H(G(z_{ij}^1)(x, y), G(z)\,(x, y)) \leqslant \frac{1}{2^2} \end{aligned} \tag{19}$$

a.e. $(x, y) \in D$, that is the condition 1) holds for $n = 1$.

Moreover, using (13), (16) and (18), it follows

$$\begin{aligned} &\| g^1(z)(x, y) - g^0(z)\,(x, y) \| \leqslant \| v_{ij}^1(x, y) - g^0(z_{ij}^1)(x, y) \| + \\ &+ \| g^0(z_{ij}^1)(x, y) - g^0(z)(x, y) \| = d(g^0(z_{ij}^1)(x, y), G(z_{ij}^1)(x, y)) + \\ &+ \| g^0(z_{ij}^1)(x, y) - g^0(z)(x, y) \| < \frac{1}{2} + \| g^0(z_{ij}^1)(x, y) - g^0(z)\,(x, y) \| \end{aligned} \tag{20}$$

that implies

$$\mu(\{(x, y) \in D \mid ||g^1(z)(x, y) - g^0(z)(x, y)|| \geqslant 1\}) \leqslant$$

$$\leqslant \mu\left(\left\{(x, y) \in D \mid ||g^0(z^1_{ij})(x, y) - g^0(z)(x, y)|| \geqslant \frac{1}{2}\right\}\right) < \frac{1}{2}, \tag{21}$$

that is the condition 2) holds for $n = 1$.

Obviously, a similar construction can be used for $n > 1$, and the theorem is proved.

4. *The Picard problem.*

Consider the multi-valued equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \ (x, y) \in D, \ z \in B, \tag{22}$$

where $F : D \times B \to \text{comp}\, X$.

The Picard problem is defined in [2], [3] and consists in determination of a solution of the equation (22) satisfying the conditions (5) in the hypotheses (H_0), (H_1), (H_2). We state the following theorem.

Theorem 2. Let $F : D \times B \to \text{comp}\, X$ *be a multi-valued map satisfying the hypothesis*

(H_3) *F is continuous on $D \times B$.*

If the hypotheses $(H_0)-(H_3)$ *is fulfilled, the Picard problem* (22) + (5) *has at least an absolutely continuous solution* $\bar{z} : D \to \mathbf{R}^n$, $\bar{z} \in C.(D; \mathbf{R}^n)$.

Proof. Using the Theorem 1 it follows that there exists a continuous selection $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ for $G(z)$ given by (7). Define, for each $z \in K$, the function $h(z) : D \to \mathbf{R}^n$ by

$$h(z)(x, y) = \alpha(x, y) + \iint_{D_3(x, y)} g(z)(u, v)\, du dv =$$

$$= P(x) + Q(y) - P(\psi)(y)) + \int_0^y dv \int_{\psi(y)}^x g(z)(u, v) du, \ (x, y) \in D. \tag{23}$$

Then, $h(z) \in C^*(D; \mathbf{R}\ )$ for each $z \in K$, [10]. One obtains $h(K) \subset K$. Using the Schauder Fixed Point Theorem, it follows that there exists $\bar{z} \in K$ such that $h(\bar{z}) = \bar{z}$, that is $h(\bar{z})(x, y) = \bar{z}(x, y)$, $(x, y) \in D$.

That implies from (23) $\bar{z}(x, 0) = P(x)$, $0 \leqslant x \leqslant a$, $\bar{z}(\psi(y), y) = Q(y)$, $0 \leqslant \leqslant y \leqslant b$, therefore (5) and (22) hold for $\bar{z}$, consequently z is a solution of the Picard problem (22)+(5), a.e. $(x, y) \in D$.

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ABSTRACT. — Astronomical Observatory of the University. The paper deals with the development of the modern astrononomical research in Cluj. The founding, endowing and activity of the Astronomical Observatory of the University are presented, as well as the difficult work of its managers along the time. The modern residence of the Observatory and the rich scientific activity within the framework of this institution are also pointed out.

Cercetările moderne de astronomie au început la Cluj odată cu înfiinţarea Universităţii româneşti (1919), avînd drept iniţiatori pe *profesorul Gheorghe Bratu* (1881—1941) şi *profesorul Gheorghe Demetrescu* (1885—1969), care au elaborat planurile celui dintîi observator modern înzestrat la Cluj şi au format primii astronomi ce urmau să ducă mai departe creaţia lor**.

Observatorul astronomic al Universităţii din Cluj a fost construit şi dotat între anii 1920—1934, în partea de sud a oraşului, unde a avut multă vreme un cîmp larg de vizibilitate. La stăruinţele profesorului Gheorghe Bratu (directorul Observatorului între anii 1919—1923 şi 1928—1941), se fac primele comenzi de aparate şi cărţi, iar profesorul Gheorghe Demetrescu (directorul Observatorului între anii 1923—1928) completează aceste planuri, care au fost realizate astfel: în anul 1924 se obţine terenul, iar în 1927 se construieşte sala meridiană, în care se montează o lunetă de treceri, transformată dintr-un teodolit vechi, şi încep lucrările practice de astronomie. După mari greutăţi materiale, legate de asigurarea fondurilor necesare, cînd sursa principală de venituri o constituiau taxele studenţeşti, între anii 1928—1931 este construită clădirea ecuatorială cu o cupolă mobilă avînd diametrul de 5 m (construită şi montată de casa Gillon din Paris). Aici au fost instalate în următorii doi ani: ecuatorialul, Prin avînd un telescop Newton (cu oglindă parabolică, D = 50 cm, F = 250 cm) şi o lunetă cu obiectiv Zeiss (D = 20 cm, F = 300 cm), ambele instrumente fiind montate de inginerul Nicolae Bratu, fiul prof. Gh. Bratu. Alte instrumente mai mici, o lunetă de treceri, două sextante, două teodolite, cronometre şi pendule (de timp mediu şi sideral) au completat înzestrarea Observatorului. În anul 1934 este terminat pavilionul central pentru bibliotecă şi laboratoare.

Rolul şi meritele profesorului Gheorghe Bratu în domeniul astronomiei sînt pregnant înfăţişate în raportul Facultăţii de Ştiinţe a Universităţii din Cluj privind transferul său de la Catedra de analiză matematică la Catedra de astronomie. Cităm:

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** *Dar preocupările de astronomie pe meleagurile transilvănene sînt foarte vechi, contopindu-se cu începuturile culturii şi civilizaţiei. Istoria astronomiei consemnează cunoştinţe şi cercetări astronomice remarcabile ale geto-dacilor, observatoare astronomice medievale înfiinţate în focarele de cultură, datînd încă din veacul al XV-lea, precum şi lucrări astronomice scrise, de mare valoare — mergînd pînă la elaborarea unor sisteme ale lumii — ce se păstrează şi astăzi în muzeele din Alba Iulia, Cluj-Napoca, Oradea şi alte oraşe (a se vedea bibliografia de la sfîrşitul articolului).*

„Dl. Profesor Gh. Bratu, de la începutul carierei sale științifice, și-a manifestat înclinarea spre Astronomie. În adevăr, bursier prin concurs al Academiei Române în 1909, Dsa a fost trimis la Paris cu specială destinație de a face studii și lucrări practice de Astronomie.

În această calitate, Dsa a obținut diploma de Astronomia aprofundată la Facultatea de Științe din Paris, fiind clasificat în fruntea candidaților.

Ca „Astronome adjoint" la Observatorul din Paris a făcut timp de trei ani (1909–1912) lucrări practice de Astronomie, trecînd succesiv cele trei servicii fundamentale, după cum se vede din certificatul eliberat la 19 Iulie 1912 de Dl. Baillaud, Directorul acelui Observator. Dsa a lucrat 1) *La serviciul meridian* (...); 2) *La serviciul equatorial* (...); 3) *La serviciul fotografic al cerului*.

Certificatul Observatorului din Paris se termină cu următoarele aprecieri: „Rezultă din aceste expuneri că Dl. Bratu, cu activitatea căruia Observatorul nostru se poate lăuda, a făcut la acest Observator un ansamblu de lucrări foarte complete".

Paralel cu aceste studii și lucrări de astronomie, Dl. Bratu a trecut și Doctoratul în Matematici la Sorbona, în iunie 1914. Subiectul tezei sale „*Asupra echilibrului firelor*" e o problemă de Mecanică Analitică în strînsă legătură cu problema de Mecanică cerească a echilibrului maselor fluide.

Întors în țară, Dl. Bratu și-a continuat activitatea paralel în cele două direcții: de Astronomie și de Matematică pură.

În 1919 fiind numit profesor de Analiză Matematică la Facultatea de Științe din Cluj, i s-a încredințat și suplinirea catedrei de Astronomie, iar de la 1 Octombrie 1920 atît Direcția cît și organizarea Observatorului Astronomic din Cluj.

În această privință situația era cu deosebire grea (...). Crearea unui Observator Astronomic la Cluj, mai ales în condițiile financiare de după război, era o problemă deosebit de anevoioasă și dacă azi putem spune că sîntem aproape de realizarea ei completă, aceasta se datorește marilor calități de organizator precum și tenacității, abnegației și muncii neobosite a Dlui Profesor Gh. Bratu. De altfel toți Colegii noștri s-au putut convinge de aceste calități în numeroșii ani în care Dsa a făcut administrația facultății noastre, fie ca decan fie ca prodecan.

Remarcăm de asemenea că Dl. Profesor Bratu a profitat de cercetările științifice făcute în străinătate, publicînd pe lîngă vreo 30 de Memorii de Matematică pură și următoarele lucrări de Astronomie: 1) Efemeridele planetei 498 Tokio; 2) Efemeridele planetei 537 Pauly (...); 3) Despre planeta Marte (...).

De la 1923 la 1928 catedra de Astronomie la Facultatea de Științe din Cluj a fost ocupată de Dl. Profesor G. Demetrescu. De la plecarea Dlui Demetrescu la București în Martie 1928 și pînă astăzi, Dl. Profesor G. Bratu a reluat Direcția Observatorului precum și catedra de Astronomie în suplinire. De atunci Dsa face neîntrerupt cursul de Astronomie la Facultatea noastră. (...)

Deoarece, prin lipsa unui Observator Astronomic, orice cercetare astronomică era imposibilă la Cluj, Dl. Prof. Bratu și-a pus ca prim scop al activității Dsale realizarea acestui Observator și de numele său va rămîne legată această creație.

Putem spune azi, cu legitimă mîndrie și mulțumire, că: Sala meridiană e complet instalată și serviciul regulat al orei este asigurat. Sala ecuatorială e clădită; marea cupolă de 6 m diametru e montată; luneta ecuatorială e complet construită și achitată la Paris și urmează să fie adusă la Cluj în cursul lunii Aprilie 1931. Telescopul aferent e și el gata și ținem să remarcăm că grație relațiilor și intervențiilor Dlui Bratu, oglinda parabolică de 0,50 m diametru a fost construită chiar în atelierele Observatorului Astronomic din Paris. Este pentru prima dată cînd acest Observator consimte să lucreze pentru un alt Observator din lume. În fine, clădirea pavilionului central cuprinde deja subsolul și parterul.

Putem spune cu drept cuvînt că activitatea Dlui Prof. Bratu s-a identificat cu crearea acestui așezămînt de cercetări științifice care, sîntem siguri, va fi o podoabă a facultății noastre.

Ca atare, socotim în unanimitate că locul Dlui Prof. Bratu este la Direcția acestui Observator pentru ca, odată instalat, să poată culege roadele științifice ale strădaniei sale neobosite de 11 ani (...)". (Semnează membrii Consiliului Facultății de Științe: Th. Angheluță – decan, N. Abramescu, A. Maior, E. Racoviță, P. Sergescu, D. Pompeiu, Gh. Spacu ș.a.; actul de arhivă nr. 1054–1930/31.)

Profesorul Gheorghe Demetrescu, fiind numit la Universitatea din Cluj la 1 iunie 1923, dar păstrînd un contact permanent cu Observatorul din București – unde a revenit definitiv la 1 martie 1928 ca prim-astronom și vice-director, a desfășurat, de asemnea, o activitate remarcabilă, atît la catedră, cît și la Observatorul astronomic, contribuind temeinic la formarea primelor promoții de absolvenți în matematică ai Universității, dintre care s-au afirmat ca astronomi valoroși Ioan Armeanca și Ioan Curea.

Activitatea științifică propriu-zisă în cadrul Observatorului din Cluj începe din 1933, cînd profesorul Gh. Bratu angajează Observatorul într-o colaborare cu Observatorul din Paris, la *„Catalogul hărții fotografice a Cerului"*, lucrare de colaborare mondială, condusă de acesta din urmă, din care primului îi revenea reducerea clișeelor fotografice pentru zona de +20° (între anii 1933—1947). Această lucrare — „operă monumentală și istorică", după caracterizarea prof. Gh. Bratu — a avut drept scop eternizarea Cerului secolului XX, imprimîndu-l pe plăci de cupru, dar, evident, înregistrarea în cataloagele cerești era suficientă pentru știință.

În cadrul unei conferințe ținute la Universitatea din Cluj, profesorul Gh. Bratu spunea despre această lucrare:

„Pentru a se putea studia schimbările ce se produc în poziția și în strălucirea stelelor în timp de veacuri, schimbări ce nu pot fi observate în viața unui om, e absolut necesar ca pozițiile și strălucirile actuale ale stelelor să fie *înscrise în cataloage cerești*, spre a se păstra și spre a se putea *compara* situația cerului de azi cu cea a cerului de peste 100, 200, 1000 de ani.

Cum facerea acestor cataloage cere un timp îndelungat și o muncă uriașă, la 1889, la un congres internațional ținut la Paris s-a decis să se *fotografieze* bucată cu bucată toată bolta cerească, făcîndu-se poze de pătrate de pe sfera cerească de 2° lungime pe 2° lățime. La această operă internațională, pusă sub direcția Observatorului din Paris, s-au angajat 24 observatoare din lume (...).

Observatorul din Paris își luase pe seama lui studierea și fotografierea a 4 zone cerești. După cererea mea, la lucrarea unei zone, zona +20°, colaborează și Observatorul din Cluj.

Observatorul din Paris ne procură clișeele și datele necesare și noi la Cluj facem calcule pentru zona +20° — ceea ce reprezintă o muncă de mai mulți ani, pentru a determina cu cea mai mare exactitate *coordonatele cerești* ale stelelor cuprinse în zona dintre paralelele de 19° și 21° latitudine cerească. Cînd lucrarea va fi terminată, rezultatele vor fi publicate la Paris într-un volum special.

Numai cu aceste sacrificii imense și cu această muncă uriașă geniile veacurilor viitoare vor putea descoperi *legi noi* în Știința încă puțin cunoscută, numită *Astrofizica* sau *Astronomia stelară"*.

Observatorul astronomic din Cluj era terminat în 1934, dar îi mai lipseau accesoriile. În următorii patru ani, tinerii colaboratori ai Observatorului realizează completările necesare: *astronomul Ioan Armeanca* (1900—1954) pune în funcțiune laboratorul de fotometrie fotografică și fotoelectrică (printre primele din lume) cu un fotometru Guthnick cu electrometru Lindemann cu cadrane; iar *astronomul Ioan Curea* (1901—1977) reinstalează vechea stație seismică, cu seismografe Mainka.

În 1938, Observatorul din Cluj, condus de Prof. Bratu, este angajat ca unitate de cercetare deja pe *trei direcții fundamentale:*

a) colaborează la „Catalogul hărții fotografice a Cerului" în continuare (prin prof. Gh. Bratu, în colaborare cu I. Armeanca, Gheorghe Chiș și Ștefan Radu);

b) studiul fotoelectric al stelelor variabile (prin astronomul I. Armeanca);

c) studiul seismelor din Transilvania (prin astronomul și seismologul I. Curea).

Profesorul I. Armeanca, socotit primul astrofizician român în sensul strict al cuvîntului, și-a început munca de specializare în domeniul astrofizicii la Göttingen, unde se dedică fotometriei stelare. Timp de trei ani aici, apoi la Observatorul din Kiel, sub conducerea profesorilor Rosenberg și Stobbe, își încheie teza sa de doctorat cu titlul *„Străluciri fotografice și fotovizuale ale stelelor* *din vecinătatea polului"* (1933), care constituie o remarcabilă contribuţie la extinderea fotometriei fotografice, fiind citată în toate lucrările de bază de fotometrie. I. Armeanca a extins secvenţa polară Nord la toate stelele dintr-o regiune de 100′ × 100′, stabilind cu multă precizie strălucirile fotografice a 260 de stele pînă la magnitudinea de 16,25 şi strălucirile fotovizuale a 220 de stele pînă la magnitudinea de 14,71, utilizînd metoda diferenţială. A făcut un studiu comparativ al fotometrului termoelectric Zeiss cu cel fotoelectric Rosenberg şi a stabilit ecuaţiile de culoare şi cele de distanţă ale obiectivelor. Pe baza secvenţei polare a lui I. Armeanca s-a obţinut o creştere a preciziei fotometriei fotografice şi vizuale şi a sporit posibilitatea de utilizare a ei.

Fotometrul achiziţionat şi asamblat în anii 1936—1938 este instalat în anii 1939—1940 la telescopul Newton al Observatorului din Cluj, dînd rezultate foarte bune.

Profesorul I. Curea, pasionat astronom şi seismolog, realizatorul de mai tîrziu — în calitatea sa de rector — al Universităţii şi Observatorului astronomic din Timişoara, ca şi al staţiilor seismice din Banat, şi-a adus o contribuţie importantă la consolidarea direcţiilor de cercetare astronomică din Cluj. În teza sa de doctorat, referitoare la determinarea polului ceresc pe cale fotografică, dă o metodă proprie, care a fost utilizată şi peste hotare în lucrările de astronomie.

O bruscă scădere a activităţii Observatorului astronomic din Cluj a avut loc odată cu declanşarea războiului şi, ca urmare a Dictatului de la Viena, Observatorul, împreună cu Facultatea de Ştiinţe căreia îi aparţinea, este mutat la Timişoara cu întreaga-i zestre, mai puţin cupola şi celelate clădiri. Luneta ecuatorială este reinstalată provizoriu în Grădina horticolă din centrul oraşului, într-o clădire de lemn, unde, la adăpostul camuflajului impus de rigorile războiului, astronomul I. Armeanca reia observaţiile fotoelectrice. Dar în urma unui bombardament este distrus întreg echipamentul fotoelectric, încît după 1945, cînd Observatorul revine la vechea matcă din Cluj, pot fi continuate doar lucrările de fotometrie fotografică.

Profesorul Gheorghe Bratu, greu lovit de evacuarea şi greutăţile de reinstalare a Observatorului la Timişoara, moare fulgerător, la 1 septembrie 1941, în deplină capacitate creatoare.

Între anii 1941—1945, directorul Observatorului din Cluj-Timişoara a fost *profesorul Constantin Pîrvulescu* (1890—1945), profesor de astronomie la Facultatea de Ştiinţe a Universităţii refugiate, care — după cum se ştie — a deschis cercetării româneşti drumul astronomiei galactice (studiul stelelor duble, al roiurilor stelare, al rotaţiei Galaxiei) şi al celei extragalactice.

În *perioada 1945—1954*, începută prin repunerea instrumentelor în stare de funcţionare, activitatea Observatorului din Cluj a fost condusă cu multă competenţă şi autoritate de *profesorul Ioan Armeanca*. Se continuă tradiţionala problemă a studiului stelelor variabile pe calea fotometriei vizuale şi fotografice şi se încheie lucrarea de colaborare cu Observatorul din Paris. Rezultatele acestei colaborări sînt inserate în lucrarea *„Catalogue de 11.755 étoiles de la zone +17° à +25° et de magnitudes 9,5 à 10,5", Publications de l'Observatoire de Paris, Ed. Gauthier-Villars, 1950.*

În anul 1951, Observatorul este transferat de la Universitate la Filiala din Cluj a Academiei R. P. Române, organizîndu-se ca unitate de cercetare. În *perioada* 1951—1961, Observatorul a primit un sprijin substanţial de la Academie, atît pentru dezvoltarea planului tematic, cît şi pentru creşterea bazei materiale şi a numărului de cercetători, căpătînd şi personal tehnic-administrativ. Noile condiţii de lucru, precum şi posibilităţile create pentru noi colaborări internaţionale — în special cu unele observatoare sovietice (Moscova, Odesa) — deschid o nouă perspectivă cercetării astronomice clujene. În afară de I. Armeanca, Gh. Chiş şi Şt. Radu, care se aflau la Observator, în perioada menţionată au venit la această instituţie succesiv: *Ioan Todoran* (1 decembrie 1951), *Elvira Botez* (1 decembrie 1951, pînă în anul 1962), Árpád Pál (1 mai 1957, după efectuarea stagiului de doctorat la Universitatea „M. V. Lomonosov" din Moscova, Institutul Astronomic „P. K. Şternberg", Catedra de Mecanică cerească).

Din 1961, Observatorul astronomic trece în cadrul Universităţii „Babeş—Bolyai" din Cluj, păstrîndu-şi structura de unitate de cercetare.

După moartea prematură a prof. I. Armeanca, conducerea Observatorului din Cluj este preluată de *profesorul Gheorghe Chiş*, elev al prof. Gh. Bratu şi colaborator al prof. I. Armeanca; timp de 23 de ani (1954—1977) el va conduce această instituţie cu acelaşi devotament ca şi predecesorii săi.

Profesorul Gh. Chiş şi-a început activitatea la Observatorul din Cluj la 1 februarie 1936, fiind numit în postul de preparator. A fost trecut asistent, în cadrul acestui Observator, la 1 februarie 1943, şi şef de lucrări în acelaşi an, la 1 decembrie. La 1 octombrie 1950 devine conferenţiar de matematici generale, iar la 1 octombie 1954 trece la specialitatea sa, astronomie şi astrofizică. În data de 1 ianuarie 1960 ocupă postul de profesor titular în această specialitate, pe care o păstrează pînă la 1 iulie 1977, cînd devine profesor consultant prin ieşirea la pensie. Între anii 1962—1968 a fost decanul Facultăţii de Matematică-mecanică a Universităţii din Cluj.

Cercetările ştiinţifice ale prof. Gh. Chiş se referă la următoarele patru domenii: a) probleme de astromertie prin: participarea la „*Catalogul hărţii fotografice a cerului*" (participare amintită mai sus), determinări de coordonate geografice, determinări de poziţii de comete, planete mici şi sateliţi artificiali, cercetări cu caracter astronomic asupra calendarului geto-dacic din vestigiile sanctuarului de la Sarmizegetusa; b) probleme de stele variabile prin studii fotometrice — fotovizuale, fotografice şi fotoelectrice — ale stelelor binare fotometrice şi de tip RR Lyrae, reintroducînd metoda fotometriei fotoelectrice la Observatorul din Cluj; c) probleme de mecanică cerească prin determinări de orbite de comete şi de sateliţi artificiali ai Pămîntului; d) probleme de cercetări spaţiale, prin înfiinţarea în cadrul Observatorului din Cluj a *Staţiei de observare a sateliţilor artificiali* (*cod COSPAR:* 1132) şi participarea la programele de colaborare internaţională INTEROBS, INTERKOSMOS, EUROBS, vizînd folosirea observaţiilor sateliţilor artificali ai Pămîntului (de poziţie şi fotometrice) la studiul variaţiilor parametrilor structurali ai atmosferei înalte a Pămîntului, în corelaţie cu variaţiile indicilor activităţii solare şi geomagnetice.

Dintre toate domeniile pe care prof. Gh. Chiş le-a îmbrăţişat, astrofizica a rămas domeniul său de predilecţie. Dovadă a importanţei lucrărilor sale şi ale colaboratorilor săi din acest domeniu, în 1974, prof. Gh. Chiş a fost ales preşedinte al *Subcomisiei nr.* 5 *(Stele duble)*, în cadrul Comisiei de colaborare internaţională între academiile de ştiinţe din ţări socialiste, în problema „*Fizica şi* *evoluția stelelor*", precum și vicepreședinte al Comitetului Național Român de Astronomie (pînă în 1980).

În anul 1976, extinderea orașului obligă Observatorul la *strămutarea instrumentelor de observații* (achiziționate de Gh. Bratu) pe dealul Feleacului, *în zona Făget* (8 km sud de Cluj-Napoca, 750 m altitudine), existînd aici condiții de astroclimat adecvate unor observații astrofizice. Conform hotărîrii Senatului Universității, clădirea și cupola Stației de observare din Făget au fost construite — după concepția profesorului Gh. Chiș, executantă fiind Întreprinderea „Electrometal" Cluj-Napoca — din resurse interne și cu forțe locale, alocînd în acest scop circa 1 000 000 lei (Fig. 1). Din anul 1977, Stația de observare din Făget trece în nomenclatorul Centrului de Astronomie și Științe Spațiale București, care în noua organizare, încadrează toate cadrele de cercetare astronomică din țară, care lucrează pentru această unitate de cercetare.

Rămînîd la catedră și la Observator și după pensionare, profesorul Gh. Chiș a condus Seminarul de cercetare „Structura și evoluția stelelor", precum și activitatea unor doctoranzi în specialitatea „Astronomie și astrofizică", pînă în ultimele zile ale vieții sale. În urma unei boli necruțătoare, el s-a stins din viață la 19 mai 1981.

Din 1977, ca director al Observatorului astronomic a fost numit *profesorul Árpád Pál*, decanul de atunci al Facultății de Matematică a Universității (1976—1984). Colectivul de astronomi clujeni, la începutul noii perioade, a avut de depășit nenumărate dificultăți și piedici, care păreau uneori de neînvins pentru activitatea astronomică: mutarea provizorie a zestrei și personalului Observatorului, inclusiv a cercetătorilor C.A.S.S., în clădirea Institutului de Matematică de pe lîngă Facultatea de Matematică a Universității (str. Republicii nr. 37), cu excepție cupolei vechi, care a fost adăpostită în Parcul Sportiv al Universității; demolarea vechilor clădiri (din str. Republicii nr. 109), în 1978 — în locul lor fiind construită moderna Întreprindere de Electronică Industrială și Automatizări; demersuri și străruințe pentru obținerea aprobărilor și fondurilor necesare reconstruirii Observatorului pe un nou amplasament, în valoare totală de 2 000 000 lei; și mai apoi, coordonarea lucrărilor de construcție și montaj (proiectantul fiind ELECTROUZINPROIECT București, iar executantul — S.C.P.C. Cluj-Napoca) și mutarea în noul edificiu.

Era pentru noi un motiv de legitimă satisfacție să consemnăm încheierea lucrărilor și darea în folosință, în vara anului 1982, a unui *modern și impunător pavilion al Observatorului astronomic*, în extremitatea sudică a Grădinii Botanice a Universității (str. Cireșilor 19), a cărei vegetație bogată condiționează aerul din jurul Observatorului (Fig. 2). Noua clădire asigură condiții superioare de desfășurare a activităților didactice și de cercetare științifică, în ea fiind amplasate: laboratoare didactice și de cercetare — printre care sala cupolei (vechi), adăpostind noul refractor Coudé ($D = 150$ mm, $F = 2250$ mm), achiziționat de la firma Zeiss din R.D.G., în 1980 (Fig. 3), sala meridiană și Stația de observare a sateliților artificiali ai Pămîntului (pe terasa clădirii), biblioteca și sala de lectură, cabinetul de astronomie, atelierele de mecanică și electronică, centrala termică și celelalte utilități.

De un deosebit sprijin am beneficiat în realizarea acestui obiectiv din partea organelor județene și municipale de partid și de stat, din partea Ministerului Educației și Învățămîntului, precum și din partea proiectanților și

Fig. 1. Stația de observare din Făget.

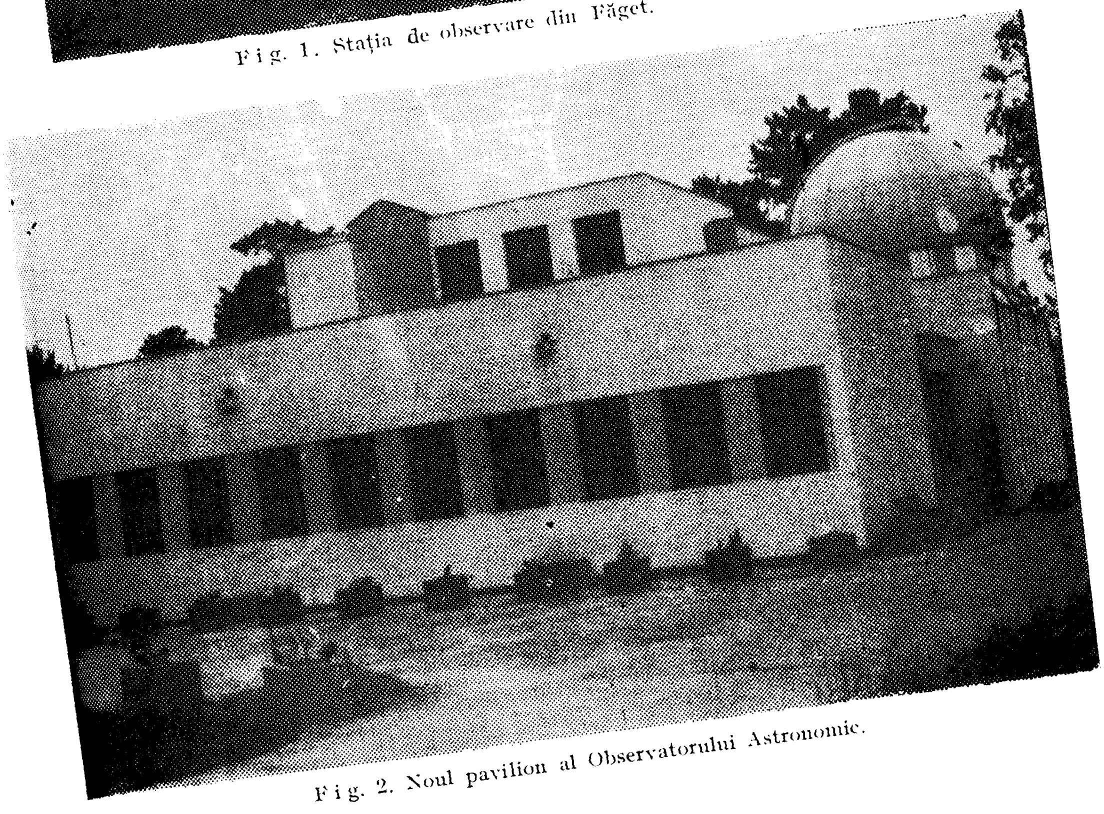

Fig. 2. Noul pavilion al Observatorului Astronomic.

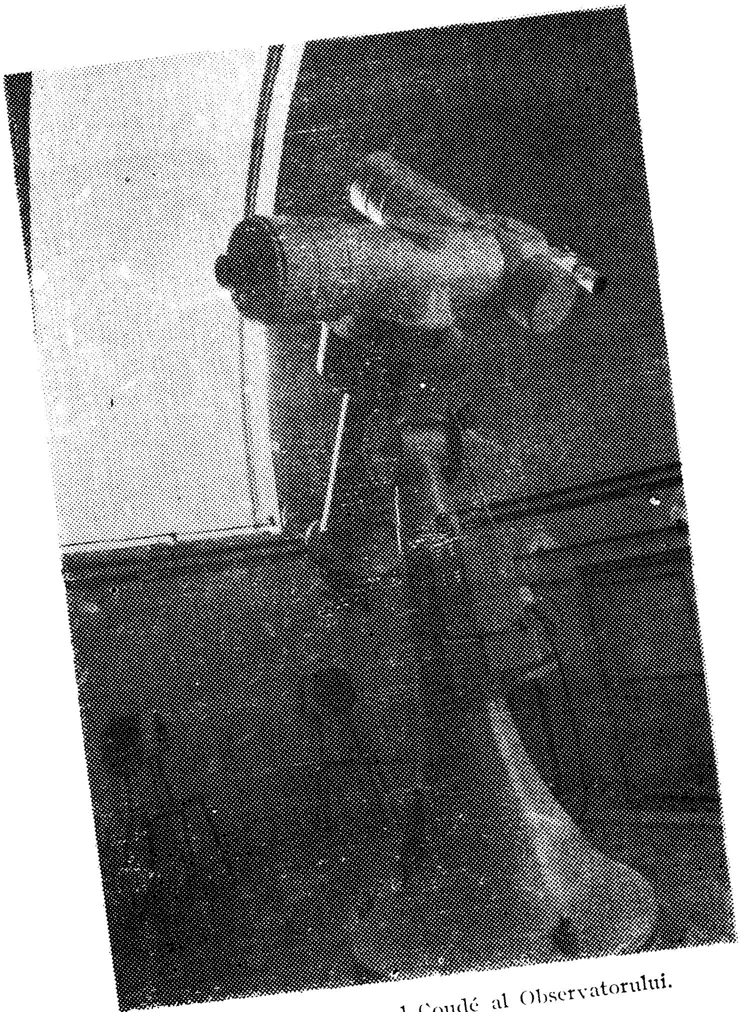

Fig. 3. Refractorul Coudé al Observatorului.

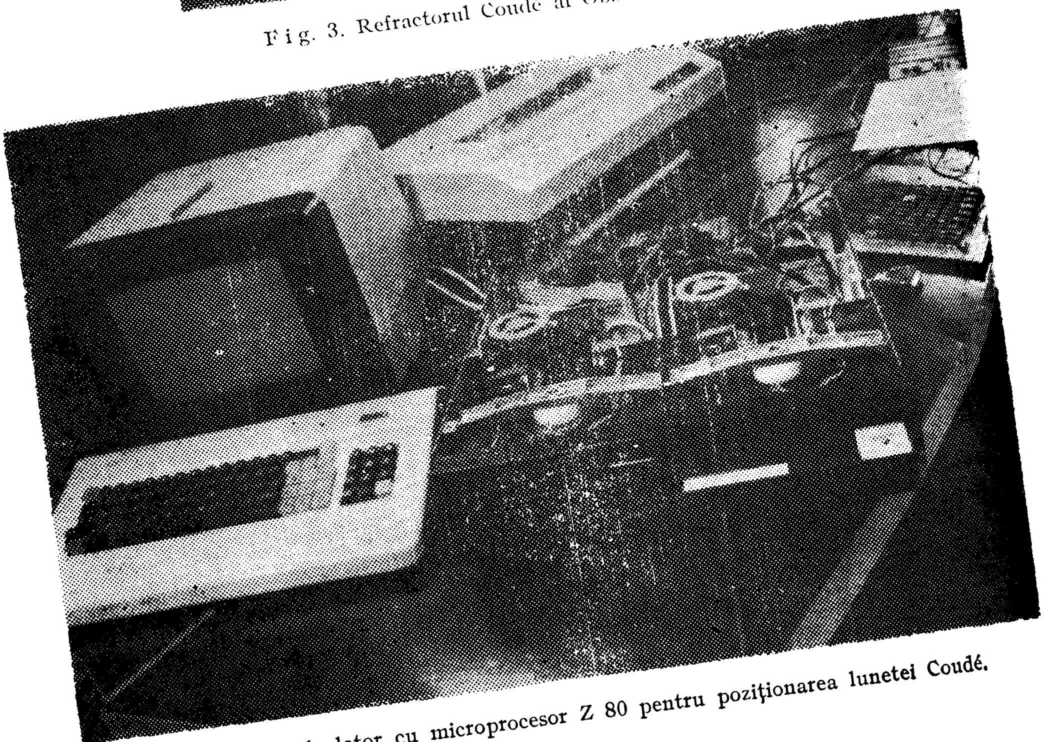

Fig. 4. Calculator cu microprocesor Z 80 pentru poziţionarea lunetei Coudé.

constructorilor întreprinderilor amintite. Tuturor le exprimăm, şi pe această cale, cele mai vii mulţumiri şi profunda noastră recunoştinţă!

În ultima vreme, în *dotarea Observatorului astronmoic* au intrat mai multe aparate noi: un complex de aparate pentru măsurarea timpului, aparate de măsurare a coordonatelor cereşti (teodolit, sextant, stereocomparator), calculatoare electronice, instalaţie pentru poziţionarea automată (pe bază de microprocesor) a lunetei Coudé (Fig. 4) ş.a., care contribuie atît la îmbunătăţirea cercetării ştiinţifice, cît şi a procesului didactic.

S-a mărit, de asemenea, fondul de cărţi şi publicaţii periodice, Observatorul astronomic dispunînd la ora actuală de o zestre de peste 16000 de volume (cărţi şi reviste), precum şi de alte materilale documentare de mare valoare culturală.

Cercetarea ştiinţifică în cadrul Observatorului nostru se realizează *în prezent* în cadrul a *două seminarii de cercetare*, ale căror tematici le vom schiţa în cele ce urmează.

Preocupări în cadrul *Seminarului „Mecanică cerească şi cercetări spaţiale"* (înfiinţat în anul 1972; conducător: dr. Á. Pál): 1. *Probleme de mecanică cerească*: varietăţi diferenţiabile şi topologice cu aplicaţii în mecanica cerească; studiul metodei medierii şi al aplicaţiilor ei în diferite probleme de mecanică cerească (orbite intermediare ale asteroizilor, cometelor, sateliţilor artificiali ş.a.); studiul problemei restrînse a celor trei corpuri şi elaborarea de modele matematice pentru cazul eliptic al acestei probleme; aplicarea metodei transformărilor Lie la studiul mişcării perturbate a corpurilor cereşti; aplicarea unor metode topologice la studiul problemei celor două şi trei corpuri; studiul soluţiilor cu ciocniri în problema a două şi n (≥ 3) corpuri. 2. *Teoria mişcării sateliţilor artificiali ai Pămîntului (SAP)*: studiul mişcării perturbate a SAP sub influenţa diferiţilor factori gravitaţionali şi negravitaţionali; studiul metodelor de integrare numerică a ecuaţiilor mişcării perturbate a SAP; elaborarea de modele matematice ale mişcării SAP, ca şi de algoritmi şi programe de calcul în vederea rezolvării acestor modele; determinări şi ameliorări de orbite ale SAP; determinări de efemeride; studii asupra mişcării sateliţilor geostaţionari; studiul mişcării de rotaţie a SAP în jurul centrului propriu de masă. 3. *Probleme privind structura atmosferei terestre în cadrul relaţiilor Soare-Pămînt*: studiul evoluţiei parametrilor de stare ai atmosferei înalte pe baza datelor observaţionale asupra frînării orbitele a SAP în atmosferă; elaborarea de formule (legi empirice) noi pentru aproximarea parametrilor de stare ai atmosferei înalte; studiul corelaţiei între evoluţia parametrilor de stare ai atmosferei înalte şi activitatea solară şi geomagnetică; elaborarea de algoritmi şi programe de calcul pentru determinarea de valori ale densităţii şi ale altor parametri de stare ai atmosferei înalte. 4. *Probleme privind observarea sateliţilor artificiali*: studiul vizibilităţii sateliţilor artificiali (condiţii de vizibilitate, vizibilitatea satelit-satelit, cazuri particulare); elaborarea de algoritmi şi programe de calcul pentru determinarea de efemeride ale SAP; studii asupra metodelor de reducere a observaţiilor de SAP; probleme privind identificarea SAP (metode, criterii).

O bună parte a rezultatelor obţinute au apărut în următoarele *publicaţii ale Seminarului*: *Visual Observations of Artificial Earth Satellites. Supplement Dedicated to the Twentieth Anniversary of the First Artificial Earth Satellite Launch.*

University Babeș—Bolyai Cluj-Napoca, Astronomical Observatory, Satellite Tracking Station No. 1132, Cluj-Napoca, 1977 ; *„Babeș—Bolyai" University, Faculty of Mathematics, Research Seminaries, Seminar of Celestial Mechanics and Space Research, Preprints* : 2/1980, 3/1982, 2/1984, 10/1985.

Preocupări în cadrul *Seminarului „Structura și evoluția stelelor"* (înființat în anul 1947, actuala denumire datînd din 1977 ; conducător : dr. V. Ureche) : *1. Interpretarea curbelor de lumină la sisteme stelare binare strînse* : modele de interpretare a curbelor de lumină ; studiul efectului de reradiație (reflexie) ; studiul efectului de elipticitate ; determinarea elementelor absolute ale componentelor. *2. Studiul variației perioadei la binare strînse* : deplasarea liniei apsidale ; efectul relativist în mișcarea liniei apsidale ; prezența celui de-al treilea corp ; evoluția sistemelor binare în faza transferului de masă ; construirea de suprafețe echipotențiale Roche și studiul stabilității. *3. Studiul fotometric al unor variabile de tip RR Lyrae* : construirea de curbe de lumină din observații ; determinări de perioade multiple ; studiul efectului Blajko ; ajustarea curbelor observate cu funcții spline ; determinarea parametrilor fizici. 4. *Studiul stabilității pulsaționale a stelelor* : efectul rotației asupra stabilității ; efectul mareic asupra stabilității. 5. *Studiul structurii și stabilității stelelor relativiste* : modele analitice și semianalitice de structură internă : omogen, liniar, politropic, de tip „stepenar" ; criterii de stabilitate a stelelor relativiste ; razele critice și masele maxime ale stelelor neutronice ; geometria continuumului spațiu-timp în interiorul și în vecinătatea obiectelor relativiste ; diagrame de imersiune ; energia gravitațională a stelelor relativiste.

Mare parte a rezultatelor obținute au apărut în următoarele *publicații ale Seminarului* : *Contributions of the Astronomical Observatory, Univ. „Babeș—Bolyai"*, Cluj-Napoca, 1976 ; *Contributions of the Astronomical Observatory. Proceedings of the Colloquium of Astronomy, Section Astrophysics, Cluj-Napoca, November 1977*, Cluj-Napoca, 1978; *„Babeș—Bolyai" University, Faculty of Mathematics, Research Seminaries, Seminar of Stellar Structure and Stellar Evolution, Preprints* : 4/1983, 2/1985, 6/1986.

În virtutea acestor preocupări, pe care îi este axată activitatea, Observatorul clujean participă la mai multe *colaborări internaționale*, sub egida Uniunii Astronomice Internaționale (IAU = International Astronomical Union) și a Comitetului de Cercetări Spațiale (COSPAR = Committee on Space Research), precum și în cadrul a două cooperări între academiile de științe din țări socialiste : „Fizica și evoluția stelelor" — în care Observatorului nostru îi revine coordonarea Subcomisiei (Subproiectului) nr. 5 „Stele duble" și „Fizica cosmică", în cadrul temei „Cercetări și experimente comune cu ajutorul observațiilor sateliților artificiali în scopuri astronomice, geofizice și geodezice". Țara noastră a găzduit ultimele reuniuni ale acestor colaborări în 1982 și, respectiv, în 1983.

Pentru dezvoltarea relațiilor internaționale ale cercetării astronomice românești, în 1930 s-a înființat Comitetul Național Român de Astronomie, care a devenit membru al U.A.I., unde rezultatele noastre au primit o binemeritată apreciere, oglindită și în faptul că numai dintre astronomii clujeni cinci au fost aleși membri individuali ai acestui for internațional.

Observatorul astronomic are și menirea de a contribui la pregătirea viitorilor profesori de matematică și de fizică, în cadrul *cursurilor și seminariilor* cuprinse în planurile de învățămînt ale secțiilor respective, *lucrările practice de obser-*

vaţii fiind singurele activităţi care pot asigura o bună înţelegere, aprofundare şi asimilare a cunoştinţelor teoretice predate la cursuri şi exersate în seminarii. La această activitate cu studenţii participă, deopotrivă, cadrele didactice şi cercetătorii Observatorului. Cursurile, culegerile de probleme, programe de calcul şi lucrări de laborator, publicate în edituri sau litografiate pe plan local de către colectivul nostru reprezintă şi ele tot atîtea ajutoare în pregătirea de specialitate a studenţilor şi în iniţierea lor în cercetarea ştiinţifică.

Ca lăcaş de ştiinţă şi cultură, destinat studiului cerului înstelat, Observatorul atrage o frecvenţă abundentă de vizitatori, în special clase de elevi, terasa lui spaţioasă de observaţii — unde se pot aşeza mai multe intrumente astronomice portabile de amatori — fiind adecvată organizării unor şedinţe demonstrative ce contribuie, începînd cu ,,trăirea emoţională a lui Galilei", însoţită de explicaţii ale specialistului, la formarea unei concepţii juste, materislist-ştiinţifice despre Univers.

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12. N. Teodorescu, G. Chiş, *Cerul — o taină descifrată* (*Astronomia în viaţa societăţii*), Ed. Albatros, Bucureşti, 1982.

RECENZII

1) Edward W. Stredulinsky, **Weighted Inequalities and Degenerate Elliptic Partial Differential Equations,** Lecture Notes in Mathematics. Vol. 1074, 143 pages, 1984.

The main purpose of this book is the investigation of various weighted spaces and weighted inequalities which are relevant to the study of solvability of the problems concerning linear or non-linear partial differential equations and in the analysis of the properties of the solutions. The usefulness of these results is illustrated in the latter part of the book where they are used to establish continuity for weak solutions of degenerate elliptic equations.

The book may be used by the specialists who work in the domain of the theory of partial differential equations and by the students who are specializing in this domain.

S. SZILÁGYI

K. Jarosz, **Perturbations of Banach Algebras,** Lecture Notes in Mathematics 1120, Springer Verlag 1985, 117 pp.

The book is dealing with three kinds of small perturbations for Banach algebras: ε-perturbations of the multiplication, ε-isomorphism and ε-isometries. The author proves stability results under small perturbations for various classes of Banach algebras. As the theory is only at the initial stage the author states many problems — the book ends with a list of 20 open problems.

S. COBZAŞ

K. Sundaresan, S. Swaminathan, **Geometry and Nonlinear Analysis in Banach Spaces,** Lecture Notes in Mathematics 1131, Springer Verlag 1985, 115 pp.

The book is concerned with differential nonlinear analysis in infinite dimensional Banach spaces. The rich and elegant finite dimensional theory do not extend automatically to the infinite dimensional case. The authors treat topics as: Smoothness classification of B-space, Smooth partitions of the unity, Smoothness and approximation, Infinite dimensional differentiable manifolds. The book is clearly written, collects together many topics scattered in various journals and will be useful to a large class of mathematicians (especially analysts).

S. COBZAŞ

Palle T. E. Jorgensen and Robert T. Moore, **Operator Commutation Relations,** Mathematics and its Applications, D. Reidel Publishing Co. 1984, 493 pp.

The authors consider infinitesimal and global commutation relations for operators showing that apparently distinct topics can be unified, in an unexpected way, through a certain analysis of operator commutation relations, leading as well to new results in diverse areas of mathematics and its applications. The book is of interest for mathematicians (both pure and applied) and for researchers in mathematical physics and quantum chemistry.

S. COBZAŞ

Jerrold Marsden, Alan Weinstein, **Calculus I, II and III,** Springer Verlag New York, Berlin, Heidelberg, Tokio 1985.

This three-volume book represents a very good introduction to real differential and integral calculus. In a didactic and rigorous manner the authors present the basic notions and results including many geometric and physical aspects of calculus with a wealth of excellent applications. Each volume contains many solved and proposed exercices and problems. The book presents interest and is useful for the students in mathematics.

D. ANDRICA

Banach Center Publications vol. 14 **Mathematical Control Theory,** Edited by C. Olech, B. Jakubczyk and J. Zabczyk, P.W.N. Warszawa 1985, 643 pp.

These are the Proceedings of the XVI-th semester of the Banach International Mathematical Center (September—December 1980). The book contains forty papers covering various topics in optical control theory, written by eminent specialists in the field as A. Bensoussan, L. D. Berkovitz, F. H. Clarke, R. Gabasov, J-L. Lions, P-L. Lions, F. Mignot, S. Rolewicz et al.

S. COBZAŞ

Jindrich Nečas, **Introduction to the Theory of Nonlinear Elliptic Equations,** Teubner-Texte für Mathematik, Band 52, Leipzig, 1983.

În prezenta carte autorul studiază probleme la limită pentru ecuaţii cu derivate parţiale de ordinul al doilea de tip eliptic. Se studiază probleme ca: spaţii Sobolev şi Morrey-Camponate, soluţii slabe, metode aproximative, regularitatea soluţiilor şi aplicaţii în teoria elasticităţii. Cartea profesorului J. Nečas reprezintă o foarte bună introducere în teoria problemelor la limită relative la ecuaţii cu derivate parţiale de tip eliptic neliniare.

I. A. RUS

Lars Hörmander: **The Analysis of Linear Partial Differential Operators**; Vol. 1: *Distribution Theory and Fourier Analysis*; Vol.2: *Differential Operators with Constant Coefficients*, Springer-Verlag, Berlin, 1983.

The volumes I and II are a systematic study of distribution theory and of partial differential operators with constant coefficients, Basic properties of distributions, Convolutions, Fourier transformation, Spectral analysis of of singularities, Hyperfunctions, Existence and approximation of solution of differential equations, Differential operators of constant strenghth, Scattering theory, Analytic function theory and differential equations, Convolution equations. These two volumes are part of a remarkable book of highest quality and of greatest importance for research workers and graduate students in mathematics.

I. A. RUS

Hideyuki Majima, **Asymptotic Analysis for Integrable Connections with Irregular Singular Points,** Lect. Notes in Math., 1075, Springer-Verlag (1984).

The book is an excellent research monograph. Using strongly asymptotic expansions of functions of several variables, the author proves existence theorems of asymptotic solutions to integrable systems of partial differential equations under certain general conditions. Other topics in this book: Riemann—Hilbert—Birkhoff problem, Poincaré's lemma and de Rham cohomology theorem.

I. A. RUS

E. Zeidler, **Nonlinear Functional Analysis and Its Applications. III.** *Variational Methods and Optimization*, Springer Verlag 1985, 662 pp.

The book is a considerably expanded version of the book of the author, „Vorlesungen über nichtlineare Funktionalanalysis III". *Variationsmethoden und Optiemierung*, Teubner Texte zur Mathematik Leipzig 1977, 239 pp., and belongs to a cycle of five books on nonlinear functional analysis: I Fixed point teohrems, II Monotone operators, IV—V Applications to mathematical physics, published originaly in German as Teubner Texte and translated (and expanded) in English and published by Springer Verlag. This is a comprehensive monograph on optimization and variational problems. The book is very well organized and very clear written. Each chapter (and there are 57 chapters) ends with a set of problems and bibliographical comments. The bibliography is very extensive (30 pages). The book ends with a list of symbols, a list of theorems and an index of notions. The book is a valuable contribution to optimization theory and related topics.

S. COBZAŞ

Jean Paul Gauthier. **Structure des systemes non-lineaires.** Éditions du CNRS, Paris, 1984, 307 p.

Dans l'Introduction du livre on présente les idées genérales, les sources et les buts du travail. Les rappels néccessaires de géométrie différentielle et de Topologie, ainsi que la théorie du controle des systèmes non-linéaires avec ses applications sont développés d'une manièer attractive et accesible d'après le schéma suivant: I. Variétés différentiables. II. Gouvernabilité. III. Observabilité et Observateurs. IV. Stabilisation. V. Découplage. VI. Bibliographie. Le livre est destiné aux étudiants qui débutent dans la recherche, autant que aux spécialistes en Automatique.

M. ŢARINĂ

Graphentheorie: eine Entwicklung aus dem 4-Farben Problem, von Martin Aigner, Stuttgart: Teubner 1984 (Teubner-Studienbücher: Mathematik) ISBN 3–519–02068–8.

Das vorliegende Buch eines bekannten Autors enthält eine sehr gute Einführung in die Graphentheorie mit nahezu allen wichtigen Begriffen und Resultate. Es wird dabei insbesondere die wichtige Rolle geschildert die das 4-Farben Problem in der Entwicklung der Graphentheorie spielte: sein Ursprung, die ersten Versuche zur Lösung des Problems mit all seinen Sackgassen und schliesslich seine ungewöhnliche Lösung mit Hilfe des Computers.

H. KRAMER

Global Analysis — Studies and Applications I, (Edited by Yu. G. Borisovich and Yu. E. Gliklikh), Lectures Notes in Mathematics vol. 1108, Springer Verlag 1984, 301 pp.

The volume contains the translations of the Voronezh University Press series ,,Novoe v global'nom analyze" for the years: 1982 — Equations on manifolds; 1983 — Topological and geometrical methods in mathematical physics; 1984 — Geometry and topology in global nonlinear problems. The aim of the series is to publish survey (expository) papers and a small number of short communications. Among the members of the editorial board and contributors there are well known specialists as A. T. Fomenko, A. S. Mishchenko, S. P. Novikov, M. M. Postnikov, A. M. Vershik et al. The translation and publication in Lectures Note Series make these important contributions to global analysis accessible to a larger set of readers.

S. COBZAŞ

Nonlinear Analysis and Optimization, Bologna 1982, Edited by C. Vinti, Lecture Notes in Mathematics vol. 1107, Springer Verlag 1984, 214 pp.

These are the Proceedings of a meeting organized in Bologna in Honour of Professor Lamberto Cesari (a similar meeting took place in 1980 at the University of Texas at Arlington). The book begins with a paper of D. Graff on L. Cesari scientific activity and a paper of J. Serrin, Applied mathematics and scientific thought. There are also ten contributed papers by eminent specialists in the field: L. Cesari himself, A. Bensoussan, J. Frehse, J. P. Gossez, P. Hess, R. Kannan, J. Mawhin et al.

S. COBZAŞ

Y. Okuyama, **Absolute Summability of Fourier Series and Orthogonal Series,** Lecture Notes in Mathematics vol. 1067, Springer Verlag 1984, 117 pp.

The absolute summability of a series is a generalization of the concept of absolute convergence just as the summability is an extension of the concept of convergence. The absolute summability methods for non-absolute convergent series (Nörlund — and Riesz — absolute summability) are given both for trigonometric series and for the Walsh orthogonal system. The book will be useful to all interested in harmonic analysis.

S. COBZAŞ

Raghavan Narasimhan, **Complex Analysis in One Variable,** Birkhäuser Verlag 1985, 266 pp.

The aim of this book is to present, from a modern point of view, the theory of functions of one complex variable, relating the subject to other branches of mathematics, especially several complex variables (a field which owes much to the author of this book). The author achieves masterly this end and the result is an excelent monograph in complex function theory. The book also contains a chapter (Chapter 8) on several complex variables but, as the author points out in the preface, as a whole, the book is about one variable.

S. COBZAŞ

P. Schapira, **Microdifferential Systems in the Complex Domain,** Grundlehren der mathematischen Wissenchaften vol. 269, Springer Verlag 1985, 214 pp.

The subject of this book involves several branches of mathematics as: microlocal analysis, linear partial differential equations, algebra and complex analysis. Its aim is to present, at an accessible level, to the analyst the algebraic methods used in this field and to the algebraist some topics from partial differential equations. The book is a very good introduction to this difficult and very active domain of research.

S. COBZAŞ

H. Schlichtkrull, **Hyperfunctions and Harmonic Analysis on Symmetric Spaces,** Progress in Mathematics vol. 49, Birkhäuser Verlag 1984, 1985 pp.

The book is divided in two parts. The first one (Chapters 1 and 2) is expository (few proofs are given) and gives an introduction to microlocal analysis and hyperfunctions. In the second part, containing also some original contributions of the author, these results are applied to symmetric spaces. The book is an outhgrowth of an essay which received a gold medal from the University of Copenhagen.

S. COBZAŞ

Albrecht Fröhlich, **Classgroups and Hermitian Modules,** Progress in Mathematics, Birkhäuser Verlag 1984, Boston—Basel—Stuttgart.

Cartea conţine o expunere sistematică şi detailată a abordării cu ajutorul omorfismului Galois a diferitelor clasgrupuri ataşate ordinelor şi în particular inelelor grupale, abordare care se dovedeşte fundamentală în cercetări recente. Cartea este utilă în cercetări de teoria numerelor algebrice, K-teorie, forme patratice şi Hermitiene şi teoria modulară.

GR. CĂLUGĂREANU

H. Junek, **Locally Convex Spaces and Operator Ideals,** Teubner Texte zur Mathematik, Band 56, Leipzig, 1983, 180 pp.

A. Pietsch was the first who studied operator ideals in Banach spaces and applied them to nuclear spaces. The author gives in this book a systematic exposition of the theory of ideals of operators ranging in locally convex spaces (LCS), showing that many properties of several classes of LCS are just consequences of some stability properties of operator ideals acting on them. A special attention is paid to $F-$, $DF-$ spaces, to spaces of differentiable and holomorphic functions and to spaces of unbounded operators.

S. COBZAŞ

Recent Trends in Mathematics, Reinhardsbrun 1982, Teubner Texte zur Mathematik, Band 50, Leipzig 1983, 329 pp.

These are the Proceedings of a Conference held in Reinhardsbrunn RDG, from October 11 to October 13, 1982, editde by H. Kurke, J. Mecke, H. Triebel and R. Thiele. The conference was attended by 62 mathematicians working in various branches of mathematics (S. V. Bochkariev, L. D. Kudryavtsev, Z. Cieselski, W. Dickmeis, R. J. Nessel, K-H. Elster, A. Göpfert, J. Necas et al.). The book contains 40 of the contributed papers, the programme of the conferenceand the list of participants.

S. COBZAŞ

Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Leipzig 1983, Teubner Texte zur Mathematik, Band 67, Leipzig 1984, 234 pp.

The book contains the contributions of the participants at this conference grouped in three sections: A. Topological algebras and their representations; B. Operator ideals and geometry of Banach spaces, and C. Algebraic approach to quantum field theory and statistical physics. Sections A + C contain 18 papers and Section B, 12 papers. The book contains valuable contributions to these fields and it is of interest for a large class of mathematicians and physicists.

S. COBZAŞ

Thomas Zink, **Cartiertheorie kommutativer formaler Gruppen,** B. G. Teubner, Leipzig, 1984.

The theory of commutative formal groups has a special importance in algebraic number theory and in algebraic geometry over a field of characteristic p. The french mathematician

P. Cartier found a new approach to this theory which is simpler and more general than others and which has interesting applications to abelian manifolds. The book of Th. Zink is for students and mathematicians interested in algebraic geometry or number theory and familiar with commutative algebra. It presents the theory in a new way based on concepts of deformation theory. During the six chapters of the book, besides the main theorems of the theory, basic facts on isogenies, deformations of p-divisible formal groups and Dieudonne's classification are treated.

RODICA COVACI

L. Lovász, M. D. Plummer: **Matching Theory**, Akadémiai Kiadó, Budapest, 1986, 544 + XXXIII pp.

This book deals with the matchings (sets of edges without common points) in graphs. In the theory of matchings a lot of applied problems can be modelled, from which the entire theory was really borne.

A complete treatment of this and related objects is divided into twelve chapters. These chapters are the followings: 1. Matchings in bipartite graphs, 2. Flow theory, 3. Size and structure of maximum matchings, 4. Bipartite graphs with perfect matchings, 5. General graphs with perfect matchings, 6. Some graph-theoretical problems related to matchings, 7. Matching and linear programming, 8. Determinants and matchings, 9. Matching algorithms, 10. The f-factor problem, 11. Matroid matching, 12. Vertex packing and covering, and References with an impressive number of titles. Algorithmical aspects are also considered.

This well-written book is recommanded to all, who are interested in matching problems.

Z. KÁSA

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