STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

1

1986

CLUJ-NAPOCA

ANUL XXXI 1986

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

1

Redacţia: 3400 CLUJ-NAPOCA, str. M. Kogălniceanu, 1 ● Telefon 1 61 01

SUMAR — CONTENTS — SOMMAIRE — INHALT

Recenzii — Book Reviews — Comptes rendus — Buchbesprechungen

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

NOTE ON THE NUMBER OF HAMILTONIAN PATHS IN TOURNAMENTS

DĂNUŢ MARCU*

Received January 12, 1982

ABSTRACT. — For a not strong tournament T we denot by $H(T)$, $H(C_i)$, $i = 1, 2, \ldots, m$, the number of Hamiltonian paths of T respectively C_i (C_i is a strong component of T). Using the results of [4] we show that

$$H(T) = \prod_{i=1}^{m} H(C_i),$$

$$H(T) \geqslant (\alpha !)^m \prod_{i=1}^{m} |C_i|,$$

$$H(T) \geqslant 2^{m(\beta-1)} \prod_{i=1}^{m} C_i,$$

where

$$\alpha = \min_{1 \leqslant i \leqslant m} \min_{x \in V(C_i)} d^+(x),$$

$$\beta = \min_{1 \leqslant i \leqslant m} \min_{x \in V(C_i)} d^-(x).$$

A *digraph* (directed graph [1]) D consists of a set $V(D)$ of *vertices* and a set $E(D)$ of ordered pairs xy of distinct vertices called *edges*. If xy is an edge of D, we say that x *dominates* y [1]. The number of vertices dominating (resp. dominated by) vertex x is called the *indegree* (resp. *outdegree*) of x and is denoted $d^-(x)$ (resp. $d^+(x)$).

A *path* [1] is a digraph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. This path is called an x_1x_n path and is denoted $x_1x_2 \ldots x_n$. A *strong component* [1], C, of a digraph D is a maximal *subgraph* [1] such that for any two vertices x, y of C, C contains an xy path and a yx path.

Digraph D is *strong* [1] if it has only one component. The *condensed* digraph D^*, of D, is the digraph for which the strong components of D are $C(D^*)$ and C_1C_2 is an edge of $E(D^*)$ if and only if there exist $x \in C_1$, $x \in C_2$ such that xy is an edge of $E(D)$. A *tournament* [2] is a digraph such that each pair of vertices is joined by precisely one edge.

A *Hamiltonian* path [1] of a digraph D is a path including just once every vertex of D.

* *University of Bucharest, Faculty of Mathematics, 7000 Bucharest, Romania*

Lemma 1. *If D contains a Hamiltonian path, then D^* contains also one.*
Proof. Trivial.

Lemma 2. *Let T be a tournament not strong and $C_1, C_2, \ldots, C_m$ its strong components. The components of T can be placed in a sequence as a form $C_{i_1}, C_{i_2}, \ldots, C_{i_m}$, such that every vertex of C_{i_k} dominates every vertex of C_{i_j} whenever $k < j$.*

Proof. By Rédei's theorem of [3], T contains a Hamiltonian path It follows from lemma 1 that T^* contains one. Let it $C_{i_1}, C_{i_2}, \ldots, C_{i_m}$ be. According with definition of T and T^* results that every vertex of C_{i_k} dominates every vertex of C_{i_j} whenever $k < j$. (q.e.d.).
For a not strong tournament T we denote by $H(T)$, $H(C_i)$, $i = 1, 2, \ldots, m$, he number of Hamiltonian paths of T respectively C_i.

Theorem 1. *For a not strong tournament T holds*

$$H(T) = \prod_{i=1}^{m} H(C_i).$$

Proof. Let $C_{i_1}, C_{i_2}, \ldots, C_{i_m}$ the Hamiltonian path of T defined by lemma 2. Retaining this order, we find the Hamiltonian paths in every component, and after that, we link these paths using the edges of T^* in all possible ways. In this way we generate all Hamiltonian paths of T. Since T is a tournament it follows (according to lemma 2) that

$$H(T) = \prod_{i=1}^{m} H(C_i). \text{ (q.e.d.)}$$

Theorem 2. (C. Thomassen [4]). *A strong tournament with minimum outdegree $\geqslant k$ has at least $k!$ Hamiltonian paths starting at any vertex.*

Theorem 3. (C. Thomassen [4]). *A strong tournament with minimum ndegree $\geqslant k$ has at least 2^{k-1} Hamiltonian paths starting at any vertex.*

Denoting

$$\alpha = \min_{1 \leqslant i \leqslant m} \min_{x \in V(C_i)} d^+(x),$$

$$\beta = \min_{1 \leqslant i \leqslant m} \min_{x \in V(C_i)} d^-(x),$$

t follows, according to theorems 1, 2 and 3, that

$$H(T) \geqslant (\alpha!)^m \prod_{i=1}^{m} |C_i|,$$

$$H(T) \geqslant 2^{m(\beta-1)} \prod_{i=1}^{m} |C_i|.$$

REFERENCES

1. Harary F., *Graph theory*, Addison—Wesley, Reading, Mass., 1969.
2. Moon J. W., *Topics on tournaments*, Holt, Reinehart and Winston, New York, 1968.
3. Rédei L., *Ein kombinatorischer Satz*, Acta Litt, Szeged, 7 (1934), 39—43.
4. Thomassen C., *On the number of Hamiltonian cycles in tournaments*, Preprint Series, 1978/79, 23, March 1979, Århus Universitet, Matematisk Institut.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

PROBLÉMES DE RÉCURRENCE POUR DES CONNEXIONS SEMI-SYMÉTRIQUES, MÉTRIQUES

P. ENGHIŞ* et P. STAVRE**

Manuscrit reçu le 13 mars 1982

ABSTRACT. — Recurence Problems for Semi-symetric, Metric Connexions. The results in [3] and [5] are further developed to semi-symetric, metric connexions given by (1), using invariants $\overline{A}^s_{ijk}$, $\overline{Z}^s_{ijk}$, $\overline{W}^s_{ijk}$ and $\overline{T}^s_{ijk}$ given by (14), (18), (31) and (41).

§ 1°. Soit L_n une variété différentiable à n dimensions, de classe C^∞ et g une métrique riemannienne sur L_n de composantes g_{ij} dans une carte locale $(U, \varphi; \varphi(x) = x^1(n) \ldots x^n(x); x \in U)$. Nous allons noter par ∇ la connexion Levi—Civita, correspondant à g, de coéficients $\left\{ {i \atop jk} \right\}$ dans la carte locale (U, φ), par R^s_{ijk} les composantes de son tenseur de courbure [2], par $R_{ij} = R^s_{ijs}$ (le tenseur de Ricci) et par $R = g^{ij}R_{ij}$, la courbure scalaire.

Soit dans L_n une connexion, D, semi-symétrique, métrique, [1], [18] de coéfficients

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \omega_j\delta^i_k - g_{jk}\omega^i \, ; \; \omega^i = g^{ir}\omega_r \tag{1}$$

Nous avons

$$T^i_{jk} = \omega_j\delta^i_k - \omega_k\delta^i_j \tag{2}$$

$$g_{ij/k} = 0 \tag{3}$$

où $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ (les composantes du tenseur de torsion) et par/on a noté la dérivée covariante par rapport à D.

Nous allons noter par $\overline{R}^s_{ijk}$ les composantes du tenseur de courbure pour la connexion D, par $\overline{R}_{ij} = \overline{R}^s_{ijs}$ le tenseur de Ricci et $\overline{R} = g^{ij}\overline{R}_{ij}$ sa courbure scalaire.

Si D a $\overline{R}^s_{ijs} = 0$, alors g est conformément plate [18] et donc si la variété riemannienne (L_n, g) a l'index $i_\nabla > 1$ alors R^s_{ijk} est récurrent [8] D'où:

* *Université de Cluj-Napoca, Faculté de Mathematique, 3400 Cluj-Napoca, Roumanie*
** *Université de Craiova, Faculté de Science Economique, 1100 Craiova, Roumanie*

PROPOSITION 1.1. *Si la connexion (1) est la connexion de K. Yano (c'est-a-dire a $\overline{R}^s_{ijk} = 0$) de courbure scalaire $R \neq cst$ et $i_\nabla > 1$, alors*

$$R^s_{ijk,r} = \varphi_r R^s_{ijk}\,;\ \varphi_r \neq 0 \tag{4}$$

où on a noté par virgule la dérivée covariante par rapport à ∇.

C'est-à-dire la variété riemannienne (L_n, g) est récurrente.

Si D est une connexion linéaire à tenseur de courbure $\overline{R}^s_{ijk}$ et on a

$$\overline{R}^s_{ijk/r} = \varphi_r \overline{R}^s_{ijk}\,;\ \varphi_r \neq 0 \tag{5}$$

on dira que L_n est D-récurrente. Il en résulte

$$\overline{R}_{ij/r} = \varphi_r \overline{R}_{ij} \tag{6}$$

c'est-à-dire L_n est D—Ricci-récurrente (la réciproque n'est pas, en général, vraie).

Pour les connexions D semi-symétriques [1], caractérisées par (2), dans [13] on a établi les propriétés équivalentes

$$\frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j} = 0 \Leftrightarrow T_{j/i} = T_{i/j} \Leftrightarrow T^s_{ij/s} = 0 \ (\mathrm{div}\ T = 0) \tag{7}$$

où $T_j = T^s_{sj}$ (le vecteur de torsion). Plus tard, dans [7], on retrouvé ces propriétés et, de plus, on montre que si D a la propriété

$$T_{j/k} = T_{k/j} \tag{7'}$$

c'est-à-dire D est une E-connexion [6] alors les premières identités de Bianchi ont lieu

$$\overline{R}^s_{ijk} + \overline{R}^s_{jki} + \overline{R}^s_{kij} = 0 \tag{8}$$

Toujours dans [13] on montre que si D est semi-symétrique métrique et a l'une des propriétés équivalentes (7) alors le tenseur de Ricci est symétrique $\overline{R}_{ij} = \overline{R}_{ji}$. La connexion peut-être réduite à une forme canonique.

Dans [4], indépendemment, on étudie un cas plus général de connexions à propriété (7'). (E-connexions [6]).

Observation 1.1. Les connexions semi-symétriques pour lesquelles ω_i est gradient (connexions semi-symétriques spéciales [7] sont des E-connexions [6]), donc elles vérifient (7') ou les propriétés équivalentes.

Dans ce qui suit, nons allons utiliser les connexions semi-symétriques métriques (1) ou semi-symétriques.

Dans [12] on montre que pour une connexion semi-symétrique on a les relations

$$\sum_{jkh} \overline{R}^s_{ijk/h} = 2 \sum_{jkh} \overline{R}^s_{ijk}\omega_h \tag{9}$$

(somme selon la permutation circulaire de j, k, h) et donc

$$\sum_{jkh} \overline{R}^s_{ijk|h} = 0 \Leftrightarrow \sum_{jkh} \overline{R}^s_{ijk}\omega_h = 0 \tag{10}$$

si

$$\overline{R}^s_{ijk|r} = \omega_r \overline{R}^s_{ijk} \tag{11}$$

alors de (9), (11) il résulte

$$\sum_{jkh} \overline{R}^s_{ijk}\omega_h = 0 \tag{12}$$

d'où

PROPOSITION 1.2. *Si la connexion D est semi-symétrique et $\overline{R}^s_{ijk}$ est D-récurrent à covecteur de récurrence ω_r alors on a (12) et donc on a le deuxième groupe d'identités Bianchi (10)*

COROLLAIRE 1.1. *Si L_n est D-symétriquement Cartan c'est-à-dire $\overline{R}^s_{ijk|r} = 0$ alors on a (12), si D est semi-symétrique.*

Pour les espaces de Walker [16] et pour la connexion D considérée plus haut, en les notant par $D - K^*_n$, on a

COROLLAIRE 1.2. *Si L_n est D-symétriquement Cartan alors il est un espace $D - K^*_n$.*

COROLLAIRE 1.3. *Dans un espace $D - K^*_n$ les relations (12) ont lieu si le vecteur de récurrence est ω_h. (Nous allons noter l'espace par $\omega - D - K^*_n$).*

Dans (12), en appliquant une contraction dans s et h, on obtient

$$[\overline{R}^s_{ijk} + (\overline{R}_{ik}\delta^s_j - \overline{R}_{ij}\delta^s_k)]\,\omega_s = 0 \tag{13}$$

D'où

PROPOSITION 1.3. *Dans un espace $\omega - D - K^*_n$ ou dans un espace D-symétriquement Cartan, ω_s satisfait (13).*

COROLLAIRE 1.4. *Si on note*

$$\overline{A}^s_{ijk} = \overline{R}^s_{ijk} + \overline{R}_{ik}\delta^s_j - \overline{R}_{ij}\delta^s_k \tag{14}$$

alors une condition nécessaire pour que L_n soit un espace D-symétriquement Carton (et donc $D - K^*_n$) ou $\omega - D - K^*_n$, est le rang $\|\overline{A}^s_{ijk}\| < n$.

COROLLAIRE 1.5.

$$\overline{A}^s_{ijk} + \overline{A}^s_{ikj} = 0$$

COROLLAIRE 1.6. *Si la connexion D, semi-symétrique métrique a la propriété (7') alors*

$$\overline{A}^s_{ijk} + \overline{A}^s_{jki} + \overline{A}^s_{kij} = 0$$

puisque $\overline{R}_{ij} = \overline{R}_{ji}$.

De la condition de complète intégrabilité pour (3) il résulte $\bar{R}_{pijk} = -\bar{R}_{ipjk}$ ùo $\bar{R}_{pijk} = g_{ps}\bar{R}^s_{ijk}$. De là et de (13) il résulte

$$[2\bar{R}^s_k - \delta^s_k\bar{R}]\,\omega_s = 0 \tag{15}$$

où

$$\bar{R}^s_k = g^{sp}\bar{R}_{pk}.$$

§ 2. Dans [11], pour une connexion D semi-symétrique métrique on a introduit les invariants $\bar{T}^s_{ijk}$ (nommé invariant D -concirculaire) et $\bar{Z}^s_{ijk}$ (nommé invariant D-coharmonique) et les transformations D-concirculaires et D-coharmoniques qui les caractérisent. Celles-ci généralisent les invariants concirculaires et coharmoniques [9].

Dans [3] on étudie les relations entre les espaces riemanniens récurrents, conformément récurrents, concirculairement récurrents et coharmoniquement récurrents. On va généraliser maitenant ces résultats en utilisant les invariants $\bar{T}^s_{ijk}$ et $\bar{Z}^s_{ijk}$.

On va noter

$$H^s_{ijk} = \frac{1}{n(n-1)}\left(g_{ij}\delta^s_k - g_{ik}\delta^s_j\right) \tag{16}$$

et évidemment $H \neq 0$. Soit

$$Z^h_{ijk} = R^h_{ijk} + \frac{1}{n-2}\left(R_{ik}\delta^h_j - R_{ij}\delta^h_k + g_{ik}R^h_j - g_{ij}R^h_k\right) \tag{17}$$

le tenseur coharmonique de courbure relatif à ∇ (invariant aux transformations coharmoniques [9] et

$$\bar{Z}^h_{ijk} = \bar{R}^h_{ijk} + \frac{1}{n-2}\left(\bar{R}_{ik}\delta^h_j - \bar{R}_{ij}\delta^h_k + g_{ik}\bar{R}^h_j - g_{ij}\bar{R}^h_k\right) \tag{18}$$

le tenseur D-coharmonique de courbure (invariant aux transformations D-coharmoniques [11]). Si C^h_{ijk} est le tenseur de courbure conforme, Weyl, pour ∇, [2] et $\bar{C}^h_{ijk}$ le tenseur D-conforme de courbure pour D, on a

$$Z^h_{ijk} + \frac{nR}{n-2}H^h_{ijk} = C^h_{ijk} \tag{19}$$

$$\bar{Z}^h_{ijk} + \frac{u\bar{R}}{n-2}H^h_{ijk} = \bar{C}^h_{ijk} \tag{20}$$

Comme on a

$$C^h_{ijk} = \bar{C}^h_{ijk} \tag{21}$$

il résulte

$$\bar{Z}^h_{ijk} - Z^h_{ijk} = \frac{n}{n-2}H^h_{ijk}(R - \bar{R}) \tag{22}$$

De (17) et (18) par contraction par rapport à h et k, il résulte

$$Z_{ij} = -\frac{R}{n-2} g_{ij}\,;\ \bar{Z}_{ij} = -\frac{\bar{R}}{n-2} g_{ij} \tag{23}$$

DÉFINITION 2.1. S'il existe un covecteur $\varphi_r \neq 0$, tel que

$$\bar{Z}^h_{ijk|r} = \varphi_r \bar{Z}^h_{ijk} \tag{24}$$

on va dire que L_n est D-coharmoniquement récurrent.

Il résulte

$$\bar{Z}_{|r} = \varphi_r \bar{Z}_{ij} \tag{25}$$

De (3), (23) et (25) il résulte

$$\bar{R}_r = \varphi_r \bar{R} \tag{26}$$

Donc :

PROPOSITION 2.1. *Si L_n est un espace D-coharmoniquement récurrent alors $\bar{Z}_{ij}$ et $\bar{R}$ sont D-récurrents avec le même covecteur de récurrence.*

De (3), (20), (24) et (26) il résulte

$$\bar{C}^h_{ijk|r} = \varphi_r \bar{C}^h_{ijk} \tag{27}$$

D'où :

PROPOSITION 2.2. *Un espace L_n, D-coharmoniquement récurrent est D-conformément récurrent (27) et on ne peut pas avoir $\bar{R} = ct \neq 0$.*

Si on a (27), de (3) et (20) il résulte

$$\bar{Z}^h_{ijk|r} = \varphi_r \bar{Z}^h_{ijk} - \frac{n}{n-2} H^h_{ijk}(\bar{R}_{|r} - \varphi_r \bar{R}) \tag{28}$$

d'où

PROPOSITION 2.3. *Si L_n $(n > 3)$ est D-conformément récurrent alors on a (28). La condition nécessaire et suffisante pour que L_nD-conformément récurrent suive D-coharmoniquement récurrent est (26).*

Évidemment, on suppose que D n'est pas la connexion de K. Yano.

Observation 2.1. Si dans (1) on fait $\omega = 0$, alors $D = \nabla$ et on obtient les résultats de [3].

Observation 2.2. Si L_n est D-récurrent (5), D étant la connexion (1), alors L_n est D-coharmoniquement récurrent et D-conformément récurrent.

On a aussi

PROPOSITION 2.4. *Une condition nécessaire et suffisante pour qu'un espace L_n, avec la connexion D semi-symétrique métrique (1), D-coharmoniquement récurrent ou D-conformément récurrent, soit D-récurrent est que L_n soit D-Ricci récurrent.*

Observation 2.3. Si L_n possède une connexion, D, K. Yano alors il est conforme plate et donc

$$Z^h_{ijk} = \frac{nR}{2-n} H^h_{ijk} \tag{29}$$

D'où la condition nécessaire et suffisante pour que (L_n, g) soit coharmoniquement récurrent est $R_{/r} = \varphi_r R$.

§ 3. Pour une connexion semi-symétrique D dans [12] on a mis en évidence un invariant $\overline{W}^h_{ijk}$ de type Weyl, du groupe de transformation

$$\tilde{\Gamma}^h_{jk} = \Gamma^h_{jk} + \delta^h_j \tau_k + \delta^h_k \tau_j \tag{30}$$

La connexion D définie par (30) est semi-symétrique.

Si la connexion semi-symétrique D a la propriété (7′) alors $\overline{W}^h_{ijk}$ a une forme analoque à celle des connexions symétriqes. Si $d_\tau = 0$ (τ est fermée) alors $\overline{W}^h_{ijk}$ a une forme analoque à l'invariant de Weyl, $\overline{W}^h_{ijk}$ de la géométrie riemannienne [2]. En cas particulier si D est (1) ou si il a aussi la propriété (7′) alors $\overline{W}^h_{ijk}$ aura une forme analogue à W^h_{ijk} de la géométrie riemannienne. Dans ce cas, comme on l'a montré, on a $\bar{R}_{ij} = \bar{R}_{ji}$. Pour (1), le tenseur de Bianchi est nul [12]. Soit donc, pour (1), le tenseur D-projectif de courbure :

$$\overline{W}^h_{ijk} = \bar{R}^h_{ijk} - \frac{1}{n-1} (\bar{R}_{ij}\delta^h_k - \bar{R}_{ik}\delta^h_j) \tag{31}$$

DEFINITION 3.1. S'il existe un covecteur φ_r tel que

$$\overline{W}^h_{ijk/r} = \varphi_r W^h_{ijk} \tag{32}$$

alors on va dire que L_n est D-projectivement récurrent.

Observation 3.1. Si la connexion $D(1)$ a la propriété (7′) alors on aura

$$\bar{R}_{sijk} = \bar{R}_{jksi}.$$

Observation 3.2. De (5), (6) et (31), il résulte que si L_n est D-récurrent, alors il est D-projectivement récurrent.

De (6), (31) ez (32) et de l'observation 3.2 il résulte

PROPOSITION 3.1. *Un espace L_n D-projectivement récurrent est D-récurrent, si et seulement s'il est D-Ricci récurrent (avec le même φ_r).*

DEFINITION 3.2. Le tenseur

$$\bar{E}_{ij} = \bar{R}_{ij} - \frac{\bar{R}}{n} g_{ij} \tag{33}$$

sera nommé tenseur D-Einstein. S'il existe un covectuer φ_r tel que

$$\bar{E}_{ij/r} = \varphi_r \bar{E}_{ij} \tag{34}$$

nous allons dire que L_n est *D-Einstein récurrent.*

Si L_n est D-conformément récurrent et D-Einstein récurrent (avec le même φ_r), alors il résulte

$$\bar{R}^h_{ijk/r} = \varphi_r \bar{R}^h_{ijk} + H^h_{ijk}(\bar{R}_{/r} - \varphi_r \bar{R}) \tag{35}$$

et réciproquement. D'où

PROPOSITION 3.2. *La condition nécessaire et suffisante pur que L_n soit D-conformément récurrent et D-Einstein récurrent est (35).*

Conséquence 3.1. Si on a (35) alors on a (28).

PROPOSITION 3.3. *Un espace L_n D-conformément récurrent et D-Einstein récurrent est D-récurrent si et seulement si on a (26).*

Notons

$$\overline{W}^h_k = g^{ij} W^h_{ijk}, \quad \overline{W}_{sk} = g_{sh} \overline{W}^h_k \tag{36}$$

De la complète intégrabilité de (3) il résulte

$$\bar{R}_{sijk} = -\bar{R}_{isjk} \tag{37}$$

et donc

$$W^h_k = \frac{n}{n-1}\left(\bar{R}^h_k - \frac{1}{n}\bar{R}\delta^h_k\right); \quad \overline{W}_{sk} = \frac{n}{n-1}\bar{E}_{sk} \tag{38}$$

De (32) et (38) il résulte

$$\overline{W}^h_{k/r} = \varphi_r \overline{W}^h_k; \quad \overline{W}_{sk/r} = \varphi_r \overline{W}_{sk}; \quad \bar{E}_{sk/r} = \varphi_r \bar{E}_{sk} \tag{39}$$

c'est-à-dire :

PROPOSITION 3.4. *Si L_n est D-projectivement récurrent alors il est aussi D-Einstein récurent.*

De (31), (32) et (38), il résulte

$$\bar{R}^h_{ijk/r} - \varphi_r \bar{R}^h_{ijk} = \overline{W}^h_{ijk/r} - \varphi_r \overline{W}^h_{ijk} + \frac{1}{2}\delta^h_k(\overline{W}_{ij/r} - \varphi_r \overline{W}_{ij}) -$$

$$- \frac{1}{n}\delta^h_j(\overline{W}_{ik/r} - \varphi_r \overline{W}_{ik}) + H^h_{ijk}(\bar{R}_{/r} - \varphi_r \bar{R}) \tag{40}$$

D'où, en employant (35), (38) et (39), il résulte

PROPOSITION 3.5. *La condition nécessaire et suffisante pour que l'espace L_n soit D-projectivement récurrent est (35).*

Des propositions 3.2 et 3.5 il résulte :

PROPOSITION 3.6. *Les espaces L_n, D-conformément récurrents et D-Einsterin récurrents, D-projectivement récurrents et respectivement ceux qui vérifient (35) (avec le meme φ_r) coincident.*

Observation 3.3. Si L_n est D-projectivement récurrent et si on a (26), il résulte de (35) que L_n est D-récurrent.

Observation 3.4. Si $\omega = 0$ alors $D = \nabla$ et on obtient les résultats de [3]. Dans ce cas, comme on le sait [10], si la métrique est positivement définie alors les espaces projectivement récurrents coincident astec les récurrents.

§ 4. Soit le tenseur

$$\bar{T}^h_{ijk} = \bar{R}^h_{ijk} - \bar{R}H^h_{ijk} \tag{41}$$

qui est invariant à une transformation D-concirculaire [11] et qui est analogue au tenseur concirculaire de courbure de la géométrie concirculaire des espaces riemanniens ce pourquoi on le nommera tenseur D-concirculaire de courbure.

DEFINITION 4.1. Si'il existe $\varphi_r \neq 0$ tel que

$$\bar{T}^h_{ijk/r} = \varphi_r \bar{T}^h_{ijk} \tag{42}$$

alors on dit que L_n est D-concirculairement récurrent.

De (33) et (41) il résulte :

$$\bar{T}_{ij} = \bar{T}^h_{ijh} = \bar{E}_{ij} \tag{43}$$

et de (42) et (43) il résulte

$$\bar{E}_{ij/r} = \varphi_r \bar{E}_{ij} \tag{44}$$

ou

PROPOSITION 4.1. *Si L_n est D-concirculairement récurrent, alors il est aussi D-Einstein récurrent.*

De (42) il résulte

$$\bar{T}^h_{ijk/r} - \varphi_r \bar{T}^h_{ijk} = \bar{R}^h_{ijk/r} - \varphi_r \bar{R}^h_{ijk} - H^h_{ijk}(\bar{R}_{/r} - \varphi_r \bar{R}) \tag{45}$$

D'où

PROPOSITION 4.2. *La condition nécessaire et suffisante pour que l'espace L_n soit D-concirculairement récurrent est (35).*

En conclusion, des propositions 3.2, 3.4 et 4.2 il résulte :

PROPOSITION 4.3. *Les espaces L_n, D-concirculairement récurrents, D-projectivement récurrents, D-conformément récurrents et D-Eistein récurrents ainsi que ceux ayant la propriété (35), avec le meme φ_r, coincident. Leur partie commune avec les espaces D-coharmoniquement récurrents avec le même φ_r est un espace D-récurrent avec le même φ_r.*

Ici encore on remarque que si on prend $\omega = 0$, alors $D = \nabla$ et on obtient les résultats de [3].

BIBLIOGRAPHIE

1. Eisenhart, L. P., *Non Riemannian Geometry*, American Math. Soc., New York, (1927).
2. Eisenhart, L. P., *Riemannian Geometry*, Princeton Univ.-Press, (1949)
3. Enghiş, P., *Relations entre des espaces Riemanniens à tenseur récurrents*, Studia Univ. Babeş Bolyai, Math., XXV, 1, 1980, 67–72.
4. Enghiş, P., *Sur des espaces A à connexion affine*, Studia Univ. Babeş Bolyai, Math. Mech. (1972) f 2, 47–53.
5. Enghiş, P., *Sur les espaces V Récurrents et Ricci récurrents*, Studia Univ. Babeş Bolyai Cluj. Ser. Math. Mec., 1, (1972) 3–6.
6. Enghiş, P., *E-conexiuni semi-simetrice*, Studia Univ. Babeş Bolyai, Cluj, Math., XXIX (1982) 66–69
7. Golab, S., *On semi-symetric and Quarter-symetric linear connections*, Tensor N. S. 29, (1975) 249–254.
8. Levin, J., Katzin, G. H., *Conformally flat spaces admitting special quadratic first integrals*, Tensor. N. S., vol. 19, (3) 1968.
9. Ishii, J., *On conharmonic transformations*, Tensor N. S., 72, (1957) 70–80.
10. Matsumoto, M., *On Riemannian space with recurrent projective curvature*, Tensor N.S., 19 (1968), 11–18.
11. Stavre, P., *On the D-concircular and D-coharmonic transformations*, (sub tipar), Tensor N.S.
12. Stavre, P., *Asupra unor conexiuni speciale pe varietăţi diferenţiabile*, Colocviu nat. de Geometrie, Buşteni 27–30 iunie 1981.
13. Stavre, P., *Asupra formei canonice a unei conexiuni*, St. şi cercet. mat., Bucureşti, (22) 1 (1970) 73–80.
14. Stavre, P., Smaranda, D., *On the associate index of a semi-symetric metric connection*, Colocviu Nat. de Geometrie Gh. Vrănceanu, 1980.
15. Stavre, P., *On the S-concircular and S-coharmonic connections*, Tensor N.S., (sub tipar).
16. Walker, A. G., *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc., 2, (52) (1950), 36–64.
17. Wong, I. C., *A clase of non Riemannian K* spaces*, Proc. London Math. Soc., 5 (1953), 118–128.
18. K. Yano, *On semi-symetric metric connection*, Rev. Roumaine de Math. Pure et Appl. Bucureşti, 9 (1979), 1579–1586.
19. K. Yano, *Concircular Geometry I*. Proc., Tokyo, 16 (1940), 195–200.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

ON A LIBERA INTEGRAL OPERATOR

VINOD KUMAR* and S. L. SHUKLA*

Received May 18, 1982

ABSTRACT. — In the present paper we study the Libera integral operator $F(z) = \frac{2}{z}\int_0^z f(t)dt$ for the class of starlike functions having negative coefficients. Our results are sharp and improve the results of Libera and Livingston.

1. **Introduction.** Let $\alpha \in [0, 1)$ and $\beta \in (0, 1]$. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular in the unit disc $U = \{z : |z| < 1\}$ is said to belong to $S^*(\alpha, \beta)$, the class of starlike functions of order α and type β, if and only if

$$|\{zf'(z)/f(z) - 1\}/\{zf'(z)/f(z) + (1 - 2\alpha)\}| < \beta, \ z \in U.$$

It is well known that such functions are univalent in U. The class $S^*(\alpha)$ of starlike functions of order α is identified by $S^*(\alpha) \equiv S^*(\alpha, 1)$. The class $S^*(0)$ is called the class of starlike functions and is denoted by S^*.

Libera [2] showed that, if $f(z) \in S^*$, then so does the function $F(z)$ defined by

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt. \tag{1}$$

Subsequently, Livingston [3] considered the converse problem and proved that, if $F(z) \in S^*$, the f(z) belongs to S^* in $|z| < 1/2$. In this paper we improve these results of Libera and Livingston for the class of starlike functions having negative coefficients.

The technique imployed by us is intirely different from those of Libera [2] and Livingston [3]. Infact, our basic tool is the following theorem due to Gupta and Jain [1].

THEOREM A. *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *belongs to* $S^*(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} \{(n - 1) + \beta(n + 1 - 2\alpha)\} |a_n| \leqslant 2\beta(1 - \alpha).$$

The result is sharp.

---

* *Department of Mathematics, Janta College, Bakewar 206124, Etawah (U.P.), India*

We shall frequently use the above result in particular for $\beta = 1$ which is due to Silverman [6].

2. Main results.

THEOREM 1. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. *If* $f(z) \in S^*(\alpha, \beta)$, *then the function* $F(z)$ *defined by* (1) *belongs to* $S^*(\rho)$, *where* $\rho = \frac{3 + \beta(1 + 2\alpha)}{3 + \beta(5 - 2\alpha)}$. *The result is sharp. Further, the converse need not be true.*

PROOF. Since $F(z) \in S^*(\alpha, \beta)$, Theorem A ensures that

$$\sum_{n=2}^{\infty} \left\{\frac{(n-1) + \beta(n+1-2\alpha)}{2\beta(1-\alpha)}\right\} |a_n| \leqslant 1. \tag{2}$$

Also, from (1) we have $F(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$, where $|b_n| = \left(\frac{2}{n+1}\right)|a_n|$.

Let $F(z) \in S^*(\sigma)$, then, by Theorem A, it holds if and only if

$$\sum_{n=2}^{\infty} \left(\frac{n-\sigma}{1-\sigma}\right) |b_n| \leqslant 1.$$

Thus we have to find the largest value of σ so that the above inequality holds. Now this inequality holds if

$$\sum_{n=2}^{\infty} \left(\frac{n-\sigma}{1-\sigma}\right) |b_n| \leqslant \sum_{n=2}^{\infty} \left\{\frac{(n-1) + \beta(n+1-2\alpha)}{2\beta(1-\alpha)}\right\} |a_n|$$

or if

$$\left(\frac{n-\sigma}{1-\sigma}\right) |b_n| \leqslant \frac{(n-1) + \beta(n+1-2\alpha)}{2\beta(1-\alpha)} |a_n|, \text{ for each } n = 2, 3, \ldots$$

which is equivalent to

$$\sigma \leqslant \frac{(n+1)\{(n-1) + \beta(n+1-2\alpha)\} - 4n\beta(1-\alpha)}{(n+1)\{(n-1) + \beta(n+1-2\alpha)\} - 4\beta(1-\alpha)} = \rho_n,$$

say, $(n = 2, 3, \ldots)$.

It is easy to verify that ρ_n is an increasing function of n. Therefore, $\rho = \inf_{n \geqslant 2} \rho_n = \rho_2$ and, hence $\rho = \frac{3 + \beta(1 + 2\alpha)}{3 + \beta(5 - 2\alpha)}$.

To show the sharpness we take the function $f(z)$ given by

$$f(z) = z - \frac{2\beta(1-\alpha)}{1 + \beta(3 - 2\alpha)} z^2.$$

Then

$$F(z) = z - \frac{4\beta(1-\alpha)}{3\{1 + \beta(3 - 2\alpha)\}} z^2$$

and, therefore

$$z\frac{F'(z)}{F(z)} = \frac{3\{1+\beta(3-2\alpha)\} - 8\beta(1-\alpha)z}{3\{1+\beta(3-2\alpha)\} - 4\beta(1-\alpha)z} = \frac{3+\beta(1+2\alpha)}{3+\beta(5-2\alpha)}, \text{ for } z = 1.$$

Hence, the result is sharp.

We now show that the converse of the theorem need not be true. To this end we consider the function

$$F(z) = z - \left(\frac{1-\rho}{3-\rho}\right) z^3.$$

Theorem A guarantees that $F(z) \in S^*(\rho)$. But the coorresponding function

$$f(z) = z - 2\left(\frac{1-\rho}{3-\rho}\right) z^3$$

does not belong to $S^*(\alpha, \beta)$, since, for this $f(z)$ the coeficient inequality of Theorem A is not satisfied.

As promised in the introduction, we now state a corollary of Theorem 1 which improves the result of Libera [2, Theorem 1] for the class of starlike functions having negative coefficients.

COROLLARY 1. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. *If* $f(z) \in S^*$, *then the function* $F(z)$ *defined by* (1) *belongs to* $S^*(1/2)$. *The result is sharp. The converse need not be true.*

REMARK. Recently, Mocanu et al. [5] have shown that, if $(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$, then the function $F(z)$ defined by (1) belongs to S^* (29435), whereas, Miller et al [4] have shown that $F(z) \in S^*\left(\frac{\sqrt{17}-3}{4}\right)$. The above corollary provides better estimate for the Taylor expansion of $f(z)$ are negative. Moreover, our result is sharp also.

THEOREM 2. *Let* $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. *If* $F(z) \in S^*(\alpha, \beta)$, *then the function* $f(z)$ *defined by* (1) *belongs to* $S^*(\rho)$ *in* $|z| < r^*(\rho, \alpha, \beta)$, *where*

$$r^*(\rho, \alpha, \beta) = \inf_{n \geq 2} \left[\left(\frac{1-\rho}{n-\rho}\right)\left(\frac{(n-1)+\beta(n+1-2\alpha)}{(n+1)\beta(1-\alpha)}\right)\right]^{1/(n-1)}$$

The result is sharp.

PROOF. Since $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, it follows from (1) that $f(z) = z - \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| z^n$. In order to establish the required result it suffices to show that

$$|zf'(z)/f(z) - 1| < (1-\rho) \text{ in } |z| < r^*(\rho, \alpha, \beta).$$

Now

$$(3) \qquad |zf'(z)/f(z) - 1| = \left| \frac{-\sum_{n=2}^{\infty} (n-1)\left(\frac{n+1}{2}\right)|a_n|z^{n-1}}{1 - \sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)|a_n|z^{n-1}} \right| \leqslant$$

$$\leqslant \frac{\sum_{n=2}^{\infty} (n-1)\left(\frac{n+1}{2}\right)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)|a_n||z|^{n-1}} < (1-\rho),$$

provided

$$(4) \qquad \sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)\left(\frac{n+1}{2}\right)|a_n||z|^{n-1} < 1.$$

But, for $F(z) \in S^*(\alpha, \beta)$, Theorem A ensures that

$$\sum_{n=2}^{\infty}\left\{\frac{(n-1) + \beta(n+1-2\alpha)}{2\beta(1-\rho)}\right\}|a_n| \leqslant 1.$$

Therefore, the inequality (4) holds if

$$\left(\frac{n-\rho}{1-\rho}\right)\left(\frac{n+1}{2}\right)|a_n||z|^{n-1} < \left\{\frac{(n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}\right\}|a_n|,$$

$$\text{for each } n = 2, 3, \ldots,$$

or if

$$|z| < \left[\left(\frac{1-\rho}{n-\rho}\right)\left(\frac{(n-1) + \beta(n+1-2\alpha)}{(n+1)\beta(1-\alpha)}\right)\right]^{1/(n-1)}, \text{ for each } n = 2, 3, \ldots$$

Hence, $f(z) \in S^*(\rho)$ in $|z| < r^*(\rho, \alpha, \beta)$.

Sharpness follows if we take the function $F(z)$ given by

$$F(z) = z - \frac{2\beta(1-\rho)}{(n-1) + \beta(n+1-2\alpha)} z^n, \; n = 2, 3, \ldots .$$

This completes the proof of theorem.

Since $r^*(\alpha, \alpha, 1) = 2/3$, we have the following corollary as an imm diate consequence of Theorem 2.

COROLLARY 2. *Let* $F(z) = z - \sum_{n=2}^{\infty}|a_n|z^n$. *If* $F(z) \in S^*(\alpha)$, *then the fun tion* $f(z)$ *defined by* (1) *belongs to* $S^*(\alpha)$ *in* $|z| < 2/3$. *The result is sha with the extremal function* $F(z) = z - \left(\frac{1-\alpha}{2-\alpha}\right)z^2$.

REMARK. It is a remarkable feature of corollary 2 that the radius of the disc, in which $f(z)$ belongd to $S^*(\alpha)$, is independent of α. When $\alpha = 0$, the corollary improves a result of Livingston [3, Theorem 1] for the class of starlike functions having negative coefficients.

REFERENCES

1. V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., 14 (1976), 409–416.
2. R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. 16 (1965), 755–758.
3. A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. 17 (1966), 352–357.
4. S. S. Miller, P. T. Mocanu and M. O. Reade, *Starlike integral operators*, Pacific J. Math. 79 (1978), 157–168.
5. P. T. Mocanu, M. O. Reade and D. Ripeanu, *The order of starlikeness of a Libera integral operator*, Mathematica (Cluj), 19(42), No. 1 (1977), 67–73.
6. H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51 (1975), 109–116.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

L'ÉNERGIE INFORMATIONNELLE RÉCIPROQUE

ELENA OANCEA*

Manuscrit reçu le mai 20, 1982

ABSTRACT. — The Reciprocal Informational Energy. The reciprocal informational energy (informational energy redundance) is introduced. Some of its properties and their use in statistical correspondence and classification are stated.

L'article introduit l'énergie informationnelle réciproque avec ses propriétés et quelques applications statistiques.

Soit X une variable aléatoire avec la répartition de probabilités: $X(p_1, \ldots, p_n)$, $p_i \geqslant 0$, $\sum_i p = 1$. On sait [1] que l'énergie informationnelle de X est:

$$E(X) = E(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i^2 \tag{1}$$

avec les propriétés

$$E(1, 0, \ldots, 0) = 1$$

$$E(1/n, \ldots, 1/n) = 1/n,$$

quelle que soit X on a:

$$1/n \leqslant E(X) \leqslant 1.$$

Soit (X, Y) la variable aléatoire bidimensionnelle avec la répartition

$X \backslash Y$	y_1	$\ldots$	y_j	$\ldots$	y_m	
x_1	p_{11}	$\ldots$	p_{1j}	$\ldots$	p_{1m}	p_1
x_i	p_{i1}	$\ldots$	p_{ij}	$\ldots$	p_{im}	p_i
x_n	p_{n1}	$\ldots$	p_{nj}	$\ldots$	p_{nm}	p_n
	q_1	$\ldots$	q_j	$\ldots$	q_m	1

$$p_{ij} \geqslant 0, \quad i = \overline{1, n}, \quad j = \overline{1, m}, \quad \sum_{i,j} p_{ij} = 1$$

$$p_{ij} = P(X = x_i \cap Y = y_j)$$

$$\sum_j p_{ij} = p_i = P(X = x_i),$$

$$i = \overline{1, n}, \quad \sum_i p_i = 1,$$

$$\sum_i p_{ij} = q_j = P(Y = y_j),$$

$$j = \overline{1, m}, \quad \sum_j q_j = 1.$$

* *Université de Cluj-Napoca, Faculté de Mathematique, 3400 Cluj-Napoca, Roumanie*

DÉFINITION 1. L'énergie informationnelle de (X, Y) est :

$$E(X, Y) = E(X \cap Y) = \sum_{i,j} p_{ij}^2 \tag{2}$$

Propriétés.

1E. Si X, Y sont indépendantes $\Leftrightarrow p_{ij} = p_i q_j,\ i = \overline{1, n},\ j = \overline{1, m} \Leftrightarrow$

$$E(X, Y) = E(X)E(Y) \tag{3}$$

2E. Dans le cas X, Y quelconques on sait : $P(X = x_i \cap Y = y_j) = P(X = x_i)P(Y = y_j \mid X = x_i)$, $\Leftrightarrow$

$$E(X, Y) = \sum_j q_j^2 E(X \mid Y = y_j) = \sum_i p_i^2 E(Y \mid X = x_i) \tag{4}$$

$$E(X \mid Y = y_j) = \sum_i (p_{ij/q_j})^2,\ E(Y \mid X = x_i) = \sum_j (p_{ij/p_i})^2$$

3E. On a immédiatement

$$0 \leqslant E(X, Y) \leqslant 1 \tag{5}$$

DÉFINITION 2. L'énergie informationnelle réciproque de X et Y ou la redondance de l'énergie informationnelle est

$$R_E(X, Y) = \frac{E(X)E(Y)}{E(X,Y)}. \tag{6}$$

Propriétés.

1R.
$$R_E(X, Y) = R_E(X, Y) \tag{7}$$

2R. Si $X = Y \Leftrightarrow n = m,\ p_i = q_j,\ i = j = \overline{1, n} \Leftrightarrow$

$$R_E(X, Y) = E(X) = E(Y) \tag{8}$$

3R. Si X, Y sont indépendentes $\Leftrightarrow p_{ij} = p_i q_j,\ i = \overline{1, n},\ j = \overline{1, m} \Leftrightarrow$

$$R_E(X, Y) = 1.$$

4R. On a toujours :

$$\frac{1}{nm} \leqslant R_E(X, Y) \leqslant \frac{1}{E(X,Y)} \tag{9}$$

Remarque. Si entre X et Y il y a une rélation :

$$Y = f(X)$$

f étant une application détérministe on a :

$$R_E(X, Y) = R_E(X, f(X)) = E(X).$$

Donc R_E est un indicateur seulement pour la dépendance aléatoire.

Applications. 1. On peut utiliser la propriété (2R) comme critézium pratique pour vérifier la correspondance entre deux variables aléatoires par exemple X, Y : Si $R_E(X, Y) = E(X) \Rightarrow X = Y$ dans le sens fixé (2R) et aussi dans le cas des deux caractéristiques statistiques.

2. La propriété (3R) peut vérifier l'indépendance entre deux variables aléatoires ou caractéristiques statistiques X, Y dans ce cas il faut avoir : $R_E(X, Y) = 1$.

3. On peut utiliser la redondance de l'énergie informationnelle pour classifier apres la dépendance aléatoire plusieurs variables aléatoires $X_1, X_2, \ldots, X_k$, $k > 2$.

BIBLIOGRAPHIE

1. O. Onicescu, *Energia informațională*, Rev. de Statistică, București, 11, 1966.
2. A. Reny, *Calcul des Probabilités avec un appendice sur la Théorie de l'Information*, Dunod, Paris, 1966.

A NOTE ON DUALITY THEORY FOR AN INDEFINITE FUNCTIONAL PROGRAMMING PROBLEM

I. M. STANCU-MINASIAN* and VIVEK PATKAR*

Received July 2, 1982

ABSTRACT. — A dual problem is constructed for an indefinite quadratic programming problem with linear constraints. The principal idea in formulating the dual problem is to reduce the primal problem to a convex programming problem and then to use the results of V. PATKAR et al [6].

1. **Introduction.** The study of the indefinite quadratic programming problem, in whose dual we are interested, was initiated by K. SWARUP [8]. It was proved that for solving this problem it is sufficient to solve an equivalent convex programming problem.

In this note an attempt is made to construct a dual for such problem defined as

(I) $$\text{Maximize } f(x) = (c^t x + \alpha)(d^t x + \beta)$$
$$\text{subject to } Ax \leqslant b\,;\ x \geqslant 0$$

where A is an $(m \times n)$ matrix, c, d and x are $(n \times 1)$ vectors, b is an $(m \times 1)$ vector, α, β are scalar constants and t denotes the transpose of a matrix. Let

$$S = \{x \in R^n \mid Ax \leqslant b\,;\ x \geqslant 0\}$$

Assume that S is regular, i.e. nonempty and bounded. Further it is assumed that $(c^t x + \alpha)$ and $(d^t x + \beta)$ are positive for all feasible solutions.

It is easy to see that the set S is a convex set and that the function f is neither convex nor concave on S. It has been shown by O. L. MANGASARIAN [4] that the objective function in (I) is pseudoconcave on S.

Problem (I) will be called the primal problem, and with this we associate another problem (II) called the dual problem, as given below.

(II) $$\text{Minimize } g(u, v, w) = \frac{1}{v}$$
$$\text{subject to } -A^t u + dv + cw^2 \leqslant 0$$
$$b^t u + \beta v + \alpha w^2 \leqslant 2w$$
$$u, v \geqslant 0$$

where u is an $(m \times 1)$ vector and v and w are real numbers

* *Bucharest, Romania, Laboratories of Economic Cybernetics Department, 7000 Bucharest, Romania*
** *Bombay, India, Bombey Metropolitan Region Development Authority*

2. Dualization Process. Making use of the variable transforma
$y = tx$, which is a homeomorphism, with the scalar $t > 0$ selected so t
$d^t y + \beta t = 1$, the problem (I) becomes

(II)
$$\text{Maximize } F(y, t) = \frac{c^t y + \alpha t}{t^2}$$
$$\text{subject to } Ay - bt \leqslant 0$$
$$d^t y + \beta t = 1$$
$$y, t \geqslant 0$$

Or, equivalently

(IV)
$$\text{Minimize } F'(y, t) = \frac{t^2}{c^t y + \alpha t}$$
$$\text{subject to } Ay - bt \leqslant 0$$
$$d^t y + \beta t = 1$$
$$y, t \geqslant 0$$

According to C. R. Bector [1] the function $F'(y, t)$ is convex. Now (IV) can be rewritten as :

(IV′)
$$\text{Minimize } F'(y, t) = \frac{\left[(0, 1)\begin{pmatrix} y \\ t \end{pmatrix}\right]^2}{(c^t, \alpha)\begin{pmatrix} y \\ t \end{pmatrix}}$$

$$\text{subject to } \begin{pmatrix} -A & b \\ d^t & \beta \\ -d^t & -\beta \end{pmatrix}\begin{pmatrix} y \\ t \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \; y, \; t \geqslant 0$$

In this form the problem (IV′) is in the same form as the convex fracti
programming problem over linear constraints considered by V. P A T K
et al [6]. Using the result of V. P A T K A R et al. [6] a dual prog
coresponding to this problem is.

(V)
$$\text{Maximize } G(u, v_1, v_2, w) = (0, 1, -1)\begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix}$$

$$\text{subject to} \begin{pmatrix} -A^t & d & -d \\ b^t & \beta & -\beta \end{pmatrix}\begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} c \\ \alpha \end{pmatrix} w^2 \leqslant 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} w$$

$$u, v_1, v_2, w \geqslant 0$$

that is,

(V′)
$$\begin{aligned} &\text{Maximize} \quad G(u, v_1, v_2, w) = v_1 - v_2 \\ &\text{subject to} \quad -A^t u + \mathrm{d}v_1 - dv_2 + cw^2 \leqslant 0 \\ &\qquad b^t u + \beta v_1 - \beta v_2 + \alpha w^2 \leqslant 2w \\ &\qquad u, v_1, v_2, w \geqslant 0 \end{aligned}$$

Substituting $v_1 - v_2 = v$ so that v is unrestricted in sign, we see that the problem (V′) reduces to

(V″)
$$\begin{aligned} &\text{Maximize} \quad G(u, v, w) = v \\ &\qquad -A^t u + \mathrm{d}v + cw^2 \leqslant 0 \\ &\text{subject to} \quad d^t u + \beta v + \alpha w^2 \leqslant 2w \\ &\qquad u, w \geqslant 0 \end{aligned}$$

Or, equivalently

(VI)
$$\begin{aligned} &\text{Minimize} \quad g(u, v, w) = \frac{1}{v} \\ &\qquad -A^t u + \mathrm{d}v + cw^2 \leqslant 0 \\ &\text{subject to} \quad b^t u + \beta v + \alpha w^2 \leqslant 2w \\ &\qquad u, w \geqslant 0 \end{aligned}$$

which is just the problem (II).
General algorithms to solve such problems are available (see, for example, [5]).

Since I is equivalent with III, III is equivalent with IV (or IV′), V (or V′ or V″) is the dual program to IV (or IV′) and VI is equivalent with V, it results that VI is the dual program to I.

Now we shall formalize this dualization process by proving the duality theorems in the next section.

3. Duality Theorems. The following two theorems are easy to prove and so we omit the proofs.

THEOREM 1. *If x and (u, v, w) are feasible solutions for (I) and (II), respectively, then*

$$f(x) \leqslant g(u, v, w) \text{ for all } x, u, v, \text{ and } w.$$

THEOREM 2. *If $\hat{x}$ and $(\hat{u}, \hat{v}, \hat{w})$ are feasible solutions for (I) and (II), respectively, such that*

$$f(\hat{x}) = g(\hat{u}, \hat{v}, \hat{w})$$

then $\hat{x}$ and $(\hat{u}, \hat{v}, \hat{w})$ are the optimal solutions of (I) and (II), *respectively.*

We now establish the main duality theorem.

THEOREM 3. *The primal problem* (I) *has an optimal solution if and only if the dual problem* (II) *has an optimal solution. In either case, their optimal values are equal.*

Proof: The problems IV (or III or IV′) and I are equivalent; therefore, a solution $x = x^*$ of (I) guarantees a solution $y^* = t^*x^*$, $t^* = 1/(d^tx^* + \beta)$ to problem (IV) (or IV′) As shown by V. P A T K A R et al. [6] a solution (y^*, t^*) to (IV′) implies the existence of a solution $(u^*, v^* = v_1^* - v_2^*, w^*)$ to (V) and their respective extreme values are equal, i.e.

$$F'(y^*, t^*) = \frac{t^{*2}}{c^ty^* + \alpha t^*} = v^*$$

or equivalently, a solution $x^* = y^*/t^*$ to (I) implies a solution (u^*, v^*, w^*) to (II) and $f(x^*) = (c^tx^* + \alpha)(d^tx^* + \beta) = F(y^*, t^*) = \frac{1}{F'(y^*, t^*)} = \frac{c^ty^* + \alpha t^*}{t^{*2}} =$ $= \frac{1}{v^*} = g(u^*, v^*, w^*)$. Conversely, using the transformation $v = v_1 - v_2$ in the dual problem (II) and applying converse duality theory from Ref. [6], we get the required result.

REMARK. The dual problem though obtained indirectly, is in an explicit form unlike the one proposed by C.R. B E C T O R and M. D A H L [2].

M. K. B E D I [3] considered a more general class of problems than problem (I), but the duality results do not hold since the converted primal problem is not necessarily a convex programing problem. See also and Ref. [7] which considers Bedi's class in a more general setting.

REFERENCES

1. C. R. B e c t o r, *Programming Problems with Convex Fractional Functions*. Operations Res. 16. pp. 383–390 (1968).
2. C. R. B e c t o r, and M. D a h l, *Simplex Type Finite Iteration Technique and Duality for a Special Type of Pseudo-Concave Quadratic Program*. Cahiers Centre Études Recherche Opér, 16 (3), pp. 207–222 (1974).
3. M. K. B e d i, *Duality for a Special Class of Quasi-Convex Programming Problems*. Z. Angew. Math, Mech. 58(3), pp. 165–166 (1978).
4. O. L. M a n g a s a r i a n, *Non-Linear Programming*. McGraw-Hill, Book Co., New York, 1969.
5. B. M a r t o s, *Nonlinear Programming. Theory and Methods*. Akadémiai Kiadó, Budapest, 1975.
6. V. P a t k a r, P. C. S a x e n a and O m P a r k a s h, *Dual Program for a Convex Fractional Function*. Econom. Cybernet. Stud. Res. 15(1), pp. 77–80 (1981).
7. C. S i n g h, *A Class of Mathematical Programming Problems*. Operations Research Verfahren. Vol. 40, pp. 157–158 (1981).
8. K. S w a r u p, *Programming with Indefinite Quadratic Function with Linear Constrains*. Cahiers Centre Études Recherche Opér., 8(2), pp. 132–136 (1966).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

ASUPRA UNOR PROPRIETĂŢI DE COMPACTITATE ÎNTR-UN SPAŢIU g-2-metric

D. BORŞAN*

Intrat în redacţie la 30 noiembrie 1982

ABSTRACT. — Some Properties of Compactness in a g-2-Metric Space. In this note the concept of precompact generalized 2-metric space is defined. It is shown that under certain conditions on the partially ordered set, which is envolved in the definiton of the g-2-metric, the sequentially compactness implies the precompactness.

1. Extinzînd noţiunea de 2-metrică definită de S. Gähler în 1963 [3], lucrarea [1] introduce conceptul de g-2-metrică (2-metrică generalizată) în modul următor.

Se consideră o mulţime parţial ordonată $(\mathfrak{M}, \leqslant)$, o operaţie ternară pe $\mathfrak{M}$, $\varphi : \mathfrak{M}^3 \to \mathfrak{M}$, supusă condiţiilor :

(φ_1) $\varphi(a, b, c) = \varphi(a, c, b) = \varphi(b, c, a)$,

(φ_2) $a \leqslant a_1, b \leqslant b_1, c \leqslant c_1$, cel puţin una dintre inegalităţi fiind strictă $\Rightarrow \varphi(a, b, c) < \varphi(a_1, b_1, c_1)$, $a, b, c, a_1, b_1, c_1 \in \mathfrak{M}$ şi $\mathcal{E} \leqslant \mathfrak{M}$, $\mathcal{E}$ nevidă şi admiţînd minoranţi în $\mathfrak{M} \setminus \mathcal{E}$.

Se numeşte g-2-metrică pe mulţimea (cu cel puţin trei elemente) X o aplicaţie $\rho : X^3 \to \mathfrak{M}$. care satisface axiomele :

(ρ_{1a}) $x \neq y \Rightarrow \exists z \in X\ \exists e \in \mathcal{E} : \rho(x, y, z) \nless e$,

(ρ_{1b}) $\forall e \in \mathcal{E} : \rho(x, y, z) < e \Rightarrow x = y$ sau $x = z$ sau $y = z$,

(ρ_2) $\rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x)$,

(ρ_3) $\rho(x, y, z) \leqslant \varphi[\rho(x, y, t), \rho(x, t, z), \rho(t, y, z)]$, $\forall x, y, z, t \in X$.

Cuplul (X, ρ) se numeşte spaţiu g-2-metric iar toplogia $\mathcal{T}_\rho$, generată de subbaza

$$\mathcal{S} = \{V_e(x, y) \mid x, y \in X, e \in \mathcal{E}\} \quad \text{unde} \quad V_e(x, y) = \{z \in X \mid \rho\ (x, y, z) < e\}$$

se numeşte topologie indusă de ρ.

Făcînd asupra mulţimii $\mathcal{E}$ anumite ipoteze suplimentare, în lucrările [1], [2] s-au stabilit o serie de proprietăţi ale topologiei $\mathcal{T}_\rho$ (s-au indicat baze de vecinătăţi avantajoase, s-au studiat proprietăţi de separaţie, s-a caracterizat în termenii g-2-metricii noţiunea de limită a unui şir Moore-Smith de puncte din X).

În prezenta notă, dăm unele proprietăţi de compactitate ale topologiei $\mathcal{T}_\rho$.

* *Universitatea din Cluj-Napoca, Facultatea de Matematică, 3400 Cluj-Napoca, România*

2. A fiind o mulţime arbitrară vom folosi notaţiile $\mathfrak{L}(A)$, pentru mulţimea părţilor lui A şi $\mathfrak{L}_c(A)$, pentru mulţimea părţilor finite şi nevide ale mulţimii A.

Într-un spaţiu g-2-metric (X, ρ) introducem noţiunea de e-reţea ($e \in \mathcal{E}$) relativ la o parte finită a spaţiului prin următoarea definiţie (vezi [3])).

DEFINIŢIA (2.1). Fie (X, ρ) un spaţiu g-2-metric, $e \in \mathcal{E}$ şi $M \in \mathfrak{L}_c(X)$. Se numeşte e-reţea a spaţiului (X, ρ) relativ la mulţimea M, o mulţime $R \subseteq X$, cu proprietăţile:

(i) $R \in \mathfrak{L}_0(X)$

(ii) $\forall x \in X \exists v \in R \forall y \in M : \rho(x, r, y) < e$.

DEFINIŢIA (2.2). Spaţiul (X, ρ) este prin definiţie, precompact dacă pentru orice $e \in \mathcal{E}$ şi orice mulţime finită nevidă $M \subseteq X$, există o e-reţea a spaţiului, relativ la M.

Dăm acum o caracterizare a noţiunii de precompactitate.

TEOREMA (2.1). *Spaţiul g-2-metric (X, ρ) este precompact dacă şi numai dacă*

$$\forall e \in \mathcal{E} \forall M \in \mathfrak{L}_c(X) \exists R \in \mathfrak{L}_0(X) : X = \bigcup_{r \in R} \bigcap_{y \in M} V_e(r, y)$$

Demonstraţie.

Necesitatea. Presupunem că (X, ρ) este precompact şi fie $e \in \mathcal{E}$ şi $M = \{y_1, y_2, \dots, y_m\} \in \mathfrak{L}_0(X)$. Există atunci o e-reţea $R = \{r_1, \dots, r_n\}$ a spaţiului X, în raport cu mulţimea M. Urmează că:

$$\forall x \in X \exists i \in \{1, 2, \dots, n\} \forall j \in \{1, 2, \dots, m\} : \rho(r_i, y_j, x) < e \Rightarrow$$

$$\Rightarrow \forall x \in X \exists i \in \{1, 2, \dots, n\} \forall j \in \{1, 2, \dots, m\} : x \in V_e(r_i, y_j) \Rightarrow$$

$$\Rightarrow \forall x \in X \exists i \in \{1, 2, \dots, n\} : x \in \bigcap_{j=1}^{m} V_e(r_i, y_j) \Rightarrow$$

$$\Rightarrow \forall x \in X : x \in \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} V_e(r_i, y_j). \text{ Deducem că } X = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} V_e(r_i, y_j).$$

Suficienţa. Presupunem că

$$\forall e \in \mathcal{E} \forall M \in \mathfrak{L}_0(X) \exists R \in \mathfrak{L}_c(X) : X = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} V_e(r_i, y_j).$$

unde $R = \{r_i \mid i = \overline{1, n}\}$, $M = \{y_j \mid j = \overline{1, m}\}$. Vom demonstra că mulţimea R este o e-reţea a spaţiului X relativ la M. Avem

$$x \in X = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} V_e(r_i, y_j) \Rightarrow \exists i \in \{1, \dots, n\} \forall j \in \{1, \dots, m\} :$$

$\rho(r_i, y_j, x) < e \Rightarrow \exists r \in R \forall y \in M : \rho(r, y, x) < e$, ceea ce arată că R este o e-reţea relativ la M.

TEOREMA (2.2). *Fie* $\rho : X^3 \to \mathfrak{M}$ *o g-2-metrică și presupunem că mulțimea* $\mathcal{E} \subseteq \mathfrak{M}$ *(care intervine în definiția g-2-metricii* ρ*), satisface ipotezele suplimentare*

$$(E_1)\ e_0 \in \mathcal{E},\ a \in \mathfrak{M},\ a < e_0 \Rightarrow \exists e_1, e_2 \in \mathcal{E} : \varphi(a, e_1, e_2) \leqslant e_0$$

$$(E')\ \forall e \in \mathcal{E} \exists e' \in \mathcal{E} : e' < e.$$

Atunci, dacă $(X, \mathcal{T}_\rho)$ *este secvențial compact spațiul* (X, ρ) *este precompact.*

Demonstrație. Raționăm prin reducere la absurd. Presupunem că spațiul $(X, \mathcal{T}_\rho)$ este secvențial compact (deci că orice secvență de puncte din X conține o subsecvență care tinde în sensul topologiei $\mathcal{T}_\rho$ către un punct din X) și nu este precompact. Urmează că există un element $e \in \mathcal{E}$ și o mulțime $M \in \mathfrak{X}_0(X)$ și nu avem nici o e-rețea a spațiului relativ la M.

Fie x_1 un punct arbitrat din X. Mulțimea $\{x_1\}$ nefiind o e-rețea relativ la M, există un element $x_2 \in X$ și un $y_1 \in M$ astfel ca $\rho(x_1, x_2, y_1) \nless e$. Mulțimea $\{x_1, x_2\}$ nu este nici ea o e-rețea relativ la M; există deci $x_3 \in X$ și $y_2^1, y_2^2 \in M$ astfel ca $\rho(x_1, x_3, y_2^1) \nless e$ și $\rho(x_2, x_3, y_2^2) \nless e$. Continuînd în acest mod, obținem o secvență de putncte $(x_n)_{n\in\mathbf{N}}$ cu proprietățile

(i) $m \neq n \Rightarrow x_m \neq x_n$;

(ii) $\forall m, n \in \mathbf{N},\ m \neq n \exists y_{(n,m)} \in M : \rho(x_n, x_m, y_{(n,m)}) \nless e$

Spațiul (X, ρ) fiind, prin ipoteză, secvențial compact, din această secvență se poate extrage o subsecvență, convergentă către un punct $x_0 \in X$. Pentru a evita complicarea notațiilor, vom nota în continuare acest subșir tot cu $(x_n)_{n\in\mathbf{N}}$. Prin urmare, conform celor de mai sus, există o secvență de puncte din spațiu, $(x_n)_{n\in\mathbf{N}}$, care se bucură de proprietățile (i) și (ii) și care are limita x_0. Să observăm acum că, în ipotezele (E_1) și (E') impuse mulțimii $\mathcal{E}$, pentru elementul $e \in \mathcal{E}$ considerat, există $e_1, e_2, e_3 \in \mathcal{E}$ astfel ca $\varphi(e_1, e_2, e_3) \leqslant e$. Într-adevăr, în baza ipotezei (E'), există $e_1 < e$; pentru $e_1 < e$, aplicînd ipoteza (E_1), deducem că există $e_2, e_3 \in \mathcal{E}$ cu $\varphi(e_1, e_2, e_3) < e$. Conform teoremei (2.1) din [2], perechilor (e_1, M) și (e_2, M) le corespund respectiv numerele naturale $n_1 = n_1(e_1, M)$ și $n_2 = n_2(e_2, M)$ astfel ca pentru $n \geqslant n_1$ să avem $\rho(x_0, x_n, y) < e_1$ și pentru $n \geqslant n_2$ să avem $\rho(x_0, x_n, y) < e_2$ oricare ar fi $y \in M$. Notînd cu $n_0 = \max\{n_1, n_2\}$ putem în definitiv scrie:

$$n \geqslant n_0 \Rightarrow \forall y \in M : \rho(x_0, x_n, y) < e_1 \text{ și } \rho(x_0, x_n y) < e_2.$$

Fie acum $A_1 = \{x_n \mid n \geqslant n_0\} \subseteq \{x_n \mid n \in \mathbf{N}\} = A$. x_0 fiind punct de acumulare pentru mulțimea A, este punct de acumulare și pentru mulțimea A_1 (în [2] se arată că $(X, \mathcal{T}_\rho)$ este un spațiu T_1). Urmează că, orice vecinătate a punctului x_0, conține puncte din A_1, diferite de punctul x_0. Prin urmare pentru elementul $e_3 \in \mathcal{E}$ pus în evidență mai sus, mulțimea $V_{e_3}(x_0, x_{n_0})$, ca vecinătate a punctului x_0, conține $x^* \in A_1$; există așadar un $m_0 > n_0$, astfel ca $x_{m_0} = x^* \in V_{e_3}(x_0, x_{n_0})$, adică $\rho(x_0, x_{n_0}, x_{m_0}) < e_3$.

Folosind proprietăţile g-2-metricii ρ şi ale operaţiei ternare φ, avem atunci:

$$\forall y \in M : \rho(x_{n_0}, x_{m_0}, y) \leqslant \varphi[\rho(x_{n_0}, x_l, y), \rho(x_{m_0}, x_l, y), \rho(x_l, x_{m_0}, x_{n_0})] <$$
$$< \varphi(e_1, e_2, e_3) \leqslant e,$$

în contradicţie cu proprietatea (ii) a secvenţei $(x_n)_{n \in \mathbf{N}}$.

Contradicţia la care am ajuns încheie demonstraţia teoremei.

BIBLIOGRAFIE

1. D. Borşan, *Spaţii 2-metrice generalizate*, Studia Univ. Babeş-Bolyai, Mathematica XXV, 2(1980), 59–64.
2. D. Borşan, *Unele proprietăţi ale topologiei induse de o g-2-metrică*, Studia Univ. Babeş-Bolyai, Mathematica, XXVI, 4, (1981), 61–64.
3. S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachrichten 26, (1963/64), 114–148
4. J. Kelley, *General Topology*, Toronto 1955.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

MÉTHODE DE L'ALTERNATIVE ET SOLUTIONS NUMÉRIQUES DE CERTAINS PROBLÈMES AUX LIMITES NON-LINÉAIRES

DAMIAN TRIF*

Manuscrit reçu le 1 décembre 1982

ABSTRACT. — The Alternative Method and Numerical Solutions of Some Problems at the Non-Linear Limit. In this paper the author deals with Cesari's alternative method [1] applied to numerical study of the periodical solutions for equations of the form $x'' = q(x,x',t)$. The obtained results extend those of [3], confirming utility of the alternative method as a numerical one.

Dans ce travail on trouve des solutions numériques périodiques d'équations différentielles $x'' = q(x, x', t)$ à l'aide de la méthode de l'alternative de Cesari [1]. Les résultats en élargissent celles que propose notre travail [3] et confirment l'utilité de cette méthode pour la résolution numérique des problèmes aux limites non-linéaires.

Soit S l'espace de Banach des fonctions $x : R \to R$, continues et 2π-périodiques et $||x|| = \sup\{|x(t)|\,|t \in [0,2\pi]\}$. Pour tout $x \in S$ on construit la série de Fourier

$$x(t) \sim \frac{a_0}{2} + \sum_{s=1}^{\infty} (a_s \cos st + b_s \sin st)$$

où

$$a_s = \frac{1}{\pi}\int_0^{2\pi} x(\tau)\cos s\tau d\tau,\ \ b_s = \frac{1}{\pi}\int_0^{2\pi} x(\tau)\sin s\tau d\tau.$$

On sait que cette série est uniformément convergente vers x si x est une fonction de Lipschitz.

Pour tout $m \in Z_+$ on introduit les opérateurs $P_m : S \to S$ et $H_m : S \to S$, où

$$P_m x(t) = \frac{a_0}{2} + \sum_{s=1}^{m} (a_s \cos st + b_s \sin st)$$

$$H_m x(t) = \sum_{s=m+1}^{\infty} \frac{1}{s} (-b_s \cos st + a_s \sin st)$$

a_s, b_s, $s = 0, 1, \ldots$ étant les coefficients de Fourier de $x \in S$.

On note $S_m^0 = \{x \in S\,|P_m x = x\}$, $S_m^1 = \{x \in S\,|P_m x = 0\}$.

On démontre par calcul

* *Université de Cluj-Napoca, Faculté de Mathematique, 3400 Cluj-Napoca, Roumanie*

THÉORÈME 1. *Pour tous* $m \in Z_+$ *et* $x \in S$ *on a*

$$\text{a) } H_m x(t) = \int_0^{2\pi} \left[\frac{\tau - t}{2\pi} + \frac{1}{2}\,\mathrm{sgn}(t-\tau) - \sum_{s=1}^{m} \frac{\sin s(t-\tau)}{s\pi}\right] x(\tau)\,d\tau$$

et

$$||H_m|| = \int_0^{2\pi} \left| -\frac{\tau}{2} + \frac{1}{2} - \sum_{s=1}^{m} \frac{\sin s\tau}{s\pi}\right| d\tau \to 0 \quad pour \quad m \to \infty.$$

$$\text{b) } H_m^2 x(t) = \int_0^{2\pi} \left[-\frac{(t-\tau)^2}{4\pi} + \frac{|t-\tau|}{2} - \frac{\pi}{6} + \sum_{s=1}^{m} \frac{\cos s(t-\tau)}{s^2\pi}\right] x(\tau)\,d\tau$$

et

$$||H_m^2|| = \int_0^{2\pi} \left| -\frac{\tau^2}{4\pi} + \frac{\tau}{2} - \frac{\pi}{6} + \sum_{s=1}^{m} \frac{\cos s\tau}{s^2\pi}\right| d\tau \to 0 \; pour \; m \to \infty.$$

On considère l'équation différentielle $x'' = q(x, x', t)$, où $q : [A, B] \times [A_1, B_1] \times R \to R$ est continue, 2π-périodique par rapport à t, fonction de Lipschitz par rapport à x est x', aux constantes L et L'.

Soient $m \in Z_+$ et $x_0 \in S_m^0$. On définite le procès itératif

$$y_1^0 = x_0 \qquad y_1^{k+1} = x_0 + H_m y_2^k$$

$$y_2^0 = x_0' \qquad y_2^{k+1} = x_0' + H_m q(y_1^{k+1}, y_2^k, t), \; k = 0, 1, \ldots$$

Si $||y_1^k - x_0|| \leqslant a$, $||y_2^k - x_0'|| \leqslant b$ pour tous $k = 0, 1, \ldots, n$ où $a > 0$, $b > 0$alors $(||y_1^{n+1} - x_0|| = ||H_m y_2^n|| = ||H_m^2 q(y_1^n, y_2^{n-1}, t)|| \leqslant \leqslant ||H_m^2|| \cdot ||(I - P_m) q(y_1^n, y_2^{n-1}, t)|| \leqslant a$ et $||y_2^{n+1} - x_0'|| = ||H_m q(y_1^{n+1}, y_2^n, t)|| \leqslant ||H_m|| \cdot ||(I - P_m) q(y_1^{n+1}, y_2^n, t)|| \leqslant b$ si m est suffisamment grand.

On a $||y_1^{k+1} - y_1^k|| \leqslant ||H_m|| \cdot ||y_2^k - y_2^{k-1}||$, $||y_2^{k+1} - y_2^k|| \leqslant ||H_m|| \cdot \cdot q(y_2^{k+1}, y_2^k, t) - q(y_1^k, y_2^{k-1}, t)|| \leqslant (||H_m||(L||y_1^{k+1} - y_1^k|| + L'||y_2^k - - y_2^{k-1}||) \leqslant (||H_m||^2 L + ||H_m||L')||y_2^k - y_2^{k-1}||$. Pour $\delta = ||H_m||^2 L + + ||H_m||L'$ on a

$$||y_2^{k+1} - y_2^k|| \leqslant \delta^k ||y_2^1 - y_2^0|| \leqslant \delta \; ||H_m|| \cdot ||q(x_0, x_0', t)||$$

$$||y_1^{k+1} - y_1^k|| \leqslant \delta^{k-1} ||H_m||^2 ||q(x_0, x_0', t)||.$$

Si m est suffisamment grand, les suites y_1^k, y_2^k sont fondamentales dans S, donc convergentes, d'où $y_1^k \to y_1 \in S$, $y_2^k \to y_2 \in S$.

On a $y_1 = x_0 + H_m y_2$, $y_2 = x_0 + H_m q(y_1, y_2, t)$. De plus, $y_1' = x_0' + (I - P_m) y_2 = y_2$, donc $y_1 = x_0 + H_m^2 q(y_1, y_1', t)$. La fonction y_1 s'appèlle *la fonction associée* à x_0 et, au fond, s'obtent par le procès itératif

$$y_1^{k+1} = x_0 + H_m^2 q(y_1^k, y_1^{k\prime}, t), y_1^0 = x_0,\ k = 0, 1, \ldots$$

Si $m < n$, $x_0 \in S_m^0$ et $y \in S$ est la fonction associée à x_0, si $x_1 = P_n y$ et y_1 est la fonction associée à x_1, alors $y = y_1$.

En effet, $y = x_1 + H_n^2 q(y, y', t)$, donc $||y' - y_1'|| = ||x_1 + H_n q)y, y', t) - x_1' - H_n q(y_1, y_1', t)|| \leqslant H_n||\cdot||q(y, y', t) - q(y_1, y_1', t)|| \leqslant ||H_n||L||y - y_1|| + ||H_n||L'||y' - y_1'||$. Mais $||y - y_1|| = ||x_1 + H_n y' - x_1 - H_n y_1'|| \leqslant ||H_n||\cdot||y' - y_1'||$ donc $||y' - y_1'|| \leqslant (||H_n||^2 L + ||H_n||L')||y' - y_1'|| = \delta||y' - y_1||$. Il en résulte $y' = y_1'$ parce que $\delta < 1$, donc $y = y_1$.

THÉORÈME 2. *L'équation* $x''(t) = q(x(t), x'(t), t)$, où $q : [A, B]x [A_1, B_1]\, xR \to R$ est continue, 2π-périodique par rapport à t, fonction de Lipschitz par rapport à x et x', aux constantes L et L', a une solution 2π-périodique si et seulement s'il existent $m \in Z_+$ et $x_0 \in S_m^0$ tels que $x_0''(t) = P_m q(y(t), y'(t), t)$ où y est la fonction associée à x_0. De plus, y est solution de cette équation.

Démonstration. a) Si x est solution, soit m suffisamment grand tel qu'il existe la fonction y associée a $x_0 = P_m x$. Alors, $x_0 + H_m^2 q(x, x', t) = x_0 + H_m^2 x'' = P_m x + (I - P_m) x = x$. Mais $y = y_1 = x_0 + H_m y_2$, $y_2 = x_0' + H_m q(y_1, y_2, t)$. De plus, $x = x_0 + H_m x_2$, $x' = x_2 = x_0 + H_m q(y_1, y_2, t)$. De plus, $x = x_1 = x_0 + H_m x_2$, $x' = x_2 = x_0'' + H_m q(x_1, x_2, t)$ d'ou $||y_2 - x_2|| \leqslant ||H_m||(L||y_1 - x_1|| + L'||y_2 - x_2||)$, $||y_1 - x_1|| \leqslant ||H_m||\cdot||y_2 - x_2||$, donc $||y_2 - x_2|| \leqslant \delta||y_2 - x_2||$, possible si et seulement si $y_2 = x_2$. Il en résulte $y_1 = x_1$, c'est à dire $y = x$ et $x'' = q(y, y', t)$, d'où $x_0 = P_m q(y, y', t)$.

b) Soient m suffisamment grand et x_0 solution du système $x_0'' = P_m q(y, y', t)$, où y est la fonction associée à x_0. Donc $y = x_0 + H_m^2 q(y, y', t)$. Par dérivation deux fois on obtient $y'' = x_0'' + (I - P_m) q(y, y', t)$ donc $y'' = q(y, y', t)$.

Si $x_0 = \frac{a_0}{2} + \sum_{s=1}^{m} (a_s \cos st + b_s \sin st)$ pour m suffisamment grand tel qu'il existe la fonction y associée à x_0, alors le système $x_0'' = P_m q(y, y', t)$ devient

$$\frac{1}{2\pi}\int_0^{2\pi} q(y(\tau), y'(\tau), \tau)\, d\tau = 0$$

$$\frac{1}{\pi}\int_0^{2\pi} q(y(\tau), y'(\tau), \tau) \cos s\tau d\tau = s^2 a_s$$

$$\frac{1}{\pi}\int_0^{2\pi} q(y(\tau), y'(\tau), \tau) \sin s\tau d\tau = s^2 b_s$$

pour $s = 1, \ldots, m$, aux inconnues $a_0, a_1, b_1, \ldots, a_m, b_m$

Par example, on considère l'équation

$$x'' = 0,1 \sin t - x + 0,1\,(1 - x^2)x'$$

On prend $x_0 = -2,4 \cos t$ (approximation Galerkin d'ordre 2) $a = 0,2$, $b = 0,75$. On trouve $\delta = 0,83$ pour $m = 3$. Les ystème $x_0'' = P_m q(y, y', t)$ a été résolu par la méthode de la recherche unidimensionelle, la série de y étant tronquée au rang $N = 13$. Les approximations y_1^k de y ont été continuées jusqu'à la précision $\varepsilon = 10^{-9}$. On truve la même solution que Urabe, Reiter [2] pour cette equation. De puls, on obtient même solution pour $m = 1$, pendant que dans [2] la solution a été obtenue à l'aide d'un système de 27 équations.

L'étude des erreurs sera faite dans un autre travail.

BIBLIOGRAPHIE

1. Cesari, L., *Functional Analysis and Galerkin's Method*, Michigan Math. J., *11*, 4, 1964, 385–414.
2. Urabe, M., Reiter, A., *Numerical Computation of Nonlinear Forced Oscillations by Galerkin's Procedure*, J. of Math. Anal. Appl., *14*, 1, 1966, 107–140.
3. Trif, D., *Rezolvarea numerică a unor probleme la limită cu ajutorul metodei de alternativă*, Studia Univ. Babeş-Bolyai, Mathematica, XXVII, 1982, 28–32.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

COORDONATIZAREA UNEI CLASE DE $\mathfrak{B}$ — STRUCTURI

ANGELA VASIU*

Intrat în redacţie la 7 decembrie 1981

ABSTRACT. — The Coordinatisation of a Class of $\mathfrak{B}$-Structures. In the present note a class of affine Barbilian structures is characterized by geometric axioms. A coordonatization by a calculus with points which defines an algebraic structure of loop is introduced.

O structură de incidenţă ($\mathfrak{P}$, $\mathfrak{D}$, I) este un ansamblu format dintr-o mulţime $\mathfrak{P}$ ale cărei elemente se numesc puncte împreună cu un subsistem de submulţimi ale lui $\mathfrak{P}$, notat $\mathfrak{D}$ ale cărui elemente se numesc drepte, împreună cu o relaţie binară simetrică I:

$$I \subset \mathfrak{P} \times \mathfrak{D}$$

numită relaţie de incidenţă. Vom nota pentru $A \in \mathfrak{P}$, $d \in \mathfrak{D}$, $I \in (A, d)$, AId şi vom spune că A este incident cu d sau că d trece prin A.

DEFINITIA 1. *A, $B \in \mathfrak{P}$ se numesc vecine dacă există a, $b \in \mathfrak{D}$ a.î., A, B I a, b şi se notează A o B. Negaţia acestei relaţii se notează prin $A \emptyset B$.*

DEFINITIA 2. *Dreptele a, $b \in \mathfrak{D}$ se numesc vecine dacă oricare ar fi AIa există BIb a.î. AoB şi invers pentru PIa există QIb a.î. PoQ.*

DEFINITIE. *O structură de incidenţă cu vecinătăţi, împreună cu o relaţie de echivalenţă definită pe mulţimea dreptelor ei, notată $||$, se numeşte o structură afină Barbilian notată ($\mathfrak{P}$, $\mathfrak{D}$, I, $||$) dacă sînt îndeplinite următoarele axiome:*

AXIOMA 1. $\forall A, B \in \mathfrak{P}$, există cel puţin o dreaptă $d \in \mathfrak{D}$, dIA, B

AXIOMA 2. $\forall A \in \mathfrak{P}$ şi $\forall d \in \mathfrak{D}$, $\exists_1 d_1 IA$, $d_1 || d$

AXIOMA 3. $AID \wedge AId_1$ atunci $d \emptyset d_1 \Rightarrow \overline{d \cap d_1} = 1$

AXIOMA 4. $d_1 o d_2$, AId, d_1, $B\, Id$, d_2, $d \emptyset d_1 \Rightarrow A$ o B

AXIOMA 5. $d_1, d_2 IA$, BId_1, $B \emptyset A$, $d \emptyset d_1$, BId, CId, d_2 a.î. $BoC \Rightarrow$ $\Rightarrow d_1 o d_2$

AXIOMA 6. $d_1 || d_2$, $d \emptyset d_1$, AId_1, $d \Rightarrow \exists BId_2$, $d \wedge d_2 \emptyset d$

AXIOMA 7. $\forall d \in \mathfrak{D}$, $\exists P, QId$, $P \emptyset Q$. Există două drepte concurente şi nevecine.

Vom numi o structură ($\mathfrak{P}$, $\mathfrak{D}$, I, $||$) pe scurt o $\mathfrak{B}$-structură.

---

* *Universitatea din Cluj-Napoca, Facultatea de Matematică, 3400 Cluj-Napoca, România*

1. Primele consecințe ale sistemului de exiome.

PROPOZIȚIA *1.1.* $P, Q \in P, P \varnothing Q \Rightarrow \exists_1 dIP, Q$.

Demonstrație. Conform axiomei A_1 există o dreaptă dIP, Q și ținîn cont de definiția 1, din $P\varnothing Q$, rezultă că d este singura dreaptă incident cu d.

Vom nota pentru $P\varnothing Q$, dreapta dIP, Q prin PQ.

PROPOZIȚIE *1.2. Dacă $d_1 \varnothing d_2$ și AId_1, d_2 și $BId_1, B\varnothing A$ atunci ori dreaptă d incidentă cu B este nevecină cu d_2.*

Demonstrație. Presupunînd că dreapta d este vecină cu dreapta d avînd în vedere ipoteza $d_1\varnothing d_2$, conform axiomei A_4 am avea AoB, în con tradicție cu ipoteza $A\varnothing B$. Deci avem $d\varnothing d_2$.

Conform definiției 2 pentru $d_1\varnothing d_2$, există un punct BId_1, nevecin c orice punct CId_2. Următoarea propoziție demonstrează că această propri tate are loc pentru orice punct BId_1 și $B\varnothing A$.

PROPOZIȚIA *1.3. Dacă $d_1 \varnothing d_2$ și AId_1, d_2 și $BId_1, B\varnothing A$ atunci $B\varnothing C$ oricare ar fi CId_2.*

Demonstrație. Conform axiomei A_1 există o dreaptă d a.î. B, CI În baza propoziției 1.2. are loc propietatea $d\varnothing d_2$. Presupunînd că BoC pentru dreptele d_1, d_2, d și punctele BC în baza axiomei A_5, am avea d_1od Această contradicție cu ipoteza $d_1\varnothing d_2$ demonstrează proprietatea $B\varnothing C$

Vom nota în continuare unica paralelă prin A la d, asigurată de axi ma A_2, prin $(A||d)$.

PROPOZIȚIA *1.4. Dacă $d_1\varnothing d_2$ și AId_1, d_2 atunci orice paralelă la intersectează orice paralelă la d_2 într-un singur punct.*

Demonstrație. Dacă $d||d_2$ atunci conform axiomei A_6 BId, d_1 și $d\varnothing d$ Analog dacă $d'||d_1$ avem $d'\varnothing d_2$ și există CID', d_2. Din $d'\varnothing d_2$ și $d||d$ în baza axiomei A_6 avem $s'\varnothing d$ și există DId', d. În baza axiomei A_3, este unicul incident cu d și d'.

2. **Coordonatizarea $\mathfrak{B}$-structurilor.** Fie d_1 și d_2 dreptele concurent nevecine, asigurate de axioma A_7 și OId_1, d_2. Conform axiomei A_7 exis $AId_1, A\varnothing O$ și există $BId_2, B\varnothing O$. Dreptele d_1, d_2, AB sînt două cîte do nevecine, avînd în vedere ipoteza $d_1\varnothing d_2$ și propoziția 1.2. Dreptele d_1, împreună cu O, A și B vom spune că formează o configurație de coord natizare.

Considerăm următoarele mulțimi:

$$[|d_1|] := \{d \in \mathfrak{D} \mid d || d_1\}$$

și

$$\{d_1\} := \{P \mid P \in \mathfrak{P}, PId_1\}$$

Într-o configurație de coordonatizare are loc:

PROPOZIȚIA 2.1. *Mulțimea $\{d_1\}$ este izomorfă cu oricare din următoar mulțimi:*

a) $[|AB|]$, *b)* $\{d_2\}$, c) $\{d\}$, $\forall d \in D, d||d_2$

d) $[|d_2|]$, e) $[|d_1|]$.

Demonstrație. a) Considerăm aplicația

$$\pi : \{d_1\} \to [\,|AB|\,]$$
$$P \to (P\,||AB)$$

Oricare ar fi $P \in \{d_1\}$ conform axiomei A_2 există o singură dreaptă dIP și $d\,||AB$.

Invers, dacă $d \in [\,|AB|\,]$, din $AB \varnothing d_1$ în baza axiomei A_6, $d\varnothing d_1$ și există QId, d_1.

Deci aplicația π este o bijecție între $\{d_1\}$ și $[\,|AB|\,]$, ceea ce demonstrează proprietatea a).

b) Din $AB\,||\pi(X)$ și $AB\varnothing d_2$, în baza axiomei A_6 au loc:

$d_2 \varnothing \pi(X)$ și există un singur punct $X'Id_2$, $\pi(X)$.

Aplicația:

$$\pi_1 : \{d_1\} \to \{d_2\}$$

$\pi_1(X) := X'$, $X' := d_2 \cap \pi(X)$ este o bijecție între $\{d_1\}$ și $\{d_2\}$.

c) Ținînd cont de b) și axioma A_6 rezultă ușor că $\{d_1\}$ și $\{d\}$, $d\,||d_2$, sînt izomorfe.

d) Considerăm aplicația:

$$\pi_2 : \{d_1\} \to [\,|d_2|\,]$$
$$P \to (P\,||d_2)$$

Conform axiomei A_2 oricărui $P \in \{d_1\}$ îi corespuunde din $[\,|d_2|\,]$ un singur element. Dacă $d \in [\,|d_2|\,]$ atunci conform axiomei A_6 rezultă că există un singur QId_1, d și deci π_2 este o bijecție între $\{d_1\}$ și $[\,|d_2|\,]$.

e) Considerăm aplicația:

$$\pi_3 : \{d_2\} \to [\,|d_1|\,]$$
$$X' \to (X'\,||d_1)$$

Această aplicație, avînd în vedere demonstrația de la punctul d) este o bijecție. Atunci aplicația

$$\pi_3 \circ \pi_1 : \{d_1\} \to [\,|d_1|\,]$$

este o bijecție între $\{d_1\}$, $[\,|d_1|\,]$. Într-adevăr

$$\pi_1(X) := (X\,||AB) \cap d_2 = X'$$

iar $\pi_3(X')$ face să corespundă lui X' o singură dreaptă $d\,||d_1$. Compunerea celor două bijiecții arată că: $\pi_4 = \pi_3 o \pi_1$ este o bijecție care stabilește izomorfismul lui $\{d_1\}$ cu $[\,|d_1|\,]$.

PROPOZIȚIA 2.2. *Intre mulțimea* $\mathfrak{P}$ *și* $\{d_1\}^2$ *există o corespondență biunivocă.*

Demonstrație. Dacă $P \in \mathfrak{L}$ atunci conform axiomei A_2 există o singu dreaptă x, $x \| d_2$ și PIx și o singură dreaptă y, $y \| d_1$ și yIP.

Fie $X := \pi_2^{-1}(x)$ și

$$Y' := y \cap d_2$$

Dacă $y := \pi_1^{-1}(Y')$, atunci fiecărui punct $P \in \mathfrak{L}$ îi corespunde un sing element $(x, y) \in \{d_1\}^2$.

Invers, dacă avem $(X, Y) \in \{d_1\}^2$ atunci fie $\pi_2(X) := x$ și $y := \pi_4($ Dreptele $x \| d_2$ și $y \| d_1$, conform propoziției 1.4., se taie într-un sing punct P.

Notăm prin $M := \{d_1\}$, a cărui elemente le notăm după cum urmea: $O := o$, $A := 1$, $P := p$, iar BId_2, îl notăm cu $1'$.

Dreapta $d_1 = OA$ o notăm prin $\overline{0, 1}$. Avînd în vedere propoziția 2.2. pute nota :

$$\{d_1\} := \overline{0,1} := \{(x, 0) \mid x \in M\}$$

$$\{d_2\} := \overline{0,1'} := \{(0, y) \mid y \in M,\ y' := \pi_1^{-1}(y)\}$$

Cu aceste notații $A \equiv (1, 0)$, $B \equiv (0, 1')$, $0 \equiv (0, 0)$

PROPOZIȚIA *2.3. Dacă $d_1 \emptyset d_2$ și OId_1, d_2 și PId_2, $P\emptyset O$ atunci $(P \| d_1)$ $\emptyset PQ$, oricare ar fi QId_1.*

Demonstrație. Conform propoziției 1.2. dreapta PQ este nevecină dreapta d_1'. Ținînd cont de axioma A_6, $(P \| d_1) \emptyset PQ$.

CONSECINȚA. Într-o *configurație de coordonatizare dreapta* $((0,1') \|$ *este nevecină cu dreapta* $\overline{(0,1')}$, $\overline{(p, 0)}$ *oricare ar fi* $p \in M$ *(corespunzător u punct* $P = (p, 0)$ *din* $\{d_1\}$*).*

Putem acum defini pe mulțimea $\{d_1\}$ două operații cu puncte : a ad nare și înmulțire.

Fie $d_1' = (B \| d_1)$, $R := (P \| d_2) \cap d_1'$.

(1) $P + Q := (R \| BQ) \cap d_1 = S$. Punctul S este univoc deter nat. Într-adevăr $BQ\emptyset d_1$ atunci $(R \| BQ)\emptyset d_1$ și deci există un singur pu $S \in (R \| BQ) \cap d_1$.

Dacă în coordonatizarea dată avem :

$$P = (p, 0),\ Q = (q, 0) \text{ și } S = (s, 0)$$

pe mulțimea M am definit o lege de compoziție internă.

(2) $p + q := s$

Din definiția (1) deducem :

(3) $A + O = O + A = A$

iar pentru $(M, +)$ avem :

(4) $a + 0 = 0 + a = a$

Față de o configurație de coordonatizare d_1, d_2, O, A, B, are loc:

PROPOZIȚIA *2.4* $\forall R$, RId_1', $d_1' = (B||d_1)$, *are proprietatea*

$$R \emptyset S, \ \forall SId_1.$$

Demonstrație. Fie $P = (R||d_2) \cap d_1$. Ținînd cont de propoziția 1.4., $(R||d_2) \emptyset d_1$ și conform axiomei A_6 $(R||d_2) \emptyset d_1'$. Are loc:

$R \emptyset P$. Într-adevăr presupunînd RoP din $RP=(R||d_2) \emptyset d_1'$ am avea conform axiomei A_5, $BPod_1'$, contradicție cu consecința propoziției 2.3.

Din $R\emptyset P$ și $RP\emptyset d_1$ avem conform propoziției 1.3. $R\emptyset S$.

CONSECINȚA. $\forall RId_1'$ *și* SId_1 *există o singură dreaptă incidentă cu ele. Dreapta* $RS \emptyset d_1$.

Putem acum arăta că ecuațiile:

(5) $P + X = S$

și

(6) $X + Q = S$

au soluții univoc determinate.

Într-adevăr avînd în vedere definiția 1 dacă $R = (P||d_1) \cap d_1'$; dreapta RS în baza propoziției 2.4. este univoc determinată. Din $RS\emptyset d_1$ rezultă în baza propoziției 1.4. că $\overline{\overline{(B||RS) \cap d_1}} = 1$. Fie $X := (B||RS) \cap$ $\cap\, d_1$ atunci are loc în baza definiției 1:

$$P + X = S$$

Din $BP\emptyset d_1$ rezultă $\overline{\overline{(S||d_2) \cap d_1'}} = 1$. Fie $R := (S||d_2) \cap d_1'$. Avem apoi $\overline{\overline{(S||d_2) \cap d_1}} = 1$ și fie X acel punct. Avînd în vedere definiția dată în (1) avem:

$$X + P = S$$

În baza celor de mai sus în $(M, +)$, ecuațiile:

(7) $p + x = s$

și

(8) $x + p = s$

au soluții unice în M și deci are loc:

TEOREMA 2.1. *$(M, +)$ este un loop în care o are rol de element neutru.*

Definim acum pe d_1 o operație multiplicativă,

Fie d_1, d_2, O, A, B o configurație de coordonatizare. Fie $P' = (P||$ $||AB) \cap d_2$ atunci definim:

(9) $P.Q := (P'||BQ) \cap d_1 = R$

În baza celor de mai sus, se deduce ușor că R este univoc determinat.

Au loc relaţiile :

(10) $A.P = P.A = P$

Dacă $P = (p, 0)$, $Q = (q, 0)$ şi $R = (r, 0)$ definim pe M operaţia :

(11) $p.q : = r$

Din $A = (1, 0)$ în baza relaţiei (10) deducem că 1 are rol de uni faţă de această operaţie definită pe M.

BIBLIOGRAFIE

1. KLINGENBERG, W.: „*Projective und affine Ebenen mit Nachbarelemen* în Math. Zeitschr., 60 (1954), 384–406.
2. LEISNER, W. : „*Affine Barbilian-Ebenen (I)*", în Journal of Geom., vol. 6/1 (1 31–57.
3. LEISSNER, W. : „*Affine Barbilian-Ebenen (II)*", în Journal of Geom., vol (1975), 105–129.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

ON THE EXPONENTIAL PENALTY FUNCTION METHODS FOR GEOMETRIC PROGRAMMING PROBLEMS

LUCIA BLAGA*

Received December 22, 1982

ABSTRACT. — In this paper an exponential penalty function method is given for the solving of geometric programming problems. It is proved that after a finite number of steps the method becomes a stable interior penalty one. Geometric programming test problems have been solved on computer by this method.

In this paper we shall show that exponential penalty function methods from [1] can be extend for geometric programming problems.

We consider the geometric programming problem:

(P) $$\inf \{p_0(x) \mid p_k(x) \leqslant 1,\ k = 1, \ldots, p;\ x > 0\}$$

where p_k are posynomials:

$$p_k(x) = \sum_{i \in I_k} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_n^{a_{in}},\ k = 0, 1, \ldots, p \tag{1}$$

and $$c_i \in R_+,\ a_{ij} \in R,\ j = 1, 2, \ldots, n.$$

The sets I_k are defined by:

$$\bigcup_{k=0}^{p} I_k = \{1, 2, \ldots, m\},\ I_k \cap I_h = \Phi,\ k \neq h \tag{2}$$

where m denotes the total number of terms that apear in the posynomials p_k, $k = 0, 1, \ldots, p$.

By cetting

$$x_j = e^{z_j},\ j = 1, 2, \ldots, n \tag{3}$$

the problem (P) is transformed into:

$$(\mathrm{P}_z):\quad \inf \{q_0(z) \mid q_k(z) \leqslant 1,\ k = 1, 2, \ldots, p\}$$

where

$$q_k(z) = \sum_{i \in I_k} c_i e^{\sum_{j=1}^{n} a_{ij} z_j},\ k = 0, 1, \ldots, p \tag{4}$$

* *Institute of Polytechnics, 3400 Cluj-Napoca, Romania*

and

$$c_i \in R_+,\ a_{ij} \in R,\ j = 1, \ldots, n.$$

We note by

$$g_k(z) = q_k(z) - 1 \quad k = 1, 2, \ldots, p$$
$$g_0(z) = q_0(z)$$

Then the program (P_z) becomes:

(P'z) $$\inf \{g_0(z) \mid g_k(z) \leqslant 0,\ k = 1, 2, \ldots, p\}$$

Let

$$\Omega = \{z \in R^n \mid g_k(z) \leqslant 0,\ k = 1, \ldots, p\}$$

We construct penalty functions in the following way:

$$F_n(z) = g_0(z) + \frac{s_n}{\sqrt{t_n}} \, ||z||^2 + s_n \sum_{k=1}^{p} \exp\,(t_n^2 g_k(z)) \quad n \in N$$

where $(s_n)_{n\in N}$, $(t_n)_{n\in N}$ are real positive sequences satisfying the followi conditions:

$$s_n t_n \geqslant 1,\ n \in N,\ \lim_{n\to\infty} t_n = \infty,\ \lim_{n\to\infty} s_n = s \geqslant 0$$

and $|| \cdot ||$ denotes the euclidian norm.

THEOREM: *If program* (P) *is superconsistent and canonic* [2, pag. 1: *then* (i) min $\{F_n(z) \mid z \in R^n\}$ *has a unique solution* z_n, $n \in N$

(ii) *There exists an* $n^* \in N$, *such that for any* $n \in N$, $n \geqslant n^*$ *impl* $z_n \in \operatorname{int} \Omega$.

(iii) *If we note by* $\Omega(g_0)$ *the solution set of the problem* (P_z) *and* z_0 *an element of* $\Omega\,(g_0)$ *with the property that.*

$$||z_0|| = \min_{z\in\Omega(g_0)} ||z||,$$

then the sequence $(z_n)_{n\in N}$ *converges to* z_0.

$$\text{(iv)}\ g_0(z_0) = \min_{z\in\Omega} g_0(z) + O\left(\frac{s_n}{\sqrt{t_n}}\right) \textit{for } n \to \infty.$$

Proof: A geometric program (P) is canonic if and only if its dual D l an admissible solution $y > 0$ (every component of y is positive) (Th rem 9.2 — /2/), where

$$(D):\ \max\left\{v(y) = \prod_{i=1}^{m}\left(\frac{c_i}{y_i}\right)^{y_i} \prod_{k=1}^{p} \lambda_k(y)^{\lambda_k(y)} \mid y \in \Omega^*\right\}$$

$$\lambda_k(y) = \sum_{i\in I_k} y_i,\ k = 1,\ 2,\ \ldots,\ p$$

and $\Omega^* \subset R^m$ is the set of those $y \in R^m$ which satisfy the conditions

$$y_1 \geqslant 0,\ y_2 \geqslant 0,\ \ldots,\ y_m \geqslant 0 \tag{8}$$

$$\sum_{i \in I_0} y_i = 1 \tag{9}$$

$$\sum_{i=1}^{m} a_{ij} y_i = 0,\ j = 1, 2, \ldots, n \tag{10}$$

If (P) is consistent and there exists an $y^* \in \Omega$, $y^* > 0$, then the program (P) has solution (Theorem 8.2., [2]).

The program (P_z) and therefore the program (P'_z) is convex, every hypotesis of the theorem (3.1) from [1] is fullfield and thus, the theorem is proved.

Remark 1. If the program (P) is only consistent, (instead of superconsistent) and in addition we assume that the dual (D) of the program (P) has every y_i, $i \in I_k$, unbounded in Ω^*, then the set Ω of admissible solutions of program (P'_z) is compact. (Consequence 8.1. [2]).

Under these conditions the statement (ii) of theorem remains valid even for non-regularized penalty functions of the following form :

$$f_n(z) = p_c(z) + s_n \sum_{k=1}^{p} \exp\,(t_n^2 g_k(z)) \qquad n \in N$$

Remark 2. In view of remark (2.4) from [1], considering the regularized penalty functions F_n with the choice of $s = 0$, as the limit poin of s_n, the method presented works at least like an exterior penalty method, where accumulation points of the (unique) trial solutions solve the problem (P'_z).

In addition, if the Slater condition is satisfied, then the method becomes a (stable) interior penalty method after a finit number of steps.

Remark 3. Choosing the parameter $s_n = t_n^{-1}$ and keepping the superconsistent condition of the program (P), we get the better estimate for the rate of convergence of the values, namely $O(t_n^{-3/2})$ for $t_n \to \infty$.

Using this method we solved the following geometric programming problemes :

1) $\min \{2x_1x_3^{-1}x_4 + 2x_1 + x_1x_4 + x_2^{-1}\}$

$$\text{on} \begin{cases} x_1^{-1}x_2 + x_1^{-1} \leqslant 1 \\ x_1 x_4^{-1} \leqslant 1 \\ x_1 x_3 \leqslant 1,\ x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0 \end{cases}$$

Starting with (1, 1, 1, 1) (which is an exterior point of Ω), a 42 iterations the method gives.

$$\min\{2x_1x_3^{-1}x_4 + 2x_1 + x_1x_4 + x_2^{-1}\} = 12{,}0032141$$

for $x_1 = 1{,}2243291$

$x_2 = 0{,}2752316$

$x_3 = 0{,}7782453$

$x_4 = 1{,}2564112$

2) $\min\{x_1 + x_2\}$

on $\begin{cases} x_1^{-1} + x_2^{-1} \leqslant 1 \\ x_1 > 0,\ x_2 > 0 \end{cases}$

After 30 iterations, with starting point (1, 1) (which is an extc point of feasible set Ω) the method gives:

$\min\{x_1 + x_2\} = 3{,}9886424$

for $x_1 = x_2 = 1{,}9943212$

For un constrained minimum we used the method of conjugate dients.

REFERENCES

1. Hartung, J., *On exponential Penalty Function Methods*, Math., Op., Statist., Ser. mization, Vol. 11, 1980, no. 1, 71–84.
2. Maruşciac, I., *Programare geometrică şi aplicaţii*, Ed. Dacia, Cluj-Napoca, 19
3. Murphy, G., H., *A class of exponential penalty functions*. SIAM J. Control, 12 (1 679–687.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

ON SOME CLASSES OF DIFFERENTIAL SUBORDINATIONS

TEODOR BULBOACĂ*

Received December 15, 1982

ABSTRACT. — The author genealizes in the present paper the results obtained in [1] on differential subordinations of the form $\psi(p(z), zp'(z)) \prec h(z)$ when $\psi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$ by using the methods in [1] and [2], then consequences and applications of these are given.

1. **Introduction.** Let $H(U)$ be the space of functions analytic in the unit disk U. If $f, g \in H(U)$ we say that f if subordinate to $g(f \prec g$ or $f(z) \prec g(z))$ if g is univalent in U, $f(0) = g(0)$ and $f(u) \subseteq g(u)$.

Let $\psi : \mathbf{C}^2 \to \mathbf{C}$ be analytic in a domain D, and let p be in $H(U)$ with $(p(z), zp'(z)) \in D$ when $z \in U$.
Let $h \in H(U)$ be univalent in U and suppose that p satisfies the differential subordination $\psi(p(z), zp'(z)) \prec h(z)$.

In [1] the authors determine conditions on ψ and h so that $p(z) \prec \prec h(z)$ in the case $\psi(p(z), zp'(z)) = \theta(p(z)) + zp'(z)\Phi(p(z))$ and they give applications of these results.
In this paper we shall study the differential subordination in the case

$$\psi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$$

and applications of these results are given.

2. **Preliminaries.** We will need the two lemmas presented in this section.
LEMMA 1. *Let $g \in H(U)$ with $g(0) = 0$ be univalent and starlike in U. If $f \in H(U)$ and*
Re $[zf'(z)/g(z)] > 0$, $z \in U$ *then f is univalent in U.*
This result is the well-known criterion of univalence of O z a k i and K a p l a n [3].
We said that $L : U \times [0, +\infty) \to \mathbf{C}$ is a subordination (or Loewner) chain if $L(\cdot, t)$ is analytic and univalent in U for all $t \geq 0$, $L(z, \cdot)$ is continously differentiable on $[0, +\infty)$ for all $z \in U$ and $L(z; s) \prec L(z; t)$ when $0 \leqslant s < t$.
LEMMA 2. [4, p. 159]. *The function $L(z, t) = a_1(t)z + \ldots$, with $a_1(t) \neq \neq 0$ for all $t \geqslant 0$ is a subordination chain if and only if* Re $\left[z \frac{\partial L}{\partial z} \Big/ \frac{\partial L}{\partial t}\right] > 0$ *for all $z \in U$ and $t \geqslant 0$.*

* *Industrial Licle no. 1, 2900 Arad, Romania*

THEOREM A [2]. *Let $h, q \in H(U)$ be univalent in U and suppose $q \in H(\bar{U})$. If $\psi : \mathbf{C}^3 \to \mathbf{C}$ satisfies:*

a) *ψ is analytic in a domain $D \subset \mathbf{C}^3$,*

b) *$(q(0), 0, 0) \in D$ and $\psi(q(0), 0, 0) \in h(U)$,*

c) *$\psi(r, s, t) \notin h(U)$ when $(r, s, t) \in D$,*

$$r = q(\zeta),\ s = m\zeta q'(\zeta),\ \mathrm{Re}(1 + t/s) \geqslant m\,\mathrm{Re}(1 + \zeta q''(\zeta)/q'(\zeta))$$

where $|\zeta| = 1$, $m \geqslant 1$,

then for all $p \in H(U)$ so that

$$(p(z),\ zp'(z),\ z^2p''(z)) \in D \text{ when } z \in U \text{ we have}$$

$$\psi(p(z),\ zp'(z),\ z^2p''(z)) \prec h(z) \Rightarrow p(z) \prec q(z).$$

3. Main results.

THEOREM 1. *Let q be convex (univalent) in U, and let α, β be analytic in a domain $D \supset q(U)$ and γ analytic in $\mathbf{C}$. Suppose that*

(i) $$\mathrm{Re}\, \frac{\alpha'(q(z)) + \beta'(q(z))\,\gamma((1+t)\,zq'(z))}{\beta(q(z))\,\gamma'((1+t)\,zq'(z))} \geqslant 0$$

or all $z \in U$ and $t \geqslant 0$

(ii) $Q(z) = zq'(z)\beta(q(z))\gamma'(zq'(z))$ *is starlike (univalent) in U.*

If p is analytic in U with $p(0) = q(0)$, $p(U) \subset D$ then

$$\alpha(p(z)) + \beta(p(z))\gamma(zp'(z)) \prec \alpha(q(z)) + \beta(q(z))\gamma(zq'(z)) \Leftrightarrow p(z) \prec q(z).$$

Proof. Without loss of generality we can assume that p and q satisfy the conditions of the theorem on the closed disk $\bar{U}$. If not, then we can replace $p(z)$ by $p_r(z) = p(rz)$ and $q(z)$ by $q_r(z) = q(rz)$, where $0 < r < 1$. Then the new functions satisfy the conditions of the theorem on $\bar{U}$; we would then prove $p_r(z) \prec q_r(z)$ for all $0 < r < 1$ and by letting $r \uparrow 1^-$ we obtain $p(z) \prec q(z)$.

The function $L(z, t) = \alpha(q(z)) + \beta(q(z))\gamma((1 + t)zq'(z))$ is continuously differentiable on $[0, +\infty]$ for all $z \in U$ and analytic in U for all $t \geqslant 0$. Because $q'(0) \neq 0$, $Q'(0) \neq 0$ from (i) for $z = 0$ we deduce that $\frac{\partial L}{\partial z}(0, t) \neq 0$ for all $t \geqslant 0$. Because q is convex, a simple calculation combined with (i) yields $\mathrm{Re}\left[z\frac{\partial L}{\partial z} \Big/ \frac{\partial L}{\partial t}\right] > 0$ for all $z \in U$ and $t \geqslant 0$. hence by Lemma 2, $L(z, t)$ is a subordination chain. If we let $h(z) = L(z, 0) = \alpha(q(z)) + \beta(q(z))\gamma(zq'(z))$ and using (i) for $t = 0$ we obtain

$\mathrm{Re}\,[zh'(z)/Q(z)] > 0$ for all $z \in U$,

hence by Lemma 1, h is univalent in U.

Let $\psi(r, s) = \alpha(r) + \beta(r)\gamma(s)$ analytic in the domain $E = D \times \mathbf{C}$; then $(q(0), 0) \in E$, $\psi(q(0), 0) = h(0) \in h(U)$ and because $L(z, t)$ is a subordination chain we deduce

$$\alpha(q(\zeta)) + \beta(q(\zeta))\gamma((1 + t)\zeta q'(\zeta)) \notin h(U) \text{ for } t \geqslant 0 \text{ and } |\zeta| = 1.$$

Using Theorem A we conclude that $p(z) \prec q(z)$. If we take $\beta(w) = 1$, $w \in \mathbf{C}$ then from Theorem 1 we obtain:

COROLLARY 1. *Let q be convex (univalent) in U, let α be analytic in a domain $D \supset q(U)$ and let γ be analytic in $\mathbf{C}$. Suppose that*

(i) $$\operatorname{Re} \frac{\alpha'(q(z))}{\gamma'((1 + t)\, zq'(z))} \geqslant 0$$

for all $z \in U$ and $t \geqslant 0$

(ii) $Q(z) = zq'(z)\gamma'(zq'(z))$ is starlike *(univalent) in U.*

If p is analytic in U with $p(0) = q(0)$, $p(U) \subset D$ then

$$\alpha(p(z)) + \gamma(zp'(z)) \prec \alpha(q(z)) + \gamma(zq'(z)) \Rightarrow p(z) \prec q(z).$$

If we take $\alpha(w) = 0$, $w \in \mathbf{C}$ then form Corollary 1 we obtain:

EXAMPLE 1.1. Let q be convex (univalent) in U and γ be analytic in $\mathbf{C}$ and suppose that $Q(z) = zq'(z)\gamma'(zq'(z))$ is starlike (univalent) in U. If p is analytic in U with $p(0) = q(0)$ then

$$\gamma(zp'(z)) \prec \gamma(zq'(z)) \Rightarrow p(z) \prec q(z).$$

If in this result we take $q(z) = z$, $z \in U$ then $Q(z) = z\gamma'(z)$ is starlike (univalent) in U if and only if γ is convex (univalent) in U, hence we obtain:

EXAMPLE 1.2. Let γ be analytic in $\mathbf{C}$ so that γ is convex (univalent) in U. If p is analytic in U with $p(0) = 0$, then $\gamma(zp'(z)) \prec \gamma(z) \Rightarrow p(z) \prec$ $\prec z$. We can easily prove that for $|\lambda| \leqslant 1$ the function $\gamma(z) = e^{\lambda z}$ is convex (univalent) in U and analytic in $\mathbf{C}$, hence we obtain:

EXAMPLE 1.3. If $|\lambda| \leqslant 1$ and p is analytic in U with $p(0) = 0$, then

$$e^{\lambda zp'(z)} \prec e^{\lambda z} \Rightarrow p(z) \prec z.$$

Since $F(z) = \frac{1}{z}\int_0^z f(t)dt$ is convex if f is convex of order $-1/2$ [5], we can easily show that $\gamma(z) = \frac{e^{\lambda z} - 1}{\lambda z}$, $|\lambda| \leqslant 3/2$ is convex (univalent) in U and analytic in $\mathbf{C}$, hence we obtain:

EXAMPLE 1.4. If $|\lambda| \leqslant 3/2$ and p is analytic in U with $p(0) = 0$, then

$$\frac{e^{\lambda zp'(z)} - 1}{\lambda zp'(z)} \prec \frac{e^{\lambda z} - 1}{\lambda z} \Rightarrow p(z) \prec z.$$

If we take $\alpha(w) = 0$, $w \in \mathbf{C}$ and $\gamma(w) = e^w$, $w \in \mathbf{C}$ then from Theorem we obtain:

COROLLARY 2. *Let q be convex (univalent) in U and β be analytic domain $D \subset q(U)$.*
Suppose that

(i) $$\mathrm{Re}\ \beta'(q(z))/\beta(q(z)) \geqslant 0,\ z \in U$$

(ii) $$Q(z) = zq'(z)\beta(q(z))e^{zg'(z)}$$

is starlike (univalent) in U.
If p is analytic in U with $p(0) = q(0)$, $p(U) \supset D$ then

$$\beta(p(z))e^{zp'(z)} \prec \beta(q(z))e^{zq'(z)} \Rightarrow p(z) \prec q(z).$$

If we take $\beta(w) = w$, $w \in \mathbf{C}$, then from Corollary 2 we obtain:
EXAMPLE 2.1. Let q be convex (univalent) in U and suppose th

(i) $$\mathrm{Re}\ q(z) > 0,\ z \in U$$

(ii) $$Q(z) = zq'(z)q(z)e^{zq'(z)}$$

is starlike (univalent) in U. If p is analytic in U, with $p(0) = q(0)$, th

$$p(z)e^{zp'(z)} \prec q(z)e^{zq'(z)} \Rightarrow p(z) \prec q(z).$$

If we take $\beta(w) = e^w$, $w \in \mathbf{C}$ then from Corollary 2 we obtain:
EXAMPLE 2.2. Let q be convex (univalent) in U so that

$$Q(z) = zq'(z)e^{q(z)+zq'(z)}$$

is starlike (univalent) in U. If p is analytic in U and $p(0) = q(0)$ th

$$e^{p(z)+zp'(z)} \prec e^{q(z)+zq'(z)} \Rightarrow p(z) \prec g(z).$$

If we take, in this example, $q(z) = \lambda z$, $|\lambda| \leqslant 1/2$we can easily prove th $Q(z) = \lambda ze^{2\lambda z}$ is starlike (univalent) in U, hence we obtain:
EXAMPLE 2.3. If $|\lambda| \leqslant 1/2$ and p is analytic in U with $p(0) = 0$, th

$$e^{p(z)+zp'(z)} \prec e^{2\lambda z} \Rightarrow p(z) \prec \lambda z.$$

An interesting case is obtained when in Theorem 1 we take $\gamma(w) = \imath$ $w \in \mathbf{C}$; we can easily show that in this case we have:

COROLLARY 3. *Let q be univalent in U and let α, β be analytic in domain $D \supset q(U)$ with $\beta(w) \neq 0$ for all $w \in q(U)$. Suppose that*

(i) $$\mathrm{Re}\left[\frac{\alpha'(q(z))}{\beta(q(z))} + \frac{zQ'(z)}{Q(z)}\right] > 0,\ z \in U$$

(ii) $Q(z) = zq'(z)\beta(q(z))$ *is starlike (univalent) in U.*
If p is analytic in U, $p(0) = q(0)$ and $p(U) \subset D$ then $\alpha(p(z)) + zp'($ $\beta(p(z)) \prec \alpha(q(z)) + zq'(z)\beta(q(z)) \Rightarrow p(z) \prec q(z)$.

This corollary represents Theorem 3 from [1], and for $\beta(w) = 1$, $w \in \mathbf{C}$ we obtain:

EXAMPLE: 3.1. Let q be convex (univalent) in U, α be analytic in a domain $D \supset q(U)$ and suppose that

$$\mathrm{Re}\left[\alpha'(q(z)) + 1 + \frac{zq''(z)}{q'(z)}\right] > 0, \ z \in U.$$

If p is analytic in U, $p(0) = q(0)$ and $p(U) \subset D$ then $\alpha(p(z)) + zp'(z) \prec \prec \alpha(q(z)) + zq'(z) \Rightarrow p(z) \prec q(z)$. This example gives us some interesting particular cases presented in the next examples.

EXAMPLE 3.2. If q is convex (univalent) in U with $|\mathrm{Im}\, q(z)| \leqslant \pi/2$, $z \in U$ and p is analytic in U with $p(0) = q(0)$ then

$$e^{p(z)} + zp'(z) \prec e^{q(z)} + zq'(z) \Rightarrow p(z) \prec q(z).$$

Proof. The function $\alpha(w) = e^w$, $w \in \mathbf{C}$ is analytic in $\mathbf{C}$ and

$$\mathrm{Re}\left[\alpha'(q(z)) + 1 + \frac{zq''(z)}{q'(z)}\right] = e^{\mathrm{Re}\, q(z)} \cos \mathrm{Im}\, q(z) + \mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0, \ z \in U$$

if q is convex in U and $|\mathrm{Im}\, q(z)| \leqslant \pi/2$, $z \in U$.

Remark. If, in this example, we take $q(z) = \lambda z$, $|\lambda| \leqslant \pi/2$, we obtain that if p is analytic in U with $p(0) = 0$, then

$$e^{p(z)} + zp'(z) \prec e^{\lambda z} + \lambda z \Rightarrow p(z) \prec \lambda z.$$

EXAMPLE 3.3. If q is convex (univalent) in U and $\mathrm{Re}\, q(z) > \beta$, $z \in U$ and if p is analytic in U with $p(0) = q(0)$, then

$$p(z)\left(\frac{1}{2}\, p(z) - \beta\right) + zp'(z) \prec q(z)\left(\frac{1}{2} q(z) - \beta\right) + zq'(z) \Rightarrow p(z) \prec q(z).$$

Proof. In Example 3.1 considering $\alpha(w) = \frac{1}{2}\, w^2 - \beta w$, $w \in \mathbf{C}$ analytic in $\mathbf{C}$ we have

$$\mathrm{Re}\left[\alpha'(q(z)) + 1 + \frac{zq''(z)}{q'(z)}\right] = \mathrm{Re}\left[q(z) - \beta + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right] > 0$$

when $z \in U$.

Remarks. 1°. If we take $\beta = 0$, then from Example 3.3 we obtain:
If q is convex (univalent) in U with
$\mathrm{Re}\, q(z) > 0$, $z \in U$ and p is analytic in U with $p(0) = q(0)$, then

$$\frac{1}{2}\, p^2(z) + zp'(z) \prec \frac{1}{2}\, q^2(z) + zq'(z) \Rightarrow p(z) \prec q(z).$$

2°. We can easily show that $q(z) < e^{\lambda z}$, $|\lambda| \leqslant 1$, is convex (univalent) in U and $\mathrm{Re}\, q(z) > 0$, $z \in U$ hence we have: If $|\lambda| \leqslant 1$ and p is analytic in U with $p(0) = 1$, then

$$\frac{1}{2}\, p^2(z) + zp'(z) \prec \frac{1}{2}\, e^{2\lambda z} + \lambda z e^{\lambda z} \Rightarrow p(z) \prec e^{\lambda z}.$$

3°. If we take, in Example. 3.3., $q(z) = \frac{1 + (2\beta - 1)z}{1 + z}$, $\beta < 1$, we c

easily show that q is convex (univalent) in U and $\mathrm{Re}\, q(z) > \beta$, $z \in$
hence we obtain:
If $\beta < 1$ and p is analytic in U with $p(0) = 1$, then

$$p(z)\left(\frac{1}{2}p(z) - \beta\right) + zp'(z) \prec \frac{1 - 2\beta - 2(2\beta^2 - 4\beta + 3)z + (1 - 2\beta)z^2}{2(1 + z)^2} \Rightarrow$$

$$\Rightarrow p(z) \prec \frac{1 + (2\beta - 1)z}{1 + z}.$$

4°. If we take $\beta = 0$ in this last case, we obtain:
If p is analytic in U with $p(0) = 1$, then

$$\frac{1}{2}p^2(z) + zp'(z) \prec \frac{1 - 6z + z^2}{2(1 + z)^2} \Rightarrow p(z) \prec \frac{1 - z}{1 + z}.$$

5°. If we take, in Example 3.3, $q(z) = \frac{z}{1 - z}$ then q is convex (univa

lent) in U, hence we obtain:
If p is analytic in U with $p(0) = 0$, then

$$\frac{1}{2}(p(z) + p^2(z)) + zp'(z) \prec \frac{3}{2}\frac{z}{(1 - z)^2} \Rightarrow p(z) \prec \frac{z}{1 - z}.$$

6°. If we take, in Corollary 3, $\alpha(w) = w$ and $\beta(w) = \frac{1}{\gamma}w$, $w \in$
$\gamma \neq 0$, $\mathrm{Re}\,\gamma \geq 0$ then we obtain the well-known result of Hallenbe
and Ruscheweyh [6].

REFERENCES

1. S. S. Miller, P. T. Mocanu, *On some classes of first-order differential subordinati* (to appear).
2. S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent function* Michigan Math. J., 28 (1981), 151–171.
3. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J., 1 (1952), 169–
4. Ch. Pommerenke, *„Univalent Functions"*, Vanderhoeck and Ruprecht, Gotti 1975.
5. G. S. Sălăgean, *Properties of starlikeness and convexity preserved by some int operators*, Romanian — Finnish Seminar of Complex Analysis, Bucharest 1976, P Notes in Math, 743, Springer — Verlag 1979, 367–372.
6. D. J. Halenbeck, S. Ruscheweyh, *Subordination by convex functions*, Proc. A Math. Soc, 52 (1975), 191–195.

APROXIMAREA FUNCŢIILOR DE DOUĂ VARIABILE CU AJUTORUL UNUI OPERATOR DE TIP BERNSTEIN

A. CIUPA*

Intrat în redacţie la 15 martie 1983

ABSTRACT — Approximation of Functions of Two Variables by Means of an Operator of Bernstein Type. In this paper a linear positive operator of Bernstein type introduced and investigated recently by Bleiman-Butzer and Hahn [1] is extended to two variables. The corresponding operator $L_{m,n}$, defined on the space C(D) of continuous functions on $D = [0, \infty) x [0, \infty)$, is given at (2.1). It is proved that if $f \in C(D)$, then $\lim L_{m,n} f$ uniformly on every compact $[0, a] x [0, b]$ $(a, b > 0)$ when $m, n \to \infty$. An extension of the well-knows asymptotic estimation of Voronovskaja from the Bernstein operator to the operator $L_{m,n}$ is given at (3.1). Theorem 2 shows an evaluation of the order of approximation of the function $f \in C(D)$ by $L_{m,n} f$, using the second order modulus of continuity ω_2. One sees that the operator $L_{m,n}$ has approximation properties similar to those of operator $B_{m,n}$ of Bernstein, defined on the space of continuous functions on a rectangle $[0, a] x [0, b]$.

1. În lucrarea [1], G. Bleiman, P. L. Butzer şi L. Hahn au introdus un operator liniar şi pozitiv, L_m, ataşat unei funcţii f, continuă pe $[0, \infty)$, definit prin

$$(L_m f)(x) = \frac{1}{(1+x)} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m-k+1}\right), \quad m \in N.$$

Scopul acestei lucrări este extinderea la două variabile a rezultatelor mai importante din lucrarea citată.

2. Notăm cu $C(D)$ spaţiul funcţiilor continue pe $D = [0, \infty) \times \times [0, \infty)$. Unei funcţii $f \in C(D)$, îi ataşăm operatorul $L_{m,n}$, definit prin:

$$(2.1) \qquad (L_{m,n}f)(x, y) = L_{m,n}(f;\, x, y) =$$

$$= \frac{1}{(1+x)^m} \cdot \frac{1}{(1+y)^n} \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{m}{k}\binom{n}{j} x^k y^j f\left(\frac{k}{m-k+1}, \frac{j}{n-j+1}\right), \quad m, n \in \mathbf{N}.$$

Operatorul acesta este liniar şi mărginit, în sensul că pentru $\forall (x, y) \in D$, avem

$$(2.2) \qquad |L_{m,n}(f;\, x, y) - f(x, y)| \leqslant ||f||_{C_B(D)} \;\forall,\; f \in C_{B(D)},$$

unde am notat cu $C_B(D)$ clasa funcţiilor definite pe D, mărginite şi uniform continue în sensul normei:

$$(2.3) \qquad ||f||_{C_B(D)} = \sup_{(x,y) \in D} |f(x, y)|.$$

* *Institute of Polytechnics, 3400 Cluj-Napoca, Romania*

Operatorul este pozitiv pe D, în sensul că dacă $f(x, y) \geqslant 0$ $\forall(x, y) \in$ atunci avem de asemenea $L_{m,n}(f;\ x, y) \geqslant 0$, pentru $\forall(x, y) \in D$.

Pentru a demonstra că $\lim\limits_{m,n\to\infty} L_{m,n}(f;\ x, y) = f(x, y)$ pentru $\forall(x, y) \in D$ și $\forall f \in C(D)$, efectuăm cîteva calcule preliminare. Aplicăm operatoi $L_{m,n}$ funcțiilor de probă 1, x, y și $x^2 + y^2$. Rezultatele sînt cuprinse

LEMA 1. *Pentru* $\forall(x, y) \in D$, *avem:*

$$(2.4)\quad L_{m,n}(1;\ x, y) = 1$$

$$(2.5)\quad L_{m,n}(t;\ x, y) = x - x\left(\frac{x}{1+x}\right)^m,\quad m \in \mathbf{N}$$

$$(2.6)\quad L_{m,n}(\tau,\ x, y) = y - y\left(\frac{y}{1+y}\right)^n,\quad n \in \mathbf{N}$$

$$(2.7)\quad L_{m,n}(t^2 + \tau^2;\ x, y) = x^2 + y^2 + \frac{2x(1+x)^2}{m+2} + \frac{2y(1+y)^2}{n+2}$$

pentru

$$(2.8)\quad m \geqslant N(x),\ \textit{unde}\ N(x) = 24(1+x)$$
$$n \geqslant N(y),\ \textit{unde}\ N(y) = 24(1+y).$$

Demonstrație: Relațiile (2.4), (2.5) și (2.6) rezultă imediat folosind formu binomului, după cum urmează:

$$L_{m,n}(1;\ x, y) = \frac{1}{(1+x)^m} \cdot \frac{1}{(1+y)^n} \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{m}{k}\binom{n}{j} x^k y^j = 1$$

$$L_{m,n}(t;\ x, y) = \frac{1}{(1+x)^m} \cdot \frac{1}{(1+y)^n} \cdot \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{m}{k}\binom{n}{j} x^k y^j \frac{k}{m-k+1} =$$

$$= \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k \cdot \frac{k}{m-k+1} = \frac{x}{(1+x)^m} \cdot [(1+x)^m - x^m] =$$

$$= x\left[1 - \frac{x^m}{(1+x)^m}\right].$$

Analog se găsește că

$$L_{m,n}(\tau;\ x, y) = y\left[1 - \frac{y^n}{(1+y)^n}\right].$$

Să calculăm rezultatul aplicării operatorului $L_{m,n}$ asupra funcţiei $x^2 + y^2$. Avem:

$$L_{m,n}(t^2 + \tau^2;\ x,\ y) = \frac{1}{(1+x)^m}\cdot\frac{1}{(1+y)^n}\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}x^k y^j\left[\frac{k^2}{(m-k+1)^2} + \frac{j^2}{(n-j+1)^2}\right] = \frac{1}{(1+x)^m}\sum_{k=0}^{m}\binom{m}{k}x^k\frac{k^2}{(m-k+1)^2} +$$

$$+\frac{1}{(1+y)^n}\sum_{j=0}^{n}\binom{n}{j}y^j\frac{j^2}{(n-j+1)^2}.$$

În [1] se demonstrează că

$$(L_m t^2)(x) = \frac{1}{(1+x)^m}\sum_{k=0}^{m}\binom{m}{k}x^k\frac{k^2}{(m-k+1)^2} \leqslant x^2 + \frac{2x(1+x)^2}{m+2}$$

pentru $m \geqslant N(x)$, unde $N(x) = 24(1+x)$.

Aplicînd aici acest rezultat, avem:

$$L_{m,n}(t^2+\tau^2;\ x,\ y) \leqslant x^2 + y^2 + \frac{2x(1+x)^2}{m+2} + \frac{2y(1+y)^2}{n+2}.$$

pentru $m \geqslant N(x)$, unde $N(x) = 24(1+x)$ şi pentru $n \geqslant N(y)$, unde $N(y) = 24(1+y)$.

Rezultatele acestei leme le folosim pentru a demonstra

TEOREMA 1. *Dacă* $f \in C(D)$, *atunci*

$$\lim_{m,n\to\infty} L_{m,n}(f;\ x,\ y) = f(x,\ y),\ \forall(x,\ y) \in D,$$

convergenţa fiind uniformă pe fiecare compact $[0,\ a]\times[0,\ b] \subset D$.

Demonstraţie: Din relaţiile (2.4), (2.5), (2.6), (2.7) rezultă că avem

$$\lim_{m,n\to\infty} L_{m,n}(f;\ x,\ y) = f(x,\ y),\ \forall(x,\ y) \in D,$$

unde $f(x,\ y)$ reprezintă pe rînd funcţiile 1, x, y, $x^2 + y^2$. Aplicînd teorema lui V.I. Volkov [4], care constituie o extindere la două variabile a cunoscutei teoreme a lui Bohman—Korovkin, rezultă că atunci cînd $m,\ n \to \infty$, şirul $L_{m,n} f$ definit la (2.1), tinde către f, $\forall(f \in C(D)$.

3. În continuare ne vom ocupa de evaluarea ordinului de aproximare a funcţiei $f \in C(D)$ prin $L_{m,n} f$. Pentru aceasta avem nevoie de unele noţiuni preliminare. Notăm

$$C_B^2(D) = \{f \in C_B(D)\ |\ f^{(i,j)} \in C_B(D),\ 1 \leqslant i,\ j \leqslant 2\},$$

(unde prin $f^{(i,j)}$ am notat derivata parţială de ordinul i în raport cu şi j în raport cu y a funcţiei f), cu norma

$$||f||_{C_B^2} = ||f||_{C_B} + ||f||_{C_B}^1 + ||f||_{C_B}^2,$$

unde

$$||f||_{C_B} = \sup_{(x,y)\in D} |f(x, y)|$$

$$||f||_{C_B}^1 = \sup_{(x,y)\in D} \{|f(x, y)|, |f^{(1,0)}_{(x,y)}|, |f^{(0,1)}_{(x,y)}|\}$$

$$||f||_{C_B}^2 = \sup_{(x,y)\in D} \{|f(x, y)|, |f^{(1,0)}_{(x,y)}|, |f^{(0,1)}_{(x,y)}|, |f^{(2,0)}_{(x,y)}|, |f^{(1,1)}_{(x,y)}|, |f^{(0,2)}_{(x,y)}|\}.$$

Vom demonstra

LEMA 2. *Dacă* $f \in C_B^2(D)$, *atunci are loc inegalitatea*

$$(3.1)\quad |L_{m,n}(f;\ x, y) - f(x, y)| \leqslant 2||f||_{C_B^2}\left[\frac{x(1+x)^2}{m+2} + \frac{y(1+y)^2}{n+2} + \right.$$
$$\left. + 2\,\frac{x(1+x)^2}{m+2}\cdot\frac{y(1+y)}{n+2}\right],$$

pentru orice $m \geqslant N(x)$ *şi orice* $n \geqslant N(y)$, *unde* $N(x) = 24(1 + x)$ *şi* $N(y) = 24(1 + y)$.

Demonstraţie: Calculele care urmează se bazează pe rezultatele Lemei şi pe liniaritatea operatorului $L_{m,n}$. Pentru început folosim relaţia (2. apoi aplicăm funcţiei f formula lui Taylor. Avem:

$$L_{m,n}(f;t,\tau) - f(x,y) = L_{m,n}[f(t,\tau) - f(x,y);\ x, y)] =$$

$$= L_{m,n}[f^{(1,0)}_{(x,y)}(t-x) + f^{(0,1)}_{(x,y)}(\tau-y) + \frac{1}{2}f^{(2,0)}_{(\xi,\eta)}(t-x)^2 + f^{(1,1)}_{(\xi,\eta)}(t-x)(\tau-y)$$

$$+ \frac{1}{2}f^{(0,2)}_{(\xi,\eta)}(\tau-y)^2;\ x, y] = f^{(1,0)}_{(x,y)}L_{m,n}(t-x;\ x,y) + f^{(0,1)}_{(x,y)}L_{m,n}(\tau-y;\ x, y)$$

$$+ L_{m,n}\left[\frac{1}{2}f^{(2,0)}_{(\xi,\eta)}(t-x)^2 + f^{(1,1)}_{(\xi,\eta)}(t-x)(\tau-y) + \frac{1}{2}f^{(0,2)}_{(\xi,\eta)}(\tau-y)^2;\ x, y\right],$$

unde $t < \xi < x$, $\tau < \eta < y$.

Trecînd la modul şi folosind definiţia dată normei în spaţiul $C_B^2(D)$, ave

$$|L_{m,n}(f;t,\tau) - f(x,y)| \leqslant [|L_{m,n}(t-x;\ x,y)| + |L_{m,n}(\tau-y;x,y)|]\,||f||_{C_B}^1$$

$$+ \frac{1}{2}L_{m,n}[(t-x+\tau-y)^2;\ x, y]\,\cdot\,||f||_{C_B}^2.$$

Folosind relațiile (2.5) și (2.6), găsim:

$$|L_{m,n}(f; t, \tau) - f(x, y)| \leqslant \left[x\left(\frac{x}{1+x}\right)^m + y\left(\frac{y}{1+y}\right)^n\right] \cdot ||f||^1_{C_B} +$$

$$+ \frac{1}{2} L_{m,n}[(t - x + \tau - y)^2; x, y)^2 \cdot ||f||^2_{C_B}.$$

Pentru calculul lui $L_{m,n}[(tx + \tau - y)^2; x, y]$ folosim relațiile (2.5), (2.6), (2.7) și (2.8). Vom avea:

$$L_{m,n}[(t - x + \tau - y)^2; x, y] \leqslant \frac{2x(1+x)^2}{m+2} + \frac{2y(1+y)^2}{n+2} +$$

$$+ 2x^2\left(\frac{x}{1+x}\right)^m + 2y^2\left(\frac{y}{1+y}\right)^n + 2xy\left(\frac{x}{1+x}\right)^m \cdot \left(\frac{y}{1+y}\right)^n.$$

Folosind această majorare și grupînd termenii în altă ordine, rezultă:

$$|L_{m,n}(f; t, \tau) - f(x, y)| \leqslant x\left(\frac{x}{1+x}\right)^m ||f||^1_{C_B} +$$

$$+ \left[\frac{x(1+x)^2}{m+2} + x^2\left(\frac{x}{1+x}\right)^m\right] ||f||^2_{C_B} + y\left(\frac{y}{1+y}\right)^n ||f||^1_{C_B} +$$

$$+ \left[y\frac{(1+y)^2}{n+2} + y^2\left(\frac{y}{1+y}\right)^n\right] ||f||^2_{C_B} + xy\left(\frac{x}{1+x}\right)^m \left(\frac{y}{1+y}\right)^n ||f||^2_{C_B}.$$

În [1] s-a demonstrat că pentru $m \geqslant N(x)$, unde $N(x) = 24(1 + x)$, avem:

$$x\left(\frac{x}{1+x}\right)^m ||f'||_{C_B} + \left[x\frac{(1+x)^2}{m+2} + x^2\left(\frac{x}{1+x}\right)^m\right] ||f''||_{C_B} \leqslant$$

$$\leqslant \frac{2x(1+x)^2}{m+2} (||f'||_{C_B} + ||f''||_{C_B}).$$

Aplicînd inegalitatea analoagă în cazul a două variabile, găsim:

$$|L_{m,n}(f; t, \tau) - f(x, y)| \leqslant \frac{2x(1+x)^2}{m+2} (||f||^1_{C_B} + ||f||^2_{C_B}) +$$

$$+ \frac{2y(1+y)^2}{n+2} (||f||^1_{C_B} + ||f||^2_{C_B}) + xy\left(\frac{x}{1+x}\right)^m \left(\frac{y}{1+y}\right)^n ||f||^2_{C_B},$$

pentru $m \geqslant N(x)$ și $n \geqslant N(y)$, unde $N(x) = 24(1 + x), N(y) = 24(1 + y)$. Folosim o altă inegalitate dată în [1]:

$$\frac{x(1+x)^2}{m+2} \geqslant x\left(\frac{x}{1+x}\right)^m \left[\frac{x(x + 2m + 4)}{m+2} + \frac{3m+1}{2}\right]$$

pentru $m \geqslant 4x$, $x \geqslant 0$,
De aici rezultă că

$$x\left(\frac{x}{1+x}\right)^m \leqslant \frac{2x(1+x)}{m+2}.$$

Folosind această majorare, obţinem :

$$|L_{m,n}(f;\ t,\ \tau) - f(x,\ y)| \leqslant 2(||f||^1_{C_B} + ||f||^2_{C_B})\left[\frac{x(1+x)^2}{m+2} + \frac{y(1+y)^2}{n+2} + \right.$$
$$\left. + 2\frac{x(1+x)^2}{m+2}\cdot\frac{y(1+y)^2}{n+2}\right].$$

Deci

$$|L_{m,n}(f;\ t,\ \tau) - f(x,\ y)| \leqslant 2||f||_{C^2_B}\left[\frac{x(1+x)^2}{m+2} + \frac{y(1+y)^2}{n+2} + \right.$$
$$\left. + 2\frac{x(1+x)^2}{m+2}\cdot\frac{y(1+y)^2}{n+2}\right].$$

În continuare vom folosi modulul de continuitate de ordinul d ω_2, definit astfel :

$$\omega_2(t,\ f) = \sup_{||h||<t} |\Delta^2_{h_1,h_2} f|,\ \text{unde}\ h = (h_1,\ h_2),\ \ ||h|| = |h_1| + |h_2|$$

TEOREMA 2. *Dacă* $f \in C_B(D)$, *avem :*

$$|L_{m,n}(f;x,\ y) - f(x,\ y)| \leqslant 2C\{\omega_2(\sqrt{A(x,\ y)},\ t) + A(x,\ y)||f||_{C_B(D)}\},$$

unde

$$A(x,\ y) = \frac{x(1+x)^2}{m+2} + \frac{y(1+y)^2}{n+2} + 2\frac{x(1+x)^2}{m+2}\cdot\frac{y(1+y)^2}{n+2},$$

$m \geqslant N(x)$, $n \geqslant N(y)$, *cu* $N(x) = 24(1 + x)$, $N(y) = 24(1 + y)$, *iar* C *e o constantă pozitivă independentă de* f *şi* $A(x,\ y)$.

Demonstraţie : Pentru $f \in C_B(D)$ *şi* $g \in C^2_B(D)$, avem următoarele inega tăţi :

3.2) $$|L_{m,n}(f;\ x,\ y) - f(x,\ y)| \leqslant |L_{m,n}(f;\ x,\ y) - L_{m,n}(g;\ x,\ y)| +$$
$$+ |L_{m,n}(g;\ x,\ y) - g(x,\ y)| + |g(x,\ y) - f(x,\ y)| \leqslant 2||f - g||_{C_B(D)} +$$
$$+ 2||g||_{C^2_B}\left[\frac{x(1+x)^2}{m+2} + \frac{y(1+y)^2}{n+2} + 2\frac{x(1+x)^2}{m+2}\cdot\frac{y(1+y)^2}{n+2}\right].$$

În continuare folosim funcţionala K, definită pentru $t \geqslant 0$, prin :

$$K(t,\ f) = \inf_{g \in C^2_B(D)} \{||f - g||_{C_B(D)} + t||g||_{C^2_B(D)}\}.$$

Legătura dintre funcţionala K şi modulul de continuitate de ordinul (este dată în [2] :

$$K(t,\ f) \leqslant C[\omega_2(t^{1/2},\ f) + \min\{1,\ t\}||f||_{C_B(D)}],$$

pentru $\forall t \geqslant 0$, $f \in C_B(D)$, cu constanta C pozitivă şi independentă d şi t.

Observînd că membrul stîng al inegalității (3.2) nu depinde de funcția $g \in C_B^2(D)$, obținem :

$$|L_{m,n}(f;\ x,\ y) - f(x,\ y)| \leqslant 2K(A(x,\ y),\ f) \leqslant$$
$$\leqslant 2C\{\omega_2(\sqrt{A(x,\ y)},\ f + \min\{1,\ A(x,\ y)\}\ ||f||_{C_B(D)}\} =$$
$$= 2C\{\omega_2(\sqrt{A(x,\ y)},\ f) + A(x,\ y)\cdot ||f||_{C_B(D)}\}.$$

Din rezultatele obținute mai sus, se constată că operatorul liniar și pozitiv $L_{m,n}$, definit la (2.1), are proprietăți de aproximare a funcțiilor de două variabile, analoage cu cele pe care le are operatorul lui Bernstein $B_{m,n}$, definit pe spațiul funcțiilor continue pe patratul unitate. Acestea vizează o teoremă de convergență uniformă de același tip pe un drepunghi de forma $[0,\ a] \times [0,\ b]$, o teoremă analoagă cu teorema lui Voronovskaja și evaluări cu ajutorul modulului de continuitate de ordinul doi, care sînt comparabile cu evaluarea dată de T. Popoviciu [3] pentru operatorii lui Bernstein.

BIBLIOGRAFIE

1. G. Bleiman, P. L. Butzer and L. Hahn, *A Bernstein type operator approximating continuous functions on the semi-axis*, Indagationes Mathematicae, 42 (1980), 255–262.
2. P. L. Butzer and H. Berens, *Semi Groups of Operators and Approximations*, Springer-Verlag, Berlin, 1967.
3. T. Popoviciu, *Despre cea mai bună aproximație a funcțiilor continue prin polinoame*, Inst. Arte Grafice, Cluj, 1937.
4. V. I. Volkov, *Convergența șirurilor de operatori liniari pozitivi în spațiul funcțiilor continue de două variabile* (l. rusă), Dokl. Akad. Nauk SSSR, 115 (1957), 17–19.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

CALCULUS OF THE SAFETY COEFFICIENT AT VARIABLI LOADING THROUGH ASYMMETRICAL CYCLES USING PARABOLIC APPROXIMATION

ŞTEFAN MAKSAY*

Recivied: April 1, 1983

ABSTRACT. — In this paper the diagram of the resistences at weariness (Haigh diagram) is approximated by a parabole and the expression of the safety coefficient at variable loading for this modelling is obtained, (12) — (13). The procedure used here makes the calculated coefficients be closer to the real ones.

The classic modellings of the diagram of resistances at wearin (Haigh diagram) through the schematizings suggested by Serensen a Gh. Buzdugan [1] lead to the following expressions of the safe coefficient at variable loadings

$$C_d = \frac{1}{\psi + \theta},$$

$$C_e = \frac{1}{\sqrt{\psi^2 + \sigma^2}},$$

where, using the acknowledged notations,

$$\psi = \sigma_v/\sigma_{-1}, \quad \theta = \sigma_m/\sigma_c.$$

The present paper has in its purpose to approximate the Haigh ty diagram by a parabole, the expression of the safety coefficient for t modelling being obtained.

Imposing on the parabole

$$\sigma_{VL} = \alpha\sigma_{ml}^2 + \beta\sigma_{mL} + \gamma$$

the condition of passing through points $A(0, \sigma_{-1})$, $B(\sigma_c/2, \sigma_c/2)$, $C(\sigma_c,$ determinations

$$\gamma = \sigma_{-1},$$

$$\alpha = -\frac{\sigma_{-1}}{\sigma_c^2} - \frac{b}{\sigma_c}$$

$$\beta = \frac{\frac{\sigma_0}{2} + \frac{\sigma_{-1}}{\sigma_c^2}\left(\frac{\sigma_0}{2}\right)^2 - \sigma_{-1}}{\frac{\sigma_0}{2} - \frac{1}{\sigma_c}\left(\frac{\sigma_0}{2}\right)^2}$$

result for coefficients α, β, γ.

* *Engineering Institule of Hunedoara, 2750 Hunedoara, Romania*

The parabole omothetic to parabole (4) to the origin, the coefficient ents being determined by relations (5), (6), (7), the points of which represent loading cycles with the same safety coefficient $c > 1$, leads to relation

$$\sigma_m^2 \frac{\sigma_{-1} + \beta\sigma_c}{\sigma_c^2} \cdot c^2 - (\beta\sigma_m - \sigma_v) \cdot c - \sigma_{-1} = 0 \qquad (8)$$

where

$$\sigma_{mL} = c\sigma_m, \qquad (9)$$

$$\sigma_{VL} = c\sigma_v. \qquad (10)$$

Introducing notation

$$B = \beta \frac{\sigma_m}{\sigma_{-1}}, \qquad (11)$$

from relation (8) results the expression of the safety coefficient at variable loading having the from

$$c_p = \frac{1}{\sqrt{\frac{1}{4}(\psi - B)^2 + \theta^2\left(1 + \beta\frac{\sigma_c}{\sigma_{-1}}\right)} + \frac{1}{2}(\psi - B)}. \qquad (12)$$

In case the exact value of σ_0 is not known, the condition that the parabole pass through point B may be given up, imposing on it the condition to admit axis $0\sigma_V$ as of symmetry, so $\beta = 0$, case in which relation (12) becomes

$$c_{pp} = \frac{1}{\sqrt{\frac{1}{4}\psi^2 + \theta^2} + \frac{1}{2}\psi}. \qquad (13)$$

In case of the numeric application characterized by $\sigma_m = 10$ daN/mm^2 $\sigma_V = 6$ daN/mm^2, $\sigma_c = 72$ daN/mm^2, $\sigma_{-1} = 43$ daN/mm^2, $\sigma_0 = 64{,}5$ daN/mm^2, the values of the four weariness coefficients, corresponding to the previous modellings, are

$$c_d = 3{,}592 \quad c_p = 4{,}284 \quad c_{pp} = 4{,}441 \quad c_e = 5{,}08. \qquad (14)$$

The method suggested here has, compared to previous methods, the advantage that the curve chosen for the limit cycles approximates better the diagram of resistances at weariness, which makes the calculated c coefficients be closer to the real ones.

REFERENCES

1. Gh. Buzdugan, „Rezistența materialelor", Ed. Tehnică, București, 1975.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

A GENERALIZATION OF FUBINI'S NUMBERS

M. MUREŞAN* and GH. TOADER*

Received May 8, 1983

ABSTRACT. — Starting from the classification method employed by D. Bushaw in [1], the problem of determining the number of non-equivalent positions obtained from a propositional function by quantification of the variables is studied.

Let $P(x_1, \ldots, x_n)$ be a propositional function with n free va bles. One gets a proposition by binding in some way the variables the universal or by the existential quantifier.

We are interested in determining the number of propositions wl results by changing the order of binding or the quantifiers (the f problem). If we have bound successively two variables by the same qu tifier, the other variables preserving the order of binding and the qua fiers, we obtain equivalent propositions, that is they do not depend the order of these two variables. How many equivalence classes obtains in this way? (the second problem). How many classes of non-e valent propositions result if the propositional function is symmetr with respect to certain groups of variables? (the third problem). Sim problems arise if we let some variables free.

These problems may appear in a classification method and their s tions show the number of related notions which can be derived fror given propositional function. Such a classification method was used D. Bushaw in [1] in connection with Lyapunov and Poisson stal ties. His paper contains only numerical results. In our paper we g the general solutions of the three above mentioned problems and t under the assumption of using k quantifiers. The solution of the seco problem may be regarded as a generalization of Fubini's numbers, beca for $k = 2$ these numbers are the double of the corresponding Fubi numbers.

We shall present now the above mentioned problems in an equival form, to introduce simpler the k quantifiers.

1. *The first problem. Let us consider n ordered sets M_i satisfying following conditions:*

$$\text{card } M_i = k, \text{ for } i = 1, \ldots, n;$$

$$M_i \cap M_j = \emptyset, \text{ if } i \neq j.$$

How many words of length n can be obtained by taking from each set exactly one element?

* *University of Cluj-Napoca, Faculty of Mathematics, 3400 Cluj-Napoca, Romania*

In other words, if we denote:

$$M_i = \{x_1^i, \ldots, x_k^i\}, \text{ for } i = 1, \ldots, n \tag{1}$$

with $x_j^i \neq x_q^p$ if $i \neq p$ or $j \neq q$, we look for the cardinal $a_{n,k}$ of the set:

$$A_{n,k} = \left\{x_{j_1}^{i_1} \ldots x_{j_n}^{i_n} : 1 \leq j_1, \ldots, j_n \leq k, (i_1, \ldots, i_n) \in P_n\right\} \tag{2}$$

where P_n denotes the set of permutations of $1, \ldots, n$.

It is easy to prove that:

$$a_{n,k} = n! \cdot k^n, \text{ for } k \geq 1 \tag{3}$$

and we take by definition $a_{0,k} = 1$.

We remark that for every k the series:

$$A_k(t) = \sum_{n=0}^{\infty} a_{n,k} \cdot \frac{t^n}{n!} = \sum_{n=0}^{\infty} (kt)^n$$

is uniformly convergent for $|t| < 1/k$, and its sum:

$$A_k(t) = (1 - kt)^{-1} \tag{4}$$

is the exponential generating function (see [3]) for the sequence $(a_{n,k})_{n=0}^{\infty}$, that is:

$$a_{n,k} = A_k^{(n)}(0).$$

Remark 1. The number of words of length not greater than n which can be obtained by taking at most one element from each set M_i, that is the cardinal $a_{n,k}^0$ of the set:

$$A_{n,k}^0 = \left\{x_{j_1}^{i_1} \ldots x_{j_m}^{i_m} : (i_1, \ldots, i_m) \in P_m, 1 \leq j_1, \ldots, j_m, \leq k, m \leq n\right\} \tag{2'}$$

is:

$$a_{n,k}^0 = \sum_{m=0}^{n} \binom{n}{m} \cdot a_{n-m,k} = \sum_{m=0}^{n} \frac{a_{n,k}}{a_{m,k}}.$$

Since

$$a_{n+1,k} = k(n + 1) \cdot a_{n,k} \tag{5}$$

we get the recurrence relation:

$$a_{n+1,k}^0 = k(n + 1) \cdot a_{n,k}^0 + 1 \tag{5'}$$

and the exponential generating function:

$$A_k^0(t) = \frac{e^t}{1 - kt}. \tag{4'}$$

2. *The second problem. If in $A_{n,k}$ we identify two words which differ by subwords with the same position and whose letters have the same lower index, how many different words result?*

In other words, we want to determine the cardinal $b_{n,k}$ of the quo
set:

$$B_{n,k} = A_{n,k}/\sim$$

where "$\sim$" means the equivalence relation defined by:

$$ux_m^i x_m^j v \sim ux_m^j x_m^i v$$

where u and v are subwords, possible empty, $m \in \{1, 2, \ldots, k\}$
$i, j - \{1, 2, \ldots, n\}$.

We may identify $B_{n,k}$ with the subset $A'_{n,k}$ of $A_{n,k}$, which con
only the words of $A_{n,k}$ satisfying the following condition: if there are
successive letters with the same lower index, then their upper ind
are ordered increasingly.

Let $B^j_{n,k}$ be the subset of $A'_{n,k}$ composed by all its words whose
letter has the lower index not greater than j. Let $b^j_{n,k} = \text{card } B^j_{n,k}$.

$$b_{n,k} = \sum_{j=1}^{k} b^j_{n,k} \quad (n = 1, 2 \ldots,)$$

and we take:

$$b_{0,k} = b^j_{0,k} = 1 \qquad (j = 1, 2, \ldots, k).$$

A word in $B^j_{n,k}$ has on the first p positions $(1 \leqslant p \leqslant n)$ letters
lower index not greater than j followed (if $p \neq n$) by a letter with l
index greater than j. It results the relation:

$$b^j_{n,k} = \sum_{p=1}^{n} \binom{n}{p} \sum_{\substack{i=1 \\ i \neq j}}^{n} b^i_{n-p,k}$$

and so we have:

$$b_{n,k} = (k-1) \sum_{p=1}^{n} \binom{n}{p} \cdot b_{n-p,k} + 1$$

where $b_{0,k} = 1$.

THEOREM 1. *The exponential generating function for the nu*
$(b_{n,k})_{n=0}^{\infty}$ *is*

$$B_k(t) = (1 - k + ke^{-t})^{-1}.$$

Proof. Since $b_{n,k} \leqslant a_{n,k}$, all the following are true (see [2]) and
radius of convergence is at least $1/k$:

$$B_k(t) = 1 + \sum_{n=1}^{\infty} \left[(k-1) \sum_{p=1}^{n} \binom{n}{p} \cdot b_{n-p,k} + 1 \right] \frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} + (k-1) \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{p=1}^{n} \binom{n}{p} \cdot b_{n-p,k} = e^t + (k-1) \cdot \sum_{p=1}^{\infty} \frac{t^p}{p!} \cdot \sum_{m=1}^{\infty} \frac{t^m}{m!} \cdot b$$

$$= e^t + (k-1) \cdot B_k(t) \cdot (e^t - 1).$$

Remark 2. It may be proved that $b_{n,2} = 2f_n$, where f_n is the Fubini's number (see [**3**]).

Remark 3. Let us consider the set of words of length not greater thann, that is consider the set:

$$B^0_{n,k} = A^0_{n,k}/\sim \tag{6'}$$

If $b^0_{n,k} = \text{card } B^0_{n,k}$, then:

$$b^0_{n,k} = \sum_{i=0}^{n} \binom{n}{i} \cdot b_{n-i,k}$$

or:

$$b^0_{n,k} = \frac{k \;\; b_{n,k} - 1}{k - 1}. \tag{7'}$$

The sequence $(b^0_{n,k})_{n=0}^{\infty}$ has the following generating function:

$$B^0_k(t) = \frac{e^t}{1 - k + k \cdot e^{-t}}. \tag{8'}$$

3. *The third problem. If in $A'_{n,k}$ we identify two words in the following relation:*

$$u x^1_i v x^2_j w \approx u x^2_i v x^1_j w \tag{9}$$

where u, v and w are subwords (if $i = j$ the subword v cannot be empty) and $i, j \in \{1, 2, \ldots, k\}$, what is the cardinal $c_{n,k}$ of the quotient set:

$$C_{n,k} = A'_{n,k}/\approx. \tag{10}$$

It is obvious that we may identify the set $C_{n,k}$ with the set $A''_{n,k}$ consisting of those words of $A'_{n,k}$ in which the upper index 1 appears before the upper index 2. If we add to the set $A'_{n,k}$ the words of the form $u x^2_i x^1_i w$, we get $2 \cdot c_{n,k}$ words. But the number of the words of the form $u x^2_i x^1_i w$ is $b_{n-1,k}$, so:

$$c_{n,k} = \frac{1}{2} \cdot (b_{n,k} + b_{n-1,k}).$$

If relations of the form (9) hold for the elements of p pairwise disjoint sets M_i, then the cardinal of the corresponding quotient set is:

$$c^p_{n,k} = \frac{1}{2^p} \sum_{i=0}^{p} \binom{p}{i} \cdot b_{n-i,k}. \tag{11}$$

For the sequence $(c_{n,k})_{n=2}^{\infty}$ we have the following exponential generating function:

$$c_k(t) = \frac{1}{2}\left[B_k(t) + \int_0^t B_k(s)\,ds\right]. \tag{12}$$

Remark 4. If we allow words of length not greater than n, we obta a set of cardinal:

$$c_{n,k}^{p,0} = \sum_{i=0}^{n-2p} \sum_{i_1=0}^{2} \cdots \sum_{i_p=0}^{2} \binom{n-2p}{i} \cdot \binom{2}{i_1} \cdots \binom{2}{i_p} c_{n-i-s_1,k}^{p-s_1} \quad (1$$

where:

$$s_1 = \sum_{j=1}^{p} \left[\frac{i_j+1}{2}\right], \; s_2 = \sum_{j=1}^{p} i_j, \; c_{n,k}^{1} = c_{n,k}, \; c_{n,k}^{0} = b_{n,k}$$

and $[x]$ represents the integer part of x.

Remark 5. Since

$$B_k'(t) = B_k(t) + (k-1) \cdot B_k^2(t),$$

we get, step by step:

$$B_k^{(n)}(t) = [P_{n,k}(B_k)](t),$$

where $P_{n,k}$ is a polynomial of degree $n+1$ which verifies the follow recurrence formula:

$$P_{n,k}(t) = [t + (k-1) \cdot t^2] \cdot P_{n-1,k}'(t), \; P_{0,k}(t) = t. \quad ($$

As $B_k(0) = 1$, we have:

THEOREM 2. *For every* n:

$$b_{n,k} = P_{n,k}(1) \quad ($$

where the polynomial $P_{n,k}$ *is defined by* (*13*).

Remark 6. Denoting

$$P_{n,k}(t) = \sum_{i=1}^{n+1} d_{n,k}^{i} \cdot t^{i}, \; d_{n,k}^{0} = 0 \quad ($$

by (13) it results:

$$d_{n,k}^{i} = i \cdot d_{n-1,k}^{i} + (k-1)(i-1) \cdot d_{n-1,k}^{i-1}, \; i = 1, \ldots, n+1. \quad ($$

Again denoting:

$$d_{n,k}^{i} = (k-1)^{i-1} d_{n}^{i}, \; i = 1, \ldots, n+1 \quad ($$

from (16) we have:

$$d_n^{i} = i \cdot d_{n-1}^{i} + (i-1) \cdot d_{n-1}^{i-1}. \quad ($$

Since:

$$d_0^1 = d_1^1, \; d_{i-1}^1 = (i-1)!, \; d_i^0 = 0, \; i = 2, 3, \ldots$$

from (16′) it results that d_n^i are independent of k. That is, we have following:

THEOREM 3. *For auy n and any k:*

$$b_{n,k} = \sum_{i=1}^{n+1} (k-1)^{-1} \cdot d_n \tag{19}$$

where d are given by (16) *and* (18).

REFERENCES

1. Bushav, D. *Stabilitiet of Lyapunov and Poisson types* SIAM Review, 11, 1969. 214 225.
2. Cvrtan, H., *Théorie elámentaire des functions analytiques d'une ou plusieurs varivbles complexes*, Hermanu, Paris. 1961.
3. Comtet, L., *Advanced combinatirics*, D. Reidel Publishing Company Dordrecht, 1974.

FONCTIONS DONT LA SOMME JOUIT DE LA PROPRIÉTÉ DE DARBOUX

IOAN MUNTEAN*

Manuscrit reçu le 5 juin 1983

ABSTRACT. — Functions Whose Sum has the Darboux Property. It is shown that the sum of two functions in the Baire first class without common point of discontinuity and having the Darboux property, possesses the Darboux property too.

1. *Introduction.* Comme H. Lebesgue l'a déjà remarqué en 1904, l semble des fonctions jouissant de la propriété de Darboux n'est pas fe par rapport à l'addition. Néanmoins, A.M. Bruckner a prouvé récemn que la somme d'une fonction continue avec une fonction jouissant d propriété de Darboux et appartenant à la première classe de B possède la propriété de Darboux. Dans la deuxième section de cette r on étend le résultat de A.M. Bruckner au cas des fonctions qui n' pas de points communs de discontinuité. La troissième section uti cette extension pour la construction d'une fonction bornée jouissant la propriété de Darboux qui est nonprimitivable et non-intégrable au s de Riemann. De pareilles fonctions interviennent dans une classifica proposée en [7] pour certains ensembles de fonctions réelles.

2. *Paires de fonctions dont la somme jouit de la propriété de Darbo* Soit I un intervalle en R. On dit qu'une fonction $f: I \to R$ jouit de propriété de Darboux si pour tous $a, b \in I$, $a < b$, et pour tout λ, si entre $f(a)$ et $f(b)$, il existe $c \in [a, b]$ tel que $\lambda = f(c)$. On dit qu' fonction $f: I \to R$ appartient à la première classe de Baire lorsqu'il ex une suite de fonctions continues $f_n: I \to R$ telles que $\lim_{n\to\infty} f_n) =$ pour chaque $x \in I$ On désigne par $\mathfrak{D}$ et $\mathfrak{B}_1$ les ensembles fonctions jouissant de la propriété de Darboux, respectivement appa nant à la première classe de Baire. Pour abréger l'écriture on p $\mathfrak{D}\mathfrak{B}_1 = \mathfrak{D} \cap \mathfrak{B}_1$.

Soient $a, b, c, \alpha \in R$, avec $a \leqslant c \leqslant b$ et $a \neq b$, et soit $s_{\alpha,c}: [a, b]$· la fonction donnée par la formule

$$s_{\alpha,c}(x) = \begin{cases} \sin \dfrac{1}{x - c} & \text{si } x \neq c, \\ \alpha & \text{si } x = c. \end{cases}$$

* *Université de Cluj-Napoca, Faculté de Mathématique, 3400 Cluj-Napoca, Roumanie*

Sans dificulté, on vérifie les propriétés qui suivent :

a) $s_{\alpha,c} \in \mathcal{B}_1$ et $s_{\alpha,c}$ est intégrable au sens de Riemann pour tout $s \in R$;

b) $s_{\alpha,c} \in \mathcal{D}$ si et seulement si $|\alpha| \leqslant 1$;

c) $s_{\alpha,c}$ est primitivable si et seulement si $\alpha = 0$.

H. Lebesgue [5], pag. 90—91, a remarqué que la somme de deux fonctions de $\mathcal{D}$ n'appartient plus nécessairement à $\mathcal{D}$. En effet, pour les fonctions $f, g: [0, 1] \to R$ données par $f(x) = -g(x) = \sin \frac{1}{x}$ si $x \neq 0$, et $f(0) = g(0) = 1$, on a $f, g \in \mathcal{D}$ et $f + g \notin \mathcal{D}$. Les fonctions f et g de cet exemple sont toutes les deux discontinues. H.W. Ellis [3], pag, 484—485, a construit une fonction $f \in \mathcal{D}$ et une fonction *continue* g, telles que $f + g \notin \mathcal{D}$. De plus, A.M. Bruckner [1], pag. 4, donne un exemple dans lequel la fonction g est *linéaire*. Un exemple possédant les mêmes propriétés que celui de A.M. Bruckner avait été construit implicitement par H. Lebesgue. À savoir, la fonction $\varphi: [0, 1] \to R$ de Lebesgue [5], pag. 90, prend toutes les valeurs de $[0, 1]$ dans tout intervalle, donc $\varphi \in \mathcal{D}$; la fonction $\psi: [0, 1] \to R$ définie dans le même livre de Lebesgue, pag. 91, par $\psi(x) = 0$ si $\varphi(x) = x$, et $\psi(x) = \varphi(x)$ si $\varphi(x) \neq x$, vérifie $\psi \in \mathcal{D}$. La somme $h(x) = \psi(x) - x$ n'appartient pas à $\mathcal{D}$, car $\varphi(1) = 1$ et il existe $a \in \left[0, \frac{1}{2}\right]$ tel que $\varphi(a) = 1$, donc $h(1) = -1$ et $h(a) = 1 - a > 0$, tandis que $h(x) \neq 0$ pour tout $x \neq 0$. Une recherche approfondie des phénomènes relevés par les derniers exemples a été entreprise par R. Švarc [11] et A. M. Bruckner et J. Ceder [2]. Finalement, rappelons que W. Sierpiński [10] a montré que toute fonction réelle sur I peut être écrite comme une somme de deux fonctions de $\mathcal{D}$.

D'autre part, il est intéressant de mettre en évidence des conditions supplémentaires assurant l'appartenance à $\mathcal{D}$ de la somme de deux fonctions de $\mathcal{D}$. Les théorèmes suivants, valables pour un intervalle compact $I = [a, b]$, $a < b$, fournissent de telles conditions.

2.1. THÉORÈME (C. Neugebauer [8]). *Si* f *est approximativement continue,* g *est primitivable et l'une des fonctions f et g est bornée, alors $f + g \in \mathcal{D}$.*

2.2. THÉORÈME (A.M. Bruckner [1], pag. 14). *Si $f \in \mathcal{D}\mathcal{B}_1$ et g est continue, alors $f + g \in \mathcal{D}\mathcal{B}_1$.*

Désignons par disc (f) l'ensemble des points de discontinuité d'une fonction donnée f. Énoncé d'une manière symétrique, le résultat qui suit constitue une généralisation du Théorème 2.2.

2.3. THÉORÈME. *Si $f \in \mathcal{D}\mathcal{B}_1$, $g \in \mathcal{D}\mathcal{B}_1$ et* disc $(f) \cap$ disc $(g) = \emptyset$, *alors $f + g \in \mathcal{D}\mathcal{B}_1$.*

La démonstration du Théorème 2.3 s'appuie sur la caractérisation suivante des fonctions de $\mathcal{D}$ dans l'ensemble $\mathcal{B}_1$:

2.4. THÉORÈME (W.H. Young [12]). *Soit $f: [a, b] \to R$ une fonction de $\mathcal{B}_1$. On a $f \in \mathcal{D}$ si et seulement si pour chaque $x \in [a, b]$ ils existent* *une suite croissante* (x_n) *et une suite décroissante* (y_n) *telles que* $a < x_n$ $< x < y_n < b$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$ *et* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(x)$. *(L que* $x = a$ *ou* $x = b$, *la suite* (x_n), *respectivement* (y_n) *ne figure pas à cet énoncé).*

Pour une démonstration du Théorème 2.4 voir aussi [1], pag. 8-

Démonstration du Théorème 2.3. On a $f + g \in \mathcal{B}_1$, car la classe est fermée par rapport à l'addition. Pour établir la relation $f + g$ nous utiliserons le Théorème 2.4. Soit $x \in [a, b]$. Admettons d'ab que $a < x < b$. Lorsque $x \notin$ disc (f), la relation $g \in \mathcal{DB}_1$ et le Théor 2.4 impliquent l'existence d'une suite croissante (x_n) et d'une suite déc ssante (y_n) telles que $a < x_n < x < y_n < b$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$ et $\lim_{n\to\infty} f(x_n)$ $= \lim_{n\to\infty} f(y_n) = f(x)$. Puisque f est continue dans le point x, on a

$$\lim_{n\to\infty} [f(x_n) + g(x_n)] = \lim_{n\to\infty} [f(y_n) + g(y_n)] = f(x) + g(x).$$

Lorsque $x \in$ disc (f), on a $x \notin$ disc (g) et on procède de la même man en utilisant la relation $f \in \mathcal{DB}_1$ et la continuité de g dans le point

Quand $x = a$ ou $x = b$ on raisonne comme plus haut en travaill seulement avec la suite (y_n), respectivement la suite (x_n). Par conséqu dans tous les cas la condition du Théorème 2.4 est remplie pour la som $f + g$, donc $f + g \in \mathcal{D}$.

2.5. REMARQUES. Comme le premier exemple de H. Lebesgu montre, la condition disc $(f) \cap$ disc $(g) = \emptyset$ est essentielle pour la valab du Théorème 2.3. En effet, dans cet exemple on a $f \in \mathcal{DB}_1$, $g \in$ et disc $(f) \cap$ disc $(g) = \{0\}$. Remarquons aussi que, en vertu du Théor 2.2, les fonctions φ et ψ du deuxième exemple de H. Lebesgue n'appar nnent pas à $\mathcal{B}_1$.

2.6. REMARQUE. Toutefois, la condition disc $(f) \cap$ disc $(g) = \emptyset$ Théorème 2.3 n'est pas nécessaire. A cette fin rappelons que la fonc $f\colon [0, 1] \to R$ introduite par R.J. Fleissner [4], pag. 18, est bor discontinue dans le point $x = 0$ et approximativement continue. La fo tion $g = s_{0,0}$, avec $a = 0$ et $b = 1$, est primitivable, donc le Théorème implique $f + g \in \mathcal{DB}_1$, bien que disc $(f) \cap$ disc $(g) \neq \emptyset$.

3. *Une fonction bornée jouissant de la propriété de Darboux, qui non-primitivable et non-intégrable au sens de Riemann.* Dans cette sec nous utiliserons le Théorème 2.3 pour la construction d'une fonc bornée de $\mathcal{DB}_1$, qui est non-primitivable et non-intégrable au sens de R mann. De pareilles fonctions interviennent dans une classification prop en [7] pour certains ensembles de fonctions réelles.

Soit $(r_n)_{n\geq 1}$ une suite dont l'ensemble de termes coïncide avec l'ensem de tous les nombres rationnels de l'intervalle [0, 1].

3.1. THÉORÈME (D. Pompeiu [9]). (i) *La série de fonctions*

$$\sum_{n=1}^{\infty} 2^{-n}(x - r_n)^{1/2}$$

est uniformément convergente sur $[0, 1]$, *donc sa somme* F *est une fonction continue sur* $[0, 1]$;

(ii) *La fonction* $F : [0, 1] \to [a, b]$ *ainsi définie, où* $a = F(0)$ *et* $b = F(1)$, *possède une dérivée positive, est bijective et son inverse* $G = F^{-1}$ *est dérivable sur* $[a, b]$;

(iii) *La dérivée* $g = G'$ *est bornée et non-intégrable au sens de Riemann sur* $[a, b]$.

Pour une démonstration du Théorème 3.1 voir aussi [6].

3.2. THÉORÈME. *Dans la classe* $\mathfrak{D}\mathfrak{B}_1$ *il existe une fonction bornée, qui est non-primitivable et non-intégrable au sens de Riemann.*

Démonstration. La fonction $g : [a, b] \to R$ du Théorème 3.1 est primitivable, donc $g \in \mathfrak{D}$ et $g \in \mathfrak{B}_1$ (cf. [5], pag. 92) ; il s'en suit qu'il existe un point de continuité $c \in [a, b]$ pour la fonction g (cf. [1], pag. 1). La fonction $f = s_{1,c}$ est non-primitivable, mais elle est intégrable au sens de Riemann et vérifie $f \in \mathfrak{D}\mathfrak{B}_1$.

Puisque dusc $(f) \cap$ disc $(g) = \emptyset$, le Théorème 2.3 entraînne $f + g \in \mathfrak{D}\mathfrak{B}_1$. La somme $h = f + g$ satisfait également le reste des propriétés de l'énoncé, car les classes des fonctions bornées, des fonctions primitivables et des fonctions intégrables au sens de Riemann sont fermée par rapport à l'addition.

3.3. REMARQUE. La fonction h de la démonstration du Théorème 3.2 prouve bien que la Théorème 2.3 est effectivement plus général que le Théorème 2.2.

BIBLIOGRAPHIE

1. Bruckner, A. M., *Differentiation of real functions*, Lecture Notes in Mathematics, 659. Springer-Verlag, Berlin, 1978.
2. Bruckner, A. M. and Ceder, J., *On the sum of Darboux functions*, Proc. Amer. Math. Soc. *51* (1975), 97–102.
3. Ellis, H. W., *Darboux properties and applications to non-absolutely convergent integrals*, Canadian J. Math. *3* (1951), 471–485.
4. Fleissner, R. J., *Multiplication and the fundamental theorem of calculus : a survey*, Real Anal. Exchange *2* (1976), 7–34.
5. Lebesgue, H., *Leçons sur l'intégration et sur la recherche des fonctions primitives*, Gauthier–Villars, Paris, 1904.
6. Marcus, S., *Funcţiile lui Pompeiu*. Stud. Cercet. Mat., *5* (1954), 413–419.
7. Muntean, I., *O clasificare a unor mulţimi de funcţii reale pe intervale compacte* Gazeta Matem. Perfecţ. Metod. şi Metodol. in Matem. şi Inform. *6* (1985), 7–12.
8. Neugebauer, C., *On a paper of M. Iosifescu and S. Marcus*, Canadian Math. Bull., (1963), 367–371.
9. Pompeiu, D., *Sur les fonctions dérivées*, Math. Ann., *63* (1907), 326–332.
10. Sierpiński, W., *Sur une propriété des fonctions réelles quelconques*, Matematiche (Catania), *8* (1953), 43–48.
11. Švarc, R., *On the range of values of the sum of a continuous and a Darboux function*, Časopis Pěst. Mat., *98* (1973), 178–180.
12. Young, W. H., *A theorem in the theory of functions of a real variable*, Rend. Circ. Mat. Palermo, *24* (1907), 187–192.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

SPAŢII CARE ADMIT CONEXIUNI SEMI-SIMETRICE, METRI(ŞI APROAPE METRICE

P. STAVRE*

Intrat în redacţie la 20 decembrie 1981

ABSTRACT. — Spaces Admitting Semi-Symetric, Metric and Nearly Metric Connections. The semi-symetric, metric connections being S — circular connections. The semi-symetric, metric connections being S — circular connections and the associated connections of the Weyl type are studied in the present paper. Then results of theorems 1—7 are obtained.

Fie L_n o varietate diferenţiabilă n-dimensională, de clasă C^∞ şi metrică Riemanniană în L_n. O conexiune liniară, D, în L_n, al cărui te de torsiune, T,

$$T(x, y) = D_x y - D_y x - [x, y], \quad x, y \in \mathfrak{X}(L_n)$$

este de forma

$$T(x, y) = \omega(y)x - \omega(x)y$$

unde ω este o 1 — formă, $\omega \in \Lambda^1(L_n)$, se numeşte semi-simetrică [8]).

Într-o hartă locală, T se va scrie

$$T^h_{ij} = \omega_i \delta^h_j - \omega_j \delta^h_i.$$

Avem

$$\omega(T(x, y)) = 0$$

Fie ∇ conexiunea Levi-Civita corespunzătoare lui $g(g_{ij})$ şi Γ^i_{jk} c cienţii conexiunii D. Conexiunile (D), metrice, sînt

$$\Gamma^l_{ij} = \left\{ {l \atop ij} \right\} + \frac{1}{2}(T^l_{ij}) + \frac{1}{2}(T^s_{ik}g_{sj} + T^s_{jk}g_{si})g^{kl}$$

Dacă se cere ca D, metrica, să fie semi-simetrică, atunci se ob

$$\Gamma^h_{jk} = \left\{ {h \atop jk} \right\} + \omega_j \delta^h_k - g_{jk}\omega^h; \quad (\omega^h = g^{hs}\omega_s)$$

Vom avea

$$g_{ij/k} = 0,$$

unde s-a mutat prin /, derivata covariantă în raport cu D.

* *Universitatea din Craiova, Facultatea de matematică, 1100 Craiova, România*

DEFINIŢIA 1. O conexiune semi-simetrică metrică (3) se va numi S-concirculară, dacă

$$\omega_{i/j} - \frac{1}{2} g_{ij}\omega^m\omega_m = fg_{ij}\,; \qquad f \in \mathfrak{F}(L_n) \tag{5}$$

Denumirea de concirculară este dată prin analogie cu transformările concirculare din geometria concirculară (a unui spaţiu V_n) a lui K. Yano. În [6] se studiază aceste conexiuni şi se arată că sînt caracterizate de invarianţii, echivalenţi,

$$R^s_{ijr} - \frac{R}{n(n-1)}(g_{ij}\delta^s_r - g_{ir}\delta^s_j) = r^s_{ijr} - \frac{r}{n(n-1)}(g_{ij}\delta^s_r - g_{ir}\delta^s_j) \tag{6}$$

$$R^s_{ijr} - \frac{1}{n-1}(g_{ij}R^s_r - g_{ir}R^s_j) = r^s_{ijr} - \frac{1}{n-1}(g_{ij}\delta^s_r - g_{ir}\delta^s_j) \tag{7}$$

(unde r^s_{ijr} este tensorul de curbură, r_{ij} tensorul lui Ricii, $r^s_j = g^{sk}r_{kj}$ iar $r = g^{sk}r_{sk}$ ([2]).

Membrul drept din (6) este tensorul de curbură concirculară din V_n, cunoscut. Se mai arată şi

TEOREMA 1. *Condiţia necesară şi suficientă ca spaţiul să aibă curbură Riemanniană constantă este*

$$R^s_{ijr} - \frac{R}{n(n-1)}(g_{ij}\delta^s_r - g_{ir}\delta^s_j) = 0 \tag{6'}$$

sau

$$R^s_{ijr} - \frac{1}{n-1}(g_{jr}R^s_i - g_{ir}R^s_j) = 0. \tag{6''}$$

Din (5), rezultă

$$\omega_{s/kl} = g_{sk}\omega^m\omega_{m/l} + f_l g_{sk}\,; \quad f_l = \frac{\partial f}{\partial x_l} \tag{8}$$

Schimbînd k cu l şi scăzînd rezultatele obţinem

$$R^h_{skl}\omega_h - T^h_{kl}\omega_{s/h} = (g_{sk}\omega^m\omega_{m/l} - g_{sl}\omega^m\omega_{m/k}) + (f_l g_{sk} - f_k g_{sl}) \tag{9}$$

Sau

$$R^{h}_{sk}{}^{l}\omega_h = f_l g_{sk} - f_k g_{sl}. \tag{10}$$

Din (5) rezultă

$$\omega_{i/j} - \omega_{j/i} = 0 \tag{5'}$$

Din (2″) (5′) rezultă

$$d\omega = 0 \tag{11}$$

(ω este închisă). Tensorul lui Ricci R este simetric iar tensorul lui Bianchi este nul ([5]).

Cum D este şi metrică, notînd

$$R_{pskl} = g_{ph} R^h_{skl}$$

obţinem şi pentru R_{pskl} proprietăţi analoage cu cele pentru r_{pskl}. (10) rezultă

$$\omega^p R_{pl} = (n-1) f_l; \qquad \omega_q R^q_l = (n-1) f_l$$

Din (10), (13) rezultă

$$\left[R^h_{skl} - \frac{1}{n-1}(g_{sk} R^h_l - g_{sl} R^h_k)\right]\omega_h = 0$$

care este condiţia de complet integrabilitate pentru (5).

Din (7), (14), rezultă

$$\left[r^h_{skl} - \frac{1}{n-1}(g_{sk} r^h_l - g_{sl} r^h_k)\right]\omega_h = 0$$

Dar (5) se mai scrie

$$\omega_{i,j} - \omega_i \omega_j + \frac{1}{2} g_{ij} \omega^m \omega_m = f g_{ij}$$

unde s-a notat prin (,) derivata covariantă în raport cu conexiun Condiţia de complet integrabilitate pentru (5″) este (15).

Dacă cerem că D, care este S-concirculară, să aibă propriet (6′), atunci (14) este satisfăcută (şi deci (15) este satisfăcută). Dar în caz, conform teoremei (1), spaţiul este cu curbură Riemanniană tantă.

Invers. Fie un spaţiu cu curbură, Riemanniană, constantă. A avem

$$r^h_{skl} - \frac{1}{n-1}(g_{sk} r^h_l - g_{sl} r^h_k) = 0$$

Deci avem (15) care este condiţia de complet integrabilitate p (5″) ([7]). Fie ω soluţia lui (5″) şi Γ definită prin (3) cu acest ω. Ea face (5), adică Γ este S-concirculară. Deci avem (5), (7). Cum avem rezultă (6′) (sau 6″). De unde

TEOREMA 2. *Ca să existe o conexiune semi-simetrică S-concirc cu proprietatea (6′) (sau 6″) este neesar şi suficient ca spaţiul să aibă cu Riemanniană constantă.*

Fie dar ω o soluţie a lui (5″) şi conexiunea

$$\bar{\Gamma}^h_{jk} = \begin{Bmatrix} h \\ jk \end{Bmatrix} - \omega_k \delta^h_j$$

Evident, conexiunea $\bar{\Gamma}$ este semi-simetrică şi are tensorul de torsiune egal cu tensorul de torsiune al conexiunii semi-simetrice Γ. Deci tensorul deformării afine $\tau^h_{jk} = \Gamma^h_{jk} - \bar{\Gamma}^h_{jk}$, este un tensor simetric,

$$\tau^h_{jk} = \omega_j \delta^h_k + {}'\omega_k \delta^h_j - g_{jk}\omega^h. \tag{18}$$

Avem, ([4)]

$$g_{jk} \| r = 2\omega_r g_{jk} \tag{19}$$

unde s-a notat prin // derivata covariantă în raport cu $\bar{\Gamma}$. Obţinem în acest fel pe varietate o structură de spaţiu Weyl, deşi $\bar{\Gamma}$ nu este simetrică. Se va numi structură D-Weyl (structură Weyl semi-simetrică). De unde

TEOREMA 3. *Dacă spaţiul are curbură Riemanniană constantă, atunci pe varietatea L există o structură de spaţiu D-Weyl.*

În general, pentru (17), avem

$$\bar{R}^h_{ijk} = r^h_{ijk} + \frac{1}{1-n}\delta^h_i(T_{j/k} - T_{k/j}) \tag{18'}$$

unde $T_k = T^s_{sk}$. Dacă ω, din (17), este ω, din (3), rezultă

$$R^h_{ijk} = r^h_{ijk} + 2f(g_{ij}\delta^h_k - g_{ik}\delta^h_j) \tag{19'}$$

În acest caz, cum ω este închisă, avem

$$T_{j/k} = T_{k/j} \tag{20}$$

Deci

$$R^h_{ijk} = r^h_{ijk}. \tag{21}$$

DEFINIŢIA 2. O transformare $\Gamma = \psi(\Gamma)$, de conexiune, este numită transformare de tip Γ_R, dacă tensorii de curbură se corespund (21).

Cum avem (21) rezultă

TEOREMA 4. *Transformarea de conexiune* (17), *cu* ω *soluţie a lui* (5''), *este o transformare de tip* Γ_R.

Din (18), (19), (21) rezultă

$$R^h_{ijk} = \bar{R}^h_{ijk} + 2f(g_{ij}\delta^h_k - g_{ik}\delta^h_j), \tag{22}$$

care dă relaţia dintre tensorii de curbură ai celor două conexiuni.

Avem,

$$R_{ij} = \bar{R}_{ij} + 2f(n-1)g_{ij} \tag{23}$$

$$R = \bar{R} + 2n(n-1)f \quad ; \quad f = \frac{R - \bar{R}}{2n(n-1)} \tag{24}$$

De unde, invariantul

$$R^s_{ijr} - \frac{R}{n(n-1)}(g_{ij}\delta^s_r - g_{ir}\delta^s_j) = \bar{R}^s_{ijr} - \frac{\bar{R}}{n(n-1)}(g_{ij}\delta^s_r - g_{ir}\delta^s_j) \tag{25}$$

Sau, dacă se cunoaşte iniţial (6), atunci din (6), (21) rezultă (25). general, avem

TEOREMA 5. *Dacă Γ este S-concirculară şi Γ, D-Weyl, cu acela din Γ, atunci un invariant al transformării* (18) *este dat de* (25).

Din (21), (25) rezultă

$$R_{ij} - \frac{R}{n} g_{ij} = \bar{R}_{ij} - \frac{\bar{R}}{n} g_{ij} = r_{ij} - \frac{r}{n} g_{ij}$$

Din (25) (26), rezultă

$$R^s_{ijr} - \frac{1}{n-1}(R_{ij}\delta^s_r - R_{ir}\delta^s_j) = \bar{R}^s_{ijr} - \frac{1}{n-1}(\bar{R}_{ij}\delta^s_r - \bar{R}_{ir}\delta^s_j) = P^s_{ijr},$$

unde P^s_{ijr} este tensorul de curbură al lui Weyl ([1]).

De unde

TEOREMA 6. *Dacă Γ este o conexiune S-concirculară iar Γ este S-l cu acelaşi convector, ω, atunci avem* (27).

În cazul că spaţiul are curbură Riemanniană constantă, existenţ ω este asigurată şi invariantul (25) este nul.

Fie dar Γ (3) S-concirculară şi $\bar{\Gamma}$ (17), cu acelaşi ω. Deoarece Γ S-concirculară avem (5′). Din (2″), (5′) rezultă

$$\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} = 0,$$

adică ω este un gradient. Deci există o hartă locală în care forma

$$ds = \omega_i \, dx^i$$

poate fi adusă la forma canonică

$$ds = dx^1$$

şi deci ω la forma canonică

$$\omega_i = \delta^1_i$$

$$\begin{cases} T^h_{1k} = \delta^h_k = -T^h_{k1}; & k > 1 \\ T^h_{ij} = 0; \quad i = j \text{ sau } i, j > 1 \end{cases}$$

De unde

TEOREMA 7. *Dacă spaţiul admite o conexiune, Γ, S-concirculară, a este un spaţiu cu torsiune constantă (în sensul că există o hartă în care siunea este constantă).*

În această hartă, $\bar{\Gamma}$ are forma canonică

$$\bar{\Gamma}^h_{j1} = \left\{ {h \atop j1} \right\} - \delta^h_j; \quad \bar{\Gamma}^h_{js} = \left\{ {h \atop js} \right\}; \quad s > 1$$

Deci, în această hartă, coeficienţii conexiunii $\overline{\Gamma}$ coincid cu coeficienţii conexiunii Levi—Civita pentru $s > 1$. Sau

$$g_{ij\,||1} = 2g_{ij}\,; \quad g_{ij\,||s} = 0, \quad s > 1$$

Adică $\overline{\Gamma}$ este aproape metrică.

BIBLIOGRAFIE

1. L. P. Eisenhart, *Non Riemannian Geometry*, American Math. Soc. New York, **27** (1927).
2. L. P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press. 1949.
3. R. Miron, *Asupra conexiunilor compatibile cu structuri conform aproape simplectice sau conform metrice*, Analele Univ. Timişoara, **V** (1967).
4. D. Smaranda, O. C. Andonie, *On semi-symmetric connection*, Ann, Fac. Sci. de Kinshasa (Zaïre), *2* (1976).
5. P. Stavre, *Asupra formei canonice a unei conexiuni*, Studii şi cercetări matematice (Bucureşti), **22**, 1 (1970), 73—80.
6. P. Stavre, *On S — concircular and S — echarmonic connection*, Tensor N. S. (to appear).
7. P. Stavre, *Asupra soluţiei unui sistem de ecuaţii cu derivate parţiale de ordinul doi*, Simpozionul Naţional de analiză funcţională şi ecuaţii, Craiova 5—7 noiembrie 1981.
8. K. Yano, *On semi-symmetric metric connection*, Rev. Roum. de math. pures et appl. (Bucureşti), **9** (1970), 1579—1586.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 1, 1986

RECEN

Algebraic and Differential Topology — Global Differential Geometry, edited by George M. Rassias, Teubner — Texte zur Mathematik, Bd. 70, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1984, 348 pp.

The volume gives an insight on various research problems and new theories in the fields of algebraic and differential topology, global differential geometry and other related topics. The papers are written by eminent scientists on the occasion of the 90-th Anniversary of Marston Morse's birth.

H. WIESLER

Perspectives in Mathematics, Edited by W. Jäger, J. Moser, R. Remmert; Birkhäuser-Verlag, Basel—Boston—Stuttgart, 1984, 587 pp.

The authors, in selected problems, report on the state of mathematics today and give directions for future development. The question treated provide a broad overall picture of the today mathematical research. The volume is published at the occasion of the 40-th anniversary of the foundation of the Mathematical Research Institute in Oberwolfach which has exercised a fundamental influence on the development of mathematics both in FRG and abroad.

H. WIESLER

Global Analysis — Analysis on Manifolds, edited by Themistocles M. Rassias, Teubner — Texte zur Mathematik, Bd. 57, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1983; 376 pp.

The papers published in the volume report on recent results and provide an account of some of the most important achievements in the areas of Marston Morse's fields of study: Analysis, Topology, Calculus of Variations, Geometry and Dynamics. The submitted works are dedicated to the memory of Marston Morse on the occasion of the ninetieth anniversary of his birth.

H. WIESLER

G. Hammer, D. Pallasc (editors), **Selected Topics in Operation search and Mathematical Economics.** Sp Verlag, Berlin, Heidelberg, New York, T 1984, X + 478 pages (Lecture Notes in nomics and Mathematical Systems, V 226).

This volume contains 37 papers senting lectures by invited speakers and munications given at the 8th Symposiu Operations Research, held at the Univ of Karlsruhe (FRG) from August 22 to A 25, 1983. The papers are grouped in parts having the following headings: mization Theory, Control Theory, Mat tical Economics, Game Theory, Graph Th Fixed Point Theory, Statistics and M Theoretic Concepts, Applications. The finds in these papers recent develop from a wide spectrum of up-to-date re fields, highlighting not only new basic retical results, but also important tech and methods of direct practical interest. refore, the book is to be recommended researchers working in the rapidly gr up area of Operations Research.

WOLFGANG W. BREC

D. H. Luecking and L. A. bel: **Complex Analysis. A Functional lysis Approach.** (Universitext). Springe lag, New York—Berlin—Heidelberg—T 1984, vi + 176 p.

The aim of the book is to prese standard material on functions of a co variable, using methods of functional a such as duality in locally convex topo vector spaces, the Hahn-Banach theorem principle of uniform boundedness. Note all the needed elements of functional an are proved in the book.

The main object of study is the l convex space $H(G)$ of all holomorphic fun on the open set G, endowed with the top of uniform convergence on compact s of G. The description of the dual of $H(G$ allow simple proofs for Runge's approxim theorem and the Cauchy integral the

The book is addressed to mathematicians and students of mathematics with some knowledge of complex variables.

V. ANISIU

Jean-Pierre Kahane, **Some Random Series of Functions,** Second edition, Cambridge Studies in Advanced Mathematics 5, Cambridge University Press 1985.

The first edition of this well known book was published in 1968. A Russian translation appeared in „Mir" Editors, Moscow 1973. With respect to the first edition there are six new sections in the old chapters and five new chapters, reflecting the progress made in this field since the appearence of the first edition.

S. COBZAŞ

P. T. Johnstone, **Stone spaces,** Cambridge Studies in Advanced Mathematics 3, Cambridge University Press, 1982.

The starting point of this book is Stone's representation theorem for Boolean algebras published in 1936, 1937. The term Stone space stands for compact Hausdorf totally disconnected spaces, characterizing the prime ideals spaces of Boolean algebras. Johnstone's book gives a systematic account, in a categorical language, of all facts related to Stone spaces, proving that mathematics still exists as a whole and not only as a collection of particular areas of research and emphasizing the dialectical interplay between „discrete" and „continuous' mathematics. The book ends with an extensive bibliography — 40 pages and can be used both by beginners, as a textbook, and by specialists as a reference book.

S. COBZAŞ

Multifunctions and Integrands, Stochastic analysis, Approximation and Optimization, Catania 1983, Edited by Gabriella Salinetti, Lecturs Notes in Mathematics 1091, Springer Verlag 1984, 234 pp.

These are the Proceedings of an international conference held in Catania, Italy, in June 1983, under the scientific direction of R. T. Rockafellar, M. Valadier and G. Salinetti. The book contains survey papers, reports on recent progress and problems for further investigation, written by eminent specialists in the field (R. T. Rockafellar, J-B. Wetts, J-P. Aubin, C. Castaing, A. Cellina, E. de Giorgi, C. Olech, L. Thibault, M. Valadier et al).

S. COBZAŞ

G. M. Henkin, J. Leiterer, **Theory of Functions on Complex Manifolds,** Akademie-Verlag, Berlin 1984, 226 pp. Published also by Birkhäuser Verlag, Basel—Boston—Stuttgart 1984.

The aim of this book is to give an introduction to the theory of functions of several complex variables based on global integral formulas (appropiate generalizations of Cauchy formula). This approach is more constructive than that based sheaf theory or $\bar{\partial}$-Neumann problem and permits to obtain the results in a strengthened form. Each section of the book ends with exercises, remarks and problems. An extensive bibliography is given. The book is a valuable contribution to several complex variables.

D. ANDRICA

J. Elschner, **Singular Ordinary Differential Operators and Pseudodifferential Operators,** Mathematical Research (Mathematische Forschung) Band 22, Akademie-Verlag, Berlin 1985, 200 pp.

A linear differential equation whose coefficient of the highest derivative vanishes at some points is called degenerate or singular. The aim of this book is to present, using the methods of linear functional analysis, a general theory of solvability of such equations (Chapters 1—3). The book also contains applications to partial differential equations (Ch. 4), index and Fredholm property of some pseudodifferential operators (Ch. 5) and Galerkin method (Ch. 6, the last). This book is a valuable contribution to differential equation theory.

S. COBZAŞ

Daniel Segal, **Polyciclic gropus, Cambridge Tracts in Mathematics,** Cambridge University Press 1983.

This book is a comprehensive account of the present state of the theory of polyciclic groups. Also, providing a connected and self-contained account of the group-theoretical background, it explains in detail how deep methods of number theory and algebraic group theory have been used to achieve some very recent an rather spectacular advances on the subject.

G. CĂLUGĂREANU

Théorie de l'itération et ses applications, Toulouse (17–22 Mai 1982), Editions du CNRS, Paris 1982, 264 pp.

Les travaux présentès au Colloque sont groupés en cinq sections. Conférences générales (L. Reich, G. Targonski, R. Thom (présentation orale)), I. Théorie mathèmatique de l'itération, II. Systèmes dynamiques discrets, bifurcations, attracteurs, III. Systémès diffèrentiels, IV. Applications.

I. PĂVĂLOIU

H. Baumgärtel, **Analytic Perturbation Theory for Matrices and Operators,** Akademie Verlag, Berlin 1985, 427 pp. Published also by Birkhäuser Verlag, Basel–Boston–Suttgart, in the series Operator Theory: Advances and Applications, vol. 15, 1985.

The book studies how the Jordan structure of a matrix changes by small analytic perturbations. The perturbation theory for isolated eigenvalues of linear operators in infinite dimensional spaces and the case of several complex variable analytic perturbations are also considered. As perturbation theory strongly influences areas beyond mathematics, especially theoretical physics, the book is of interest for mathematicians, physicists, engineers, a.o.

S. COBZAŞ

Hans-Ulrich Schawrz, **Banach Lattices and Operators,** Teubner-Texte zur Mathematik Band 71, Leipzig 1984.

The purpose of this book is to present the theory of Banach lattices and some aspects of operators between them. The volume is divided into three parts — vector l… Banach lattices, and operators. The th… vector lattices is developed as far a… needed for further investigations of … lattices. The main aim of the second … is presentation of various classes of … lattices. This chapter also contains the … sentation theorems for $C(K)$ — and L… spaces.

The third part deals with the ord… perties of bounded linear operators b… Banach lattices. The results are used to … the structure of certain types of Bana… tices.

The proofs of the basic facts are … in full detail, whereas the proofs of … results are more concise; thus the b… accessible and useful not only for the sp… but also for research workers in other…

I. …

Mathematical Research, Band 18, Aka… Berlin 1984. *Structural Induction on … Algebras,* by H. Reichel.

This book represents the second … the Introduction to Theory and Appli… of Partial Algebras. As partial algebr… usefull in computer science, the autho… lopes a special theory for applications … direction, where the structural inducti… free algebric structures plays an im… role.

N. …

G. Scheja, U. Storch, **Le… der Algebra,** Teil 1, Teil 3, B. G. T… Stuttgart, 1980, 1981.

The book contains the lectures d… by the authors in Bochum and Osn… Universities. It is an introduction in … algebra, useful for students in math… and physics. The first volume deals wit… elements of set theory and a basic st… groups, rings and modules. Volume 3 co… some special questions, supplements t… of the six chapters of the first volume. … exercises are proposed to the reader…

R. CO…

B. **Huppert**, N. **Blackburn**, **Finite Groups II, III**, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

Volume II and III are completing the first one (appeared in 1967). They give descriptions of the recent development of certain important parts of the subject. Volume II deals with relations between finite groups and linear algebra, giving some elements of general representation theory and presenting the linear methods which have proved useful in questions involving nilpotent groups and soluble groups. The final volume is concerned with some of the developments of the subject in the 1960's: local finite group theory, Zassenhaus groups and multiply transitive permutation groups. The book is of special interest for mathematicians who study and research finite group theory. More than any other work, it reflects the status of research in this field.

RODICA COVACI

J.-P. Aubin A. Cellina, **Differential Inclusions**, Grundlehren der mathematischen Wissenschaften 264, Springer Verlag, 1984, 342 pp.

Differential inclusions arise naturally in the study of dynamical systems having velocities not uniquely determined, i.e. the differential equations $x' = f(x)$ is replaced by the differential inclusion $x' \in F(x)$, $f \cdot$ a set valued mapping. The topics the book is dealing with are: set valued mappings, existence theorems (via selection and fixed-point theorems), applications to optimal control and viability theory, Lyapunov functions. The book is self-contained and can be recommended to all those interested in these problems (mathematicians, economists, biologists etc.).

S. COBZAŞ

W. H Schikhof, **Ultrametric calculus, An introduction to p-adic analysis**, Cambridge Studies in Advanced Mathematics 4, Cambridge University Press, 1984, 366 pp.

The analysis on the fields with non-archimedean valuation, i.e. a valuation for which the triangle inequality is replaced by the strong triangle (ultrametric) inequality is called ultrametric (non-archimedean, p-adic) analysis. The book is dealing with the familiar notions of continuity, differentiability, (power) series, integrations etc. Going on this way there are things looking like in the classical case (analysis over R or C), but, as the author points out in the preface, „the strong triangle inequality causes fascinating deviations from the classical analysis". The book is an excelent introduction to non-archimedean analysis.

S. COBZAŞ

S. Wagon, **The Banach—Tarski Paradox**, Encyclopedia of Mathematics and its Applications, Cambridge Univ. Press 1985, 251 pp.

In 1924 S. Banach and A. Tarski proved in Fundamenta Mathematicae an astonishing results which is often stated as: It is possible to cut up a pea into a finitely many pieces that can be rearranged by rigid motions to obtain a ball the size of the sun. Such a construction is possible in every space R^n with $n \geq 3$ and it is impossible in R^1 and R^2. Since our world is at least three dimensional, a practical application of this striking result of pure mathematics will solve, probably, the food problem on our planet. The construction uses the axiom of choice (AC) and this result has determined many mathematicians to look critically at the AC. The final chapter of the book contains a discussion on the philosophical and technical aspects of the usage of AC. The aim of the book is two-fold: the first one is to present as simple as possible the Banach—Tarski paradox and related results and the second one is to present some very recent and deep results of Gromov, Margulis, Rosenblatt, Sullivan, Tits and others. The result is an excellent monograph which everyone will enjoy to read. A word is to be said on the elegant typographical shape of the book.

S. COBZAŞ

J. D. Dollard, Ch. D. Friedman, **Product Integration with Applications to Differential Equations**, Encyclopedia of Mathematics and Its Applications, Cambridge Univ. Press 1984, 253 pp.

The product integration (called sometimes multiplicative integration) is a tool discovered in 1887 by V. Volterra for solving systems of differential equations. The monograph treats systematically the product integral, its properties and applications to differential equations. The book also contains an extensive Apendix by P. R. Masani (34 pages) entitled „The place of Multiplicative Integration in Modern Analysis". The book will be useful to all interested in differential equations, especially in complex setting.

S. COBZAŞ

Julio R. Bastida, **Field Extensions and Galois Theory,** Encyclopedia of Mathematics and its Applications, vol. 22, Addison--Wesley Publishing Co., 1984, 294 pp.

The book is a very good presentation of an important branch of algebra: the theory of field extensions and Galois theory. The author presents, in an accessible way, the main ideas and results of Galois theory. Each section is completed with many proposed problems. The book ends with an extensive bibliography. The book is useful for the algebraists and students in mathematics.

D. ANDRICA

N. Z. Shor, **Minimization Methods for Non-Differentiable Functions,** Springer-Verlag, Berlin—Heidelberg—New York—Tokyo 1985, VIII + 162 p.

This monograph summarizes some extensions of the author's results concering generalized gradient methods for nonsmooth minimization. It is the translation oftheRussian original: „Methody minimizatsii nedifferentsiruemykh funktsij i ikh prilozheniya", published by Naukova Dumka, Kiev, 1979. We warmly recommend the book not only to those who are interested in minimization methods, but to all who are interested in new methods of investigation in optimization theory.

D. I. DUCA

Paul Erdős, András Hajnal, tila Máté, Richard Rado, **natorial Set Theory: Partition Relati Cardinals,** Akadémiai Kiadó, Budapest

This book presents the most im ideas and results in combinatorial set The book contains 12 chapters: Introd Preliminaries, Fundamentals about p relations, Trees and positive ordinary p relations, Negative ordinary partition r and the discutions on the finite ca canonization lemmas, Large cardinals, ssion on the ordinary partition relatio superscript 2, Discussion on the ordina tition relation with superscript ≥ 3, applications of combinatorial methods, survey of square bracket relation. At t it is given an extended literature and an index. The book is an excellent guide interested in partition theory.

D. AN

John B. Conway, **A Co Functional Analysis,** Springer-Verlag, York—Berlin—Heidelberg—Tokyo, 1985 + 404 pp.

Unlike many modern treatment book begins with the particular and its way to the more general, helping the to develop an intuitive approach to the Thus the first two chapters are on spaces, the third is on Banach space the fourth is on locally convex spaces. V treats the weak and weak-star top The following four chapters look at bounded operators, Banach algebras, bras and spectral theorem. Unbounded tors and Fredholm theory for bounded tors on a Hilbert space are examined last two chapters.

I. MU

ÎNTREPRINDEREA POLIGRAFICĂ CLUJ,
Municipiul Cluj-Napoca, Cd. nr. 450/1986

Revista științifică a Universității din Cluj-Napoca, **STUDIA UNIVERSITATIS BABEȘ-BOLYAI**, apare începînd cu anul 1986 în următoarele condiții:

matematică — trimestrial
fizică — semestrial
chimie — semestrial
geologie-geografie — semestrial pentru geologie și anual pentru geografie
biologie — semestrial
filozofie — semestrial
științe economice — semestrial
științe juridice — semestrial
istorie — semestrial
filologie — semestrial

STUDIA UNIVERSITATIS BABEȘ-BOLYAI, the scientific journal of the University of Cluj-Napoca, starting with 1986 is issued se follows:

mathematics: quarterly
physics: biannually
chemistry: biannually
geology-geography: biannualy on geology and yearly on geography
biology: biannually
philosophy: biannually
economic sciences: biannually
juridical sciences: biannually
history: biannually
philology: biannually

43875

Abonamentele se fac la oficiile poştale, prin factorii poştali şi prin difuzorii de presă, iar pentru străinătate prin „ROMPRESFILATELIA", sectorul export-import presă, P. O. Box 12—201, telex. 10376 prsfir, Bucureşti, Calea Griviţei nr. 64—66.

STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

2

1986

CLUJ-NAPOCA

ANUL XXXI 1986

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

2

Redacţia: 3400 CLUJ NAPOCA, Str. M. Kogălniceanu 1 ● Telefon 1 61 01

SUMAR — SUMMARY — SOMMAIRE

Recenzii — Book Rewiews — Comptes rendus

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

SPAŢII D—H-RECURENTE

P. STAVRE*, P. ENGHIŞ**

Intrat în redacţie: 14.I.1983

ABSTRACT. — **D—H—Recurrent Spaces.** The results in [3], [4] and [9] are further carried in the present paper. The D-H-recurrences (2,1), (3,4), (4,1), (5,2), (6,1) establisched and the relationships between D-H-recurrences tensors $\underset{s}{H}^{k}_{ijk}$ (3,9), (4,5), (5,7), (5,8), (6,4), (6,8), (6,9) are evidenced, showing that they are analogue to those between the tensors for which the D-H-recurrence has been defined.

Introducere. În prezenta lucrare se fac extinderi a rezultatelor din [3], [4], [9].

Se stabilesc $D—H$-recurenţele şi se pun în evidenţă relaţiile dintre tensorii de $D—H$—recurenţă $\underset{s}{H}^{h}_{ijk}$ arătînd că ele sînt analoge cu cele dintre tensorii pentru care s-a definit $D—H$—recurenţă.

§ 1. Fie L o varietate diferenţiabilă de calsă C^∞ înzestrată cu o metrică riemanniană g de componente g_{ij} într-o hartă locală $(u\,;\,x^i)$. Vom nota cu ∇, conexiunea Levi-Civita corespunzătoare, de coeficienţi $\left\{ {i \atop jk} \right\}$ în harata locală $(u,\ x^i)$, prin R^i_{jkh} componentele tensorului de curbură [1], prin $R_{ij} = R^s_{ijs}$ tensorul lui Ricci iar prin $R = g^{ij} R_{ij}$ curbura scalară.

O conexiune D semisimetrică metrică în L_n [2], [14], în harta locală considerată $(u,\ x^i)$, are coeficienţii:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \omega_j \delta^i_k - g_{jk}\omega^i \tag{1.1}$$

$$\omega^i = g^{is}\omega_s \tag{1.2}$$

şi

$$T^i_{jk} = \omega_j \delta^i_k - \omega_k \delta^i_j \tag{1.3}$$

$$g_{ij/k} = 0 \tag{1.4}$$

unde T este tensorul de torsiune a lui D iar prin $/$ s-a notat derivarea covariantă în raport cu D.

*Universitatea din Cluj-Napoca, Facultatea de Matematică-Fizică, Catedra de matematică, 3400 Cluj-Napoca România

**Universitatea din Craiova, Facultatea de Ştiinţe economice, 1100 Craiova, România

Fie $\bar{R}^i_{jkh}$ tensorul de curbură a lui D, $\bar{R}_{ij} = \bar{R}^s_{ijs}$ tensorul lui Ric $\bar{R} = g^{ij}\bar{R}_{ij}$ curbura scalară, $\bar{T}^i_{jkh}$ tensorul D-concircular de curbură [1] $\bar{Z}^i_{jkh}$ tensorul D—coharmonic de curbură [11], $\bar{P}^i_{jkh}$ tensorul D-pr iectiv de curbură [10] şi $\bar{C}^i_{jkh}$ respectiv tensorul de curbură conform lui D.

Dacă D este o conexiune K. Yano (adică $\bar{R}^i_{jkh} = 0$) atunci g e conform plată [14] şi avem:

$$\omega_{i,j} - \omega_{j,i} = 0 \quad (1.$$

unde prin virgulă s-a notat derivata covariantă în raport cu ∇.

Din (1,5) rezultă relaţiile echivalente [6] $\omega_{i/j} - \omega_{j/i} = 0$, $\partial_i \omega_j - \partial_j \omega_i = 0$, $T_{i/j} - T_{j/i} = 0$, $T^k_{ij/k} = 0$, (div T=0), $d\omega = 0$, (ω închisă

(1.

Dacă D este mai generală ca o conexiune K. Yano, $\bar{R}^i_{jkh} \neq 0$, d tensorul D-concircular de curbură [11], $\bar{T}^i_{jkh}$ este nul, atunci pent $n \geqslant 3$, g este conform plată [7] şi avem

$$\bar{R}_{/r} - 2\omega\bar{R} = 0,\ \bar{R}_{/r} = \partial_r \bar{R} \quad (1.$$

deci [4], [7] rezultă $\bar{R}^s_{ijk}$ recurent respectiv D—recurent [4] cu cove tor de recurenţă 2ω

$$\bar{R}^i_{jkh,r} = 2\omega_r \bar{R}^i_{jkh} = \bar{R}^i_{jkh/r} \quad (1.$$

Avem deci

PROPOZIŢIA 1.1. *D-conexiunile ce nu sînt conexiuni K. Yano şi a tensorul D-circular nul, sînt D-recurente cu covector* 2ω.

§ 2. *DEFINIŢIA* 1. Vom spune că spaţiul (L_n, D) este *D—H-re curent,* dacă

$$\bar{R}^i_{jkh/r} = \sigma_r \bar{R}^i_{jkh} + \underset{1}{H}{}^i_{jkh}(\bar{R}_{/r} - \sigma_r R) \quad (2.1)$$

unde σ_r este un convector iar $\underset{1}{H}{}^i_{jkh}$ un tensor de tip (1 ; 3).

Din (2.1) rezultă

$$\bar{R}_{jk/r} = \sigma_r \bar{R}_{jk} + \underset{1}{H}{}_{jk}(\bar{R}_{/r} - \sigma_r \bar{R}) \quad (2.2)$$

unde $\underset{1}{H}{}_{jk} = \underset{1}{H}{}^i_{jki}$ şi spaţiul se va numi *D—H-Ricci-recurent.*

Observaţia 1. Un spaţiu D—H-recurent este şi D—H-Ricci recurent, reciproca nefiind în general adevărată.

Notînd $\underset{1}{H} = g^{jk}\underset{1}{H}_{jk}$ din (2.2) rezultă

$$\bar{R}_{/r} - \sigma_r\bar{R} = \underset{1}{H}(\bar{R}_{/r} - \sigma_r\bar{R}) \tag{2.3}$$

şi dacă (L_n, g) este $D-H$-recurent propriu $(R_{/r} - \sigma_r R \neq 0)$ atunci $\underset{1}{H} = 1$ şi avem

PROPOZIŢIA 2.1. *Intr-un spaţiu (L_n, D) $D-H$-recurent propriu avem $\underset{1}{H} = g^{ij}\underset{1}{H}_{ij} = 1$.*

Este evident că un spaţiu (L_n, D), $D-H$-recurent în care $\bar{R}_{/r} - \sigma_r\bar{R} = 0$ este D-recurent [4] cu acelaşi covector σ_r.

Să presupunem că (L_n, D) este $D-H$-recurent de covector σ şi D-recurent de covector $\bar{\sigma} \neq \sigma$

$$\bar{R}^i_{jkh/r} = \bar{\sigma}_r\bar{R}^i_{jkh} \tag{2.4}$$

Din (2.1) şi (2.4) rezultă

$$(\bar{\sigma}_r - \sigma_r)\bar{R}^i_{jkh} = \underset{1}{H}^i_{jkh}(\bar{R}_{/r} - \sigma_r\bar{R}) \tag{2.5}$$

şi cum din (2.4) avem $\bar{R}_{/r} = \bar{\sigma}_r\,\bar{R}$, din (2.5) rezultă

$$(\bar{\sigma}_r - \sigma_r)\bar{R}^i_{jkh} = \bar{R}\underset{1}{H}^i_{jkh}(\bar{\sigma}_r - \sigma_r) \tag{2.6}$$

şi in ipoteza în care lucrăm $\bar{\sigma} \neq \sigma$, deducem:

$$\bar{R}^i_{jkh} = \bar{R}\underset{1}{H}^i_{jkh} \tag{2.7}$$

iar din (2.1) şi (2.7) pentru $\bar{\sigma}_r$ rezultă

$$\bar{\sigma}_r = \partial_r \ln \bar{R} \tag{2.8}$$

Avem deci

PROPOZIŢIA 2.2. *Intr-un spaţiu (L_n, D) $D-H$-recurent de vector σ şi D-recurent de vector $\bar{\sigma} \neq \sigma$, tensorul de $D-H$-recurenţă este dat de (2.7), iar $\bar{\sigma}$ verifică (2.8).*

COROLAR 2.1. *Din (2.1) şi (2.7) rezultă pentru $R \neq 0$,*

$$\underset{1}{H}^i_{jkh/r} = 0 \tag{2.9}$$

COROLAR 2.2. *Dacă spaţiul (L_n, D) este D-recurent de vector $\bar{\sigma}_r$ dat de (2.8) el este şi $D-H$ recurent de tensor de H recurenţă dat de (2.7) iar σ_r arbitrar.*

În particular dacă $\omega = 0$, avem $D = \nabla$ şi se obţin rezultatele din [9].

Considerînd tensorul D-concircular de curbură

$$\bar{T}^i_{jkh} = \bar{R}^i_{jkh} - \bar{R}\,\underset{1}{H}^i_{jkh} \tag{2.10}$$

unde

$$H^i_{jkh} = \frac{1}{n(n-1)}(g_{jk}\delta^i_h - g_{jh}\,\delta^i_k) \qquad (2.11)$$

dacă spaţiul (L_n, D) are proprietatea $\bar{T}^i_{jkh} = 0$ şi $n \geqslant 3$, atunci confor[m] propoziţiei 1.1 rezultă (1.8) şi dacă (L_n, D) este $D-H$ recurent propr[iu] avem $\sigma \neq 2\omega$, iar din (2.7) rezultă

$$\underset{1}{H^i_{jkh}} = H^i_{jkh} \qquad (2.12)$$

Avem deci:

PROPOZIŢIA 2.3. *Dacă D este o conexiune semi-simetrică mai gener[ală] cu o conexiune K. Yano, dar cu tensorul D-concircular de curbură nul[,] spaţiul (L_n, D) este $D-H$-recurent, atunci avem (2.12).*

Observaţia 2. $\bar{R} \neq 0$, deoarece din anularea tensorului D-concirc[ular] de curbură ar rezulta $\bar{R}^i_{jkh} = 0$ şi conexiunea ar fi K. Yano. La fel $\bar{R}$ $\neq$ const. deoarece din (1.7) ar rezulta $\omega_r = 0$ şi D ar fi egală cu ∇.

Observaţia 3. Proprietatea 2.3 are loc pentru $D \neq \nabla$. Pentru $D =$ [∇] adică $\omega = 0$ din anularea tensorului concircular de curbură pentru $n \geqslant$ [3] din faptul că curbura riemanniană este constantă, rezultă $R =$ const. deci $R^i_{jkh,r} = 0$, de unde:

COROLAR 2.3. *Dacă (L_n, ∇) este $\nabla-H$ recurent, atunci spaţiul nu po[ate] fi recurent cu convector de recurenţă $\bar{\sigma} \neq \sigma$.*

Fie pentru D, tensorul lui Einstein

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij} \qquad (2.13)$$

Evident, dacă $\bar{T}^i_{jkh} = 0$ avem $\bar{E}_{ij} = 0$. Să presupunem deci $\bar{T}^i_{jkh} \neq 0$ [şi] că $\bar{E}_{ij}$ este D-recurent

$$\bar{E}_{ij|r} = \sigma_r \bar{E}_{ij} \qquad (2.14)$$

Din (2.2), (2.13), (2.14) rezultă

$$(\bar{R}_{|r} - \sigma_r \bar{R})\left(\underset{1}{H_{ij}} - \frac{1}{n} g_{ij}\right) = 0 \qquad (2.15)$$

de unde avem

$$\underset{1}{H_{ij}} = \frac{1}{n} g_{ij} = H_{ij} \qquad (2.16)$$

Invers, din (2.2) şi (2.16) rezultă (2.14), deci avem:

PROPOZIŢIA 2.4. *Dacă (L_n, D) este $D-H$-Ricci recurent, atunci condiţ[ia] necesară şi suficientă ca tensorul lui Einstein să fie D-recurent este (2.16)*

COROLAR 2.4. *Dacă (L_n, D) este $D-H$ recurent atunci (2.16) este condiția necesară și suficientă ca tensorul E (2.13) să fie D-recurent cu același covector de recurență.*

§3. Fie Z^i_{jkh} tensorul coharmonic de curbură [5] și $\bar{Z}^i_{jkh}$ tensorul D-coharmonic de curbură [8], [11]. Avem [4],

$$C^i_{jkh} = Z^i_{jkh} + \frac{n}{n-2} R \, H^i_{jkh} \tag{3.1}$$

$$\bar{C}^i_{jkh} = \bar{Z}^i_{jkh} + \frac{n}{n-2} \bar{R} \, H^i_{jkh} \tag{3.2}$$

unde H^i_{jkh} este dat de (2.11).

Cum avem [12] $C^i_{jkh} = \bar{C}^i_{jkh}$, din (3.1) și (3.2) rezultă

PROPOZIȚIA 3.1. *Intre tensorul coharmonic de curbură al lui ∇ și D, există relația*

$$\bar{Z}^i_{jkh} - Z^i_{jkh} = \frac{1}{n-2} H^i_{jkh}(R - \bar{R}) \tag{3.3}$$

DEFINIȚIA 2. Vom spune că spațiul (L_n, D) este *$D-H$-coharmonic recurent* dacă

$$\bar{Z}^i_{jkh/r} = \sigma_r \bar{Z}^i_{jkh} + \underset{2}{H^i_{jkh}}(\bar{R}_{/r} - \sigma_r \bar{R}) \tag{3.4}$$

unde σ_r este un convector și $\underset{2}{H^i_{jkh}}$ un tensor de tip (1.3) cu proprietatea

$$\underset{2}{H^s_{ijs}} = \underset{2}{H_{ij}} = -\frac{1}{n-2} g_{ij}. \tag{3.5}$$

Din (3.4) rezultă

$$\left(-\frac{1}{n-2} g_{ij} - \underset{2}{H_{ij}}\right)(R_{/r} - \sigma_r \bar{R}) = 0 \tag{3.6}$$

și dacă H_{ij} ar fi diferit de $-\frac{1}{n-2} g_{ij}$ ar rezulta $\bar{R}_{/r} - \sigma_{/r} \bar{R}_n = 0$ și deci [4] spațiul (L_n, D) ar fi D-coharmonic recurent.

Din (3.5) rezultă

$$\underset{2}{H} = g^{ij} \underset{2}{H_{ij}} = -\frac{n}{n-2} \tag{3.7}$$

Să presupunem spaţiul (L_n, D), $D-H$-recurent (2.1). Derivînd covariant în raport cu D tensorul D-coharmonic $\bar{Z}^h_{ijk}$ şi ţinînd seama de (2.1) şi (2.2) avem:

$$\bar{Z}^h_{ijk|r} = \sigma_r \bar{Z}^h_{ijk} + \left[\underset{1}{H}{}^h_{ijk} + \frac{1}{n-2}\left(\underset{1}{H}_{ik}\delta^h_j - \underset{1}{H}_{ij}\delta^h_k + g_{ik}\underset{1}{H}{}^h_j - g_{ij}\underset{1}{H}{}^h_k\right)\right] \cdot [\bar{R}_{|r} - \sigma_r \bar{R}] \qquad (3.8)$$

unde $H^h_j = g^{hs}H_{sj}$

Din (3.8) rezultă:

PROPOZIŢIA 3.2. *Un spaţiu (L_n, D), $D-H$-recurent este şi $D-H$-coharmonic recurent cu acelaşi covector σ şi cu tensor $\underset{2}{H}{}^i_{jkh}$ de H-recurenţă dat de*

$$\underset{2}{H}{}^h_{ijk} = \underset{1}{H}{}^h_{ijk} + \frac{1}{n-2}\left(\underset{1}{H}_{ik}\delta^h_j - \underset{1}{H}_{ij}\delta^h_k + g_{ik}\underset{1}{H}{}^h_j - g_{ij}\underset{1}{H}{}^h_k\right) \qquad (3.9)$$

Reciproc, din (3.4) şi (2.2) rezultă:

PROPOZIŢIA 3.3. *Un spaţiu (L_n, D), $D-H$-coharmonic recurent este $D-H$-recurent, dacă şi numai dacă este $D-H$-Ricci-recurent cu acelaşi covector σ_r şi tensor de $D-H$-Ricci-recurenţă $\underset{1}{H}_{ij}$. Tensorul $\underset{1}{H}{}^i_{jkh}$ de $D-H$-recurenţă fiind dat de (3.9).*

Dacă $\omega = 0$, atunci $D = \Delta$ şi $\bar{Z}^i_{jkh} = Z^i_{jkh}$ şi se obţin rezultatele din [9].

Există un $\underset{2}{H}{}^i_{jkh}$ cu proprietatea (3.5) dat de

$$\underset{2}{H}{}^i_{jkh} = -\frac{n}{n-2}H^i_{jkh} \qquad (3.10)$$

§4. Fie $\bar{T}^i_{jkh}$, tensorul D-concircular de curbură (2.10).

DEFINIŢIA 3. Spaţiu. (L_n, D) este $D-H$ *concircular recurent,* dacă

$$\bar{T}^i_{jkh|r} = \sigma_r \bar{T}^i_{jkh} + \underset{3}{H}{}^i_{jkh}(\bar{R}_{|r} - \sigma_r \bar{R}) \qquad (4.1)$$

cu $\underset{3}{H} = 0$, unde $\underset{3}{H} = g^{ij}\underset{3}{H}_{ij}$; $\underset{3}{H}_{ij} = \underset{3}{H}{}^s_{ijs}$.

Din (4.1) rezultă

$$\bar{T}_{ij|r} = \sigma_r \bar{T}_{ij} + \underset{3}{H}_{ij}(\bar{R}_{|r} - \sigma_r \bar{R}) \qquad (4.2)$$

unde $\bar{T}_{ij} = \bar{T}^s_{ijs} = \bar{E}_{ij}$ (4.3) iar din (4.2) avem

$$\underset{3}{H}(\bar{R}_{/r} - \sigma_r \bar{R}) = 0$$

şi dacă $\underset{3}{H}$ ar fi diferit de zero, ar rezulta, [4], spaţiul (L_n, D), D-concircular recurent. Prin urmare condiţia $\underset{3}{H} = 0$ în (4.1) este esenţială.

Dacă presupunem spaţiul $D-H$-recurent, derivînd covariant (2.10) n raport cu D şi ţinînd seama de (2.1) obţinem:

$$\bar{T}^i_{jkh/r} = \sigma_r \bar{T}^i_{jkh} + (\underset{1}{H}^i_{jkh} - H^i_{jkh})(\bar{R}_{/r} - \sigma_r \bar{R}) \tag{4.4}$$

şi reciproc. Avem deci:

PROPOZIŢIA 4.1. *Orice spaţiu (L_n, D), $D-H$-recurent este şi $D-H$ concircular recurent şi reciproc. Intre tensorii de $D-H$-recurenţă şi $D-H$-concircular recurenţă avem relaţia*

$$\underset{3}{H}^i_{ikh} = H^i_{jkh} - \underset{1}{H}^i_{jkh} \tag{4.5}$$

COROLAR 4.1. *Condiţia necesară şi suficientă ca (L_n, D) să fie $D-H$-concircular recurent este ca (L_n, D) să fie $D-H$-recurent.*

COROLAR 4.2. *Condiţia necesară şi suficientă ca spaţiul (L_n, D) concircular recurent să aibă tensorul lui Einstein recurent, este ca $\underset{3}{H}_{ij} = 0$.*

§. 5 Pentru o conexiune semi-simetrică D în [10] se stabilesc transformările proiective de conexiune care au ca invariant tensorul

$$\bar{W}^s_{ijk} = \bar{R}^s_{ijk} - \frac{1}{n-1}(\delta^s_k \bar{R}_{ij} - \delta^s_j \bar{R}_{ik}) \tag{5.1}$$

analog cu tensorul proiectiv de curbură a lui Weyl, iar în [9] se stabilesc condiţiile în care $\bar{W}^s_{ijk}$ este egal cu tensorul proiectiv de curbură a lui Weyl pentru ∇.

DEFINIŢIA 4. Vom spune că spaţiul (L_n, D) este *$D-H$-proiectiv recurent*, dacă

$$\bar{W}^s_{ijk/r} = \sigma_r \bar{W}^s_{ijk} + \underset{4}{H}^s_{ijk}(\bar{R}_{/r} - \sigma_r \bar{R}) \tag{5.2}$$

cu $\underset{4}{H} = 0$, unde $\underset{4}{H} = g^{ij} \underset{4}{H}_{ij}$, $\underset{4}{H}_{ij} = g_{si} \underset{4}{H}^s_j$ iar $\underset{4}{H}^s_k = g^{ij} \underset{4}{H}^s_{ijk}$.

Dacă în (5.1) înmulţim contractat cu g^{ij} avem

$$\bar{W}^s_k = \frac{n}{n-1}\left(\bar{R}^s_k - \frac{\bar{R}}{n}\delta^s_k\right) \tag{5.3}$$

unde

$$\bar{W}^s_k = g^{ij} \bar{W}^s_{ijk} \tag{5.4}$$

$$\bar{W}_{jk} = g_{js} W^s_k = \frac{n}{n-1} \bar{E}_{jk} \tag{5.5}$$

Din (2.13) şi (5.1) rezultă

$$\overline{W}^s_{ijk/r} - \sigma_r \overline{W}^s_{ijk} = \overline{R}^s_{ijk/r} - \sigma_r \overline{R}^s_{ijk} - \tag{5.6}$$
$$- \frac{1}{n-1}\left[\delta^s_k(\overline{E}_{ij/r} - \sigma_r \overline{E}_{ij}) - \delta^s_j(\overline{E}_{ik/r} - \sigma_r \overline{E}_{ik})\right] - H^s_{ijk}(\overline{R}_{/r} - \sigma_r \overline{R})$$

iar folosind (4.2), (4.3), (4.4) şi (5.6) rezultă:

PROPOZIŢIA 5.1. *Orice spaţiu D—H-concircular recurent este D—H-proiectiv recurent cu*

$$\underset{4}{H^s_{ijk}} = \underset{3}{H^s_{ijk}} - \frac{1}{n-1}(\delta^s_k \underset{3}{H_{ij}} - \delta^s_j \underset{3}{H_{ik}}) \tag{5.7}$$

Reciproc, dacă în (5.2) înmulţim contractat cu g^{ij}, în baza lui (4.3) şi (5.5) avem:

PROPOZIŢIA 5.2. *Orice spaţiu (L_n, D) D—H-proiectiv recurent este şi D—H-concircular recurent cu*

$$\underset{3}{H^s_{ijk}} = \underset{4}{H^s_{ijk}} + \frac{1}{n}(\delta^s_k \underset{4}{H_{ij}} - \delta^s_j \underset{4}{H_{ik}}) \tag{5.8}$$

COROLAR 5.1. *Spaţiile (L_n, D) sînt în acelaşi timp, D—H-recurente, D—H-concircular recurente şi D—H-proiectiv recurente. Intre tensorii de D—H-recurenţă existînd relaţiile: (4.5), (5.7), (5.8).*

COROLAR 5.2. *Dacă în spaţiul (L_n, D) tensorul lui Einstein este D-recurent, atunci între tensorii de D—H-proiectiv recurenţă şi D—H concircular recurenţă avem*

$$\underset{4}{H^i_{jkh}} = \underset{3}{H^i_{jkh}} \tag{5.9}$$

În particular pentru $\omega = 0$, $D = \Delta$, obţinem rezultatul din [9], iar pentru $\underset{3}{H^i_{jkh}} = 0$, obţinem rezultatele din [4].

§ 6. Fie $\overline{C}^i_{jkh}$ tensorul D-conform de curbură.

DEFINIŢIA 5. Vom spune că spaţiul (L_n, D) este *D—H-conform recurent* $(n > 3)$, dacă

$$\overline{C}^s_{ijk/r} = \sigma_r \overline{C}^s_{ijk} + \underset{5}{H^s_{ijk}}(\overline{R}_{/r} - \sigma_r \overline{R}) \tag{6.1}$$

cu $\underset{5}{H_{ij}} = \underset{5}{H^s_{ijs}} = 0$.

Dacă D este mai generală decît o conexiune K. Yano şi tensorul D-concircular de curbură nu este nul $\overline{T}^i_{jkh} \neq 0$, între $\overline{C}^s_{ijk}$ şi $\overline{T}^s_{ijk}$ există o relaţie [11] analoagă cu cea dintre $\overline{C}^s_{ijk}$ şi T^s_{ijk} [3] şi anume:

$$C^s_{ijk} = \overline{T}^s_{ijk} + \frac{1}{n-2}(\overline{T}_{ik}\delta^s_j - \overline{T}_{ij}\delta^s_k + g_{ik}\overline{T}^s_j - g_{ij}\overline{T}^s_k) = C^s_{ijk} \tag{6.2}$$

unde $\overline{T}^s_j = g^{jk}\overline{T}_{kj}$. De unde

$$\overline{C}^s_{ijk} - \sigma_r\overline{C}^s_{ijk} = \overline{T}^s_{ijk/r} - \sigma_r\overline{T}^s_{ijk} + \frac{1}{n-2}[\delta^s_j(\overline{E}_{ik/r} - \sigma_r\overline{E}_{ik}) -$$

$$- \delta^s_k(\overline{E}_{ij/r} - \sigma_r\overline{E}_{ij}) + g_{ik}(\overline{E}^s_{j/r} - \sigma_r\overline{E}^s_j) - g_{ij}(\overline{E}^s_{k/r} - \sigma_r\overline{E}^s_k)] \tag{6.3}$$

Dacă spaţiul (L_n, D) este $D-H$-concircular recurent, din (4.1), (4.2), (4.3) şi (6.3) rezultă (6.1) cu

$$\underset{5}{H^s_{ijk}} = \underset{3}{H^s_{ijk}} + \frac{1}{n-2}(\delta^s_j \underset{3}{H_{ik}} - \delta^s_k \underset{3}{H_{ij}} + g_{ik}\underset{3}{H^s_j} - g_{ij}\underset{3}{H^s_k}) \tag{6.4}$$

De unde

PROPOZIŢIA 6.1. *Dacă spaţiul (L_n, D) este $D-H$ concircular recurent, atunci este $D-H$-conform recurent cu tensor de $D-H$ recurenţă dat de (6.4).*

COROLAR 6.1. *Orice spaţiu (L_n, D) $D-H$-concircular recurent cu tensorul lui Einstein D-recurent cu acelaşi covector σ, este în acelaşi timp $D-H$ conform recurent şi $D-H$ proiector recurent cu*

$$\underset{5}{H^s_{ijk}} = \underset{4}{H^s_{ijk}} = \underset{3}{H^s_{ijk}}$$

COROLAR 6.2. *Dacă spaţiul (L_n, D) este $D-H$-recurent, atunci din propoziţia 4.1 şi corolarul 6.1 rezultă că spaţiul (L_n, D) este $D-H$-conform recurent cu $\underset{5}{H^s_{ijk}}$ dat de (6.4) şi $\underset{3}{H^s_{ijk}}$ dat de (4.5).*

Din (3.2) rezultă

$$\overline{C}^s_{ijk/r} = \overline{Z}^s_{ijk/r} + \frac{n}{n-2} R_{/r}H^s_{ijk} \tag{6.6}$$

sau

$$\overline{C}^s_{ijk/r} - \sigma_r\overline{C}^s_{ijk} = \overline{Z}^s_{ijk/r} - \sigma_r\overline{Z}^s_{ijk} + \frac{n}{n-2} H^s_{ijk}(\overline{R}_{/r} - \sigma_r\overline{R}) \tag{6.7}$$

De unde

PROPOZIŢIA 6.2. *Condiţia necesară şi suficientă ca un spaţiu (L_n, D) să fie $D-H$-conform recurent $(n > 3)$ este ca spaţiul (L_n, D) să fie $D-H$-coharmonic recurent cu*

$$\underset{5}{H^s_{ijk}} = \underset{2}{H^s_{ijk}} + \frac{n}{n-2} H^s_{ijk} \tag{6.8}$$

COROLAR 6.3. *Un spaţiu D—H conform recurent ($n > 3$) este D-recurent, dacă şi numai dacă este D—H-Ricci recurent cu acelaşi covector*

COROLAR 6.4. *Condiţia necesară şi suficientă ca spaţiul (L_n, D) fie D-conform recurent este ca (L_n, D) să fie D—H coharmonic recurent $\underset{2}{H^s_{ijk}}$ dat de (3.10),* [4].

COROLAR 6.5. *Dacă (L_n, D) este D—H-concircular recurent, atunci* (*D—H coharmonic recurent cu*

$$\underset{2}{H^s_{ijk}} = \underset{5}{H^s_{ijk}} - \frac{n}{n-2} H^s_{ijk} \qquad (6$$

unde

$\underset{5}{H^s_{ijk}}$ este dat de (6.4).

Observaţia 4. Relaţiile între tensorii de $D—H$-recurenţă (3.9), (4. (5.7), (4.8), (6.4), (6.8) sînt analoage cu relaţiile ce există între tens(pentru care s-a definit $D—H$-recurenţa.

BIBLIOGRAFIE

1. Eisenhart L. P., *Riemannian Geometry*, Princeton Univ. Press, 1949.
2. Eisenhart L. P., *Non Riemannian Geometry*, American Math. Soc. New York (1927.
3. Enghiş P., *Relations entre des espaces riemenniens a tenseur récurrents*, Studia Un Babeş-Bolyai. Math. XXV. 1 (1980), 67—72.
4. Enghiş P., Stavre P., *Problèmes de récurrence pour connexions semi-symétriq métriques*, Univ. Babes-Bolyai, Sub tipar. Studia
5. Ishii I. *On conharmonic transformations*, Tensor N.S. 7,2 (1957), 70—80.
6. Stavre P., *Asupra formei canonice a unei conexiuni*, Studii şi cercetări mat. Bu reşti, tom 22,1 (1970), 73—80.
7. Stavre P., *Proprietăţi ale conexiunilor semi-simetrice metrice cu tensor D-concircul nul*, (Va apare).
8. Stavre P., *On t e S-concircular and S-coharmonic connections*, Tensor N.S. Japan (appar).
9. Stavre P., *L'espaces Riemanniens H-récurrents*, Studia Univ. Babeş-Bolyai, Mat matica, Cluj-Napoca (va apare).
10. Stavre P., *Asupra nuor conexiuni speciale pe varietăţi diferenţiabile*, Colocviul naţi nal de geometrie şi topologie, Buşteni iunie 1981.
11. Stavre P., *On the D-concircular and D-coharmonic transformations*, Tensor N.S. (apare).
12. Stavre P., Smaranda D., *On the associate index of a semisymetric metric connexi* Colovium Naţ. de Geometrie Gh. Vrînceanu, 1980.
13. Yano K , *Concircular Geometry* I., Math. Ist. Tokyo., Imperial Univ., vol 16 (194 195—200.
14. Yano K., *On semi-symetric metric connections*, Rev. Roumaine de Math. Purres et 9, (1970), 1579—1586.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

ON SOME CLASSES OF DIFFERENTIAL SUBORDINATIONS (II)

TEODOR BULBOACĂ*

Received: July 7, 1983

ABSTRACT. — The study on differential subordinations of the form $\psi(p(z), zp'(z)) \succ h(z)$, begun in [1] and [2] is further carried in the present paper for the case in which $\psi(p(z), zp'(z)) = \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$ by employing the admissible functions method of [2], obtaining generalizations of the results in [6], some consequences and examples being then presented.

Introduction. Let f and g be analytic in the unit disk U and let $H(U)$ be the space of functions analytic in U. We say that f is subordinate to g ($f \succ g$ or $f(z) \succ g(z)$) if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

If $\psi : C^2 \to C$ is analytic in a domain D, if h is univalent in U and if p is analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then p is said to satisfy the first-order differential subordination

$$\psi(p(z), zp'(z)) \prec h(z), \ z \in U.$$

In [1] the authors determine conditions on ψ and h so that $p(z) \prec h(z)$ in the case

$$\psi(p(z), zp'(z)) = \theta(p(z)) + zp'(z)\Phi(p(z))$$

and they give applications of these results in univalent function theory.

În [3] the author study the differential subordination in the case

$$\psi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$$

and applications of these results are given.

In this paper we shall study the differential subordination when

$$\psi(p(z)), zp'(z)) = \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$$

and we give some particular interesting cases.

Preliminaries. We will need the next two lemmas to prove our theorem.

LEMMA 1. [4] *Let $g \in H(U)$, with $g(0) = 0$, be univalent and starlike in U. If $f \in H(U)$ and $Re[zf'(z)/g(z)] > 0$, $z \in U$, then f is univalent in U.*

We said that $L : U \times [0, +\infty) \to C$ is a subordination (or Loewner) chain if $L(\cdot, t)$ is analytic and univalent in U for all $t \geqslant 0$, $L(z, \cdot)$ is continuosly differentiable on $[0, +\infty)$ for all $z \in U$ and $L(z, s) \prec L(z, t)$ when $0 \leqslant s < t$.

* *University of Cluj-Napoca, Faculty of Mathematics, 3400 Cluj-Napoca, Romania*

LEMMA 2. [5, p. 159] *The function* $L(z, t) = a_1(t)z + \ldots$ *with* $a_1(t) \neq$ [illegible] *for all* $t \geqslant 0$ *is a subordination chain if and only if*

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right] > 0$$

for all $z \in U$ *and* $t \geqslant 0$.

THEOREM A. [2] *Let* $h, q \in H(U)$ *be univalent in* U *and supp*[illegible] $q \in H(\bar{U})$. *If* $\psi: C^3 \to C$ *satisfies:*

a) ψ *is analytic in a domain* $D \subset C^3$,

b) $(q(0), 0, 0) \in D$ and $\psi(q(0), 0, 0) \in h(U)$,

c) $\psi(r, s, t) \notin h(U)$ *when* $(r, s, t) \in D$, $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re}(1 + t/s) \geqslant m \operatorname{Re}(1 + \zeta q''(\zeta)/q'(\zeta))$ *where* $|\zeta| = 1$, $m \geqslant 1$,

then for all $p \in (H(U)$ *so that* $(p(z), zp'(z), z^2p''(z)) \in D$, *when* $z \in$ [illegible] *we have:*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \quad \textit{implies that} \quad p(z) \prec q(z).$$

MAIN. RESULTS. THEOREM. *Let* q *be convex (univalent) in* U, *a*[illegible] *let* α, β *be analytic in* C *and* γ *analytic in a domain* $D \supset q(U)$. *Supp*[illegible] *that*

$$\text{(i)} \qquad \operatorname{Re}\frac{\beta((1+t)zq'(z))\gamma'(q(z))}{\alpha'((1+t)zq'(z)) + \beta'((1+t)zq'(z))\gamma(q(z))} \geqslant 0$$

for all $z \in U$ *and* $t \geqslant 0$

(ii) $Q(z) = zq'(z)(\alpha'(zq'(z)) + \beta'(zq'(z))\gamma(q(z)))$ *is starlike (univalent) in* [illegible]

If p *is analytic in* U *with* $p(0) = q(0)$, $p(U) \subset D$ *and* $\alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z)) \prec \alpha(zq'(z)) + \beta(zq'(z))\gamma(q(z))$ *then* $p(z) \prec q(z)$.

Proof. Without loss of generality we can assume that p and q sati[illegible] the conditions of the theorem on the closed disk $\bar{U}$; if not, then we [illegible] replace $p(z)$ by $p_r(z) = p(rz)$ and $q(z)$ by $q_r(z) = q(rz)$ where $0 < r <$ [illegible] The new functions satisfy the conditions of the theorem on $\bar{U}$ and we wo[illegible] then prove that $p_r(z) \prec q_r(z)$, for all $0 < r < 1$. By letting $r \uparrow 1^-$ we o[illegible] tain $p(z) \prec q(z)$.

The function

$$L(z, t) = \alpha((1+t)zq')z)) + \beta((1+t)zq'(z))\gamma(q(z))$$

is continuously differentiable on $[0, +\infty)$ for all $z \in U$ and analytic [illegible] U for all $t \geqslant 0$. Because $q'(0) \neq 0$, $Q'(0) \neq 0$ from (i), for $z = 0$ we dedu[illegible] that

$$\frac{\partial L}{\partial z}(0, t) = q'(0)(\alpha'(0) + \beta'(0)\gamma(q(0)))\left(1 + t + \frac{\beta(0)\gamma'(q(0))}{\alpha'(0) + \beta'(0)\gamma(q(0))}\right)$$

and

$$\frac{\partial L}{\partial z}(0, t) \neq 0 \text{ for all } t \geqslant 0.$$

Because q is convex in U, a simple calculation combined with (i) yields

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right] > 0 \text{ for all } z \in U \text{ and } t \geqslant 0$$

hence by Lemma 2, $L(z, t)$ is a subordination chain.

If we let
$h(z) = L(z, 0) = \alpha(zq'(z)) + \beta(zq'(z))\gamma(q(z))$ and using (i) for $t = 0$ we obtain $\operatorname{Re}[zh'(z)/Q(z)] > 0$ for all $z \in U$, hence by Lemma 1, h is univalent in U.

Let $\psi(r, s) = \alpha(s) + \beta(s)\gamma(r)$ analytic in the domain $E = D \times C$; then $(q(0), 0) \in E$, $\psi(q(0), 0) = h(0) \in h(U)$ and because $L(z, t)$ is a subordination chain we have

$$\alpha((1+t)\zeta q'(\zeta)) + \beta((1+t)\zeta q'(\zeta))\gamma(q(\zeta)) \notin h(U)$$

for $t \geqslant 0$ and $|\zeta| = 1$. Using Theorem A we conclude that $p(z) \prec q(z)$.

This theorem give us some particular cases presented in the next corollaries.

If we take $\gamma(w) = 1$, $w = C$ then from Theorem we obtain:

COROLLARY 1. *Let q be convex (univalent) in U, α and β be analytic in C and suppose that*
$Q(z) = zq'(z)(\alpha'(zq'(z)) + \beta'(zq'(z)))$ is starlike (univalent) in U. If p is analytic in U with $p(0) = q(0)$, then
$\alpha(zp'(z)) + \beta(zp'(z)) \prec \alpha(zq'(z)) + \beta(zq'(z))$ implies that $p(z) \prec q(z)$.

If we take $\alpha(w) = w$, $\beta(w) = aw^2$, $w \in C$ then from Corollary 1 we obtain:

Example 1.1. Let q be convex (univalent) in U, $a \in C$ and suppose that $Q(z) = zq'(z)(1 + 2azq'(z))$ is starlike (univalent) in U. If p is analytic in U with $p(0) = q(0)$, then
$zp'(z) + a(zp'(z))^2 \prec zq'(z) + a(zq'(z))^2$ implies that $p(z) \prec q(z)$.

If we take in this example $a = 0$ we obtain the well-known result of T. J. Suffridge [6].

This example give us some interesting particular cases if we replace qq by simple convex functions.

Example 1.1.1. If $a, \lambda \in C$ so that $|a\lambda| \leqslant 1/4$ and p is analytic in U with $p(0) = 0$, then
$zp'(z) + a(zp'(z))^2 \prec \lambda z + a(\lambda z)^2$ implies that $p(z) \prec \lambda z$.

Proof. If we take in Example 1.1., $q(z) = \lambda z$, $\lambda \in C$, $z \in U$ we obtain that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\frac{1 + 4a\lambda z}{1 + 2a\lambda z} > \frac{1 - 2|2a\lambda z|}{1 - |2a\lambda z|} \geqslant 0$$

when $|2a\lambda z| \leqslant 1/2$, $z \in U$ and this last inequality is equivalent with $|a\lambda| \leqslant 1/4$.

Example 1.1.2. Let $a, \lambda \in C$ so that $|\lambda| = r_0$ where $r_0 \in (0, 1)$ the root of the equation

$$1 - r - \rho r e^r(3 + 3r + 2\rho r^2 e^r) = 0, \quad \rho = 2|a|.$$

If p is analytic in U, $p(0) = 1$ then

$$zp'(z) + a(zp'(z))^2 \prec \lambda z e^{\lambda z} + a(\lambda z e^{\lambda z})^2 \text{ implies that } p(z) \prec e^{\lambda z}.$$

Proof. We can easily prove that $q(z) = e^{\lambda z}$ is convex in U wh $|\lambda| \leqslant 1$. By letting $\lambda z = \zeta = re^{it}$, $0 \leqslant r < 1$ and $c = 2a = \rho e^{i\varphi}$, $\rho = 2$ a simple calculation yields

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{1}{|1+c\zeta e^{\zeta}|^2\zeta}(1 - r - \rho r e^r(3 + 3r + 2\rho r^2 e^r))$$

and
$$|1 + c\zeta e^{\zeta}|^2 \geqslant (1 - \rho r e^{r \cdot \cos t})^2.$$

Let $\varphi : [0, 1] \to R$, $\varphi(r) = 1 - r - \rho r e^r(3 + 3r + 2\rho r^2 e^r)$. Since $\varphi'(r) < 0$, $\varphi(0) = 1$, $\varphi(1) = -2\rho e(3 + \rho e) < 0$, we conclude that there exi $r_0 \in (0, 1)$ such that $\varphi(r_0) = 0$. Moreover r_0 is the only root of the fun tion φ and for all $r \in [0, r_0)$ we have $\varphi(r) > 0$.

Let $\psi : [0, 1] \to R$, $\psi(r) = 1 - \rho r e^r$. Because $\psi'(r) < 0$, $\psi(0) = \varphi(0) =$ and $\varphi(r) = \psi(r) - r(1 + \rho e^r(2 + 3r + 2\rho r^2 e^r))$ then $\varphi(r) \leqslant \psi(r)$ for $r \in [0, 1]$, we obtain that $\psi(r) > 0$ for all $r \in [0, r_0]$, hence $|1 + c\zeta e^{\zeta}|^2 \geqslant (1 - \rho r e^r)^2 > 0$ and $\operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0$ for all $z \in U$ when $|\lambda| \leqslant$

Example. 1.1.3. Let $a, \lambda \in C$ so that $|\lambda| \leqslant \min\{r_0, r_1\}$ whe $r_0 = \min\{|r| : r^2 + 2(1 + a)r + 1 = 0\}$ and

$$r_1 = \min\{r;\ r > 0,\ r^6 + 2(3|a| - 2)r^5 + (8|a|^2 - 12|a| + 1)r^4 -$$
$$- (8|a|^2 - 16|a| + 1)r^2 + 2(3|a| + 2)r - 1 = 0\}.$$

If p is analytic in U with $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}$$

Proof. We can easily prove that $q(z) = \dfrac{\lambda z}{1 + \lambda z}$ is convex in U wh $|\lambda| \leqslant 1$. By letting $\lambda z = \zeta = re^{it}$, $0 \leqslant r < 1$ and $c = 2a = \rho e^{i\varphi}$, we obta

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{-r^6 - (3\rho - 4)r^5 - (2\rho^2 - 6\rho + 1)r^4 + (2\rho^2 - 8\rho + 1)r^2 - (3\rho + 4)r + 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1|^2}$$

Let $\varphi : [0, 1] \to R$, $\varphi(r) = -r^6 - (3\rho - 4)r^5 - (2\rho^2 - 6\rho + 1)r^4 + (2\rho^2 - 8\rho + 1)r^2 - (3\rho + 4)r + 1$; because $\varphi(0) = 1$, $\varphi(1) = -8\rho <$ there exists $r' \in (0, 1)$ so that $\varphi(r') = 0$; hence $\varphi(r) > 0$ for all $r \in [0,$ where r_1 is the smallest positive root of the equation $\varphi(r) = 0$. A sim calculation yields

$\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1 \neq 0$, for all $z \in U$, when $|\lambda| \leqslant \min\{1, r_0\}$, hence $\operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0$ for all $z \in U$ when $|\lambda| \leqslant \min\{r_0, r$

Remarks. From the proof of Example 1.1.3. we observed that this result is not sharp; better upper bounds may be found in the case when $a \in R$.

Case 1. Let $0 < a < 1$ and $\lambda \in C$ with

$$|\lambda| \leqslant \min\{1 + a - \sqrt{a^2 + 2a},\ r_*\} \text{ where}$$

$$r^* = \min\{r : r > 0,\ r^4 - 2(3a + 2)r^3 + 2(4a^2 + 8a + 1)r^2 - 2(3a + 2)r - 1 = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}.$$

Proof. In the case $0 < a < 1$ we deduce that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2 - 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1^3|}(-r^4 + 2(3a + 2)r^3 - 2(4a^2 + 8a + 1)r^2 + 2(3a + 2)r + 1)$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$ and the right-hand term is defined for all $z \in U$ when $|\lambda| \leqslant r_0 = 1 + a - \sqrt{a^2 + 2a} \in (0, 1)$. If we let $\psi : [0,1] \to R$ $\psi(r) = -r^4 + 2(3a + 2)r^3 - 2(4a^2 + 8a + 1)r^2 + 2(3a + 2)r - 1$ we have

$\psi(0) = -1$, hence $\operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0$ for all $z \in U$ if $|\lambda| \leqslant \min\{r_0, r_*\}$.

Case 2. Let $a \geqslant 1$ and $\lambda \in C$ with

$$|\lambda| \leqslant \min\{1 + a - \sqrt{a^2 + 2a};\ r_*\} \text{ where}$$

$$r_* = \min\{r : r > 0,\ r^4 - 2(3a + 2)r^3 + 2(4a^2 + 6a + 3)r^2 - 2(3a + 2)r + 1 = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + a\left(\frac{\lambda z}{(1 + \lambda z)^3}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}.$$

Proof In the case $a \geqslant 1$ we deduce by using the proof of Example 1.1.3., that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^3 - 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1|^2}(-r^4 + 2(3a + 2)r^3 - 2(4a^2 + 6a + 3)r^2 - 2(3a + 2)r - 1),$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$ and as in the Case 1 we deduce the above result.

When $a = 1$ we can easily show the following result.

Case 2'. If $|\lambda| \leqslant 3 - 2\sqrt{2}$ and p is analytic in U, $p(0) = 0$, then

$$zp'(z) + (zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + \left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2$$

implies that $p(z) \prec \dfrac{\lambda z}{1 + \lambda z}$.

Case **3**. Let $-2/3 < a < 0$ and $\lambda \in C$ with $|\lambda| \leqslant \min\{1, r_0, r_*\}$ where $r_0 = \min\{|r| : r^2 + 2(1 + a)r + 1 = 0\}$ and

$$r^* = \min\{|r| : r^4 - 2(3a + 2)r^3 + 2(4a^2 + 8a + 1)r^2 - 2(3a + 2)r + 1 = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2$$

implies that $p(z) \prec \dfrac{\lambda z}{1 + \lambda z}$.

Proof. If $-2/3 < a < 0$ we can easily show that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2 - 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1|^2}(-r^4 + 2(3a + 2)r^3 -$$
$$- 2(4a^2 + 8a + 1)r^2 + 2(3a + 2)r - 1)$$

where $\zeta = \lambda z = re^{it}, 0 \leqslant r < 1$. We have $\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1 \neq 0$ for all $z \in U$ when $|\lambda| \leqslant \min\{1, r_0\}$ and the right-hand term is positive when $|\lambda| \leqslant \min\{1, r_0, r_*\}$.

Case **4**. Let $a \leqslant -2/3$ and $\lambda \in C$ with $|\lambda| \leqslant \min\{1, r_0, r_*\}$ where $r_0 = \min\{|r| : r^2 + 2(1 + a)r + 1 = 0\}$ and

$$r_* = \min\{|r| : r^4 + 2(3a + 2)r^3 + 2(4a^2 + 8a + 1)r^2 + 2(3a + 2)r + 1 = 0\}$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}.$$

Proof. As in the Case 3, we can show that if $a \leqslant -2/3$

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2 - 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1|^2}(-r^4 - 2(3a + 2)r^3 -$$
$$- 2(4a^2 + 8a + 1)r^2 - 2(3a + 2)r - 1)$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$ and $c = 2a$ and the right-hand term is positive for all $z \in U$ when $|\lambda| \leqslant \min\{1, r_0, r_*\}$.

When $a = -2/3$ we can easily deduce the following result:
Case 4'. If $|\lambda| \leqslant (4 - \sqrt{7})/3$ and p is analytic in U, $p(0) = 0$ then

$$zp'(z) - \frac{2}{3}(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} - \frac{2}{3}\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2$$

imlpies that $p(z) \prec \frac{\lambda z}{1 + \lambda z}$.

Example 1.1.4. Let $a, \lambda \in C$ with $|\lambda| \leqslant r_0$, for $a = -1/2$ and $|\lambda| \leqslant \min\left\{r_0, \frac{1}{|2a + 1|}\right\}$ for $a \neq -1/2$, where

$$r_0 = \min\{r : r > 0, -(8|a|^2 + 6|a| + 1)r^3 + (8|a|^2 - 12|a| + 1)r^2 - - 3(2|a| + 1)r + 1 = 0\}.$$

If p is analytic in U, $p(0) = \log 1 = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{1 + \lambda z} + a\left(\frac{\lambda z}{1 + \lambda z}\right)^2 \text{ implies that } p(z) \prec \log(1 + \lambda z).$$

Proof. If $|\lambda| \leqslant 1$ the function $q(z) = \log(1 + \lambda z)$, $\log 1 = 0$ is convex (univalent) in U; if we let $c = 2a = \rho e^{i\varphi}$ and $\lambda z = \zeta = re^{it}$, $0 \leqslant r < 1$, by using Example 1.1. we deduce

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{(-2\rho^2 - 3\rho - 1)r^3 + (2\rho^2 - 6\rho + 1)r^2 - (3\rho + 3)r + 1}{|1 + \zeta|^2|1 + \zeta + c\zeta|^2}.$$

The right-hand term is defined and positive when $|\lambda| \leqslant r_0$ in the case $a = -1/2$ and for $|\lambda| \leqslant \min\left\{r_0, \frac{1}{|2a + 1|}\right\}$ in the case $a \neq -1/2$ for all $z \in U$, and using Example 1.1. we obtain the above result.

Example 1.2. Let q be convex (univalent) in U and $a \in C\setminus\{-1\}$ so that $Q(z) = zq'(z)(1 + ae^{zq'(z)})$ is stalike in U. If p is analytic in U, $p(0) = q(0)$, then

$$zp'(z) + ae^{zq'(z)} \prec zq'(z) + ae^{zq'(z)} \text{ implies that } p(z) \prec q(z).$$

Proof. If we take, in Corollary 1, $\alpha(w) = w$ and $\beta(w) = ae^w$, $w \in C$, then we obtain the above result.

Example 1.2.1. Let $a \in C\setminus\{-1\}$ and $\lambda \in C$ so that $|\lambda| \leqslant \min\{r_0, r^*\}$ where $r_0 = \min\{|r| : 1 + ae^r = 0\}$ and

$$r_* = \min\{r : r > 0, 1 - 2|a|e^r - |a|re^r - |a|^2re^{2r} + |a|^2e^{-2r} = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + ae^{zp'(z)} \prec \lambda z + ae^{\lambda z} \text{ implies that } p(z) \prec \lambda z.$$

Proof. We use Example 1.2. in the case $q(z) = \lambda z$, $z \in U$. The funct $Q(z) = \lambda z(1 + ae^{\lambda z})$ is starlike in U if

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \frac{a^{-1} + e\zeta + \zeta e\zeta}{a^{-1} + e\zeta} > \frac{\rho^2 - 2\rho e^r - \rho r e^r - r e^{2r} + e^{-2r}}{|a^{-1} + e\zeta|^2} \geqslant 0,$$

where $a^{-1} = \rho e^{i\varphi}$ and $\zeta = \lambda z = re^{it}$, $r \geqslant 0$. We can easily show that inequality is satisfied under the conditions of the example.

If we take, in Corollary 1, $\alpha(w) = w$ and $\beta(w) = aw^n$, $w \in C$ obtain:

Example 1.3. Let q be convex (univalent) in U, $a \in C$, $n \in N^*$ suppose that $Q(z) = zq'(z)(1 + an(zq'(z))^{n-1})$ is starlike in U. If p is an tic in U, $p(0) = q(0)$, then

$$zp'(z) + a(zp'(z))^n \prec zq'(z) + a(zq'(z))^n \text{ implies that } p(z) \prec q(z).$$

Remark. If $n = 1$ or $a = 0$, this example yields the well-known re of T. J. Suffridge [6], and for $n = 2$ we obtain the Example 1.1.

Example 1.3.1. Let $a \in C$ and $\lambda \in C$ with $|\lambda| \leqslant (n^2|a|)^{\frac{1}{1-n}}$, is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^n \prec \lambda z + a(\lambda z)^n \text{ implies that } p(z) \prec \lambda z.$$

Proof. If we let $\zeta = \lambda z = re^{it}$ and $a = \rho e^{i\varphi}$ we obtain, for $q(z) =$ $z \in U$ that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{n^2\rho^2 r^{2(n-1)} - n(n+1)\rho r^{n-1} + 1}{|1 + na\zeta^{n-1}|^2}$$

and if $|\lambda| \leqslant (n^2|a|)^{\frac{1}{n-1}}$ we can prove that the right-hand term is posi for all $z \in U$.

Remark. For $a = 0$ this result holds for all $\lambda \in C$, and for n we obtain the Example 1.1.1.

Example 1.3.2. Let $a \in C$ and $\lambda \in C$ so that $|\lambda| \leqslant r_0$ where r $\in (0, 1]$ is the root of equation

$$1 - r - n|a|r^{n-1}e^{(n-1)r}((n+1)(r+1) + n^2|a|r^n e^{(n-1)r}) = 0.$$

If p is analytic in U, $p(0) = 1$, then

$$zp'(z) + a(zp'(z))^n \prec \lambda z e^{\lambda z} + a(\lambda z e^{\lambda z})^n \text{ implies that } p(z) \prec e^{\lambda z}.$$

Proof. The function $q(z) = e^{\lambda z}$, $|\lambda| \leqslant 1$ is convex (univalent) in If we let $2a = \rho e^{i\varphi}$ and $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$, then

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{\varphi(r)}{|1 + na\zeta^{n-1} e^{(n-1)\zeta}|^2} \text{ where}$$

$$\varphi(r) = 1 - r - \frac{n}{2}\rho r^{n-1}e^{(n-1)r}((n+1) + (n+1)r + \frac{n^2}{2}\rho r^n e^{(n-1)r}).$$

A simple calculation yields

$$|1 + na\zeta^{n-1}e^{(n-1)\zeta}|^2 \geqslant \left(1 - \frac{n}{2}\,\rho r^{n-1}e^{(n-1)r\cdot\cos}\right)^2 =: \theta(r)$$

and if we let $\psi(r) = 1 - \frac{n}{2}\,\rho r^{n-1}e^{(n-1)r}$, then $\varphi(r) \leqslant \psi(r)$ for all $0 \leqslant r < 1$. Because $\varphi'(r) < 0$, $0 \leqslant r < 1$, $\varphi(1) \leqslant 0$ and $\varphi(0) = 1 > 0$, there exists $r_0 \in (0, 1]$ so that $\varphi(r_0) = 0$ and for all $r \in [0, r_0)$ we have

$$\psi(r) \geqslant \varphi(r) > 0. \text{ If } r \in [0, r_0) \text{ then } \theta(r) \geqslant \psi^2(r) > 0$$

and by using Example 1.3. we obtain the above result.

Remark. For $n = 2$ we obtain the Example 1.1.1. and for $a = 0$ this result holds for all $\lambda \in C$.

REFERENCES

1. S. S. Miller, P. T. Mocanu, *On some classes of first-order differential subordinations*, (to appear).
2. S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), 151–171.
3. T. Bulboacă, *On some classes of differential subordinations* (I), STUDIA UNIV. Babeş-Bolyai, Math. XXXI, 1, 1986, 45–51.
4. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J., 1(1952), 169–185.
5. Ch. Pommerenke, „*Univalent Functions*", Vanderhoeck and Ruprecht, Gottingen, 1975.
6. T. J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J., 37 (1970), 775–777.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

A FIXED POINT THEOREM FOR DECREASING FUNCTIONS

MARIAN DEACONESCU*

Recevied: July 15, 1983

It is well-known that an increasing funcţiont on a complete lattice has at least a fixed point (see [1]). An analogous result for decreasing functions does not hold. Indeed, if B is a complete boolean lattice and $f: B \to B$ where $f(x) = \bar{x}$ is the complement of x for every $x \in B$, f is a decreasing funciton without fixed points. Consequently, we must give other conditions for the lattice or (and) for the function to assure the existence of a fixed point. The aim of this note is to give sufficient conditions for a decreasing function $f: L \to L$ where L is a chain to have a fixed point.

In what follows, L will be a chain and $f: L \to L$ will be a decreasing function. The fixed point set of f will be denoted by F_f. If $a, b \in L$ and $a \leqslant b$, we will denote the set $\{x \in L \mid a < x < b\}$ by (a, b). The chain L is dense if $(a, b) \neq \Phi$ for every $a, b \in L$ with $a < b$. We also consider the two subsets of $L: D = \{x \in L \mid x \leqslant f(x)\}$ and $I = \{x \in L \mid x \geqslant f(x)\}$. We have obviously $F_f = D \cap I$ and $L = D \cup I$.

LEMMA 1. *Let L be a chain and $f: L \to L$ a decreasing function. Then*

(i) *f has at most one fixed point.*

(ii) *$x \leqslant y$ for every $x \in D$, $y \in I$.*

(iii) *$f(D) \subseteq I$ and $f(I) \subseteq D$.*

Proof. (i) Let x, y be fixed points, say $x \leqslant y$. Then $f(y) \leqslant f(x)$, that is $y \leqslant x$, hence $x = y$.

(ii) Suppose $x > y$ for some $x \in D$ and $y \in I$. Then $f(x) \leqslant f(y)$ and $x \leqslant f(x)$, $f(y) \leqslant y$, hence $x \leqslant y$ by transitivity; but this contradicts the hypothesis.

(iii) Obvious.

Remark. If, moreover, f is surjective, then $f(D) = I$ and $f(I) = D$.

Proof. To show, e.g., that $f(D) = I$, we take $x \in I$ and prove that $x \in f(D)$. But $x = f(y)$ for some $y \in L$. If $y \in D$ then $x \in f(D)$. If $y \in I$, then $x \in f(I) \subseteq D$, hence $x \in D \cap I$ is a fixed point, therefore $x = f(x) \in f(D)$.

LEMMA 2. *Let L be a complete chain and $f: L \to L$ a decreasing surjective function. Set $a = \sup D$ and $b = \inf I$. Then*

(i) *$a \leqslant b$ and $(a, b) \neq \Phi$.*

(ii) *$a = f(b) \in D$ and $b = f(a) \in I$.*

* *General School nr. 3, 2700 Deva, Romania*

Proof. (i) $a \leqslant b$ follows from Lemma 1, (ii). Now suppose $a < c < b$. If $c \in D$ then $a \neq \sup D$ and if $c \in I$ then $b \neq \inf I$, contradiction.

(ii) From $b \leqslant x$ for every $x \in I$, it follows that $f(x) \leqslant f(b)$ for every $x \in I$, hence $y \leqslant f(b)$ for every $y \in D$ because $f(I) = D$. As $a = \sup D$ it follows that $a \leqslant f(b)$ and similarly $f(a) \leqslant b$. On the other hand $a \leqslant b$ implies $f(b) \leqslant f(a)$, therefore $a \leqslant f(b) \leqslant f(a) \leqslant b$, which shows that $a \in D$ and $b \in I$. Moreover, $f(b) \in D$ and since $a = \sup D$ it follows that $a = f(b)$ and similarly $b = f(a)$.

We can now state the main result of this note:

THEOREM. *Let L be a complete chain and $f: L \to L$ a decreasing surjective function. Set $a = \sup D$ and $b = \inf I$. Then f has a fixed point if and only if $a = b$, in which case the fixed point is $a = b$.*

Proof. If $a = b$ then $a = f(b) = f(a)$ by Lemma 2, (ii). Conversely, if c is a fixed point then $c \in D \cap I$, hence $c \leqslant a \leqslant b \leqslant c$, therefore $a = b = c$.

COROLLARY. *Let L be a complete dense chain and $f: L \to L$ a decreasing surjective function. Then f has an unique fixed point.*

Proof. This follows immediately from Lemma 2 (i) and the Theorem: $a = b$ by the density assumption and Lemma 2 (i), so $a = b$ is the unique fixed point of f by the Theorem.

REFERENCES

1. A. Tarski, *A lattice-theoretical fixed point theorem and its applications*, Pacific J. Math. 5 (1955), 161–170.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

A GENERALIZATION OF A COINCIDENCE THEOREM OF HADŽI

DO HONG TAN*

Received: June 14, 1983

ABSTRACT. — The purpose of this note is to generalize a coincidence theorem of Hadzic in [1] and to show that the remark in [2] about the mentioned theorem is not true.

1. In the sequel we shall use the following notations. For a metr space (X, d), $CB(Y)$ $(Cl(Y))$ stands for the family of all nonempty clos bounded (closed, resp.) of $Y \subset X$, $d(x, Y)$ — the nearest distance fro a point x to a set Y, $H(Y, Z)$ — the Hausdorff distance between tw sets Y and Z, N — the set of all natural numbers.

In [1] Hadžić has proved the following

THEOREM H. *Let X be a complete metric space, S and T continu mappings from X into itself, A a closed mapping from X into $CB(SX \cap T$ such that $ATx = TAx$, $ASx = SAx$ for every $x \in X$ and*

$H(Ax, Ay) \leqslant q\, d(Sx, Ty)$ for every $x, y \in X$, where $q \in (0, 1)$. Th there exists a sequence $\{x_n\}$ such that

1) *For every $n \in N$, $Sx_{2n+1} \in Ax_{2n}$, $Tx_{2n} \in Ax_{2n-1}$,*

2) *There exists $z = \lim Tx_{2n} = \lim Sx_{2n+1}$,*

3) *$Tz \in Az$, $Sz \in Az$.*

Theorem H can be generalized as follows

THEOREM 1. *Let X be a complete metric space, S, T continuous ma pings from X into itself, A, B, closed mappings from X into $Cl(X)$. Supp that*

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$, $SA = AS$, $TB = BT$,

(ii) *There is an upper semicontinuous from the right function $q: [0, \infty) \to [0, 1)$ such that*

$$H(Ax, By) \leqslant q(d(Sx, Ty)) \cdot \max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\}$$

for every $x, y \in X$.

Then there exists a $z \in X$ such that $Sz \in Az$, $Tz \in Bz$.

---

* *Current address: (1983—1985): Instiytut Matematyczny PAN, ul. Sniadeckich 8, skr. pocztowa 137, 00- Warszawa, Poland*

PROOF. Take $x_0 \in X$ and put $y_0 = Sx_0$ then fix $r > d(y_0, Ax_0)$ and choose $y_1 \in Ax_0$ so that $d(y_0, y_1) < r \cdot By$ (i), there is an $x_1 \in X$ with $y_1 = Tx_1$. From (ii) we have

$$d(y_1, Bx_1) \leqslant H(Ax_0, Bx_1) \leqslant q(d(y_0, y_1)) \cdot \max\Big\{d(y_0, y_1), d(y_0, Ax_0),$$

$$d(y_1, Bx_1), \frac{1}{2}[d(y_0, Bx_1) + d(y_1, A_0)]\Big\}.$$

In view of $y_1 \in Ax_0$, $d(y_0, Bx_1) \leqslant d(y_0, y_1) + d(y_1, Bx_1)$ and $q(d(y_0, y_1))<1$ from this we get $d(y_1, Bx_1) \leqslant q(d(y_0, y_1))d)y_0, y_1)$ and hence

$$d(y_1, Bx_1) < \min\{d(y_0, y_1), q(d(y_0, y_1))r\} = t.$$

Select $y_2 \in Bx_1$ so that $d(y_1, y_2) < t$. By (i), there is an $x_2 \in X$ with $Sx_2 = y_2$. Analogously, there is an $y_3 \in Ax_2$ with

$$d(y_2, y_3) < \min\{d(y_1, y_2), q(d(y_1, y_2))q(d(y_0, y_1))r\}$$

and $y_3 = Tx_3$.

Generally, we can construct two sequences $\{x_n\}$, $\{y_n\}$ with the following properties

$$y_{2n} = Sx_{2n} \in Bx_{2n-1}, \; y_{2n+1} = Tx_{2n+1} \in Ax_{2n}, \tag{1}$$

$$c_{n+1} < \min\{c_n, q(c_n) \dots q(c_0)r\}, \text{ where } c_n = d(y_n, y_{n+1}). \tag{2}$$

From (2), $c_n \to c \geqslant 0$. By the upper semicontinuity of q, $\overline{\lim}\, q(c_n) \leqslant q(c)$. Fix k with $q(c) < k < 1$, there is an $n_0 \in N$ such that $q(c_n) \leqslant k$ for $n \geqslant n_0$. Hence, for $n \geqslant n_0$ we have from (2) $c_{n+1} \leqslant k^n R$, where $R = k^{-n_0} q(c_{n_0}) \dots q(c_0)r$. Since $k < 1$, $\{y_n\}$ is a Cauchy sequence and hence $y_n \to z$. By continuity of S and T, $Ty_{2n} \to Tz$, $Sy_{2n+1} \to Sz$. From (i) and (1) we have $Ty_{2n} \in BTx_{2n-1} = By_{2n-1}$, $Sy_{2n+1} \in ASx_{2n} = Ay_{2n}$. By closedness of A and B we get $Tz \in Bz$, $Sz \in Az$. The proof is complete.

2. In [2] Sanderson claims that „the truth of Theorem H is in doubt as the proof is incomplete". But Theorem 1 shows that Theorem H is true and it seems to me that the proof in [1] is standard and clear enough. Moreover, the counter-example in [2]:

$$X = 1, \dots, 2^{-n}, \dots, 0\}, \; S = T = \text{identity}, \; A(0) = 1, \; A(1) = A(2^{-n}) \equiv X$$

is not true. In fact, A is not contractive, for

$$H\left(A(0), A\left(\frac{1}{2}\right)\right) = H(1, X) = 1 > d\left(0, \frac{1}{2}\right).$$

Besides, A is not closed, for $2^{-n} \in A(2^{-n}) \equiv X$, but $0 \notin A(0) = 1$. So this counter-example has no relations with Theorem H.

3. The following result shows that closedness of A and B can be replaced by commutativity of S and T. Namely, we have

THEOREM 2. *Let X be complete, S, T continuous on X, A and B multi-valued mappings from X into $Cl(X)$. Suppose that each of S, T commutes with the there others, $A(X) \cup B(X) \subset ST(X)$ and Condition (ii) in Theorem 1 is satisfied. Then the conclusion of Theorem 1 still holds.*

Proof. Denote $U = ST$, take $x_0 \in X$, put $y_0 = Ux_0$, fix $r > d(y_0, ATx_0)$, choose $y_1 \in ATx_0$ with $d(y_0, y_1) < r$, then select $x_1 \in X$ with $y_1 = Ux_1$. From (ii) we have

$$d(y_1, BSx_1) \leqslant H(ATx_0, BSx_1) \leqslant q(d(y_0, y_1)) \max\Big\{d(y_0, y_1),\ d(y_1, BSx_1),\ \frac{1}{2} d(y_0, BSx_1)\Big\} = q(d(y_0, y_1))d(y_0, y_1).$$

Choose $y_2 \in BSx_1$ so that $d(y_1, y_2) < \min\{d(y_0, y_1), q(d(y_0, y_1))r\}$ then select x_2 with $y_2 = Ux_2$.

Repeat this process, we get two sequences $\{x_n\}$, $\{y_n\}$ with

$$y_{2n} = Ux_{2n} \in BSx_{2n-1},\ y_{2n+1} = Ux_{2n+1} \in ATx_{2n} \tag{3}$$

and for which (2) still holds. So $y_n \to y \in X$. Now by (ii), we have

$$d(Uy, ATy) \leqslant d(Uy, Uy_{2n}) + d(Uy_{2n}, ATy) \leqslant d(Uy, Uy_{2n}) + H(BSy_{2n-1}, ATy) \leqslant d(Uy, Uy_{2n}) + q(d(Uy, Uy_{2n-1})) \max\Big\{\{d(Uy, Uy_{2n-1}), d(Uy, ATy), d(Uy_{2n-1}, Uy_{2n}), \frac{1}{2}[d(Uy, Uy_{2n}) + d(Uy_{2n-1}, ATy)]\Big\}$$

Since $d(Uy, Uy_{2n-1}) \to 0$ and $q(0) < 1$, we have $q(d(Uy, Uy_{2n-1})) \leqslant k < 1$ for n large enough. From this by letting $n \to \infty$ we get $d(Uy, ATy) \leqslant kd(Uy, ATy)$. This shows that $Uy \in ATy$ in view of closedness of ATy. Similarly, we have $Uy \in BSy$. Putting $z = Uy$, from this we get the desired result: $Sz \in Az$, $Tz \in Bz$.

REMARK When $S = T$=the identity, Theorem 2 reduces to Theorem 1 in [3].

REFERENCES

1. O. Hadzic. *A coincidence theorem for multivalued mappings in metric spaces*, Studia Univ Babeş-Bolyai, Math. 26, 4 (1981), 65–67.
2. D. E. Sanderson, 54038, Zentralblatt fur Math. 491 (1983), 347.
3. D. H. Tan, D. T. Nhan, *Common fixed points of two mappings of contractive type*, Acta Math. Vietnam. 5, 1 (1980), 150–160.

A FIXED POINT THEOREM FOR MULTI-VALUED FUNCTIONS OF CONTRACTION TYPES WITHOUT HYPOTHESIS OF CONTINUITY

NGUYEN HUY VIET*

Received: July 15, 1983

ABSTRACT. — In this paper we present a fixed point theorem for multi-valued functions of contraction type. The class of all multivalued functions which satisfy our condition is more large than those classes considered in [1], [2], [4], and [6].

Definition and notations. In the sequel we shall use the following notations. For a metric space X by $CL(X)$ we denote the class of all non-empty closed subsets of X. By H we denote the Hausdorff distance on $CL(X)$ generated by the metric

$$H(A, B) = \max \{\sup_{b \in B} \inf_{a \in A} d(a, b), \sup_{a \in A} \inf_{b \in B} d(a, b)\}$$

for all $A, B \in CL(X)$

and, as usual, $d(x, A) = \inf \{d(x, y), y \in A\}$

Let $F : X \to CL(X)$ be a multi-valued function.

DEFINITION. A sequence $\{x_n, n = 0, 1, 2, \ldots\}$ is called an orbit of F at x iff $x_0 = x$, $x_{n+1} \in Fx$, $n = 0, 1, 2, \ldots$

THEOREM. *Let X be a metric space; $F : X \to CL(X)$ be a function satisfying the following conditions:*

i) There is an orbit of F at a point x_0, containing two successive convergent subsequences

$$x_{ni} \xrightarrow[i\to\infty]{} x_*, \qquad x_{n_i+1} \xrightarrow[i\to\infty]{} x_*$$

ii) There exist real numbers q_1 and q_2:

$$q_2 < 1 \text{ such that}$$

$H(Fx, Fy) \leqslant q_1 d(x, y) + q_2 \max \{d(x, Fx) + d(y, Fy), d(x, Fy) + d(y, Fx)\}$ *for all x, y in X.*

Then
$$x_* \in Fx_*$$

Proof. Suppose $x_* \notin Fx_*$. Since Fx_* is nonempty and closed we have $d(x_*, Fx_*) = r > 0$. From the condition *i)* of the theorem it follows that for every $\varepsilon > 0$, there is a non-negative integer $i(\varepsilon)$ such that for all

* *Institute of Mathematics, Ha-noi, Viet-nam*
Present address: Faculty of Mechanics-Mathematics, Moscow, MGU. B. 234, URSS

$i \geqslant i(\varepsilon)$ both x_{n_i} and x_{n_i+1} belong to the opên ball centered at x_* of radius $\varepsilon : x_{n_i} \in 0 \ (x_*, \varepsilon)$, $x_{n_i+1} \in 0 \ (x_*, \varepsilon)$.

And hence for all $i \geqslant i(\varepsilon)$ we have

$$d(x_{n_i}, x_{n_i+1}) \leqslant d(x_{n_i}, x_*) + d(x_*, x_{n_i+1}) \leqslant 2\varepsilon$$

From here we have:

$$d(x_{n_i}, Fx_{n_i}) \geqslant 2 \qquad (1)$$

From the definition of the distance between a point and a set in metric space, it follows:

$$d(x_{n_i}, Fx_*) \leqslant d(x_{n_i}, x_*) + d(x_*, Fx_*)$$

And thus, for all $i \geqslant i(\varepsilon)$ we have

$$d(x_{n_i}, Fx_*) \leqslant \varepsilon + r \qquad (2)$$

From the condition *ii)* of the theorem and using (1) and (2) have for $i \geqslant i(\varepsilon)$

$$H(Fx_{n_i}, Fx_*) \leqslant q_1\varepsilon + q_2 \max \{2\varepsilon + r, (\varepsilon + r) + \varepsilon\}$$

Hence

$$H(Fx_{n_i}, Fx_*) \leqslant q_1\varepsilon + q_2(r + 2\varepsilon) \qquad (3)$$

In the other hand

$$d(x_*, Fx_*) \leqslant d(x_*, X_{n_i+1}) + d(x_{n_i+1}, Fx_*)$$

From this for $i \geqslant i(\varepsilon)$ we have

$$d(x_{n_i+1}, Fx_*) \geqslant r - \varepsilon \qquad (4)$$

Since $q_2 < 1$, it is clear that (3) contradicts (4) when ε is chioseen sufficiently small and $i \geqslant i(\varepsilon)$

Thus $x_* \in Fx_*$

Remark 1. In the condition *ii)* of the theorem q_1 is arbitrary and q_2 can be more than $\frac{1}{2}$.

Remark 2. In the proof of the theorem the condition *ii)* need be fulfilled only for all pairs of tipe (x_{n_i}, x_*)

By considering the simple-valed function we have the following.

COROLLARY *Let X be a metric space, $f: X \to X$ be a mapping satisfying the following conditions:*

i) There is an orbit of f at a point x_0 containing two succesive convergent subsequences

$$x_n \xrightarrow[i \to \infty]{} x_*, \quad x_{n_i+1} \xrightarrow[i \to \infty]{} x_*$$

ii) There exist real numbers q_1 and q_2, $q_2 < 1$ such that

$$d(f(x_{n_i}), f(x_*)) \leqslant q_1 d(x_{n_i}, x_*) + q_2 \max \{d(x_{n_i}, f(x_{n_i})) +$$

$$+ d(x_*, f(x_*)), d(x_{n_i}, f(x_*)) + d(x_*, f(x_{n_i}))\} \text{ for all integers } i.$$

Then x_* is a fixed point of f.

The following example shows that the theorem does not hold if q_2 is replaced by 1.

Example. $X = \left\{-\frac{1}{2^n}, n = 0, 1, 2, \ldots\right\} \cup \{0\} \cup \{1\}$,

$$f: X \to X \text{ defined by } f\left(-\frac{1}{2^n}\right) = -\frac{1}{2^{n+1}},$$

$n = 0, 1, 2, \ldots, f(0) = 1 : f(1) = -1.$

The reader can verify the fulfilment of all conditions of the theorem with $q_1 = q_2 = 1$ and f has no fixed point.

REFERENCES

1. Ciric L., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. 45, 2 (1974), 264—273.
2. Nadler S., *Multi-valued contractive mappings*, Pacific J. Math., 30 (1969) 475—488.
3. Reich S., *Kannan's fixed point theorems*, Boll. un. Math. I tal. (1971), 1—11.
4. Smithson R., *Fixed points for contractive multifunctions*, Proc. Amer. Math. Soc., 21 (1971), 192—194.
5. Do Hong Tan, *On the contraction principle*, Acta, Math. Viet. 4, 2 (1979), 88—102.
6. R. Kannan, *Some results on fixed points* II, Amer. Math. Monthly, v. 76 (1969), 405—407.

SOME NONNEGATIVE DETERMINANTS IN INNER PRODUCT SPACES

ALEXANDER ABIAN*

Received: July 30, 1983

ABSTRACT. — Cauchy — Schwarz inequality is generalized in the paper, under the form of a n-order dererminant.

A fertile source of inequalities is provided by the notion of the inner product of a vector with itself in a finite dimensional vector space over the field of the real numbers **R**. Let u and v be vectors in such an n-dimensional vector space $\mathbf{R}^n$. Thus, u is identified with an n-tuple of real numbers, say, $(a_1, a_2, \ldots, a_n)$ and v is identified with an n-tuple of real numbers, say $(b_1, b_2, \ldots, b_n)$. Denoting the *inner product* of u and v by $\langle u, v\rangle$, we have according to the usual definition:

$$\langle u, v\rangle = a_1b_1 + a_2b_2 + \ldots + a_nb_n \qquad (1)$$

Replacing in (1) the vector v by u, we have

$$\langle u, u\rangle = a_1^2 + a_2^2 + \ldots + a_n^2 \qquad (2)$$

Since the right side of the equality sign in (2) is a sum of squares of the elements of **R** (the set of all real numbers), we have:

$$\langle u, u\rangle \geqslant 0 \quad \textit{for every vector } u \textit{ in } \mathbf{R}^n \qquad (3)$$

Obviously, (3) is an inequality and as shown below, it is the motivating factor behind many inequalities. For instance, let us take instead of u the sum $v + w$ of vectors v and w. But then we have:

$$\langle v + w, v + w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \qquad (4)$$

which by (2) yields the following inequality:

$$\langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \geqslant 0 \textit{ for every } v, w \textit{ in } \mathbf{R}^n \qquad (5)$$

The inequality (5) itself can be rewritten in various ways, each giving rise to an inequality. Thus, from (5) the following two inequalities follow immediately:

$$\langle v, v\rangle + \langle w, w\rangle \geqslant -2\langle v, w\rangle \textit{ for every } v, w \textit{ in } \mathbf{R}^n \qquad (6)$$

and

$$\langle v, v\rangle + \langle v, w\rangle \geqslant -\langle v, w\rangle - \langle w, w\rangle \textit{ for every } v, w \textit{ in } \mathbf{R}^n \qquad (7)$$

True, that (5), (6), (7) are inequalities, however, most probably they are neither too interesting nor too useful. For instance, neither seems to be as interesting or as useful as the Cauchy-Schwarz inequality. A reason

* *Department of Mathematics, Iowa University, Ames, Iowa 50011, U.S.A.*

for this perhaps lies in the fact that $v + w$ is a quite trivial linear combination of v and w and in a way one should not expect to obtain an interesting inequality by merely replacing u in (3) by $v + w$.

Let us now consider a less trivial linear combination involving v and w. For instance, let us consider a linear combination involving v and w which is also orthogonal to v. In particular, let us consider the linear combination of v and w given by:

$$\langle v, v\rangle w - \langle v, w\rangle v \tag{8}$$

which is orthogonal to v. Indeed, it is trivial to verify that the inner product of $\langle v, v\rangle w - \langle v, w\rangle v$ with v is 0. Now, let us replace u in (3) by (8). Thus,

$$\langle (\langle v, v\rangle w - \langle v, w\rangle v), (\langle v, v\rangle w - \langle v, w\rangle v) \rangle \geqslant 0 \tag{9}$$

Applying the distributivity law to the above inner product and observing that $r\langle v, w\rangle = \langle w, v\rangle r$ for every v, w in $\mathbf{R}^n$ and every r in $\mathbf{R}$, we obtain, after obvious simplification:

$$\langle v, v\rangle\langle v, v\rangle\langle w, w\rangle - \langle v, v\rangle\langle v, w\rangle\langle v, w\rangle \geqslant 0 \tag{10}$$

If $v \neq 0$ then $\langle v, v\rangle \neq 0$ and therefore upon dividing both sides of the inequality (10) by $\langle v, v\rangle$ we have:

$$\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle\langle v, w\rangle \geqslant 0 \text{ for every } v, w \text{ in } \mathbf{R}^n \tag{11}$$

regardless whether $v = 0$ or $v \neq 0$.

Inequality (11) is quite interesting and quite useful. Indeed, it is the Cauchy-Schwarz inequality. Thus, starting with an interesting linear combination (8) of v and w and using it in an obvious (but very basic) inequality (3), we obtained a trather interesting inequality (11).

We may rewrite inequality (11) in the determinant form as follows:

$$\begin{vmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle v, w\rangle & \langle w, w\rangle \end{vmatrix} \geqslant 0 \qquad \text{for every } v, w \text{ in } \mathbf{R}^n \tag{12}$$

Thus, the Cauchy-Schwarz inequality lends itself to be expressed as a nonnegative determinant.

Looking at $\langle x, y\rangle$ as an entry in a matrix indicating the entry at the x-row and y-column, we rewrite (12) in the following form:

$$\begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix} \geqslant 0 \qquad \text{for every } x, y \text{ in } \mathbf{R}^n \tag{13}$$

An immediate generalization of (13) to any finite number of vectors is known in the literature as the Gramian of these vectors. Thus, for vectors x, y, z the inequality corresponding to (13) is

$$\begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle & \langle x, z\rangle \\ \langle y, x\rangle & \langle y, y\rangle & \langle y, z\rangle \\ \langle z, x\rangle & \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} \geqslant 0 \qquad \text{for every } x, y, z \text{ in } \mathbf{R}^n \tag{14}$$

Clearly, (14) is another example of nonnegative determinants.

Gramian type nonnegative determinants are known in the literature.

Below, pursuing our approach of considering the inner product with itself of an interesting linear combination of vectors, we obtain a new class of non-negative determinants.

Let us observe that in the case of vectors v and w the nonnegative determinant (12) is obtained as a result of considering the inner product with itself of a nontrivial linear combination of v and w which is orthogonal to v. Motivated by this, for vectors u, v, w let us consider a nontrivial linear combination which is orthogonal to both u and v. Such is for instance the linear combination of u, v, w given by:

$$(\langle u, v\rangle\langle v, w\rangle - \langle u, v\rangle\langle v, v\rangle u + (\langle u, v\rangle\langle u, w\rangle - \langle u, u\rangle\langle v, w\rangle)v + \\ + (\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle\langle u, v\rangle)w \tag{15}$$

It is not difficult to verify that the inner product of the vector given by (15) with itself (which is a nonnegative real number) can be written as the following determinant (which, accordingly, is also nonnegative):

$$\begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, w\rangle & \langle w, w\rangle \end{vmatrix} \end{vmatrix} \geqslant 0 \tag{16}$$

Let us observe that the nonnegative determinant given by (16) is a 2 by 2 determinant whose entries, in their turn, are also 2 by 2 determinants. Moreover, the (1, 1) entry in that determinant is the 2 by 2 determinant given by (12), and, the (1, 2) as well as the (2, 1) entry in that determinant is obtained from determinant given by (12) by substituting w for v in every occurence of v in the rightmost column of the table of (12), and, the (2, 2) entry in that determinant is obtained from determinant given by (12) by substituting w for v in every occurence of v in (12).

Appliyng our scheme to four vectors u, v, w, z we obtain the following (quite nontrivial) 2 by 2 nonnegative determinant:

$$\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, w\rangle & \langle w, w\rangle \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, w\rangle & \langle w, z\rangle \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, w\rangle & \langle w, z\rangle \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, z\rangle & \langle z, z\rangle \end{vmatrix} \end{vmatrix} \end{vmatrix} \geqslant 0$$

Naturally, all the results mentioned above are equally well applicable for the case of the real *inner product spaces*, and, more generally, for the case of the *unitary spaces*.

We summarize the method of construction of our (new class) of 2 by 2 nonnegative determinants as follows.

Let $v_1, v_2, v_3, \ldots$ be elements of a real inner product (or a unitary) space with $\langle v_i, v_j \rangle$ indicating the inner product of v_i and v_j. For every inteber $n \geqslant 2$ we define inductively a 2 by 2 symmetric matrix S_n as follows:

$$S_2 = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle \end{pmatrix} \text{ and } \quad S_{n+1} = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} = S_n$ and a_{21} is obtained from S_n by substituting v_{n+1} for v_n in every occurence of v_n in the rightmost column of the table of S_n and a_{22} is obtained from S_n by substituting v_{n+1} for v_n in every occurrence of v_n in S_n.

Replacing every matrix S_i which occurs in S_n by its determinant $|S_i|$ we obtain a nonnegative determinant $|S_n|$, i.e., $|S_i| \geqslant 0$ for every integer $n \geqslant 2$.

HYPERVALUABILITY OF A RING

J. PAPADOPOULOS*

Received: August 2, 1983

ABSTRACT. — The aim of this paper in to whow that there exist a semigroup S which although without zero divisors yet is not cancellative, and moreover a Ring exists that is hypervaluated by such a semigroup.

§ 1. **Introduction.** We wish to consider the following question: Is it possible to have a semigroup S that has no zero divisors and non-cancelative, and a ring **R** that can be hypervaluated by this semigroup?

We mean here our semigroup S to have a zero element 0 and a unit element 1, is we have $0 \cdot s = s \cdot 0 = 0 \ \forall s \in S$ and $1 \cdot s = s \cdot 1 = s \ \forall s \in S$. We remark that 1 and 0 are unique.

DEFINITION 1 *We say that a semigroup S is ordered if it is supplied with an order $\leqslant$ such that*

1. *if* $a, b, c \in S$ *then* $a \leqslant b \Rightarrow c \cdot a \leqslant c \cdot b$ *and* $a \cdot c \leqslant b \cdot c$
2. $0 \leqslant 1$ (*hence* $0 = 0 \cdot c \leqslant 1 \cdot c = c \ \forall c \in S$)

If the order is total, S is called totally ordered

DEFINITION 2 *An hypervaluation on a ring R is a function (II) from* **R** *onto a totally ordered semigroup S satisfying the following conditions:*

1. $|a| = 0 \Leftrightarrow a = 0 \qquad \forall a \in \mathbf{R}$
2. $|a| = |-a| \ \forall a \in R$
3. $|a + b| \leqslant \text{Max}\{|a|, |b|\} \ \forall a, b \in \mathbf{R}$
4. $|a \cdot b| = |a||b| \ \forall a, b \in \mathbf{R}$

Remarks 1. If the semigroup S does not have any zero divisors then the ring R does not have any either. Indeed suppose $a, b \in \mathbf{R}$ $a \neq 0$ $b \neq$ but with $a \cdot b = 0$. We then have $|a \cdot b| = |0| = 0$. So $|a \cdot b| = |a| \cdot |b| =$ But $a \neq 0$ implies $|a| \neq 0$ and $b \neq 0$ implies $|b| \neq 0$ and yet $|a||b| =$ contradicting our hypothesis that S has no zero divisors.

2. We easily see that a cancellative semigroupe has no zero divisors, however the converse is not true in general as we shall see in what follow

We are able to give an affirmativ answer to our question thus proving the following theorem:

THEOREM: *There exists a totally ordered a semigroup S, with no zero divisors and yet non cancellative, and a ring* **R** *that can be hypervaluated by this semigroup.*

---

* *John Papadopoulos, Ioulianou 41—43, Athens, Greece*

The theorem was proved by constructing an example in steps. We construct first a semigroup S_2 with the desired properties (i.e. totally ordered, no zero divisors, and not cancellative) starting out from a given but arbitrary totally ordered semigroup S_1. Then we construct a ring R that is hypervaluated by S_2.

§ 2. **Construction of** S_2. We begin with an arbitrary given totally ordered semigroup $\{S_1, \cdot, >\} = \{0_1, a, b, \ldots\}$ where 0_1 its absorbent (zero) element. Consider now the set $S_2 = S_1 \cup \{0_2\}$ that we get if we adjoint a new element 0_2 to the set S_1 and an operation $*$ defined on S_2 by $a * b = a \cdot b$ if $a, b \in S_1$ and $0_2 * a = a * 0_2 = 0_2 \ \forall a \in S_2$. (In particular $0_2 * 0_1 = 0_1 * 0_2 = 0_2$).

PROPOSITION 1 *$\{S_2, *\}$ is a semigroup.* The proof of this is straight forward. Let's show for example the associativity: Let be $a, b, c \in S_2$. If $a, b, c \in S_1$ the associativity results from the associativity in S_1. And if for emample $a = 0_2$ we then have $(0_2 * b) * c = 0_2 * c = 0_2 * (b * c)$.

PROPOSITION 2 *$(S_2 *)$ does not have any zero divisors*

Proof: Indeed it is impossible to have $a, b \in S_2$ $a, b \neq 0_2$ with $a * b = 0_2$, because since $a, b \neq 0_2$ it follows that $a, b \in S_1$ and so their product in S_2 (which coincides with their product in S_1) $a * b = a \cdot b$ also belongs to $S_1 \cdot S_0$ $a \cdot b \in S_1 \Rightarrow a \cdot b \neq 0_2$ because $0_2 \notin S_1$.

PROPOSITION 3 *$(S_2, *)$ is not cancellative*

Proof: Indeed suppose $a, b, \in S_1$ $a, b, \neq 0_1$ $a \neq b$.
We have $0_1 * a = 0_1 \cdot a = 0_1 = 0_1 \cdot b = 0_1 * b$ since 0_1 is the absorbent (zero) element in S_1 (but not in S_2). So in S_2 we can have $0_1 * a = 0_1 * b$ without having $a = b$ (we chose $a \neq b$).

PROPOSITION 4 *There is a compatible total ordering $\succ$ on S_2 that makes $(S_2, *, \succ)$ into a totally ordered semigroup.*

Proof: Define $0_2 \prec a \ \forall a \in S_1$ and $a \prec b$ iff $a < b \ \forall a, b \in S_1$. The conclusion follows immediately.

§ 3. PROPOSITION 5 *Let I be a two-sided ideal of an integral domain* $\mathbf{R}$. *If $\mathbf{R}/I$ can be hypervaluated by S_1 then* $\mathbf{R}$ *can be hypervaluated by S_2.* Before proving this proposition let's make two remarks:

1. In what follows, we employ, with no risk of confusion the symbol $\cdot$ to denote the composition in S_1 as well as in S_2.
2. The term „ideal" in a ring not necessarilly commulative signifies a two-sided ideal.

Proof of proposition **5**. Suppose we have the valuation $|\,|$:

$$R/I \xrightarrow{|\ |} S_1 = \{0_1, a, b \ldots\}$$

We construct a valuation $\|\ \|$:

$$R \xrightarrow{\|\ \|} S_2 = \{0_2\} \cup S_1 \text{ by posing:}$$

If $a \in \mathbf{R}$ $a = 0$ then $\|a\| = 0_2$
If $a \in \mathbf{R}$ $a \neq 0$ then $\|a\| = |a + I|$

This implies that if $a \in I$, then $||a|| = 0_1$

We show now that $|| \quad ||$ is a valuation of R onto $S_2 = \{0_2\} \cup S_1$.

1) $\forall a \in \mathbf{R}$ we have $||a|| = 0_2$ if and only if $a = 0$ by definition of

2) *We show that* $||-a|| = ||a|| \; \forall a \in \mathbf{R}$

i) if $a \neq 0$ then $-a \neq 0$ and we have

$||a|| = |a + I| = |-a + I|$ (because $| \quad |$ is an (hyper) valuatic $\mathbf{R}/I$ and $-a + I = -(a + I)$ in $\mathbf{R}/I = ||-a||$

ii) if $a = 0$ then $-a = 0$ and so $||a|| = ||-a|| = 0_2$.

3) *We show that* $||a + b|| \prec Max \{||a||, \, , ||b||\} \; \forall a, b \in \mathbf{R}$. Indeed

i) if $a = b = 0$ then $a + b = 0$ evident case

ii) if $a = 0 \; b \neq 0$ then $a + b = b$

$$||a|| = 0_2 \;\; ||b|| \prec 0_2 \text{ and } ||a + b|| = ||b||$$

iii) if $a, b \neq 0$ we can have $a + b \neq 0$ or $a + b = 0$

α) if $a + b = 0$ then $||a + b|| = 0_2 \prec ||a||, ||b||$ so $\prec$Max $\{||a||$

β) if $a + b \neq 0$ then we have

$$||a|| = |a + I| \qquad ||b|| = |b + I|$$

$$||a + b|| = |a + b + I| \text{ (by definition)}$$

$$= |(a + I) + (b + I)| \leqslant \text{Max}\{|a + I|, |b + I|\}$$

(since $| \quad |$ is an hypervaluation for R/I)

$$= \text{Max}\{||a||, ||b||\}.$$

4) *We show that* $||a \cdot b|| = ||a|| \cdot ||b|| \; \forall a, b \in \mathbf{R}$. Indeed, we disti the following cases:

i) if $a = b = 0$ then $ab = 0$ evident

ii) if $a = 0 \; b \neq 0$ then $a \cdot b = 0$
so $||a|| = 0_2 \; ||b|| \neq 0_2$ and $||a|| \cdot ||b|| = 0_2$.
Also $||a \cdot b|| = ||0|| = 0_2$ so $||a \cdot b|| = ||a|| \cdot ||b||$

iii) $a \neq 0, b \neq 0$ we have $a \cdot b \neq 0$ (since $\mathbf{R}$ is taken to be an in domain)

and so we have $||ab|| = |ab + I|$ (by definition)
and so $||a|| = |a + I|$
$||b|| = |b + I|$

$||a|| \cdot ||b|| = |a + I| \, |b + I| = |ab + I|$ (because)$| \quad |$ is an hypervalı $= ||ab||$ for $\mathbf{R}/I$)

We have verified that the condition 1), 2), 3), 4) that define an (l valuation are satisfied by the function $|| \quad || : R \to S_2$

So $|| \quad ||$ defines an (hyper) valuation from R onto S_2 and Proposi is thus proved.

§ 4.**Coffi's condition for hypervaluability of a ring.** DEFINITION 3. *Let A be a ring $a \in A$. We call the set of left annulators of a to be the set $\{x| \ x \in A | x \cdot a = 0\}$ and we denote it by $Ng(a)$. In an analogous way we define the set of the right annulators of a denoted by $Nd(a)$.*
Coffis theorem of valuability of a ring. Let A be a ring with a unit element 1.
A can be hypervaluated by a totally ordered semigroup S if and only if it satisfies the following conditions:

1. For all $a \in A$ $Ng(a) = Nd(s)$ (and we denote this set by $N(a)$)
2. For all a, b, $\in A$ we have $N(a \cdot b) = N(b \cdot a)$
3. The family $N = \{N(a) | a \in A\}$ is totally ordered by inclusion.

In particular, A posesses an hyprvaluation $\mathrm{st}|a| \to N(a)$ is a one-to-one correscpondence between S and N.

We remark that Coffi in his construction suppose the semigroup Commutative. The ring A is not supposed necessarily commutative, but with an indentilty element 1. The details can be found in [1]. The idea is the following:

For each $a \in A$ its „value'' $|a|$ is $N(a)$.
So, $|| : A \to N = S_1$. Moreover S_1 is totally ordered by the total order defined by:

$$a, \ b, \neq A \quad a \leqslant b \ \text{ iff } \ N(a) \supseteq N(b)$$

Now according to our previous discussion, we can take a ring A of the form $\mathbf{R}/I$ (ie $A = \mathbf{R}/I$ where I is non-zero two-sided ideal of the ring R) and $s \cdot t\, A$ satisfies the Coffi theorem conditions. By Coffi's theorem then we have an hypervaluation $||_1 : A \simeq \mathbf{R}/I \to N = S_1$, which in turn induces (according to our proposition 5 in § 3) an hypervaluation $||_2 : \mathbf{R} \to S_2$ (where S_2 is the semigroup with the desired properties, as it was constructed in § 2) and this provides us with the desired example.

§ 5 **A concrete case** Let's take $\mathbf{R} = Z$ the ring of integers. $I = (16)$ the ideal generated by the integer 16, $A = \mathbf{R}/(16)$ and A satisfies the Coffis theorem conditions as we can easily verify (we observe that $\forall\, b \in A$ $N(b) = \{x \in Z | 16 | b \ x\}$. So A is hypervaluated by a certain semigroup $S_1 = N = \{N(a) | a \in A\}$. By our discussion in § 3 then, $\mathbf{R}$ is hypervaluated by $S_2 = \{0_2\} \cup S_1$ which has the desired properties.

REFERENCES

1. C o f f i, Comptes rendus Academie des Sciences, Paris, 254, III, 1980.
2. J. P a p a d o p o u l o s, These 3 eme Cycle, Universite Paris VI, 1980.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

ON THE CONVERGENCE OF A METHOD OF INTEGRATING CAUCHY'S PROBLEMS

ERNEST SCHEIBER*

Received: October 14, 1983

ABSTRACT. — For integrating Cauchy's problems

$$\dot{x} = f(t, x) \quad x(t_0) = x^0 \tag{1}$$

on each interval t_k, t_{k+1} of the division $t_0 < t_1 < \ldots < t_n$ ($t_i = ih$, $i = 0, 1, \ldots, n$), another Cauchy problem $y = g_k(t, y)$, $y(t_k) = y_{k-1}(t_k)$ is formulated, with the solution $y_k(t)$, $k = 0, 1, \ldots, n$ ($y(t_0) = x^0$)). The function x_h, defined by $x_h(t) = y_k(t)$ if $t \in [t_k, t_{k+1}]$, is an approximation of solution (1). The relationships between $f(t, x)$ and $g_k(t, x)$, $k = 0, 1, \ldots, n-1$, are establis hed, which ensure the discrete convergence of the approximative solution x_h twerds x.

Let be the Cauchy's problem

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x^0 \end{aligned} \tag{1}$$

with the solution $x(t)$ defined on $[t_0, T]$. We supose that $f: [t_0, T] \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function. To integrate this problem we use the following method. Let be $h = (T - t_0)/n$ and $t_i = t_0 + ih$, $i = 0, 1, \ldots, n$. On each interval $[t_k, t_{k+1}]$ it's defined another Cauchy's problem

$$\begin{aligned} \dot{y} &= g_k(t, y) \\ y(t_k) &= y_{k-1}(t_k) \end{aligned} \tag{2}$$

with the solution $y_k(t)$ on $[t_k, t_{k+1}]$. For $k = 0$ we use $y(t_0) = x^0$. We note by x_h the function defined by $x_h(t) = y_k(t)$, if $t \in [t_k, t_{k+1}]$ x_h is called an approximation of the solution of problem (1). This method was used by I x a r u L. [2] and P a v e l G. [4] to integrate linear diferential equation with variable coefficients. M a r i n e s c u C. [3] consider such method to integrate linear systems of differential equations with variable coefficients. A direct proof of convergence is given.

We are interesed to establish a connection between the equations (1) and (2) which assure the convergence of the approximation x_h to x. We say that x_h concerges discretely to x if $\lim\limits_{h \searrow 0} \max\limits_{k = \overline{0,n}} ||x_h(t_k) - x(t_k)|| = 0$.

THEOREM 1. *If* $||f(t, x) - g_k(t, y)|| \leqslant a_k(t)||x - y|| + c|t - \bar{t}_k|^\gamma$ *on* $[t_k, t_{k+1}]$, *where* $\bar{t}_k \in [t_k, t_{k+1}]$, $a_k(t)$ *is a non-negative continuous function on* (t_k, t_{k+1}) *and* $c, \gamma > 0$, $k = 0, 1, \ldots, n-1$, *then the approximation* x_h *converges discretely to* x, *the solution of the Cauchy's problem.*

* *University of Braşov, Faculty of Mathematics, 2200 Braşov, Romania*

Proof. Let $a : [t_0, t_n] \to \mathbf{R}$ be the function defined by $a(t) = a_k(t)$ if $t \in [t_k, t_{k+1})$. For $t \in [t_k, t_{k+1}]$ we have

$$x(t) = x(t_k) + \int_{t_k}^{t} f(s, x(s))ds,$$

$$y_k(t) = y_k(t_k) + \int_{t_k}^{t} g_k(s, y_k(s))ds$$

and, further

$$x(t) - y_k(t) = x(t_k) - y_k(t_k) + \int_{t_k}^{t} [f(s, x(s)) - g_k(s, y_k(s))]ds$$

Using the hypothesis of the theorem we obtain

$$||x(t) - y_k(t)|| \leqslant ||x(t_k) - y_k(t_k)|| + \int_{t_k}^{t} a_k(s)||x(s) - y_k(s)||ds +$$

$$+ C\int_{t_k}^{t} |s - t_k|^{\gamma} ds \leqslant ||x(t_k) - y_k(t_k)|| + \frac{2Ch^{\gamma+1}}{\gamma + 1} + \int_{t_k}^{t} a_k(s)||x(s) - y_k(s)||ds.$$

Applying Gronwall's lemma it results

$$||x(t) - y_k(t)|| \leqslant \left(||x(t_k) - y_k(t_k)|| + \frac{2Ch^{\gamma+1}}{\gamma + 1}\right) e^{\int_{t_k} a(s)ds}$$

and particullary for $t = t_{k+1}$

$$||x(t_{k+1}) - y_k(t_{k+1})|| \leqslant \left((||x(t_k) - y_k(t_k)|| + \frac{2Ch^{\gamma+1}}{\gamma + 1}\right) e^{\int_{t_k}^{t_{k+1}} a(s)ds} \qquad k = 1, 2, \ldots, (\sqrt{n} - 1)$$

For $k = 0$ one has the inequality

$$||x(t_1) - y_1(t_1)|| \leqslant \frac{2Ch^{\gamma+1}}{\gamma + 1} e^{\int_{t_n}^{t_1} a(s)ds} .$$

For $k = 1$ we deduce

$$||x(t_2) - y_2(t_2)|| \leqslant \left(||x(t_1) - y_1(t_1)|| + \frac{2Ch^{\gamma+1}}{\gamma+1}\right) e^{\int_{t_1}^{t_2} a(s)ds} \leqslant$$

$$\leqslant \left(\frac{2Ch^{\gamma+1}}{\gamma+1} e^{\int_{t_0}^{t} a(s)ds} + \frac{2Ch^{\gamma+1}}{\gamma+1}\right) e^{\int_{t_1}^{t_2} a(s)ds} =$$

$$= \frac{2Ch^{\gamma+1}}{\gamma+1}\left(e^{\int_{t_0}^{t_2} a(s)ds} + e^{\int_{t_1}^{t_2} a(s)ds}\right).$$

Inductively, it results

$$||x(t_k) - y_k(t_k)|| \leqslant \frac{2Ch^{\gamma+1}}{\gamma+1}\left(e^{\int_{t_0}^{t_k} a(s)ds} + e^{\int_{t_1}^{t_k} a(s)ds} + \ldots + e^{\int_{t_{k+1}}^{t_k} a(s)ds}\right) \leqslant$$

$$\leqslant \frac{2kCh^{\gamma+1}}{\gamma+1} e^{\int_{t_0}^{t_k} a(s)ds} \leqslant \frac{2(t_n - t_0)Ch^{\gamma}}{\gamma+1} e^{\int_{t_0}^{t_u} a(s)ds} \quad (k = 0, 1, \ldots, n).$$

and hence $\lim_{h \downarrow 0} \max_{k=\overline{0,n}} ||x(t_k) - y_k(t_k)|| = 0.$ *

We apply this theorem to prove the convergence of mentioned meth used in [3] to integrate linear sysrems of differential equations with va ble coefficients :

$$\dot{x}_i = \sum_{j=1}^{p} a_{ij}(t)x_j + b_i(t) \quad i = 1, 2, \ldots, p.$$

We attach to the system (3) on each interval $[t_k, t_{k+1}]$ the system wi constant coefficients

$$\dot{y}_i = \sum_{j=1}^{p} a_{ij}(\tilde{t}_k)y_j + b_i(\tilde{t}_k) \quad i = 1, 2, \ldots, p,$$

where $\tilde{t}_k \in [t_k, t_{k+1}]$. We suppose that $a_{ij}(t)$, $i, j = 1, 2, \ldots, p$ and $b_i(t)$ $i = 1, 2, \ldots, p$ are continuous wirh their derivatives on $[t_0, t_n]$. This met hod is known as the step method. In this case $f(t, x) = A(t)x + b(t)$ and $g_k(t, y) = A(\tilde{t}_k)y + b(\tilde{t}_k)$ where

$$A(t) = \begin{pmatrix} a_{11}(t) \ldots a_{1p}(t) \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{p1}(t) \ldots a_{pp}(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_p(t) \end{pmatrix}$$

Then

$$\|f(t, x) - g_k(t, y)\| = \|A(t)x + b(t) - A(\tilde{t}_k) y - b(\tilde{t}_k)\| \leqslant$$
$$\leqslant \|A(t)x - A(\tilde{t}_k)x + A(\tilde{t}_k)x - A(\tilde{t}_k)y\| + \|b(t) - b(\tilde{t}_k)\| \leqslant$$
$$\leqslant \|A(\tilde{t}_k) - A(t)\| \cdot \|x\| + \|A(\tilde{t}_k)\| \|x - y\| + \|b(t) - b(\tilde{t}_k)\| \leqslant$$
$$\leqslant \|A(\tilde{t}_k)\| \|x - y\| + C|t - \tilde{t}_k| \leqslant (\max_t \|A(t)\|) \|x - y\| + C|t - \tilde{t}_k|,$$

where $C = (p + 1)M$ with $|a'_{ij}(t)|r \leqslant M$, $|b'_i(t)| \leqslant M$ for all $t \in [t_0, t_n]$, $i, j = 1, 2, \ldots p$, and $\|x(t)\| \leqslant r$, $t \in [t_0, t_0 + nh]$, $x(t)$ being the solution of the equation $\dot{x} = A(t)x + b(t)$.

The conditions of THEOREM 1 can be weakened. Let k be a compact set which contains the sets $\{x(t) : t \in [t_0, t_0 + nh]\}$ and $\{x_k(t) : t \in [t_0, t_0 + nh]\}$. Then we have

THEOREM 2. If $\left\| \int_{t'}^{t''} [f(s, x) - g_k(s, x)] ds \right\| \leqslant \xi(h)$ *for every* $t_k \leqslant t' < t'' \leqslant t_{k+1}$ *and every* $x \in K$, *such that* $\lim_{h \downarrow 0} \frac{\xi(h)}{h} = 0$ *and* $\|g_k(t, x) - g_k(t, y)\| \leqslant L\|x - y\|$, $k = 0, 1, \ldots, n - 1$ *then* x_k *converges discretely to* x.

Proof. For every $t \in [t_i, t_{i+1}]$ one has the equalities

$$x(t) - x_k(t) = x(t_i) - x_k(t_i) + \int_{t_i}^{t} [f(s, x(s)) - g_i(s, x_k(s)] ds =$$

$$= x(t_i) - x_k(t_i) + \int_{t_i}^{t} [f(s, x(s) - g_i(s, x(s)] ds + \int_{t_i}^{t} [g_i(s, x(s) - g_i(s, x_k(s))] ds$$

and hence

$$\|x(t) - x_k(t)\| \leqslant \|x(t_i) - x_k(t_i)\| + \xi(h) + L \int_{t_i}^{t} \|x(s) - x_k(s)\| ds.$$

Using Gronwall's lemma we obtain

$$\|x(t) - x_k(t)\| \leqslant [\|x(t_i) - x_k(t_i)\| + \xi(h)] e^{L(t - t_i)} \tag{4}$$

In the same way (for $t = t_{i+1}$) we find

$$x(t_{i+1}) - x_k(t_{i+1}) = x(t_i) - x_k(t_i) + \int_{t_i}^{t_{i+1}} [f(s, x(s)) - g_i(s, x_k(s)(] ds$$

and adding up these equalities for $i = 0, 1, \ldots, k-1$ we obtain

$$x(t_k) - x_h(t_k) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [f(s, x(s)) - g_h(s, x_i(s))]ds =$$

$$= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [f(s, x(s)) - g_i(s, x(s))]ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [g_i(s, x(s)) - g_i(s, x_h(s))]ds.$$

Futher we deduce

$$||x(t_k) - x_h(t_k)|| \leqslant \sum_{i=0}^{k-1} \left\| \int_{t_i}^{t_{i+1}} [f(s), x(s)) - g_i(s, x(s))]ds \right\| +$$

$$+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} ||g_i(s, x(s)) - g_i(s, x_h(s))||ds \leqslant k\xi(h) + L\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} ||x(s) - x_h(s)||ds$$

Using (4) the last inequality becomes

$$||x(t_k) - x_h(t_k)|| \leqslant k\xi(h) + L\sum_{i=0}^{k-1} [||x(t_i) - x_h(t_i)|| + \xi(h)] \int_{t_i}^{t_{i+1}} e^{L(s-t_i)}ds =$$

$$= k\xi(h) + \sum_{i=0}^{k-1} [||x(t_i) - x_h(t_i)|| + \xi(h)](e^{L(t_{i+1}-t_i)} - 1) =$$

$$= (e^{Lh} - 1)\sum_{i=0}^{k-1} ||x(t_i) - x_h(t_i)|| + e^{Lh}\xi(h)k.$$

Applying the discretely form of Gronwall's lemma we obtain

$$||x(t_k) - x_h(t_k)|| \leqslant e^{Lhk}\xi(h)k.$$

Finally

$$\max_{k=\overline{0,n}} ||x(t_k) - x_h(t_k)|| \leqslant n\xi(h)e^{Lhn} = (t_n - t_0)e^{L(t_n-t_0)}\frac{\xi(h)}{h}$$

and hence $\lim\limits_{h \downarrow 0} \max\limits_{k=\overline{0,n}} ||x(t_k) - x_h(t_k)|| = 0.$ *

Now, we show that the theorem 2 implies theorem 1. Indeed, if $\|f(t, x) - g_i(t, y)\| \leqslant a_i(t)\|x - y\| + C|t - \tilde{t}_i|^\gamma$ then for every $t_i \leqslant t' < t'' \leqslant t_{i+1}$ one have the inequalities

$$\left\| \int_{t'}^{t''} [f(t, x) - g_i(t, x)]dt \right\| \leqslant \int_{t'}^{t''} \|f(t, x) - g_i(t, x)\|dt \leqslant$$

$$\leqslant C \int_{t'}^{t''} |t - \tilde{t}_i|^\gamma dt \leqslant \frac{2C}{\gamma + 1}(t'' - t')^{\gamma+1} \leqslant \frac{2Ch^{\gamma+1}}{\gamma + 1}.$$

Taking $\xi(h) = \frac{2Ch^{\gamma+1}}{\gamma + 1}$ we observe that $\lim_{h \downarrow 0} \frac{\xi(h)}{h} = 0$.

REFERENCES

1. Halanay A., „Euaţii diferenţiale", E.D.P., Bucureşti, 1972.
2. Ixaru L. Gr., „Metode numerice pentru ecuaţii diferenţiale cu aplicaţii", Ed. Acad. R.S.R., 1979. Bucureşti.
3. Marinescu C., „O soluţie aproximativă a unui sistem de ecuaţii diferenţiale", *Bul. Univ. Braşov, Seria C*, XXII (1980), pp. 71–74.
4. Pavel G., Asupra aproximaţiei soluţiei problemei lui Cauchy relativă la o ecuaţie diferenţială liniară de ordin *n*. *Studia Univ. Babeş-Bolyai, Math.*, 23, (2), 1978, pp. 49–53.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

ON THE COMMUTATIVITY OF SOME FAMILIES OF CLOSED OPERATIONS IN A HETEROGENEOUS CLONE

IOAN CĂLUGĂR*

Received: October 26, 1983

ABSTRACT. — The heterogeneous algebras, introduced by Birkhoff and Lipson [1], play a very important role in computer science and essentialy in the study of abstract types [4,7]. We introduce the concepts of heterogeneous clone and abstract hetrogeneous clone of operations, and a commutativity property between two families of closed heterogenous operations. This commutativity is generaly complex and restrictive, but in the particular forms is very powerful in the specification of abstract types.

1. **Introduction.** Following the notations in [4], let S be a nonvoid the elements of which will be called sorts. Each indexed family of $A = (A_s)_{s \in S}$ will be called S-sorted family of sets and each inde family of mappings $f = (f_s)_{s \in S}$, where $f_s : A_s \to B_s$ is a mapping ($s \in$ will be called S-sorted mapping. An S-sorted operator domain (signat consist of a set Σ equipped with two mappings: $d : \Sigma \to S^*$ and $c : \Sigma$ called domain and respectively codomain; For each $\sigma \in \Sigma$ with $d(\sigma)$ $= w \in S^*$ and $c(\sigma) = s \in S$, we say that σ has functionality $(w, s) \in S^*$ $\times S$. So we can see Σ as a disjoint union

$$\Sigma = \bigcup_{(w,s) \in S^* \times S} \Sigma_{w,s} = \bigcup_{w \in S} \Sigma_w = \bigcup_{s \in S} \Sigma_s$$

where $\Sigma_w = \{\sigma \in \Sigma / d(\sigma) = w \in S^*],\ \Sigma_s = \{\sigma \in \Sigma / c(\sigma) = s \in S^*\}$ a

$$\Sigma_{w,s} = \Sigma_w \cap \Sigma_s$$

Let $\sigma \in \Sigma_{w,s}$, $w = s_1 \ldots s_n$; If $s \in \{s_1, \ldots, s_n\}$ we will say that σ closed; Otherwise σ is called open.

A Σ-algebra (or heterogeneous algebra) A consist of an S-sorted fam of sets $(A_s)_{s \in S}$ called carrier sets, and for each $(w, s) \in S^* \times S$ and ea $\sigma \in \Sigma_{w,s}$, there is a function $\sigma_A : A^w \to A_s$ named operation of typ and sort s, where $A^w = A_{s_1} \times \ldots \times A_{s_n}$. If $w = \varepsilon$ is the unit elem of the free monoid S^*, then σ_A is a nullary operation. If at most one ope tion σ_A is a partial function, A will be called partial Σ-algebra.

A Σ-algebra B is called Σ-subalgebra of A if $B_s \subseteq A_s$ for all $s \in$ and for all $\sigma \in \Sigma$, $\sigma_B = \sigma_{A/B}$, where $\sigma_{A/B}$ is the restriction of σ_A to

* *Energomontaj Trust Bucharest, Calculation Center, Calea Dorobanţi str, no. 105, Sect. 1. 7000 Buch Romania*

Let A, B two Σ-algebras. An S-sorted function $f: A \to B$ is called Σ-homomorphism if for all $\sigma \in \Sigma_{w,s}$ ($w \in S^*$, $s \in S$) the following diagram commute :

$$\begin{array}{ccc} A^w & \xrightarrow{\sigma_A} & A_s \\ {\scriptstyle f^w}\downarrow & & \downarrow{\scriptstyle f_s} \\ B^w & \xrightarrow{\sigma_B} & B_s \end{array} \qquad (1.1)$$

Let $X = (X_s)_{s \in S}$ an S-sorted family of variables and $W_\Sigma(X)$ the word Σ-algebra freely generated by X([1], [2], [5], [6]). The properties of Σ-algebras can be expressed by formulas builts over equations of the form $(t, t')_s$, $s \in S$, $t, t' \in [W_\Sigma(X)]_s$, using the firstorder predicate calculus. In the most general case we can consider sentences in the prenex normal form

$$Q_1 s_1 x_1 \ldots Q_n s_n x_n \bigwedge_{1 \leq i \leq k} \Big(\big(\bigvee_{1 \leq j \leq 1} p_{ij} \neq q_{ij} \big) \bigvee \big(\bigvee_{1+1 \leq j \leq m} r_{ij} = t_{ij} \big) \Big) \qquad (1.2)$$

where $Q_i \in \{\forall, \exists\}$.

The specification of an abstract type consist of a triple $SP = (S, \Sigma, E)$ where Σ is an S-sorted signature and E a set of sentences in the form (1, 2) called axioms. If the axiom in E are simple equations, the type will be called equational and then the category Alg_{SP} of all Σ-algebras satisfiyng E form a variety in the sense of [2].

For the formal description of an abstract type let take the following example :

type DATA *is* (BOOL) +
sort *data*
opns E_{data} : $\to$ *data*
COND_{data} : (*bool*, *data*, *data*) $\to$ *data*
EQ_{data} :(*data*, *data*) $\to$ *bool*
OK_{data} :(*data*)$\to$ *bool*
axioms $\forall$ *data* D, D_1, D_2
(1) $\mathrm{COND}_{data}(T, D, D_1) = D$
(2) $\mathrm{COND}_{data}(F, D, D_1) = D_1$
(3) $EQ_{data}(D, D) = T$
(4) $EQ_{data}(D, D_1) = EQ_{data}(D_1, D)$
(5) $EQ_{data}(D, D_1) = T \,\&\, EQ_{data}(D_1, D_2) = T \Rightarrow EQ_{data}(D, D_2) = T$
(6) $OK_{data}(D) = \mathrm{COND}_{data}(EQ_{data}(D, E_{data}), F, T)$
eot

where BOOL is the usual type of the truth values (see [4]).
Now $S = \{bool, data\}$, $\Sigma = \{T, F, \&, \vee, \Rightarrow, \mathrm{EQ}_{bool}, \mathrm{COND}_{bool}, \mathrm{E}_{data}$ $\mathrm{COND}_{data}, \mathrm{EQ}_{data}, \mathrm{OK}_{data}\}$, E contains the axioms of BOOL and, the axioms (1) ... (6) above.

2. Clones of Heterogeneous Operations. Let A be a novoid S-sorted family of sets and denote by $H(A)$ the set of all finitary heterogeneous operations on A. Then $H(A)$ can be viewd as a disjoint union:

$$H(A) = \bigcup_{w,s\, S^*\times S} H_{w,s}(A), \text{ where } H_{w,s}(A) = \{\sigma \in H(A)/d(\sigma) = w, c(\sigma) = s\}.$$

Let $w = s_1 \dots s_n \in S^*$, $\tau_i \in H_{u,s_i}(A)$, $(i = 1, \dots, n)$, $u \in S^*$ and $\sigma \in H_{w,s}(A)$. Then there is a unique operation θ on $H(A)$ defined by:

$$\theta(\sigma, \tau_1, \dots, \tau_n)(a) = \sigma(\tau_1(a), \dots, \tau_n(a)) \text{ for all } a \in A^u \quad (2.1)$$

and let denote $\theta(\sigma, \tau_1, \dots, \tau_n) = \sigma[\tau_1, \dots, \tau_n]$, calling it „composition" of $\tau_1, \dots, \tau_n$ with σ.
For any $w \in S^+$, where $S^+ = S^* - \{\varepsilon\}$, with $w = s_1 \dots s_n$, there are n operations on A denoted $1^{w,s_i}$ and defined by:

$$1^{w,s_i}(a) = a_i \text{ for all } a \in A^w, \quad i = 1, \dots, n. \quad (2.2)$$

Let call $1^{w,s_i}$ unit operations or projections on the i-th coordinate. We can now regard $H(A)$ as a partial heterogeneous algebra with sorts $S^* \times S$, nullary operations the units and the other operations defined in (2.1) with $d(\theta) = (w, s)(u, s_1) \dots (u, s_n)$ and $d(\theta) = (u, s)$.

Definition A set H of heterogeneous operations on an S-sorted family A of sets, containing the unit operations defined in (2.2) and closed under the compositions (2.1) is called heterogeneous clone of operations on A. This notion was introduced by P. H a l l (1958) and studied for the homogeneous case ([2], [6]).
Generaly, giving an S-sorted family A of sets and the clone $H(A)$ of all operations on A, we will call H_1 a clone on A if H_1 is a subclone of $H(A)$, i.e. a subalgebra in the sense mentioned above.

Let H_1, H_2 be heterogeneous clones on A; An $S^* \times S$-sorted mapping $f: H_1 \to H_2$ with peoperties:

(i) $d(f(\sigma)) = d(\sigma)$ and $c(f(\sigma)) = c(\sigma)$ for all $\sigma \in H_1$

(ii) $f(1^{w,s_i} = 1^{w,s_i}$, $w \in S^*$, $i = 1, \dots, n$ and

(iii) $f(\sigma[\tau_1, \dots, \tau_n]) = f(\sigma)[f(\tau_1), \dots, f(\tau_n)]$, for all composable operations $\sigma, \tau_1, \dots, \tau_n \in H_1$,

will be called homomorphism of heterogeneous clones on A.
The set of all heterogeneous clones on an S-sorted family A of sets with theirs homomorphisms, form a category.

Let Σ an S-sorted signature and A a Σ-algebra. The actions of the operations in Σ determines a heterogeneous clone on A, denoted Σ_A^* and called heterogeneous clone of action of Σ on A.

DEFINITION. An abstract heterogenous clone is a partial heterogenous algebra H defined as follows:
1° there are two mappings $d: H \to S^*$ and $c: H \to S$ which associates to each $\sigma \in H$ the domain $d(\sigma)$ and the target $c(\sigma)$ and

2° with each $w \in S^+$, $w = s_1 \ldots s_n$, H contains the unit operators $1^{w,s_i}: w \to s_i$, $i = 1, \ldots, n$ and

3° for $\sigma, \tau_1, \ldots, \tau_n$ in H with $d(\tau_1) = \ldots = d(\tau_n)$ and $d(\sigma) = c(\tau_1) \ldots c(\tau_n) \in S^*$, there is an operation on H denoted by $\sigma[\tau_1, \ldots, \tau_n] : d(\tau_1) \to c(\sigma)$ with properties:

(i) $(\sigma[\tau_1, \ldots, \tau_n])[\eta_1, \ldots, \eta_m] = \sigma[\tau_1[\eta_1, \ldots, \eta_m], \ldots, \tau_n[\eta_1, \ldots, \eta_m]]$ where $d(\tau_1) = c(\eta_1) \ldots c(\eta_m) \in S^*$

(ii) $1^{w,s_i}[\tau_1, \ldots, \tau_n] = \tau_i$, $i = 1, \ldots, n$

As a consequence of this definition we have the

DEFINITION *Every heterogeneous clone of operations is abstract.*

3. **(ii j) — Commutativity of Two Families of Operations.** Let Σ an S-sorted signature, A a Σ-algebra and $\tau = (\tau_1, \ldots, \tau_n)$ $\sigma = (\sigma_1, \ldots, \sigma_m)$ two families of operations in Σ_A^*, with:

$$\begin{aligned} \sigma_1 &: s_{11} \ldots s_{1j} \ldots s_{1n} \to s_{1j} \\ &\vdots \\ \sigma_i &: s_{i1} \ldots s_{ij} \ldots s_{in} \to s_{ij} \\ &\vdots \\ \sigma_m &: s_{m1} \ldots s_{mj} \ldots s_{mn} \to s_{mj}, \text{ and} \end{aligned} \tag{3.1}$$

$$\begin{aligned} \tau_1 &: s'_{11} \ldots s'_{i1} \ldots s'_{mj} \to s'_{i1} \\ &\vdots \\ \tau_j &: s'_{1j} \ldots s'_{ij} \ldots s'_{mj} \to s'_{ij} \\ &\vdots \\ \tau_n &: s'_{1n} \ldots s'_{in} \ldots s'_{mn} \to s'_{in} \end{aligned} \tag{3.2}$$

We call $\tau = (\tau_1, \ldots, \tau_n)$ in (3.2) compsable with σ_i in (3.1) if $d(\sigma_i) = c(\tau_1) \ldots c(\tau\ ) \in S^*$.

DEFINITION. Giving two families of closed operations in Σ_A^*, $\sigma = (\sigma_1, \ldots \ldots, \sigma_m)$ and $\tau = (\tau_1, \ldots, \tau_n)$, we will say that σ and τ commute (i, j) if:

(i) σ is composable with τ_j

(ii) τ is composable with σ_i, and

(iii) for all $a_{ij} \in A_{s_{ij}}$, $(i = 1, \ldots, m;\ j = 1, \ldots n)$ we have the identity:

$$\begin{aligned} &\sigma_i(\tau_1(a_{11}, \ldots, a_{m1}), \ldots, \tau_j(a_{1j}, \ldots, a_{mj}), \ldots, \tau_n(a_{1n}, \ldots, a_{mn})) = \\ &= \tau_j(\sigma_1(a_{11}, \ldots, a_{1n}), \ldots, \sigma_i(a_{in}, \ldots, a_{in}), \ldots, \sigma_m(a_{m1}, \ldots, a_{mn})) \end{aligned} \tag{3.3}$$

Let now $w'_j = s_{1j} \ldots s_{mj}$ for $j = 1, \ldots, n$ and take $\tau_j = 1^{w'_j}, s_{ij}$; We the have:

$$\begin{aligned} &\sigma_i(1^{w'_i, s_{i1}}(a_{11}, \ldots, a_{m1}), \ldots, 1^{w'_n, s_{in}}(a_{1n}, \ldots, a_{mn})) = \sigma_i(a_{i1}, \ldots, a_{in}) = \\ &= 1^{w'_j, s_{ij}}(\sigma_1(a_{11}, \ldots, a_{1n}), \ldots, \sigma_i(a_{i1}, \ldots, a_{in}), \ldots, \sigma_m(a_{m1} \ldots, a_{mn})) \end{aligned} \tag{3.4}$$

and therefore we can state:

PROPOSITION. *Every family of closed o perations of the form* (3.1) *co mmute* (i, j) *with a cooresponding family of units.*

If we denote $w_i = s_{i1} \ldots s_{in}$ for $i = 1, \ldots, m$, and take

$$\sigma_1 = 1^{w_1, s_{1j}}, \ldots, \sigma_{i-1} = 1^{w_{i-1}, s_{i-1,j}}, \sigma_{i+1} = 1^{w_{i+1}, s_{i+1,j}}, \ldots, \sigma_m = 1^{w_m, s_{mj}}$$ and $$\tau_1 = 1^{w'_1, s_{i1}}, \ldots, \tau_{j-1} = 1^{w'_{j-1}, s_{i,j-1}}, \tau_{j+1} = 1^{w'_{j+1}, s_{i,j+1}}, \ldots, \tau_n = 1^{w'_i, s_{in}}$$, then (3.3) becomes:

$$\sigma(a_{i1}, \ldots, a_{ij-1}, \tau(a_{1j}, \ldots, a_{mj}), a_{ij+1}, \ldots, a_{in}) = \tau(a_{1j}, \ldots, a_{i-1j}, \sigma(a_{i1}, \ldots, a_{in}), a_{i+1j}, \ldots, a_{mj}) \tag{3.5}$$

where $\sigma = \sigma_i$ and $\tau = \tau_j$. When this is the case, we call (3.5) (i, j)-commutativity of σ and τ.

The (i, j)-comutativity of two families of closed operations defined above, generalises the commutativity of two operations in the homogeneous case, ([2], III, 3), and is powerfull in the specifications of the abstract types ([3], [4]).

4. Examples

4.1. Let M a monoid acting on a set A. This is a heterogeneous algebra (see [1]) with $\Sigma = \{1_M, *, \circ\}$, where $1_M : M \to M, * : M \times M \to M$ and $0 : M \times A \to A$ and axioms:

$1_M \circ a = a$ for all $a \in A$, and

$(m * n) \circ a = m \circ (n \circ a)$ for all $m, n \in M$, $a \in A$

Let take first the families of operations $(*, \circ)$ and $(*, \circ)$. The for $(i, j) = (2, 2)$ the relation (3, 3) becomes:

$$(m * n) \circ (p \circ a) = (m * p) \circ (n \circ a) \text{ for all } m, n, p \in M, a \in A \tag{4.1.1}$$

If $p = 1_M$ then using the first axiom we have:

$$(m * n) \circ a = m \circ (n \circ a) \tag{4.1.2}$$

which is the second axiom stated above. On the other hand, taking $m = 1_M$ in (4.1.1.) and using the first axiom we obtain:

$$n \circ (p \circ a) = p \circ (n \circ a) \tag{4.1.3}$$

Now if M is the monoid of all functions $A \to A$, $*$ is the composition and $\circ$ is the value function, then (4.1.3) establishes the center of M.

Secondly, taking the (1,1) commutativity of the families $(*, *)$, $(*, *)$ we have:

$$(m * n) * (p * q) = (m * p) * (n * q) \tag{4.1.4}$$

which contains simultaneousely the associativity and commutativity of $*$ in M.

4.2. As a second example, let take the type CIRCULAR–LIST (DATA) (see [3], Fig. 4.2.2.). The last axiom assert:

$$\text{JOIN (C, INSERT (C1, D))} = \text{INSERT (JOIN (C, C1), D)} \qquad (4.2.1)$$

for all circular_list C, C1 ; data D
where INSERT : (circular_list, data) → circular_list and

JOIN : (circular_list, circular_list) → circular_list

The (4.2.1) aseert the (2.1) commutativity of JOIN and INSERT in the sense of (3.4).
Finaly, the axiom 17 in the same specification is:

$$\text{RIGHT (INSERT (INSERT (C, D), D1))} = \text{INSERT (RIGHT (INSERT (C, D1)D)} \qquad (4.2.2)$$

where RIGHT : circular_list → circular_list.
We have RIGHT ∘ INSERT : (circular_list, data) → circular_list and then (4.2.2) express the (1,1)-commutativity of RIGHT ∘ INSERT and INSERT.

Naturaly, for more complicated types the commutativity relations are more complicated.

REFERENCES

1. G. Birkhoff, J. D. Lipson, *Heterogeneous algebras*, J. of Comb. Theory **8**, 115–133 (1970).
2. P. M. Cohn, *Universal algebra*, Harper & Row, New York, Evanston and London (1965).
3. J.V. Guttag, E. Horowitz, D.R. Musser, *The Design of Data Type Specification*, USC Information Science Institute, RR 76–49 (1976).
4. J.A. Goguen, J. W. Thatcher, E. G. Wagner, J. B. Wright, *An Intitial Algebra Aporoach to the Specification, Correctness and Implementation fo Abstract Data Types*, in Current Trends in Programming Methodology, IV: Data structuring, (R.T. Yeh, Ed.), Prentice-Hall, New-Jersey, 80–149 (1977).
5. Pic Gh., Purdea I., *Tratat de algebră modernă*, Vol. I, Ed. Academiei R.S.R., București (1977).
6. T. Rus, *Data Structures and Operating Systems*, Ed. Academiei R.S.R., București & John Wiley&Sons, New York (1979).
7. Wand, M., *Final Algebra Semantics and Data Type Extensions*, J. Comp. Syst. Sci. Vol 19 (1), Aug. (1979).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

SUR CERTAINES FORMULES DE QUADRATURE OPTIMALES

PARASCHIVA PAVEL*

Manuscrit reçu le 16 novembre 1983

ABSTRACT. — On Certain Optimal Quadrature Formulas. The quadrature formulae of (1)-type form are studied, with an exactness degree of (3), for which the rest is minimum with the functionscloss $W^{r+1}M$; x_0, x_m. It is proved that such formulae are only extent in the case when $m = 2p + 1$, and such formulas are effectively construed when $p = 3$ and $p = 4$ (formulae (9) and (11)), also estimations of their rest (formulae (10) and (12), respectively) being given.

Soit $W^{r+1}[M; x_0, x_m]$ l'ensemble des fonctions définies sur l'intervalle $[x_0, x_m]$, qui satisfont aux conditions: $f \in C^r[x_0, x_m]$, $f^{(r+1)}$ segmentaire continue et $|f^{(r+1)}(x)| \leqslant M$, $x \in [x_0, x_m]$.

On considère la formule de quadrature

$$\int_{x_0}^{x_m} f(x)\mathrm{d}x = A_0[f(x_0) + f(x_m)] + A_1[f(x_1) + f(x_{m-1})] + \ldots + A_p[f(x_p) + + f(x_{p-1})] + h[f(x_{p+1}) + \ldots + f(x_{m-p-1})] + R_{m+1}[f], \quad (1)$$

où $f \in W^{r+1}[M; x_0, x_m]$, $A_0, A_1, \ldots, A_p$ sont les coefficients, $0 \leqslant p \leqslant \left[\frac{m-1}{2}\right]$ et $x_i = x_0 + ih$, $i = 0, 1, \ldots, m$ les noeuds de la formule.

Dans ce travail, nous proposons de determiner les coefficients A_i, $i = 0, 1, \ldots, p$ de manière à ce que le degré d'exactitude soit égale a 3 et le reste $R_{m+1}(f)$ soit minime, quand $f \in W^4[M; x_0, x_m]$.

Ce problème a été aussi considéré par D u r a n d [9] dans le cas $r = 1$, $p = 1$; G. C o u l m y [3] dans le cas $r = 1$, $p = 3$; L a c r o i x [9] pour $r = 3$, $p = 2$. Dans ces articles le problème du reste n'a pas été posé.

D. V. I o n e s c u [5], [6], [7], [8] a déterminé les restes de formules (1) dans le cas $r = 1, 3, 5$, $p \leqslant 4$, en supposant que $f \in C^r[x_0, x_m]$

G h. C o m a n [1], [2] a détérminé les formules de quadrature (1) optimales pour la classe $W^2[M; x_0, x_m]$ dans l'hypothèse que le degré d'exactitude de la formule est $r = 1$.

En appliquant al mèthode de ,,la fonction φ'' donnée par le prof. D. V. I o n e s c u [4], nous prenons sur l'intervalle $[x_0, x_m]$ les noeuds $x_0, x_1, \ldots, x_m$; $x = x_0 + ih$; $i = 0, 1, \ldots, m$. Nous attachons aux intervalles

* *Université du Cluj-Napoca, Faculté de Mathematique-Physique, 3400 Cluj-Napoca, Romania*

$[x_0, x_1], [x_1, x_2), \ldots, [x_{m-1}, x_m]$ les fonctions $\varphi_1, \varphi_2, \ldots, \varphi_m$ solutions des équations différentielles :

$$\varphi_i^{(IV)}(x) - 1,\ i = 1, 2, \ldots, m, \tag{2}$$

avec les conditions aux limites

$$\begin{aligned}
&\varphi_1(x_0) = \varphi_1'(x_0) = \varphi_1''(x_0) = 0\,;\ -\varphi_1'''(x_0) = \varphi_m'''(x_m) = A_0\\
&\varphi_m(x_m) = \varphi_m'(x_m) = \varphi_m''(x_m) = 0\\
&\varphi_k^{(s)}(x_k) = \varphi_{k+1}^{(s)}(x_k)\,;\ s = 0, 1, 2\,;\ k = 1, 2, \ldots, m-1\\
&\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k) = \varphi_{m-k}'''(x_{m-k}) - \varphi_{m-k+1}'''(x_{m-k}) = A_k\\
&\qquad k = 1, 2, \ldots, p\\
&\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k) = h\,;\ k = p+1,\ p+2, \ldots, m-p-1.
\end{aligned} \tag{3}$$

On obtient la formule

$$\int_{x_0}^{x_m} f(x)\mathrm{d}x = -\varphi_1'''(x_0)f(x_0) + \sum_{k=1}^{m-1}[\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k)]\, f(x_k) +$$

$$+ \varphi_m'''(x_m)f(x_m) + \int_{x_0}^{x_m}\varphi(x)f^{(IV)}(x)dx,$$

avec le reste

$$R_{m+1}[f] = \int_{x_0}^{x_m}\varphi(x)f^{(IV)}(x)\,dx. \tag{4}$$

Les fonctions

$$\varphi_k(x) = \frac{(x-x_0)^4}{4!} - \sum_{i=0}^{p} A_i \frac{(x-x_i)_+^3}{3!} - h\sum_{i=p+1}\frac{(x-x_i)_+^3}{3!}, \tag{5}$$

où

$$u_+ = \begin{cases} u \text{ si } u > 0\\ 0 \text{ si } u \leqslant 0\end{cases}.$$

vérifient les équations différentielles (2) et les conditions aux limites (3) relativement aux points $x_0, x_1, \ldots, x_{m-1}$.
Il reste à déterminer les constantes A_k, de manière à satisfaire aussi les conditions au point x_m.

On obtient

$$\begin{aligned}
&\sum_{k=0}^{p} A_p = \frac{2p+1}{2}h\\
&\sum_{k=1}^{p} k^2A_k - m\sum_{k=1}^{p} kA_k = \frac{h}{6}p(p+1)(2p+1) - \frac{mh}{6}(6p(p+1)+1).
\end{aligned} \tag{6}$$

On observe que pour la détermination complète de la solution du prol (2)+(3), $p - 2$ conditions sont encore nécessaires

D'après (4) nous obtenons l'évaluation

$$|R_{m+1}(f)= \leqslant MJ,$$

où

$$J = \int_{x_0}^{x_m} |\varphi(x)|dx = \sum_{k=1}^{m} I_k \text{ et } I_k = \int_{x_{k-1}}^{x_k} |\varphi_k(x)|dx.$$

De cette manière le problème posé se reduit à la détermination des cients de la formule (1) tels que les intégrales

$$I_k = \int_{x_{k-1}}^{x_k} |\varphi_k(x)|dx; \; k = 4, 5, \ldots, p+1,$$

soient minimes

LEMME *Le polynome de Tchébychev de seconde espèce*

$$h_1^r Q_r\left(\frac{x-a}{h_1}\right); \; Q_r(x) = \frac{\sin(r+1)\arccos x}{2^r\sqrt{1-x^2}}; \; -1 \leqslant x \leqslant 1,$$

est l'unique polynome pour lequal l'intégrale $\int_{a-h_1}^{a+h_1} |P_r(x)|dx$

atteint son minimum. Ici $P_r(x)$ est un polynome arbitraire de degré r, lequel le coefficient de la puissance la plus élevée est égal à l'unité. Pour la démonstration de ce lemme, voy [10].

De cette manière le problème posé se reduit à tels que les poly φ_k, $k = 4, 5, \ldots, p+1$ coincident avec le polynome de Tchéb $h_1^4 \cdot Q_4\left(\frac{x-a}{h_1}\right)$, sur l'intervalle $[a - h_1, \; a + h_1]$, où

$$a = \frac{x_{k-1} + x_k}{2}, \; h_1 = \frac{x_k - x_{k-1}}{2}.$$

On obtient le système d'équations

$$\sum_{i=0}^{k-1} A_i = \frac{2k-1}{2} h$$

$$\sum_{i=1}^{k-1} iA_i = \frac{32k(k-1)+7}{64} h$$

$$\sum_{i=1}^{k-1} i^2 A_i = \frac{2k-1}{2} \cdot \frac{32k(k-1)+5}{96} h$$

$$\sum_{i=1}^{k-1} i^3 A_i = \frac{4(2k-1)^2(16k(k-1)+1)+1}{1024} h.$$

En tenant compte des conditions (6) + (7) il resulte que

$$m = 2p + 1. \tag{8}$$

La solution du système d'équations (7), avec la condition (8) est :

1° pour $p = 3$ ($m = 7$)

$$A_0 = \frac{1995}{6144} h, \ A_1 = \frac{8255}{6144} h, \ A_2 = \frac{4481}{6144} h, \ A_3 = \frac{6773}{6144} h.$$

La formule de quadrature correspondente este

$$\int_{x_0}^{x_7} f(x)dx = \frac{h}{6144} [1995(f(x_0) + f(x_7)) + 8255(f(x_1) + f(x_6)) +$$

$$4481(f(x_2) + f(x_5)) + 6773(f(x_3) + f(x_4))] + \int_{x_0}^{x_7} \varphi(x) f^{(IV)}(x) dx, \tag{9}$$

où

$$|R_7(f)| \leqslant 0{,}0899494 \ h^5 M. \tag{10}$$

2°. Dans le cas $p = 4$, le système d'èquations (7) a la solution

$$A_0 = A_4 - \frac{4469}{6144}, \ A_1 = -4A_4 + \frac{33951}{6144} h, \ A_2 = 6A_4 - \frac{33663}{6144} h,$$

$$A_3 = -4A_4 + \frac{31829}{6144} h, \ A_4 \text{ arbitraire}$$

ainsi la formule de quadrature optimale est :

$$\int_{x_0}^{x_9} f(x)dx = \left(A_4 - \frac{4469}{6144} h\right)(f(x_0) + f(x_9)) + \left(-4A_4 + \frac{33951}{6144} h\right) \cdot$$

$$\cdot (f(x_1) + f(x_8)) + \left(6A_4 - \frac{33663}{6144} h\right)(f(x_2) + f(x_7)) +$$

$$+ \left(-4A_4 + \frac{31829}{6144} h\right)(f(x_3)) + f(x_6) + A_4(f(x_4) + f(x_5)) + R_9[f].$$

En choisissant $A_4 = \frac{4469}{6144} h$, on obtient une formule de quadrature du type ouvert

$$\int_{x_0}^{x_9} f(x)dx = \frac{h}{6144} [16075(f(x_1) + f(x_8)) - 6849(f(x_2) + f(x_7)) +$$

$$+ 13953(f(x_3) + f(x_6)) + 4469(f(x_4) + f(x_5)) + \int_{x_0}^{x_9} \varphi(x) f^{(IV)}(x)dx,$$

avec le reste

$$|R(f_9)| \leqslant 0{,}7032546\ Mh^6.$$

BIBLIOGRAPHIE

1. C o m a n G h., *Formule practice de cuadratură, optimale pentru o clasă de funcţii*, St Univ. Babeş—Bolyai, fs. 1 (1971), 73—79.
2. C o m a n G h., *Asupra unor formule practice de cuadratură*, Studia Univ. Babeş—B fs. 1 (1972), 61—65.
3. C o u l m y G., *Opérations sur les courbes expérimentales*, C.R. de Seances de l'Ac Paris, 246 (1958), 1799—1800.
4. I o n e s c u D. V., *Cuadraturi numerice*, Bucureşti, 1957.
5. I o n e s c u D. V., *Citeva formule practice de cuadratură*, Comunicările Acad. R. 13, 8 (1963), 689—695.
6. I o n e s c u D. V., *Construirea unor formule practice de cuadratură*, Studii şi Cerc. tem., 15, 6 (1964). 757—769.
7. I o n e s c u D. V., *Nouvelles formules practiques de quadrature*, C.R. Acad. Sci. 259 (1964), 504—507.
8. I o n e s c u D. V., C o ţ i u A., *Une extension de la formule de quadrature de La* Mathematica, 9 (32), fs. 1 (1967), 49—52.
9. M i n e u r H., *Téchniques du calcul numérique*, Librairie Polytechnique Ch. Bér Paris, 1952.
10. N i k o l s k i S. M., *Formule de cuadratură*, Bucureşti, 1964.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

FUNDAMENTAL THEOREM OF ALGEBRA FOR GENERALIZED POLYNOMIAL MONOSPLINES

DIANE L. JOHNSON*

Received: September 17, 1984

ABSTRACT. — This paper presents a Fundamental Theorem of Algebra for Generalized Monosplines as introduced by Braess and Dyn [5]. Such monosplines generated by the polynomial spline kernel are of primary interest, but similar results are obtained for totally positive generalized monosplines where the corresponding kernel satisfies the cone condition of Burchard [6].

I. **Introduction.** An extended totally positive (ETP) kernel $K(x, y)$ is a function $K:[a, b]\times[c, d]\to\mathbf{R}$ such that for any set of points $a\leqslant x_1\leqslant x_2\leqslant\ldots\leqslant x_n\leqslant b$ and $c\leqslant y_1\leqslant y_2\leqslant\ldots\leqslant y_n\leqslant d$, the corresponding determinant $\det\{K(x_i, y_j)\}_{i,j=1}^{n}=K\begin{pmatrix}x_1 & x_2 & \ldots & x_n\\ y_1 & y_2 & \ldots & y_n\end{pmatrix}>0$. We call K a totally positive (TP) kernel if this determinant is nonnegative. Where the points coincide, we replace the function by increasing partial derivatives of the function and require sufficient smoothness of the kernel.

Let $K(x, y)$ be an ETP kernel on

$[a, b]\times[c, d]$ and define $Z_m^+=\{\nu=(\nu_0,\ldots,\nu_{m+1}):\nu_i\geqslant 0$

for $i=0, 1,\ldots, m+1\}$. For $\nu\in Z_n^+$, $\omega\in Z_m^+$ let $\sum_{i=0}^{m}\omega_i=N=\sum_{i=1}^{n}\nu_i$.

Define $\Delta^n[a, b]=\{x=(x_1,\ldots,x_n): a=x_0<x_1<\ldots<x_n<x_{n+1}=b\}$,

let $1=(0, 1,\ldots, 1, 0)\in Z_m^+$ and let $K_j(x, t)=\dfrac{\partial^j}{\partial t^j}K(x, t)$.

Define the sign function $\underset{t,\omega+1}{\sigma}(t)$ to be

$$\underset{t,\omega+1}{\sigma}(t)=(-1)^{\sum_{j=1}^{i}(\omega_j+1)}\quad\text{for } t_i\leqslant t<t_{i+1},\ i=0, 1,\ldots, m.$$

Here $t_0=c$ and $t_{m+1}=d$. The sign is normalized by

$\sigma_{t,\omega+1}(t)=+1$ for $c<t<t_1$ in accordance with Braess and Dyn [5]. We define the generalized monospline $M(x)$ by

$$M(x)=\int_c^d K(x, t)\underset{t,\omega+1}{\sigma}(t)\,dq(t)-\sum_{i=0}^{m}\sum_{j=0}^{\omega_i-1}a_{ij}K_j(x, t_i),\qquad(*)$$

where $d\rho$ is a nonnegative, nonatomic measure.

* *University of Cluj-Napoca, Faculty of Mathematics, 3400 Cluj-Napoca, Romania*

A generalized monospline can also be defined in the case where the generating kernel $K(x, y)$ on $(*)$ is only totally positive. If the kernal satisfies certain cone conditions, a fundamental theorem of algebra can be obtained (see Section II). The most studied kernel of this type is the polynomial spline kernel

$$K_n(x, t) = (x - t)_+^{n-1} \text{ where } x_+^r = \begin{cases} x^r & x \geqslant 0 \\ 0 & x < 0. \end{cases}$$

In this case we define the „Generalized Polynomial Monospline" M_n for $n - \omega_i \geqslant 1$

$$\text{by } M_n(x) = \int_c^d (x - t)_+^{n-1} \underset{t,\omega+1}{\sigma}(t)\, d\rho(t) - \sum_{i=0}^{m} \sum_{j=1}^{\omega_i} a_{ij}(x - t_i)_+^{n-j}. \qquad (1)$$

Then $M_n(x)$ is a polynominal of degree n on each of the intervals (t_i, t_{i+1}), $i = 0, \ldots m - 1$, and $M \in C^{n-\omega_i-1}$ in a neighborhood of t_i

It is necessary to study the zeros of such monosplines and as a result obtain a bound on the coefficients of generalized polynomial monosplines with a full set of zeros.

Throughout the following we will count multiplicities in the manner of Micchelli [13].

The following theorem which arises from the theory of generalized signs, will be of use in the following:

Theorem A [9]. *If the number of sign changes of a monospline of the form* $(*)$ *is given by* Z, *then* $Z \leqslant N$. *Moreover, if* $Z = N$, *then if the generalized sign vector is of the form* $(S_1, S_2, \ldots, S_{N+1})$, *then*

$$S_j = sgn\ M(x_j),\ j = 1,2, \ldots, N + 1.$$

Here $a < x_1 < x_2 < \ldots < x_{N+1} < b$ define the sign changes of $M(x)$.

II. The Zero Structure of Generalized Polynomial. Monosplines.

LEMMA 1: *Let* $M_n(x)$ *be as defined in* (1). *Then* M_n *has at most*

$$\sum_{i=0}^{m} \omega_i + m \text{ zeros, counting multiplicities.}$$

Proof. We first consider the case $n = 1$. Then $\underset{t,\omega+1}{\sigma}(t) = (-1)^{i+1} = +1$ and so this reduces to lemma 2.2 of Karlin and Schumacher [11] which states that M_1 has at most $2m + 1$ zeros, noting that $\omega_i = 1$ for all i.

As a monospline of the above type is of class $C^{n-2}(-\infty, \infty)$, for $n \geqslant 2$ we may use the theory of generalized signs. Therefore, using Theorem A the result is shown.

LEMMA 2: *Let M_n be a monospline of the form* (1) *which vanishes at* $x_1 < x_2 < \ldots < x_N$ *where* $N = \sum_{i=0}^{m} (\omega_i + 1) - 1$. *If* $\omega_i < n$ *then* $x_{k(i)} < t_i < x_{n+k(i-1)+1}$ *where* $k(i) = \sum_{j=1}^{i} (\omega_j + 1)$. *In the case that* $\omega_i = n$ *then* $t_i = x_{k(i)}$

Proof. Suppose $\omega_i < n$ and $t_i \leqslant x_{k(i)}$. Define M_+ to be the monospline which agress with M_+ to the right of t_i and has no knots to the left. Then M_+ has at least $\omega_0 + \sum_{j=i+1}^{m} (\omega_i + 1) + 1$ zeros since M_n is continuous at t_i, but M_+ has only $m - i$ knots. By lemma 1, M_+ can haveat-most $\omega_0 + \sum_{j=i+1}^{m} (\omega_j + 1)$ zeros, so the first inequality must hold. The reamining assertions follow in a similar manner.

PROPOSITION 1: *Given any* $K>0$ *there exists a* $\lambda > 0$ *such that whenever* $M(x)$ *is of the form* (1)

and M *has* $\sum_{i=0}^{m} \omega_i + m$ *distinct zeros in* $(-K, K)$ *then* $|a_{ij}| \leqslant \lambda$. for $i = 0, \ldots, m$, $j = 0, \omega_i - 1$.

Proof. The proof follows that of Micchelli [13, pg. 426]. It proceeds by simultaneous induction on n and m. The case $m = 0$, $n \geqslant 1$ is obvious. If $n = 1$ and $m \geqslant 1$ then $\omega_i = 1$, $i = 1, \ldots, r$ and this case is handled by Karlin and Schumacher [11].

Now suppose the proposition is true for all generalized monosplines of the form (1) with degree n and $m - 1$ knots. Let M be a monospline of form (1) of degree n with m knots. Consider first the case $\omega_i < n$, $i = 1, \ldots, m$.

Define $D_+ M(x) = \lim_{h \to 0^+} \frac{M(x+h) - M(x)}{h}$. Then $D_+ M$ is of the form (1) and by Rolle's theorem and lemma 1, $D_+ M$ has $\sum_{i=0}^{m} \omega_i + m - 1$ distinct zeros. Therefore the induction hypothesis implies that all coefficients of $D_+ M$ are bounded. Hence the same is true for M except possibly for the constant term λ_0. Since M has certainly one zero and all of its knots are in $(-K, K)$, we see that λ_0 is also bounded.

In the case of $m = n$ for some i, we can appeal to lemma 2 to conclude that the two monosplines M_+ and M_-, as defined above, both have a maximum number of zeros in $(-K, K)$. Applying the induction hypothesis to M_+ and M_- we again conclude that M has bounded coefficients.

We now include the possibility of multiple zeros, using a limiting procedure similar to that of Karlin and Schumacher [11].

PROPOSITION 2: *Given any $K > 0$ there exists a $\lambda > 0$ such that whenever M is of form* (1) *with* $\sum_{i=0}^{m} \omega_i + m$ *zeros up to order n in* $(-K, K)$ *then* $|a_{ij}| \leqslant \lambda$ *for* $i = 0, \ldots, m$ *and* $j = 0, \ldots, \omega_i - 1$.

Proof. Let ν_i be the multiplicity of the zero x_i, where $\sum_{i=1}^{n} \nu_i = N = \sum_{i=0}^{m} \omega_i + m$. We then „spread apart" the multiple zero x_i by defining $S_{m_i+j}(l) = x_i + j\varepsilon/2^l$ for $j = 0, 1, \ldots, \nu_i - 1$ and $m_i = \sum_{j=1}^{i-1} \nu_j + 1$, where ε is a sufficiently small positive number to insure that $-\infty < S_1 < S_2 < \ldots < S_N < \infty$.

By proposition 1, given any $K > 0$ there exists a $\lambda > 0$ such that whenever M is of the form (1) with zeros $S_i(l)$ in $(-K, K)$ then the corresponding coefficients a_{ij}^l satisfy $|a_{ij}^l| \leqslant \lambda$ for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, \omega_i - 1$. Noting that λ is independent of l, there must be a subsequence of coefficients converging as $l \to \infty$, where, in the limit, $|a_{ij}| \leqslant \lambda$ for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, \omega_i - 1$. By Rolle's theorem the resulting monospline $M(x)$ has zeros at the x_i with the desired multiplicities ν_i.

III. Generalized Gaussian Quadrature Formulas with Multiple Nodes for Weak Chebysev Systems. In this section we discuss multiple node Gaussian quadrature formulas for weak Chebysev systems where the integral contains a sign function as in the previous section. This will later be used to obtain a fundamental theorem of algebra for totally positive kernels.

An N-dimensional space of functions is called a weak Chebysev space if $u \in U$ implies that u has at most $N - 1$ sign changes.

Let $\{u_i\}_{i=1}^{N}$ be a basis for U, where the domain of U is $[-\delta, 1 + \delta]$ for some $\delta > 0$. Given a set of positive integers $\{\omega_i\}_{i=1}^{m}$ and two non-negative integers ω_0 and ω_{m+1}, we have the following two relationships:

$$\text{(a)}\ N = \sum_{i=0}^{m+1} \omega_i + m$$

and (b) U is a subspace of $C^k[-\delta, 1 + \delta]$, where

$$k \geqslant \max\{\max_{1 \leqslant i \leqslant m} \omega_i, \max_{i=0,\, m+1} (\omega_i - 1)\}.$$

Notice that if $\omega_0 \leqslant 1$ and $\omega_{m+1} \leqslant 1$, we can set $\delta = 0$.

Define the convexity cone $K(U)$ by

$$K(U) = \left\{ f \varepsilon C^k[-\delta, 1 + \delta] : 0 < t_1 < \ldots < t_{N+1} \Rightarrow U\begin{pmatrix} 1, \ldots, N f \\ t, \ldots, t_{N+1} \end{pmatrix} > 0 \right\}.$$

We then have the following assumption on the cone:

For each set $0 < t_1 < t_2 < \ldots < t_m < 1$,

$$U[t_1, \ldots, t_m] = \{(f(0), \ldots, \overset{(\omega_0-1)}{f}(0), f(t_1), \ldots, \overset{(\omega_1)}{f}(t_1), f(t_2), \ldots, \overset{(\omega_2)}{f}(t_2), \ldots, \overset{(\omega_m)}{f}(t_m),$$
$$f(1), \ldots, \overset{(\omega_{m+1})-1)}{f}(1) \quad) : f \in K(U)\}$$ contains a basis for $\mathbf{R}^N$.

Consider now a measure $d\alpha$ which has the property: For each subspace U_f generated by the functions $\{u_1, \ldots, u_N, f\}$ where $f \in K(U)$, $d\alpha$ is a positive measure. By this we mean that for every nontrivial nonnegative $u \in U_f, \int_0^1 u d\alpha > 0$. Let $\sigma_{t,\omega+1}(t)$ be defined on $[-\delta, 1+\delta]$ as in Section I.

A quadrature formula of the form

$Q(u) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i-1}, a_{ij} u^j(t_i)$ where $0 = t_0 < t_1 < \ldots < t_m < t_{m=1} = 1$, such that

$$Q(u) = \int_0^1 u(t) \sigma_{t,\omega+1}(t) \, d\alpha(t) \text{ for all } u \in U$$

will lead us to a fundamental theorem of algebra as desired.

Consider, therefore, the Gaussian transform of $u_i(x)$, defined by

$$u_i(x; \varepsilon) = \frac{1}{|\varepsilon|\sqrt{2\pi i}} \int_{-\delta}^{1+\delta} \exp\left(-\frac{1}{2}\left(\frac{y-x}{\varepsilon}\right)^2\right) u_i(y) dy$$

for each $\varepsilon \neq 0$ and $i = 1, \ldots, N$. For each $\varepsilon \neq 0$, it is well known that $\{u(x; \varepsilon) : i = 1, 2, \ldots, N\}$ forms an N-dimensional extended Chebyshev system. A result of Dyn [8]) tells us that for each $\varepsilon \neq 0$ there is a unique quadrature formula of the type

$$Q_\varepsilon(f) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i-1} a_{ij}(\varepsilon) \overset{(j)}{f}(t_i)$$

$$\text{so that } Q_\varepsilon(u_i(\cdot\,; \varepsilon)) = \int_0^1 u_i(x; \varepsilon) \sigma_{t((\varepsilon)),\omega+1}(x) d\alpha(x)$$

$$\text{for } i = 1, 2, \ldots, N,$$

$$\text{where } 0 = t_0(\varepsilon) < t_1(\varepsilon) < \ldots < t_m(\varepsilon) < t_{m+1}(\varepsilon) = 1.$$

By going to an appropriate subsequence we can assume that as $\varepsilon \downarrow 0$,

$$t_i(\varepsilon) \to t_i, \text{ where } 0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant t_{m+1} = 1.$$

Actually, for these limit points it is true that:

LEMMA 3: *The limit points satisfy* $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} =$

Proof. Assume, for example, that $0 = t_0 < t_1 = t_2 < t_3 < \ldots < t_m <$
A sequence of functions $\{u_\varepsilon\}$ will be constructed where u_ε in the sp of $\{u_i(\cdot\,;\,\varepsilon)\}_{i=1}^{N}$ is such that $Q_\varepsilon(u) = 0$ for each $\varepsilon > 0$. Further, as $\varepsilon \downarrow$

$$u \rightarrow u_\varepsilon \text{ uniformly where } \int u(t)\sigma_{t,\omega+1}(t)\, d\alpha \neq 0,$$

a contradiction.

To accomplish this, select a u which satisfies

$u_\varepsilon^{(j)}(0) = 0 \quad j = 0, \ldots, \omega_0 - 1$

$u_\varepsilon(\varepsilon) = 0,\ u_\varepsilon'(\varepsilon) > 0,\ \|u_\varepsilon\| = \max_{x\in[0,1]} |u_\varepsilon(x)| = 1$

$u_\varepsilon^{(j)}(t_i(\varepsilon)) = 0 \quad j = 0, \ldots, \omega_i - 1 \quad i = 1, 2$

$u_\varepsilon^{(j)}(t_i(\varepsilon)) = 0 \quad j = 0, \ldots, \omega_i \quad i = 3, 4, \ldots, m$

$u_\varepsilon^{(j)}(1) = 0 \quad j = 0, \ldots, \omega_{m+1} - 1.$

Recall that $\sigma_{t(\varepsilon),\omega+1}(t)$ is normalized so that $\sigma_{t(\varepsilon),\omega+1}(t) = +1$ for $0 < t < t_1(\varepsilon)$.

Notice that u_ε has $\sum_{i=0}^{2} \omega_i + \sum_{i=3}^{m} (\omega + 1) + \omega_{m+1} + 1 = N - 1$ zeros, allowing the certainty that u has no further sign changes.

By going to a subsequence it can be assumed that $u_\varepsilon \rightarrow u \in U$ u formly, where $\|u\| = 1$ and $u(t)\sigma_{t,\omega+1}(t) \geqslant 0$. Clearly $Q_\varepsilon(u_\varepsilon) = 0$ for each $\varepsilon >$ but $\int u\sigma d\alpha > 0$, which is the desired contradiction.

LEMMA 4: *For these limit knots,* $0 < t_1 < t_2 < \ldots < t_m < 1$, the det minant $D([t_1, \ldots, t_m])$ of

$$\begin{pmatrix} u_1(t_0)u_1'(t_0) & \ldots & u_1^{(\omega_0-1)}(t_0)u_1(t_1) & \ldots & u_1^{(\omega_1)}(t_1), \\ \vdots \\ u_N(t_0)u_N'(t_0) & \ldots & u_N^{(\omega_0-1)}(t_0)u_N(t_1) & \ldots & u_N^{(\omega_1)}(t_1) \\ u_1(t_2) \ldots u_1(t_m) & \ldots & u_1^{(\omega_m)}(t_m)u_1(t_{m+1}) & \ldots & u^{(\omega_{m+1}-1)}(t_m) \\ \vdots \\ u_N(t_2) \ldots u_N(t_m) & \ldots & u_N^{(\omega_m)}(t_m)u_N(t_{m+1} & \ldots & u_N^{(\omega_{m+1}-1)}(t_{m+1}) \end{pmatrix}$$

is positive, where $t_0 = 0$ *and* $t_{m+1} = 1.$

Proof. Assume that the conclusion is not valid. Then there is a set $\{c_i, c_{ij}, d_i\}$ of elements not all zero such that

$$F(u_l) = \sum_{i=0}^{\omega_0-1} c_i u_l^{(i)}(0) + \sum_{i=1}^{m} \sum_{j=0}^{\omega_i} c_{ij}\, u_l^{(j)}(t_i) + \sum_{i=0}^{\omega_{m+1}-1} d_i u_l^{(i)}(t_{m+1}) = 0 \qquad (2)$$

$$\text{for } l = 1, 2, \ldots, N.$$

Since we have assumed that $U[t_1, \ldots, t_m]$ contains a basis for $\mathbf{R}^N$, there is an $f_0 \in K(U)$ such that $F(f_0) \neq 0$. Define $f_0(x; \varepsilon)$ to be the Gaussian transform of f_0 and define

$$\hat{U}(\varepsilon) \equiv \left\{u(x; \varepsilon) = \sum_{i=1}^{N} \alpha_i u_i(x; \varepsilon) : \sum_{i=1}^{N} \alpha_i^2 \leqslant 1\right\},$$

$$\hat{U}(f_0, \varepsilon) \equiv \left\{v(x; \varepsilon) = \alpha_0 f_0(x; \varepsilon) + \sum_{i=1}^{N} \alpha_i u_i(x; \varepsilon) : \sum_{i=0}^{N} \alpha_i = 1\right\}$$

and

$$F_\varepsilon(g) = \sum_{i=0}^{\omega_i-1} c_i g^{(i)}(0) + \sum_{i=1}^{m} \sum_{j=0}^{\omega_i} c_{ij} g(t_i(\varepsilon)) + \sum_{i=0}^{\omega_{m+1}-1} d_i g^{(i)}(1).$$

As we have assumed that $F(u) = 0$ for $u \in U$, letting $t_i(\varepsilon) \to t_i$ we can secure for each $\eta > 0$ and $\varepsilon(\eta) > 0$ with the property $\varepsilon < \varepsilon(\eta)$ and $u(\cdot\,; \varepsilon) \in \hat{U}(\varepsilon) \Rightarrow |F_\varepsilon(u(\cdot\,; \varepsilon))| < \eta$. (3)

Also, as $F(f_0) \neq 0$, for small $\varepsilon > 0$, $F_\varepsilon(f_0(\cdot\,; \varepsilon))$ is bounded away from zero. Thus, for small $\varepsilon > 0$ we can find a C_ε which is uniformly bounded so that

$$\int_1^0 f_0(x; \varepsilon)\sigma_{t(\varepsilon),\omega+1}(x)d\alpha(x) = Q_\varepsilon(f_0(\cdot\,; \varepsilon)] + C_\varepsilon F_\varepsilon[f_0(\cdot\,; \varepsilon)]. \qquad (4)$$

On the other hand, (3) and the properties of Q_ε and F imply

$$\int_0^1 u_\varepsilon \sigma_{t(\varepsilon),\omega+1}\, r dx = Q_\varepsilon(u_\varepsilon) = Q_\varepsilon(u_\varepsilon) + C_\varepsilon F_\varepsilon(u_\varepsilon) + o(1) \qquad (5)$$

for all $u_\varepsilon \in \hat{U}(\varepsilon)$ as $\varepsilon \downarrow 0$.

Now for each $\varepsilon > 0$, choose a v_ε in $\hat{U}(f_0, \varepsilon)$ which satisfies

$v_\varepsilon^{(j)}(0) = 0$	$j = 0, \ldots, \omega_0 - 1$
$v_\varepsilon^{(j)}(t_i(\varepsilon)) = 0$	$j = 0, 1, \ldots, \omega_i,\ i = 1, \ldots, m$
$v_\varepsilon^{(j)}(1) = 0$	$j = 0, 1, \ldots, \omega_{m+1} - 1$
$v_\varepsilon(x) > 0$	for $x \in (0, t(\varepsilon))$.

Hence, as v_ε is a function in the span of $\{u_i(\cdot\,;\varepsilon)\}_{i=1}^{N}$ and $f_0(\cdot\,;\varepsilon)$, we can use equations (4) and (5) and the fact that $Q_\varepsilon(v_\varepsilon) = F_\varepsilon(v_\varepsilon) = 0$ by the construction of v_ε to conclude that

$$\int_0^1 v_\varepsilon \, \sigma_{t(\varepsilon),\omega+1} \, d\alpha = 0(1) \tag{6}$$

On the other hand, by going to a subsequence we can find a function v in $\hat{U}(f_0, 0)$ which is the uniform limit of $\{v_\varepsilon\}$. Moreover, v has sign $+1$ on $(0, t_1)$ and sign $(-1)^{\sum_{j=1}^{i}\omega_j}$ for $x \in (t_i, t_{i+1})$.

Thus

$$\int_0^1 v(x) \underset{t,\omega+1}{\sigma(x)} \, d\alpha(x) = \lim_{\varepsilon\to 0} \int_0^{t_1(\varepsilon)} |v_\varepsilon(x)| \, (+1)\,(+1)\, d\alpha(x) +$$

$$+ \lim_{\varepsilon\to 0} \int_{t_1(\varepsilon)}^{t_2(\varepsilon)} |v_\varepsilon(x)| (-1)^{\omega_1+1} (-1)^{\omega_1+1} \, d\alpha(x) + \ldots$$

$$+ \lim_{\varepsilon\to 0} \int_{t_m(\varepsilon)}^{1} |v_\varepsilon(x)| \, (-1)^{\sum_{i=1}^{m}\omega_i+m} \, (-1)^{\sum_{i=1}^{m}\omega_i+m} \, d\alpha(x)$$

which is strictly positive, contradicting (6).

We now prove a general existence theorem for generalized Gaussian quadrature formulas with respect to weak Chebyshev systems.

THEOREM 1. *There exists a generalized Gaussian quadrature formula of the form*

$$Q(u) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i - 1} a_{i_j} \, u^{(j)}(t_i) \ \textit{such that} \tag{7}$$

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma(t)} \, d\alpha(t) \ \textit{for all} \ u \in U,$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$.

Proof. By the result of Dyn, for each $\varepsilon \to 0$ there is a unique quadrature formula Q_ε with the property

$$Q_\varepsilon[u_i(\cdot\,;\varepsilon)] = \int_0^1 u_i(t;\varepsilon) \underset{t,\omega+1}{\sigma(t)} \, d\alpha(t) \qquad i = 1, \ldots, N.$$

Letting $\varepsilon \downarrow 0$, the coefficients associated with Q_ε must be bounded uniformly, for otherwise one could construct a set of coefficients $\{c_i, c_{i_j}, d_i\}$ not all zero such that the corresponding F [see (2)] $Fu = 0$ for all $u \in U$. To construct such a set of coefficients, assume that exists a coefficient $a_i \in \{c_i, c_{i_j}, d_i\}$ which is unbounded. Upon dividing by this coefficient, in the limit one obtains a non-zero F for which $Fu = 0$ for all $u \in U$.

As such a relationship contradicts lemma 4, the coefficients of $\{Q_\varepsilon\}$ are bounded as $\varepsilon \to 0$. Therefore, by compactness, there exists a limit for the coefficients of $Q = \lim_{\varepsilon \to 0} Q_\varepsilon$. Hence we have the existence of the desired quadrature formula.

LEMMA 5. *If* $Q(u) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i - 1} a_{i_j} u^{(j)}(t_i)$ *is such that*

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) \text{ for all } u \in U,$$

where $0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1,\ t_0 = 0,\ t_{m+1} = 1,$

then (a) $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1$ *and*

(b) $$\operatorname{sgn} a_{i,\omega_i - 1} = (-1)^{\sum_{j=1}^{i} (\omega_j + 1)}$$

Proof. Assume that (a) is not correct. Then a contraction can be reached as in lemma 3.

To prove (b), for each $1 \leqslant k \leqslant m$ and $\varepsilon > 0$ one can find a function $u(\cdot\,;\,\varepsilon)$ in the span of $\{(u_i(\cdot\,;\,\varepsilon)\}_{i=1}^{N}$ which satisfies

$$\|u(\cdot\,;\,\varepsilon)\| = \max_{x \in [0,1]} |u(x\,;\,\varepsilon)| = 1$$

$$\begin{aligned}
&u^{(j)}(0\,;\,\varepsilon) = 0 \quad && j = 0 \ldots, \omega_0 - 1\\
&u(\varepsilon\,;\,\varepsilon) = 0 &&\\
&u^{(j)}(t_i\,;\,\varepsilon) = 0 && j = 0, 1, \ldots, \omega_i, \quad i = 1, \ldots, k-1, \quad k+1, \ldots m\\
&u^{(j)}(t_k\,;\,\varepsilon) = 0 && j = 0, \ldots, \omega_k - 2\\
&(-1)^{\sum_{i=1}^{k} (\omega_i + 1)} u^{(\omega_i + 1)}(t_k\,;\,\varepsilon) > 0 &&\\
&u^{(j)}(1\,;\,\varepsilon) = 0 && j = 0, 1, \ldots, \omega_{m+1}
\end{aligned}$$

By letting $\varepsilon \downarrow 0$ we can secure a uniform limit function on $[0, 1]$, $u_k \in U$. As t_i is a zero of multiplicity $\omega_i + 1$ for $i \neq k$, $i = 1, \ldots, m$, the sign of the integrand $u_k(t) \underset{t,\omega+1}{\sigma}(t)$ remains constant over $[0, 1]$. As $u^{(\omega_k-1)}(t_k\,;\ \varepsilon) > 0$, $u_k \underset{t,\omega+1}{\sigma}(t)$ is a nonnegative product. Consider, then, the result of the fact that

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) \text{ for all } u \in U\,;$$

$$0 < \int_0^1 u_k(t) \underset{(t,\omega+1)}{\sigma(t)}\, d\alpha(t) = Q(u_k) = a_{k,\omega_k-1}\, u_k^{(\omega_k-1)}(t_k)$$

implies that

$$\operatorname{sgn} a_{k,\omega_k-1} = \operatorname{sgn} u_k^{(\omega_k-1)}(t_l) = (-1)^{\sum_{i=1}^{k}(\omega_i+1)} .$$

THEOREM 2. *There is a unique quadrature formula of the form* (7) *such that*

$$Q(u) = \int u \underset{t,\omega+1}{\sigma}(t)\, d\alpha \tag{8}$$

for all $u \in U$. [Note that by Lemma 3 all the t_i are distinct and lie in $(0, 1)$.]

Proof. By Theorem 1 there exists a formula Q^* of the form (7) which satisfies (8). Let

$$A^* = [a_{00}^*, \ldots, a_{0,\omega_0-1}^*, a_{10}^*, \ldots, a_{1,\omega_1-1}^*, a_{20}^*, \ldots, a_{m,\omega_m-1}^*, a_{m+1,0}^*, \ldots, a_{m+1,\omega_{m+1}-1}^*, t_1^*, \ldots, t_m^*] \subset \mathbf{R}^N$$

be the set of values which define Q^*, and for $|\varepsilon| > 0$, let $u_i(\cdot\,;\ \varepsilon)$ be the Gaussian transform of $u_i(\cdot\,;\ 0) \equiv u_i$. Consider the nonlinear system of N equations

$$Q\,[u_i(\cdot\,;\ \varepsilon),\ A] = \int_0^1 u_i(t\,;\ \varepsilon) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) \tag{9}$$

with the vector of N unknowns:

$$A = [a_{00}, \ldots, a_{0,\omega_0-1}, a_{01}, \ldots, a_{1,\omega_1-1}, a_{20}, \ldots, a_{m,\omega_m-1}, a_{m+1,0}, \ldots, a_{i,+1,\omega_{m+1}-1}, t_1, \ldots, t_m]$$

associated with any quadrature of the form (7). For $\varepsilon = 0$, clearly $Q^* = Q(\cdot\,;\ A^*)$ satisfies (9). We indicate his dependence by letting $A^* = A^*(\varepsilon)$ for $\varepsilon = 0$. We would like to aaply the implicit function theorem to (9) with the parameter ε near $\varepsilon = 0$. At $A = A^*$ and $\varepsilon = 0$, the

Jacobian determinant at $A = A^*$ and $\varepsilon = 0$ is $\pm \prod_{i=1}^{m} a^*_{i,\omega_i-1} D[t^*_1, t^*_2, \ldots, t^*_m]$ where $D(t_1, \ldots, t_m)$ is as defined in lemma 4. By lemma 5,

$$\operatorname{sgn} a_{i,\omega_i-1} = (-1)^{\sum_{j=1}^{i}(\omega_j+1)} \qquad i = 1, \ldots, m,$$

hence $a_{i,\omega_i-1} \neq 0$ for $i = 1, \ldots, m$. Further, an argument fashioned after Lemma 4 shows that $D(t^*_1, \ldots, t^*_m) > 0$. By the implicit function theorem, for small $|\varepsilon| > 0$ one can find a solution $A^*(\varepsilon)$ of the nonlinear system (9) close to A^*.

Assume now that there is another solution to (9) at $\varepsilon = 0$, say $\hat{A} = \hat{A}(0)$. Then by the same reasoning one could find a solution $\hat{A}(\varepsilon)$ of (9) close to $\hat{A}$ for small $|\varepsilon| > 0$. For even smaller $|\varepsilon| > 0$, $\hat{A}(\varepsilon) \neq \neq A^*(\varepsilon)$. Thus far such ε, (9) has at least two solutions, contradicting the result of Dyn.

IV. Fundamental Theorem of Algebra for Generalized Polynomial Monosplines. Consider the polynomial spline kernel $\Phi_p(x, t) = \frac{(x-t)_+^{p-1}}{(p-1)!}$, where $p \geqslant 3$. We are given a nonnegative integer $\omega_0 \leqslant p$, positive integers $\{\omega_i\}_{i=1}^{m}$ and positive integers $\{\nu_i\}_{i=1}^{n}$ which satisfy the relation ships

(a) $$N = \sum_{i=0}^{m} \omega_i + m = \sum_{i=1}^{n} \nu_i,$$

(b) If $M_1 = \max_{1 \leqslant i \leqslant m} \omega_i$ and $M_2 = \max_{1 \leqslant i \leqslant m} \nu_i$,

then $$M_1 + M_2 \leqslant p - 1.$$

THEOREM 3. *For each set of n distinct numbers,* $0 < x_1 < x_2 < \ldots \ldots < x_n < 1$, *there is a unique generalized monospline of the form*

$$M(x) = \int_0^1 \Phi_p(x, t) \underset{t,\omega+1}{\sigma(t)}\, d\alpha(t) - \sum_{j=0}^{\omega_0-1} a_j \Phi_p^{(j)}(x, 0) - \sum_{i=1}^{m} \sum_{j=0}^{\omega_i-1} a_{i_j} \Phi_p^{(j)}(x, t_i)$$

such that M_p *has a zero of order* ν_i *at* x_i, $i = 1, \ldots, n$.

Here
$$\Phi_p^{(j)}(x, t) = \frac{\partial^j}{\partial t_j} \Phi_p(x, t) \quad \textit{and} \quad 0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1.$$

Indeed, for this monospline, $0 < t_1 < t_2 < \ldots < t_m < 1$ *and*

$$\operatorname{sgn} a_{i,\omega_i-1} = (-1)^{\sum_{j=1}^{i}(\omega_j+1)}, \quad i = 1, \ldots, m.$$

Proof:

We set $S(k) = \sum_{j=0}^{k-1} \nu_j$, $k = 1, \ldots, n$ where $\nu_0 = 0$, and define

$$u_{S(k)+l}(t) = \frac{\partial^{l-1}}{\partial x_k - 1} \Phi_p(x_k, t) \qquad \begin{array}{l} l = 1, \ldots, \nu_k \\ k = 1, \ldots, n. \end{array}$$

So $M^{(l-1)}(x_k)$ translates into

$$\int_0^1 u_{S(k)+l}(t) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) = \sum_{j=0}^{\omega_0-1} a_j\, u^{(j)}_{S(k)+l}(0) + \sum_{i=1}^{m} \sum_{j=0}^{\omega_i-1} a_{i_j}\, u^{(j)}_{S(k)+l}(t_i) \qquad \begin{array}{l} l = 1, \ldots, \nu_k \\ k = 1, \ldots, n \end{array}$$

From the fundamental theorem of determinants for polynomial splines [12], one can infer that

(a) $\left\{u_i(t)\right\}_{i=1}^{N}$ form a weak Chebyshev system.

(b) For each $x \in [0, 1]$ one of the functions $f(t) = \pm \Phi_p(x, t)$ is in the convex cone $K(U)$ of $\left\{u_i(t)\right\}_{i=1}^{N}$.

(c) For each sequence $0 < t_1 < \ldots < t_m < 1$, the set of N functions

$$\Phi_p(x, 0), \ldots, \Phi_p^{(\omega_0-1)}(x, 0), \Phi_p(x, t_1), \ldots, \Phi_p^{(\omega_1)}(x, t),$$

$$\Phi_p(x, t_2), \ldots, \Phi_p^{(\omega_{m-1})}(x, t_{m-1}), \Phi_p(x, t_m), \ldots, \Phi_p^{(\omega_m)}(x, t_m)$$

is independent.

The fact that $\pm\Phi_p(x, t)$ is in the convexity cone of $\{u_i(t)\}_{i=1}$ combined with the independence in (c) tells us that for each $0 < t_1 < \ldots < t_m < 1$, $U[t_1, \ldots, t_m]$ contains a basis for $\mathbf{R}^N$.
Therefore, the result follows directly from theorem 2.

One can also obtain similar results by including the right hand end point $t_{m+1} = 1$. We seek an expression of the type

$$Q(u) = \sum_{i=1}^{m} \sum_{j=0}^{\omega_i-1} a_{i_j}\, u^{(j)}(t_i) + \sum_{i=0}^{\omega_{m+1}-1} b_i\, u^{(i)}(1), \tag{10}$$

where $0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1$, such that

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) \tag{11}$$

for all u in the N-dimensional subspace generated by $\left\{u_i\right\}_{i=1}^{N}$.
The $\{\omega_i\}$, $\{\nu_i\}$ and ω_{m+1} satisfy the same restraints as in the first application.

Proceeding exactly as before, we can show:

THEOREM 4. *There exists a unique Q of the form* (10) *which satisfies* (11). *Furthermore, for such a Q, $0 < t_1 < \dots < t_m < 1$ and a_{i,ω_i-1} has sign $(-1)^{\sum_{j=1}^{i}(\omega_j+1)}$ for $i = 1, \dots, m$.*

Remark:

For the cases $p = 1$ and $p = 2$ in the defining polynomial spline kernel

$$\Phi_p(x, t) = \frac{(x-t)_+^{p-1}}{(p-1)!}$$

similar results can be obtained. In the case that $p = 1$, the relationships (a) and (b) preceeding Theorem 3 reduce to the restriction that

(a)' $\qquad N = 2m + 1$ where $\omega_i = 1$ for $i = 0, \dots, m$

and

(b)' $\qquad M_1 = M_2 = 1. \qquad \nu_i = 1$ for $i = 1, \dots, n$

In this setting $\underset{t,\omega+1}{\sigma(t)} = +1$ for all t and therefore the generalized monospline reduces to the Tchebycheffian (T^-) monospline of degree 1 with m knots considered by Karlin and Schumacher [11]. The fundamental theorem of algebra for the case $p = 1$ is found in Theorem 1.1 of that paper.

Consider the case $p = 2$, where

$$M(x) = \int_{c=t_0}^{d} (x-t)_+^1 \underset{t,\omega+1}{\sigma(t)}\, dt - \sum_{i=0}^{m}\sum_{j=1}^{\omega_i} a_{ij}(x-t)_+^{2-j}. \tag{12}$$

We wish to show the existence and uniqueness of such a generalized monospline where $\{x_i\}_{i=1}^{N}$ are given and we require that $M(x_i) = 0$ for $i = 1, 2, \dots, N$. We assume first that $a < x_1 < x_2 < \dots < x_n < b$.

Consider first the monospline $M_1(x)$ which is the restriction of M to (t_0, t_1). By lemma 4 Section III, $x_{2i} < t_i < x_{2i+1}$ for $i = 1, 2, \dots, m$. By definition, therefore

$$M_1(x) = \frac{(x-c)^2}{2} - a_{01}(x-c) - a_{02}.$$

In the case $\omega_0 = 1$, $M_1(x) = (x-c)\left(\frac{(x-c)}{2} - a_{01}\right)$ and so $M_1(x_1) = M_1(x_2) = 0$ implies that we must have $x_1 = c$ and $a_{01} = \frac{x_2 - c}{2}$.

If $\omega_0 = 2$, then a_{01} and a_{02} are given by unique solutions to a linear system induced by the zero structure. The determinant is nonzero as a result of the fact that $c < x_1 < x_2$. The right hand side is nonzero for the same reason, giving a unique set of defining coefficients for $M(x)$.

In considering $M(x)$ in the interval $t_1 < x < t_2$, as $x_4 < t_2 < x_5$ by lemma 2 of Section I

$$M(x) = M_1(x) + [(-1)^{\omega_1} + 1]\left(\frac{(x-t_1)^2}{2}\right) - a_{11}(x - t_1) - a_{12}.$$

We then use the fact that $M(x_3) = M(x_4) = 0$ to determine the unknowns a_{11}, a_{12} and t_1.

If $\omega_1 = 1$ then we are in the classical case and Theorem 1 of Micchelli [13] gives the desired result.

If $\omega_1 = 2$ then by lemma 4 of Section III, $t_1 = x_3$. Thus we can solve for a_{11} and a_{12} using the equations $M(x_3) = 0 = M_1(x_3) - a_{12}$ and $M(x_4) = 0 = M_1(x_4) + (x_4 - x_3)^2 - a_{11}(x_4 - x_3) - a_{12}$. Therefore $a_{12} = M_1(x_3)$ and $a_{11} = M_1(x_3, x_4] + (x_4 - x_3)$
where $M_1[x_3, x_4]$ denotes the divided difference of M_1 with respect to x_3 and x_4 (see ref. [7], page 195). Note that $M(x)$ is not identically zero for if it were $M_1(x)$ would be identically equal to $p_2(x) = -(x - x_3)^2 + a_{11}(x - x_3) + a_{12}$. Upon examining the roots of $p_2(x)$ one finds that it has one real root to the right of x_3. Thus M has a third root and must be identically zero, a contradiction.

The above process may be repeated to recursively determine the set $\{a_{k1}, a_{k2}, t_k\}_{k=1}^{m}$ and so follows the existence and uniquences of a generalized polynomial monospline when $p = 2$ and we have simple $\{x_i\}_{i=1}^{N}$.

To allow the zeros to have multiplicity two when $p = 2$ we employ a limiting argument similar to that of Karlin and Schumacher ([II], page 267). In this case

$$N = \sum_{i=0}^{m} \omega_i + m = \sum_{i=1}^{r} n_i \text{ where if } \omega = \max_{0 < i \leqslant m} \omega_i \text{ and}$$

$n = \max\limits_{1 \leqslant i \leqslant r} n_i$, then $\omega + n \leqslant 2$, and we have prescribed zeros of $M(x)$ at x_i of order n_i, $i = 1, \ldots, r$. Here $M(x)$ is of the form (1).

For each $l \geqslant 1$ consider a set of points $\{y_i(l)\}_{i=1}^{N}$ formed from $\{x_i\}_{i=1}^{r}$ by „spreading apart'' the multiple zeros. Specifically, if $x_{m-1} < x_m = x_{m+1} < x_{m+2}$ for some $2 \leqslant m \leqslant r - 1$, then define $y_{m+1}(l) = x_m + \frac{\varepsilon}{2l}$ where ε is a sufficiently small positive number to insure that $y_i(l) < y_{i+1}(l)$ for $i = 1, \ldots, N - 1$. For each l there exists a generalized monospline of the form (12), call it M_l, with zeros $\{y_i(l)\}_{i=1}^{N}$. Suppose that M_l has the representation

$$M_l(x) = \int_c^d (x - t)_+^{n-1} \underset{l,\omega+1}{\sigma}(t)\, dt - \sum_{i=0}^{m}\sum_{j=1}^{\omega_i} a_{ij}^{l}(x - t_i^l)_+^{n-j}.$$

The sequence of coefficients $\left\{a_{ij}^{l}\right\}_{i=0\,j=1}^{m\ \ \omega_i}$ and the sequence of knots $\left\{t_i^l\right\}_{i=0}^{m}$ depend continuously on the variable x and hence on l. By Proposition 1 of Section II the coefficients are uniformly bounded as $l \to \infty$. The knots $\left\{t_i\right\}_{i=0}^{m}$ are trivially bounded as noted by lemma 2, Section II.

Thus there exists a subsequence $\{l_k\}$ such that all coefficients and knots converge. By continuity and Rolle's theorem, the limit generalized monospline has the desired zeros $\left\{x_i\right\}_{i=1}^{r}$.

We can now state an extension to theorem 3:

THEOREM 5. *The results of theorem* **3** *remain valid if* $p = 1$ *or* $p = 2$.

REFERENCES

1. R. B. Barrar and H. Loeb, „*On a Nonlinear Characterization Problem for Monosplines*", *Journal of Approximation Theory* 18, 220–240 (1976).
2. „*Multiple Zeros and Applications to Optimal Linear Functionals*", *Numer. Math.* 25, 251, 262 (1976).
3. „*Fundamental Theorem of Algebra for Monosplines and Related Results*", *SIAM J. Numerical Analysis*, Vol. 17, No. 6, December 1980.
4. D. L. Barrow, „*On Multiple Node Caussian Quadrature Formulae*", *Mathematics of Computation*, Vol. 32, No. 142, 431–439, April 1978.
5. D. Braess and N. Dyn, „*On the Uniqueness of Monosplines and Perfect Splines of Least L—1 and L—2 Norm*", *Journal D'Analyse Mathematique*, Vol. 41 (1982).
6. H. Burchard, *Interpolation and Approximation by Generalized Convex Functions*, Ph. D. dissertation, Purdue University, West Layfayette, IB., 1968.
7. S. D. Conte and C. de Boor, *Elementary Numerical Analysis, An Algorithmic Approach*, McGraw Hill, New York, 1965.
8. N. Dyn, „*Existence and Uniqueness of Canonical Points for best L—1 Approximation with Prescribed Multiplicites of Interpolation Conditions*", to appear.
9. D. Johnson, „*Fixed Point Evaluation in Caussian Quadrature and Generalized Monosplines of Minimal Norm*", Ph. D. dissertation, University of Oregon, Eugene, 1983.
10. S. Karlin and A. Pinkus, „*Caussian Quadrature Formulae with Multiple Nodes*", *Studies in Spline Funcions and Approximation Theory*, Academic Press, New York, 1976.
11. S. Karlin and L. Schumaker, „*The Fundamental Theorem of Algebra for Tchebycheffian Monosplines*", *J. Analyse Math.*, 20, 233–270 (1967).
12. S. Karlin and W. J. Studden, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Interscience Publischers, New York, 1966.
13. C. A. Micchelli, „*The Fundamental Theorem of Algebra for Monosplines with Multiplicites*", *Linear Operators and Approximation, Proc. Conf. in Oberwolfach*, P. L. Butzer, J. P. Kahane and B. Sz. Nagy, eds., Burkhauser Verlag, Basel, Switzerland, 14–22, 1971.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 2, 1986

ON SOME CLASSES OF BI-UNIVALENT FUNCTIONS

D.A. BRANNAN*, T.S. TAHA**

Received: September 26, 1984

ABSTRACT. — Let the functions $f(z) = z + a_2z^2 + \ldots$ and its inverse f^{-1} be analytic and univalent in the unit disc. The authors obtain upper bounds for $|a_2|$ and $|a_3|$ under various additional hypotheses — namely, that f and f^{-1} are both (i) strongly starlike of order α, (ii) starlike of order β, (iii) convex of order β.

1. **Introduction.** In this note we discuss several classes of functic

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.}$$

that are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. T class of all such functions we denote by S. We denote by σ the class all functions of the form (1.1) that are analytic and bi-univalent in t unit disc, that is $f \in S$ and f^{-1} has a univalent analytic continuation $\{|w| < 1\}$. We also introduce the following classes:

(i) The class $S^*_\sigma[\alpha]$ of strongly bi-starlike functions of order α, $0 < \alpha \leqslant$

(ii) The class $S^*_\sigma(\beta)$ of bi-starlike functions of order β, $0 \leqslant \beta < 1$.

(iii) The class $C_\sigma(\beta)$ of bi-convex functions of order β, $0 \leqslant \beta < 1$.

For the above classes we give bounds for $|a_2|$, $|a_3|$; also for the cla $C_\sigma(0)$ we give the bound for $|a_n|$ and the extremal function.

The class σ was first investigated by Lewin [1]; the showed th $|a_2| < 1.51$. Later Brannan [2, Problem 6.82] conjectured that $|a_2| \leqslant \sqrt{2}$. The class $S^*_\sigma[\alpha]$ and the class $C_\sigma(0) \equiv C_\sigma$ were first introduced in [

2. The class $S^*_\sigma[\alpha]$

A function $f(z)$ of the form (1.1) belongs to the class $S^*_\sigma[\alpha]$, $0 < \alpha <$ if it satisfies the following set of conditions:

$$f \in \sigma, \tag{2.}$$

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \quad |z| < 1, \tag{2.}$$

$$\left|\arg \frac{wg'(w)}{g(w)}\right| < \frac{\alpha\pi}{2}, \quad |w| < 1, \tag{2.}$$

---

* *The Open University, Milton Keynes, U.K.*

** *University of Kuwait, Kuwait*

where

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \dots, \tag{2.4}$$

is the extension of f^{-1} to the whole of $|w| < 1$.

THEOREM 2.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belong to $S_\sigma^*[\alpha]$. *Then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{1+\alpha}} \quad and \quad |a_3| \leqslant 2\alpha.$$

Proof. We are going to follow the notation used in [4]; namely, we denote by P_α, $0 < \alpha \leqslant 1$, the class of functions

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

that are analytic in the unit disc U and subordinate to the function $\left[\frac{1+z}{1-z}\right]^\alpha$. Now, $P(z) \in P_\alpha$ if an only if $P(z) = [h(z)]^\alpha$, where $h(z) \in P_1$; and P_1 is the class of functions of positive real part in U.

Conditions (2.2) and (2.3) can be written as

$$\frac{zf'(z)}{f(z)} = [Q(z)]^\alpha \tag{2.5}$$

and

$$\frac{wg'(w)}{g(w)} = [P(w)]^\alpha, \tag{2.6}$$

respectively, where $Q(z)$, $P(w)$ belong to P_1 and have the forms

$$Q(z) = 1 + c_1z + c_2z + \dots$$

and

$$P(z) = 1 + p_1w + p_2w^2 + \dots$$

If $f(z) \in S_\sigma^*(\alpha)$, then by (2.5)

$$\frac{zf'(z)}{f(z)} = [Q(z)]^\alpha = [1 + c_1z + c_2z^2 + \dots]^\alpha.$$

From this, it follows that

$$a_2 = \alpha c_1$$

$$2a_3 = a_2^2 + \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2.$$

Also by (2.6)

$$\frac{wg'(w)}{g(w)} = [p(w)]^\alpha = [1 + p_1 w + p_2 w^2 + \dots]^\alpha.$$

This gives

$$a_2 = -\alpha p_1$$

$$3a_2^2 = 2a_3 + \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2.$$

Combining the set of equations for a_2, a_3 we obtain

$$a_2^2 = \frac{\alpha^2(c_2 + p_2)}{\alpha + 1}.$$ $|p_n| \le 2$

By a well known theorem due to Carathéodory [5, page 41], $|p_n| \le 2$, $|c_n| \le 2$. Hence

$$|a_2| \le \frac{2\alpha}{\sqrt{1 + \alpha}}.$$

For a_3 we have

$$4a_3 = \alpha(p_2 + 3c_2) + 2\alpha(\alpha - 1)c_1^2. \quad (2.7) \tag{2.7}$$

If $\alpha = 1$, then $|a_3| \le 2$. So we consider the case $0 < \alpha < 1$. By (2.7)

By (2.7) $$4 \operatorname{Re} a_3 = \alpha \operatorname{Re} \{p_2 + 3c_2 - 2(1 - \alpha)c_1^2\}. \quad (2.8) \tag{2.8}$$

For the functions $Q(z)$, $P(w)$, Herglotz's representation formula [5, page 40] states that

$$Q(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu_1(t)$$

and

$$P(w) = \int_0^{2\pi} \frac{1 + we^{-it}}{1 - we^{-it}}\, d\mu_2(t),$$

where $\mu_i(t)$ are increasing on $[0, 2\pi]$ and $\mu_i(2\pi) - \mu_i(0) = 1$, $i = 1, 2$.

We also have

$$c_n = 2\int_0^{2\pi} e^{-int}\, d\mu_1(t), \quad n = 1, 2, \dots,$$

and

$$p_n = 2\int_0^{2\pi} e^{-int}\, d\mu_2(t), \quad n = 1, 2, \dots.$$

Now (2.8) becomes

$$4\,\mathrm{Re}\, a_3 = 2\alpha\int_0^{2\pi}\cos 2t\, d\mu_2(t) + 6\alpha\int_0^{2\pi}\cos 2t\, d\mu_1(t) -$$

$$- 8\alpha(1-\alpha)\left\{\left[\int_0^{2\pi}\cos t\, d\mu_1(t)\right]^2 - \left[\int_0^{2\pi}\sin t\, d\mu_1(t)\right]^2\right\},$$

$$\leqslant 2\alpha\int_0^{2\pi}\cos 2t\, d\mu_2(t) + 6\alpha\int_0^{2\pi}\cos 2t\, d\mu_1(t) + 8\alpha(1-\alpha)\left[\int_0^{2\pi}\sin t\, d\mu_1(t)\right]^2 =$$

$$= 2\alpha\left\{1 - 2\int_0^{2\pi}\sin^2 t\, d\mu_2(t) + 3 - 6\int_0^{2\pi}\sin^2 t\, d\mu_1(t) + \right.$$

$$\left. + 4(1-\alpha)\left[\int_0^{2\pi}\sin t\, d\mu_1(t)\right]^2\right\}.$$

By Jensen's inequality [6, page 61], we have that

$$\left[\int_0^{2\pi}|\sin t|\, d\mu(t)\right]^2 \leqslant \int_0^{2\pi}\sin^2 t\, d\mu(t).$$

Hence

$$4\,\mathrm{Re}\, a_3 \leqslant 2\alpha\left\{4 - 2\int_0^{2\pi}\sin^2 t\, d\mu_2(t) - 2(1+2\alpha)\int_0^{2\pi}\sin^2 t\, d\mu_1(t)\right\}.$$

Therefore $\mathrm{Re}\, a_3 \leqslant 2\alpha$, which implies that

$$|a_3| \leqslant 2\alpha.$$

The effect of the bi-univalency condition can be easily seen by looking at the coefficients of the corresponding class $S^*[\alpha]$ introduced in [4]; this is the class of functions f of the form (1.1) univalent in $|z| < 1$ and satisfying the condition (2.2). There the sharp coefficient bounds are

$$|a_2| \leqq 2\alpha,$$

and

$$\text{if } 0 < \alpha < \frac{1}{3}, \text{ then } |a_3| \leqq \alpha,$$

$$\text{if } \frac{1}{3} < \alpha \leqq 1, \text{ then } |a_3| \leqq 3\alpha^2,$$

and

$$\text{if } \alpha = \frac{1}{3}, \text{ then } |a_3| \leqslant \frac{1}{3}\cdot$$

In each case the stated coefficient bound is sharp.

It would be of interest to know what the sharp bounds on the coefficients a_2, a_3 are in the class $S_\sigma^*[\alpha]$.

3. The class $S_\sigma^*[\beta]$

We define the class $S_\sigma^*(\beta)$, $0 \leqslant \beta < 1$, to be the class of functions of the form (1.1) satisfying the following conditions:

$$f \in \sigma,$$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \ |z| < 1, \tag{3.1}$$

and

$$\operatorname{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > \beta, \ |w| < 1, \tag{3.2}$$

where $g(w)$ is the same function as in (2.4). We call $S_\sigma^*(\beta)$ the class of bi-starlike functions of order β.

THEOREM 3.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belong to $S_\sigma^*(\beta)$, $0 \leqslant \beta < 1$. *Then*

$$|a_2| \leqslant \sqrt{2(1-\beta)} \ \textit{and} \ |a_3| \leqslant 2(1-\beta).$$

Proof: Let $P(\beta)$ be the class of functions $V(z)$ analytic in $|z| < 1$ with $V(0) = 1$, Re $>$ $(z) > \beta$ in $|z| < 1$.

In fact $P(0)$ is just the class of functions

$$P(z) = 1 + p_1 z + p_2 z^2 + \dots$$

for which Re $P(z) > 0$.

Note that $V(z) \in P(\beta)$ if and only if

$$P(z) = \frac{1}{1-\beta}(V(z) - \beta) \text{ belongs to } P(0).$$

Hence, it follows that there exists a unique $P(z) \in P(0)$ such that

$$V(z) = \beta + (1-\beta)p(z), \tag{3.3}$$

for $V(z)$ in $P(\beta)$.

Now conditions (3.1) and (3.2) are equivalent to

$$\frac{zf'(z)}{f(z)} = \beta + (1 - \beta)Q(z) \tag{3.4}$$

and

$$\frac{wg'(w)}{g(w)} = \beta + (1 - \beta)P(w), \tag{3.5}$$

respectively, where $Q(z)$, $P(w)$ belong to $P(0)$ and have the forms

$$Q(z) = 1 + c_1z + c_2z^2 + \dots$$

and

$$P(z) = 1 + p_1w + p_2w^2 + \dots .$$

Now, it follows from (3.4) that

$$a_2 = (1 - \beta)c_1 \tag{3.6}$$

and

$$2a_3 = a_2(1 - \beta)c_1 + (1 - \beta)c_2. \tag{3.7}$$

Also from (3.5) it follows that

$$a_2 = -(1 - \beta)p_1 \tag{3.8}$$

and

$$4a_2^2 = 2a_3 - a_2(1 - \beta)p_1 + (1 - \beta)p_2. \tag{3.9}$$

The four equations give

$$2a_2^2 = (1 - \beta)(c_2 + p_2).$$

Using the bounds for $|c_2|$ and $|p_2|$, we obtain

$$|a_2| \leqslant \sqrt{2(1 - \beta)}$$

and

$$|a_3| \leqslant 2(1 - \beta).$$

In comparison, let $S^*(\beta)$, $0 < \beta \leqq 1$, denote the class of functions starlike of order β in $|z| < 1$; this is the class of functions f of the form (1.1) univalent in $|z| < 1$ and satisfying the condition (3.1). It was shown in [7] that the sharp coefficient bounds for a_2, a_3 are

$$|a_2| \leqq 2(1 - \beta),$$

$$|a_3| \leqq (1 - \beta)(3 - 2\beta).$$

It would be of interest to know what are the sharp bounds on the coefficients a_2, a_3 in the class $S_\sigma^*(\beta)$.

4. The class $C_\sigma(\beta)$

A function $f(z)$ of the form (1.1) belongs to the class $C_\sigma(\beta)$ it satisfies the following set of conditions:

$$f \in \sigma, \tag{4.1}$$

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \beta, \quad |z| > 1, \tag{4.2}$$

$$\operatorname{Re}\left\{\frac{wg''(w)}{g'(w)} + 1\right\} > \beta, \quad |w| > 1, \tag{4.3}$$

where $g(w)$ is the function defined in (2.4).

THERОREM 4.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belong to $C_\sigma(\beta)$. *Then*

$$|a_2| \leqslant \sqrt{1-\beta} \text{ and } |a_3| \leqslant 1 - \beta.$$

Moreover, for the class $C_\sigma(0)$, *the extremal function is given by* $f(z) = \dfrac{z}{1-z}$ *and its rotations.*

Proof. Using the same notation as in Theorem 3.1, conditions (4.2), (4.3) give

$$\frac{zf''(z)}{f'(z)} + 1 = \beta + (1-\beta)Q(z) \tag{4.4}$$

and

$$\frac{wg''(w)}{g'(w)} + 1 = \beta + (1-\beta)P(w), \tag{4.5}$$

where $Q(z),\ P(w) \in P(0)$.

Equation (4.4) gives us that

$$2a_2 = (1-\beta)c_1 \tag{4.6}$$

and

$$6a_3 = (1-\beta)c_2 + 2a_2(1-\beta)c_1. \tag{4.7}$$

from these two equations we obtain

$$6a_3 = 4a_2^2 + (1-\beta)c_2. \tag{4.8}$$

Now, by (4.5) we obtain that

$$2a_2 = -(1-\beta)p_1 \tag{4.9}$$

and

$$12a_2^2 = 6a_3 + (1-\beta)p_2 - 2a_2(1-\beta)p_1. \tag{4.10}$$

The two equations give

$$8a_2^2 = 6a_3 + (1 - \beta)p_2. \tag{4.11}$$

Combining (4.8) and (4.11) and using the bounds for $|p_2|$ and $|c_2|$, we obtain that

$$|a_2| \leqslant \sqrt{1 - \beta}$$

and

$$|a_3| \leqslant 1 - \beta.$$

In the case $\beta = 0$, we have $C_\sigma(0) \subset C$, where C is the class of all λ normalised functions convex in the unit disc. This implies that

$$|a_n| \leqslant 1, \quad n = 2, 3, \ldots,$$

which is sharp as seen from the function

$$f(z) = \frac{z}{1 - z}, \tag{4.12}$$

which is in $C_\sigma(0)$.

The question arises whether the class $C_\sigma(0)$ and the class C are the same. The function

$$f(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right], \qquad \frac{1}{2} < \alpha < 1,$$

belongs to C; since it is not bi-univalent, it is not in $C_\sigma(0)$ — consequently $C_\sigma(0)$ is a proper subclass of C.

We emphasize that it is *not* true that: A function $f(z)$ is bi-convex in U if and only if $zf'(z)$ is bi-starlike in U. This is clear from the function in (4.12) which is bi-convex; however for that function $zf'(z)$ is the Koebe function which is not bi-starlike (since it is not bi-univalent).

REFERENCES

1. M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. 18 (1967), 63–68. MR 34 # 6074.
2. D. A. Brannan and J. G. Clunie, *Aspects of Contemporary Complex Analysis*, Academic Press, 1980. MR 82 f: 30001.
3. T. S. Taha, *Topics in Univalent Function Theory*, Ph. D. Thesis, University of London, 1981.
4. D. A. Brannan, J. Clunie and W. E. Kirwan, *Coefficient estimates for a class of star-like functions*, Can. J. Math. 22 (1970), 476–485. MR 41 # 5614.
5. C. Pommerenke, *Univalent Functions*, Vandenhoek and Ruprecht, Gottingen, 1975. MR 58 # 22526.
6. W. Rudin, *Real and Complex Analysis*, McGraw–Hill, New York, 1966. MR 35 # # 1420.
7. M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. 37 (1936), 374–408. Zbl. 14–165.

RECENZII

Measure Theory and Its Applications. Proceedings, Sherbrooke, Canada 1982, Lectures Notes in Mathematics, vol. 1033, Springer-Verlag, Berlin Heidelberh New York, 1983, 317 pp.

These are the Proceedings, edited by J—M. Belley, J. Dubois and P. Morales, of the Workshop on Measure Theory and its Applications, held at the Université de Sherbrooke from June 7 to 18, 1982. The Workshop was attented by 87 mathematicians from 12 countries presenting new and significant results in Ergodic Theory, Choquet Representation Theory, Vector Measures and Topology. The book contains 29 contributions of the participants. Let us remark the survey papers by G. Choquet, „Representation integrale", p. 114—143 and by J. Diestel and J.J. Uhl, Jr., „Progress in Vector Measures" — 1977—1983, p. 144—192. There are also other valuable papers written by leading specialists in measure theory as M. Ackoglu, J. Batt, N. Dinculeanu, A. Bellow, J.K. Brooks, G.A. Edgar, P. Greim, J. Oxtoby, F. Topse. By presenting the State of the Art, new results and putting problems which open new ways of investigations, this book is a valuable contributions to measure thory and related fields.

S. COBZAŞ

Complex Analysis — Methods, Trends and Applications, Edited by E. Lanckau and W. Tutschke, Akademie Verlag, Berlin 1983, 398 pp.

This is the first book in a series initiated by the organizers of the conference on „Complex Analysis and Its Applications to Partial Differential Equations", regularly held at the Halle-Wittenberg-Martin Luther University. The aim of the series is to present surveys giving a comprehensive explanation of complex analysis.

The book is divided into two parts: I. *Complex Analysis and Its, Relations to Other Spheres in Mathematics*, and; II. *Complex Methods in Partial Differential Equations and Other Applications of Complex Analysis*, and contains twenty-two papers written by eminent specialists in the field as W. Tu- chke, E. Lanchau, B. Bojarski, J. Law- nowicz, S. Prössdorff V.S. Vnogradov, L. Wolfersdorff et al. The papers present vario- aspects of the holomorphy in the whole ar- of mathematics and its applications, empha- sising the new concepts of generalized analy- tic functions of I.N. Vekua, pseudo-analyti- functions of L. Bers, (p. q)-analytic functi- of G.N. Položii, having deep and fruitf- applications to PDE. The book is a valu- ble contribution to the modern comp- function theory and its applications, a- we recommend it warmly to all people- rested in this field.

S. CO-

M. M. Rao, **Probability Theory and Applications,** Academic Press, New Y- 1984, 495 pp.

The book is designed as a graduate co- on probability theory and its applicati- All the proofs are given in detail and so- key results are given multiple proofs.

The author avoids excessive generali- ons (for instance Banach space valued rand- variables have not been included), the p- requisites being a knowledge of Lebe- integral. The necessary results from - analysis are reviewed in Chapter I and s- of them, usually not covered in stand- courses, are given with proofs. The boo- very well and carefully written. The a- explains the special character of the sub- the notions are gradually introduced- the need of an abstract theory is very- motivated on apparently simple real-w- problems. The book also contains very in- resting historical and philosophical comme- on the evolution of ideas and concepts - probability theory. Many classical proble- are discussed in detail and others are p- sented in the problems at the end of ea- chapter. Some of these problems are ro- tine but there are also some more diffic- problems, usually provided with hints.

The result is a fine book on probabili- theory and we recommend it warmly to - people interested in learning, applying - teaching probability theory.

S. COBZ-

H u g o S t e i n h a u s, **Selected Papers,** PWN Warszawa, 1985, 899 pp.

The book contains 84 from 255 papers of the eminent Polish mathematician Hugo Steinhaus (1887—1972), one of the founders (together with S. Banach, J. Schauder *et al.*) of functional analysis. The articles were chosen to cover the wide area of interests of H. Steinhaus, the fields he made fundamental achievements being: trigonometric and orthogonal series, functional analysis, probability theory, game theory, topology, applications of mathematics, popularization. The papers are arranged chronologically, in order to help the reader in following the developement of scientific ideas of H. Steinhaus. The book also contains an article on the life and work of H. Steinhaus written by E. Marczewski, a list of scientific publications of H. Steinhaus and some of his polemics, pamphlets and programmatic talks.

S. COBZAŞ

Conference on Applied Mathematics, Ljubljana, September 2—5, 1986, Edited by Z. Bohte, University of Ljubljana, Ljubljana, 1986

Prezenta carte cuprinde 27 de lucrări prezentate la a „V-a Conferinţă de matematici aplicate" ţinută la Ljubljana în 2—5 septembrie 1986. La această conferinţă au participat 126 matematicieni din universităţi şi centre de cercetare din Jugoslavia. Cele 27 de lucrări, menţionate mai sus, tratează probleme actuale din următoarele domenii: Analiză numerică' Informatică, Ecuaţii diferenţiale ordinare şi Ecuaţii cu derivate parţiale. Recomandăm această carte tuturor cercetătorilor antrenaţi în aceste domenii.

I.A. RUS

Discrete Geometry and Convexity, Editors Jacob E. Goodman, Erwin Lutwak, Joseph Malkevitch, Richard Pollack, *Annals of the New York Academy of Sciences*, Vol. 440, The New York Academy of Sciences, New York, 1985. (XII+392 pages).

The amin of the volume is to collect under one cover some representative current work in the areas of geometry which could be subsummed under the heading Discrete Geometry and Convexity. These areas include a rather wide spectrum of problems including purely combinatorial questions involving the geometry of finite sets of points on one extreme and integral geometry at the other. The contained 35 papers, signed by outstanding specialists of the field are distributed as follows:
1. *Discrete Problems* (8); 2. *Quantitative Convexity* (7);
3. *Qualitative Convexity* (5); 4. *Polyhedral Geometry* (5);
5. *Tilling, Packing, Covering and Weaving* (5); 6. *Computational Aspects* (5). Most of the papers have both expository and research paper characteristics. The reader can find in them an extended literature and an important amount of open problems as well. The volume ends with *Index of Contributors, Author Index* and *Subject Index*.

A. B. NÉMETH

D. P. P a r e n t, **Exercices in Number Theory,** Springer — Verlag, New York, Berlin, Heidelberg, Tokyo, *Problem Books in Mathematics*, 1984, pp.

This problem book ia a very good and attractive introduction to number theory. The book contains ten chapters in the following order: *Prime Numbers; Arithmetic Functions; Selberg's Sine; Additive Theory; Rational Series; Algebraic Theory; Distribution Modulo 1; Transcendal Numbers; Congruences Mod p; Modular Forms; Quadratic Forms; Continued Fractions; p-Adic Analysis.*

Each chapter is divided in three sections: introduction and basic results, problems, solutions. The solutions are complete and contain many remarks and bibliographycal comments.

The book is useful for all interested in number theory and related fields.

D. ANDRICA

A. L a n g e n b a c h, **Vorlesung zur höheren Analysis. Hochschulbücher für Mathematik.** Band 84, VEB Deutscher Verlag der Wissenschaften Berlin 1984, 280 pages.

The book presents some fundamental methods of linear and nonlinear functional analysis, useful for those students and specialists, (mathematicians, physicists etc.), who use analytic methods in their research domain as the theory of differential and partial differential equations, maximum and minimum problems, optimization and control theory, approximation and numerical methods etc.

To read the book one needs relatively few previous knowledge, a very clear way of presentation is chosen, too general results are not discussed. The book is written with very much pedagogical sense, so it is available to the students of mathematics, physics and engineering of lower years.

The titles of the chapters and appendices are: *Metric and Normed Linerar Spaces, Topological Spaces, Functionals and Minimum problems, Hilbert Spaces, Constructive Methods for Minimum Problems and Equations, Application of Prolongation and Completion Methods, Classification of Partial Differential Equations, Theory of Elliptic Equations, Linear Parabolic and Hyperbolic Equations, Theory of Evolution Equations, The Stone-Weierstrass Theorem, Measure-theoretical Basis of Integration of Continuous Functions.*

P. SZILÁGYI

ÎNTREPRINDEREA POLIGRAFICĂ CLUJ, Comanda nr. 502/1986

Revista științifică a Universității din Cluj-Napoca, **STUDIA UNIVERSITATIS BABEȘ-BOLYAI,** apare începînd cu anul 1986 în următoarele condiții:

matematică — trimestrial

fizică — semestrial

chimie — semestrial

geologie-geografie — semestrial pentru geologie și anual pentru geografie

biologie — semestrial

filozofie — semestrial

științe economice — semestrial

științe juridice — semestrial

istorie — semestrial

filologie — semestrial

STUDIA UNIVERSITATIS BABEȘ-BOLYAI, the scientific journal of the University of Cluj-Napoca, starting with 1986 in issued as follows:

mathematics: quarterly

physics: biannually

chemistry: biannually

geology-geography: biannually on geology and yearly on geography

biology: biannually

philosophy: biannually

economic sciences: biannually

juridical sciences: biannually

history: binnually

philology: biannually

43 875

Abonamentele se fac la oficiile poștale, prin factorii poștali și prin difuzorii de presă, iar pentru străinătate prin „ROMPRESFILATELIA“, sectorul export-import presă, P.O. Box 12—201, telex 10 376 prsfir, București, Calea Griviței nr. 64—66.

Lei 35

STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

3

1986

CLUJ-NAPOCA

ANUL XXXI 1986

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

3

Redacţia: 3400 CLUJ-NAPOCA, str. M. Kogălniceanu, 1 ● Telefon 1 61 01

SUMAR – CONTENTS – SOMMAIRE – INHALT

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

THE IRRATIONALITY OF CERTAIN INFINITE PRODUCTS

C. BADEA*

Received: January 19, 1986

ABSTRACT. — The aim of this paper is to prove the irrationality of a class of infinite products. The main theorem generalizes an old theorem of Cantor [3] and a recent result of Sándor [7]. Also, a counter-example for an assertion of Froda [5] and an application of the main result are given.

1. Introduction. An old theorem of Cantor [3] asserts that if (n_k) is a sequence of positive integers with $n_{k+1} > n_k^2$ for all large k, then the value of the product

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{n_k}\right) \tag{1}$$

is irrational.

Recently Sándor [7] gave, among other things, some conditions for which the value of the infinite product

$$\prod_{k=1}^{\infty}\left(1+\frac{m_k}{n_k}\right) \tag{2}$$

is irrational, where (m_k) is a sequence of primes and (n_k) is a sequence of positive integers. Namely, he proved that if (m_k) is a sequence of primes with $\lim_{k\to\infty} m_k = \infty$ and (n_k) is a sequence of positive integers which verify the inequalities

$$n_{h+k} \geqslant m_{h+k} \cdot n_h^{2^k}\ ;\ h \geqslant 1,\ k \geqslant 1 \tag{3}$$

then the value of the product (2) will be irrational.

The purpose of this paper is to give some conditions for which the value of the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right) \tag{4}$$

is irrational, where (a_n) and (b_n) are two sequences of positive integers. Our theorem extends Cantor's theorem which is obtained for $b_n = 1,\ n \geqslant 1$ and Sándor's result.

In the case when the infinite product (4) is divergent the problem of the rationality of his value is needless. Thus we shall assume in what follows that all infinite products which appear are convergent. Another way in which

* University of Craiova, Department of Mathematics, 1100 Craiova, Romania

we can avoid this trivial case is to make the convention that ∞ is irrational and rational in the same time.

In the proof of the main result we shall use a criterion for irrationality due to Brun [2]. A generalization of Brun's criterion was given by Froda [5] but, as well shall see in the third section of the present paper, Froda's [5] generalization is not true.

As an application of the main result we shall prove that every convergent infinite product of rational numbers greater than [1] has an infinitely many disjoint subproducts (to be defined) with irrational values.

We note in ending that the same method of proof for the main theorem was used in [1] to obtain some criteria for the irrationality of certain series.

2. Main result. The main result of the present paper is the following

THEOREM. *Let (a_n) and (b_n) be two sequences of positive integers such that*

$$a_{n+1} > \frac{b_{n+1}}{b_n} a_n^2 + \frac{b_{n+1}(b_n - 1)}{b_n} a_n + 1 - b_{n+1} \tag{5}$$

holds for all large n. Then the value of the product (4) *is an irrational number.*

For $b_n = 1$ we obtain Cantor's theorem. Also, the above Theorem is more general than Sándor's result. Indeed, if the sequences (m_k) and (n_k) verifie Sándor's conditions then, using (3) for $k = 1$, we shall find that

$$n_{k+1} \geqslant m_{k+1} n_k^2.$$

Because $\lim\limits_{k\to\infty} m_k = \infty$, for all large h we have

$$m_{k+1} n_k^2 > \frac{m_{k+1}}{m_k} n_k^2 + \frac{m_{k+1}(m_k - 1)}{m_k} + 1 - m_{k+1}$$

Thus Sándor's conditions imply the requirements of the main result, i.e. our Theorem is more general than Sándor's result.

Proof of Theorem. We have

$$\prod_{n=1}^{\infty} \left(1 + \frac{b_n}{a_n}\right) = \lim_{n\to\infty} \frac{(a_1 + b_1) \ldots (a_n + b_n)}{a_1 \ldots a_n}. \tag{6}$$

Brun's criterion asserts that a positive real number α which is the limit of an increasing sequence of rationals

$$\alpha = \lim_{r\to\infty} \frac{y_r}{x_r}, \tag{7}$$

where x_r and y_r are positive integers, is irrational when

$$\frac{y_{r+2} - y_{r+1}}{x_{r+2} - x_{r+1}} < \frac{y_{r+1} - y_r}{x_{r+1} - x_r}, \tag{8}$$

for all large r.

Taking into account this theorem we shall prove the inequality (8) for $y_r = (a_1 + b_1) \ldots (a_r + b_r)$ and $x_r = a_1 \ldots a_r$. Because the b_r's are positive integers, we get than (y_r/x_r) is an increasing sequence and thus, keeping in mind Brun's theorem, we shall find that the value of the infinite product (4) is an irrational number.

The inequality (8) in our situation $y_r = (a_1 + b_1) \dots (a_r + b_r)$ and $x = a_1 \dots a_r$ is equivalent with the following inequality

$$\frac{y_{r+1}(a_{r+2} + b_{r+2} - 1)}{x_{r+1}(a_{r+2} - 1)} < \frac{y_r(a_{r+1} + b_{r+1} - 1)}{x_r(a_{r+1} - 1)} \tag{9}$$

and thus with

$$\frac{(a_{r+1} + b_{r+1})(a_{r+2} + b_{r+2} - 1)}{a_{r+1}(a_{r+2} - 1)} < \frac{a_{r+1} + b_{r+1} - 1}{a_{r+1} - 1}. \tag{10}$$

From (10) we deduce, by routine calculations, the following equivalent relation

$$b_{r+2}a_{r+1}^2 + a_{r+1}b_{r+1}b_{r+2} + b_{r+1} < b_{r+1}a_{r+2} + b_{r+2}a_{r+1} + b_{r+1}b_{r+2} \tag{11}$$

Hence we have

$$a_{r+2} > \frac{b_{r+2}}{b_{r+1}} a_{r+1}^2 + \frac{b_{r+2}(b_{r+1} - 1)}{b_{r+1}} a_{r+1} + 1 - b_{r+2}. \tag{12}$$

We see that (12) is just (5) with $r + 1$ instead of n.

Therefore, from the assumptions of the Theorem, it follows that (8) holds for every sufficiently large r and thus (via Brun's criterion) the proof of the the main result is complete.

A simple consequence of the main theorem is the following

COROLLARY. *Let k be a positive integer and (a_n) a sequence of positive integers such that*

$$a_{n+1} > a_n^2 + (k - 1)a_n + 1 - k$$

for every sufficiently large n. Then the value of the product $\prod_{n=1}^{\infty}\left(1 + \frac{k}{a_r}\right)$ is an irrational number.

Proof. We take in the above Theorem $b_n = k$.

For $k = 1$ the condition of the above Corollary reduces to $a_{n+1} > a_n^2$, i.e. we obtain Cantor's [3] theorem.

3. A counter-example. A generalization of Brun's irrationality criterion was given by Froda [5]. Froda proved that Brun's criterion is also true when y_r and x_r are positive real numbers such that (8) holds. The same method of proof of our theorem remains valid to show, with the help of Froda's generalization, that our theorem is also true for positive numbers a_n and b_n. However, this is not valid because Froda's generalization is not correct. A counter-example is given in what follows. We note that the fact that Froda's proof for his generalization is not correct was previously known.

Let us define the sequence (c_n) by $c_1 = 2$ and by the recursive relation

$$c_{n+1} = c_n^2 - c_n + 1 \tag{13}$$

We note in passing that this sequence is in many situations a counter-example for some irrationality assertions (see for instance Erdös and Straus [4]).

Let us consider the following two sequences (a_n) and (α_n) given by

$$a_n = \frac{\log 1.5}{2^{1/c_n} - 1}$$

and

$$\alpha_n = \frac{(a_1 + \log 1.5)\dots(a_n + \log 1.5)}{a_1 \dots a_n}.$$

Let us assume that Froda's assertion is true. Then, because (α_n) is increasing, we deduce that

$$\lim_{n\to\infty} \alpha_n = \prod_{k=1}^{\infty}\left(1 + \frac{\log 1.5}{a_n}\right)$$

is irrational whether

$$v_{n+1} < v_n \tag{14}$$

for every sufficiently large n, where

$$v_n = \frac{(a_1 + \log 1.5)\dots(a_{n+1} + \log 1.5) - (a_1 + \log 1.5)\dots(a_n + \log 1.5)}{a_1 \dots a_{n+1} - a_1 \dots a_n}.$$

Following the same steps as in the proof of our main result and of the Corrollary, we get that (14) is equivalent with the inequality from the statement of the above Corollary with log 1.5 instead of k, i.e. with

$$a_{n+1} > a_n^2 + (\log 1.5 - 1)a_n + 1 - \log 1.5 \tag{15}$$

We denote $b_n = 1/(2^{1/c_n} - 1)$, then $b_n = a_n/\log 1.5$. With these notations we rewrite (15) as

$$b_{n+1} > b_n^2 \log 1.5 + (\log 1.5 - 1)b_n - 1 + (\log 1.5)^{-1} \tag{16}$$

Because

$$\prod_{n=1}^{\infty}\left(1 + \frac{\log 1.5}{a_n}\right) = \prod_{n=1}^{\infty}\left(1 + \frac{1}{b_n}\right) = \prod_{n=1}^{\infty} 2^{1/c_n} = 2^{\sum_{n=1}^{\infty} 1/c_n}$$

is convergent (see Gleason, Greenwood and Kelly [6, pp. 429–430]) we obtain that b_n tends with n to infinity. Hence

$$(\log 1.5 - 1)b_n - 1 + (\log 1.5)^{-1} < 0 \tag{17}$$

for every sufficiently large n.

On the other hand we have

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n^2} = \lim_{n\to\infty}\left(\frac{2^{1/c_n}-1}{1/c_n}\right)^2 \frac{c_{n+1}}{c_n^2} \frac{1/c_{n+1}}{2^{1/c_{n+1}}-1} =$$

$$= (\log 2) \lim_{n\to\infty} \frac{c_{n+1}}{c_n^2} = \log 2 > \log 1.5$$

because $\lim_{x\to 0} \frac{2^x-1}{x} = \log 2$ and c_n tends with n to infinity by (13). It results that

$$b_{n+1} > b_n^2 \log 1.5 \tag{18}$$

for every sufficiently large n.

From (17) and (18) it follows That (16) holds for all large n. Therefore, if Froda's assertion is true, the number

$$\lim_{n\to\infty} \alpha_n = \prod_{n=1}^{\infty}\left(1 + \frac{\log 1.5}{a_r}\right) = \prod_{n=1}^{\infty} 2^{1/c_n}$$

is an irrational number. Passing to series we find that

$$\prod_{n=1}^{\infty} 2^{1/c_n} = 2^{\sum_{n=1}^{\infty} 1/c_n}$$

is irrational. But $\sum_{n=1}^{\infty} 1/c_n = 1$ (see [6, pp. 429–430)], so we get that 2 is an irrational number, which is of course a contradiction.

Hence Froda's assertion is not true in general.

4. An application. We say that $\prod_{k=1}^{\infty} v_k$ is a *subproduct* of a given product $\prod_{k=1}^{\infty} u_k$ if (v_k) is a subsequence of the sequence (u_k). Two subproducts $\prod_{k=1}^{\infty} v_k$ and $\prod_{k=1}^{\infty} w_k$ of the same product $\prod_{k=1}^{\infty} u_k$ are *disjoint* if (w_k) is a subsequence of the sequence (u_k) from which has been taken the subsequence (v_k).

As an application of the main result we shall prove the following unexpectedly

PROPOSITION. *Every convergent infinite product of rational numbers greater than* 1 *has an infinitely many disjoint subproducts with irrational values.*

This proposition is similar with a result from [1] where we proved that every convergente infinite serie of positive rationals has an infinitely many disjoint subseries with irrational sums (the notions of subseries and disjoint subseries are defined similary).

Proof of Proposition. Let $P = \prod_{n=1}^{\infty} c_n$ be a convergent infinite product with $c_n = 1 + b_n/a_n$, where b_n and a_n are positive integers, $n = 1, 2, \ldots$. Because the product P is convergent, the sequence (b_n/a_n) tends to zero when n tends to infinity. Because a_n and b_n are positive integers we get that the sequence (a_n) tends with n to infinity.

Hence there are an infinitely many disjoint subsequences $(a_{n(k)}^{(p)})_k$ and $(b_{n(k)}^{(p)})_k$, $p = 1, 2, \ldots$ of the sequences (a_n) and (b_n) respectively, such that

$$a_{n(k+1)} > \frac{b_{n(k+1)}}{b_{n(k)}} a_{n(k)}^2 + \frac{b_{n(k-1)}(b_{n(k)} - 1)}{b_{n(k)}} a_{n(k)} + 1 - b_{n(k+1)}$$

for all large k.

Now, using the main result, we find that the subproducts of P generated by the subsequences $(a_{n(k)}^{(p)})_k$ and $(b_{n(k)}^{(p)})_k$ have irrational values.

The proof is complete.

Finally, we propose the following

QUESTION. *Is there a convergent infinite product of rationals greater than 1 such that all its subproducts have irrational values?*

An affirmative answer to this question would provide a negative answer to the problem of replacing in the above Proposition the word *„irrational"* by *„rational"*. We note that the corresponding question for the series has an affirmative answer [1]. Thus the above problem has a negative answer for series and this negative result may be explained by the fact that the set of irrationals is uncountable while the set of rationals is „only" denumerable.

REFERENCES

1. C. Badea, *The irrationality of certain infinite series*, (to appear).
2. V. Brun, *Ein Satz über Irrationalität*, Arch. for Math. og Naturvidenskab (Kristiania), (1910), 3.
3. G. Cantor, *Zwei Satze über ein gewisse Zerlegung der Zahlen in unendliche Producte*, Gesammelte Abhandlungen, 43–50, Springer, 1932.
4. P. Erdös and E. G. Straus, *On the irrationality of certain Ahmes series*, J. Indian Math. Soc. *27* (1963), 129–133.
5. A. Froda, *Sur l'irrationalité du nombre 2*, Rend. Acad. dei Lincei, Ser. 8, *35* (1963), 472–473.
6. A. M. Gleason, R. E. Greenwood and L. M. Kelly, *The W. L. Putnam Mathematical Competition — Problems and Solutions: 1936–1964*, The Math. Assoc. of America, 1965.
7. J. Sándor, *Some classes of irrational numbers*, Studia Univ. Babeş-Bolyai Math. *29* (1984), 3–12.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3 1986

GENERALIZATIONS OF AN ASYMPTOTIC FORMULA OF RAMANUJAN

LÁSZLÓ TÓTH*

Received: September 1?, 1986

REZUMAT. — Generalizări ale formulei Ramanujan asimptotice. În lucrare se arată că dacă f este o funcţie numerică total multiplicativă mărginită, atunci pentru $k > 0$ are loc evaluarea asimptotică (3.4). Ca un caz particular se obţine relaţia (3.5) iar în cazul $k = 1$ se obţine rezultatul lui Ramanujan (3.6).

1. Introduction. Let $\sigma_k(n)$ denote, as usual, the sum of the k-th powers of all positive divisors of n and let $\sigma_1(n) \equiv \sigma(n)$ denote the sum of all positive divisors of n.

In 1916 Srinivasa Ramanujan ([5], eq. 19) stated without proof the following asymptotic formula

$$\sum_{n \leqslant x} \sigma^2(n) = \frac{5}{6} \zeta(3) x^3 + 0(x^2 \log^2 x), \tag{1.1}$$

where $\zeta(s)$ is the Riemann Zeta function. Several years later B. M. Wilson ([7], § 7.) mentioned that using analytical methods another formula of Ramanujan, namely

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\, \sigma_b(n)}{n^s} = \frac{\zeta(s)\ \zeta(s-a)\ \zeta(s-b)\ \zeta(s-a-b)}{\zeta(2s-a-b)} \tag{1.2}$$

([5], eq. 15) leads to an asymptotic formula for the more general sum $\sum_{n \leqslant x} \sigma_a(n) \cdot \sigma_b(n)$, which reduces to (1.1) in case $a = b = 1$.

The aim of this paper is to establish an asymptotic formula for the sum $\sum_{1 \leqslant n} \sigma_k^2(n)$, $k > 0$ (it is the case $a = b = k > 0$) using a simple elementary method based on two convolutional identities (corollaries 2.1. and 2.3.). In fact we will deduce a slightly more general result (theorem 3.2.) and obtain as a consequence the asymptotic formula for $\sum_{n \leqslant x} \sigma_k^2(n)$ (corollary 3.3.).

2. Preliminaries. Throughout this paper x is assumed real and $\geqslant 2$, and $n \geqslant 1$ denotes an integer. Let $*$ denote the Dirichlet convolution of arithmetical functions defined by

$$(f * g)(n) = \sum_{d \mid n} f(d) \cdot g\left(\frac{n}{d}\right)$$

* 3900 Satu Mare, N. Golescu str., no. 5, Romania

We recall that an arithmetical function is multiplicative if $f(mn) = f(m)f(n)$ when m and n are coprime and completely multiplicative if $f(mn) = f(m)(n)$ for every integers $m, n \geqslant 1$.

It is well-known that the Dirichlet convolution of two multiplicative functions is also multiplicative and the multiplicative functions form a commutative group under the Dirichlet convolution.

Let μ denote the Möbius function and let $E_k(n) = n^k$, $E_0(n) \equiv U(n) = 1$ for every n. We have $\mu * U = I$, where $I(n) = 1$ or 0 according as $n = 1$ or $n > 1$ is the Dirac function (the unit element of the group) and $\sigma_k = U * E_k$. It is also well-known that the functions μ, $\mu^2 = \mu \cdot \mu$ the characteristic function of the square-free numbers, σ_k are multiplicative and E_k, U, I are completely multiplicative.

LEMMA 2.1. *If f and g are completely multiplicative functions, then*

$$(f * g)\circ E_2 = f^2 * g^2 * \mu^2 fg, \tag{2.1}$$

where o denotes the ordinary composition of functions.

Proof. Both sides of (2.1) are mutiplicative, being the Dirichlet convolution of multiplicative functions (the composition by E_2 preserves the multiplicativity). So it is enough to verify the above identity for $n = p^i$, a prime power. Noting $f(p) = a$ and $g(p) = b$ we have

$$(f^2 * g^2 * \mu^2 fg)(p^i) = (f^2 * g^2)(p^i) + (f^2 * g^2)(p^{i-1})\, f(p)\, g(p) =$$

$$= (f^2(p^i) + f^2(p^{i-1})\, g^2(p) + \dots + f^2(p) g^2(p^{i-1}) + g^2(p^i)) + (f^2(p^{i-1}) +$$

$$+ f^2(p^{i-2})\, g^2(p) + \dots + f^2(p)\, g^2(p^{i-2}) + g^2(p^{i-1}))\, f(p)\, g(p) =$$

$$= (a^{2i} + a^{2i-2} b^2 + \dots + a^2 b^{2i-2} + b^{2i}) + (a^{2i-2} + a^{2i-4} b^2 + \dots + a^2 b^{2i-4} +$$

$$+ b^{2i-2})\, ab = a^{2i} + a^{2i-1} b + a^{2i-2} b^2 + \dots + a^2 b^{2i-2} + a b^{2i-1} + b^{2i} =$$

$$= f(p^{2i}) + f(p^{2i-1})\, g(p) + \dots + f(p)\, g(p^{2i-1}) + g(p^{2i}) = (f * g)(p^{2i})$$

and the proof is complete.

COROLLARY 2.1. $(g = E_k)$ *If f is completely multipicative*

$$(f * E_k) \circ E_2 = f^2 * E_{2k} * \mu^2 f E_k \tag{2.2}$$

COROLLARY 2.2. $(g = E_k,\ f = U)$

$$\sigma_k \circ E_2 = \sigma_{2k} * \mu^2 E_k, \text{ that is}$$

$$\sigma_k(n^2) = \sum_{d\delta = n} \sigma_{k2}(d)\, \mu^2(\delta)\, \delta^k \text{ for all } n \geqslant 1.$$

LEMMA 2.2. *If f and g are completely multiplicative functions, then*

$$(f * g)^2 = f^2 * g^2 * fg * \mu^2 fg \tag{2.3}$$

Proof. The function f is completely multiplictive so its inverse under the Dirichlet convolution is $f^{-1} * = \mu f$ (see for example [1], theorem 2.) and

$f(k_1 * k_2) = fk_1 * fk_2$ for arbitrary functions k_1 and k_2 (see [1], theorem 5). So the above identity is equivalent to

$$(f * g)^2 * \mu f^2 = g^2 * fg * \mu^2 fg \text{ or}$$

$$(f * g)^2 * \mu f^2 = g(f * g * \mu^2 f).$$

Let denote $h = f * g$, where $g = h * f^{-1} * = h * \mu f$ and we have to prove

$$h^2 * \mu f^2 = g(h * \mu^2 f)$$

Because of multiplicativity it is enough to verify this identitiy for $n = p^i$,

$$(h^2 * \mu f^2)(p^i) = h^2(p^i) + h^2(p^{i-1})\mu(p)f^2(p) = h^2(p^i) - h^2(p^{i-1})f^2(p) =$$

$$= (h(p^i) - h(p^{i-1})f(p))(h(p^i) + h(p^{i-1})f(p)) = g(p^i)(h * \mu^2 f)(p^i) = (g(h * \mu^2 f))(p^i)$$

which proves the lemma.

Remark 2.1. Using that $U * \mu^2 = 2^\nu$, where $\nu(n)$ denotes the number of distinct prime factors of n, formula (2.3) leads to

$$(f * g)^2 = f^2 * g^2 * (U * \mu^2)fg \quad \text{or}$$

$$(f * g)^2 = f^2 * g^2 * 2^\nu fg,$$

which is analogous to the elementary formula $(a + b)^2 = a^2 + b^2 + 2ab$

Remark 2.2. For particular functions f and g (2.3) gives interesting formas. We mention a single example: if $f = g = U$, $f * g = U * U = \tau$ the divisor function and we have

$$\tau^2 = \tau * U * \mu^2 \text{ or } \tau^2 * \mu = \tau * \mu^2. \qquad [8]$$

For a further generalization of (2.3) see [4], theorem 2.

An immediately consequence of lemmas 2.1. and 2.2 is

LEMMA 2.3. *If f and g are completely multiplicative functions, then*

$$(f * g)^2 = (f * g) \circ E_2 * fg \tag{2.4}$$

COROLLARY 2.3. $(g = E_k)$ *If f is completely multiplicative*

$$(f * E_k)^2 = (f * E_k) \circ E_2 * fE_k \tag{2.5}$$

COROLLARY 2.4. $(g = E_k,\ f = U)$

$$\sigma_k^2 = \sigma_k \circ E_2 * E_k, \textit{ that is}$$

$$\sigma_k^2 (b) = \sum_{d\delta = n} \sigma_k(d^2)\delta^k.$$

Define $D(f, s)$ by the Dirichlet series

$$D(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \ s > 1.$$

We have $D(U, s) = \zeta(s)$, $s > 1$. f is bounded and completely multipicative the series $D(f, s)$ is absolutely convergent for $s > 1$ and

$$D(f, s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}, \tag{2.6}$$

where the product extends over the primes p (cf. [2], ch. 7. theorem 5).

LEMMA 2.4. *If f is bounded and completely multiplicative, then*

$$D(\mu^2 f, k) = \frac{D(f, k)}{D(f^2, 2k)}, \quad k > 1 \tag{2.7}$$

Proof. The series $D(\mu^2 f, k)$ is absolutely convergent for $k > 1$ and the general term is a multiplicative function of n so it can be expanded into an infinite product of Euler type (cf. [2] ch. 7., theorem 5):

$$D(\mu^2 f, k) = \prod_p \left(\sum_{i=0}^{\infty} \frac{\mu^2(p^i) f(p^i)}{p^{ki}}\right) = \prod_p \left(1 + \frac{f(p)}{p^k}\right) =$$

$$= \prod_p \left(1 - \frac{f^2(p)}{p^{2k}}\right) \Big/ \prod_p \left(1 - \frac{f(p)}{p^k}\right) = \frac{D(f, k)}{D(f^2, 2k)} \text{ by (2.6)}$$

We will use the following familiar estimates:

LEMMA 2.5.

$$\sum_{n \leq x} n^k = \frac{x^{k+1}}{k+1} + 0(x^k), \quad k \geq 0 \tag{2.8}$$

$$\sum_{n \leq x} \frac{1}{n} = 0\ (\log x) \tag{2.9}$$

$$\sum_{n \leq x} \frac{1}{n^k} = 0(x^{1-k}), \quad 0 < k < 1 \tag{2.10}$$

$$\sum_{n > x} \frac{1}{n^k} = 0\left(\frac{1}{x^{k-1}}\right), \quad k > 1 \tag{2.11}$$

LEMMA 2.6. (cf. [3], lemma 2.3) *If f is a bounded arithmetical function, then for $k > 0$*

$$\sum_{n \leq x} (f * E_k)(n) = \frac{D(f, k+1)}{k+1} x^{k+1} + 0(A_k(x)), \tag{2.12}$$

where $A_k(x) = x^k$, $x \log x$ or x according as $k > 1$, $k = 1$ or $k < 1$.

Proof. By (2.8) we have

$$\sum_{n\leq x}(f * E_k)(n) = \sum_{n=d\delta\leq x} f(d)\delta^k = \sum_{d\leq x} f(d) \sum_{\delta\leq \frac{x}{d}} \delta^k =$$

$$=\sum_{d\leq x} f(d)\left\{\frac{1}{k+1}\left(\frac{x}{d}\right)^{k+1} + 0\left(\left(\frac{x}{d}\right)^k\right)\right\} = \frac{x^{k+1}}{k+1}\sum_{d\leq x}\frac{f(d)}{d^{k+1}} + 0\left(x^k\sum_{d\leq x}\frac{1}{d^k}\right) =$$

$$=\frac{x^{k+1}}{k+1}\sum_{d=1}^{\infty}\frac{f(d)}{d^k} - \frac{x^{k+1}}{k+1}\sum_{d>x}\frac{f(d)}{d^{k+1}} + 0\left(x^k\sum_{d\leq x}\frac{1}{d^k}\right) =$$

$$=\frac{x^{k+1}}{k+1}D(f, k+1) + 0\left(x^{k+1}\sum_{d<x}\frac{1}{d^{k+1}}\right) + 0\left(x^k\sum_{d\leq x}\frac{1}{d^k}\right).$$

The first 0-term is $0\left(x^{k+1}\frac{1}{x^k}\right) = 0(x)$ by (2.11) and the second 0-term is for $k > 1$ $0(x^k)$; for $k = 1$ $0(\log x)$ by (2.9) and for $k < 1$ it is $0(x^k \cdot x^{1-k}) = 0(x)$ using (2.10) and the proof is complete.

3. Main rezult.

THEOREM 3.1. *If f is completely multiplicative and bounded, then for $k > 0$*

$$\sum_{n\leq x}(f * E_k)(n^2) = \frac{D(f^2, 2k+1)\, D(f, k+1)}{(2k+1)\, D(f^2, 2k+2)} x^{2k+1} + 0(B_k(x)), \tag{3.1}$$

where $B_k(x) = x^{2k}$, $x^2\log x$, $x^{\frac{3}{2}}\log x$, or x^{k+1} according as $k > 1$, $k = 1$, $k = \frac{1}{2}$ or $k < 1$ and $k \neq \frac{1}{2}\cdot$

Proof. By corollary 2.1. and lemma 2.6. we have

$$\sum_{n\leq x}(f * E_k)(n^2) = \sum_{n\leq x}(f^2 * E_{2k} * \mu^2 f E_k)(n) =$$

$$=\sum_{d\delta=n\leq x}\mu^2(d)f(d)d^k(f^2 * E_{2k})(\delta) = \sum_{d\leq x}\mu^2(d)f(d)d^k\sum_{\delta\leq\frac{x}{d}}(f^2 * E_{2k})(\delta) =$$

$$=\sum_{d\leq x}\mu^2(d)f(d)d^k\left\{\frac{D(f^2, 2k+1)}{2k+1}\left(\frac{x}{d}\right)^{2k+1} + 0\left(A_{2k}\left(\frac{x}{d}\right)\right)\right\} =$$

$$=\frac{D(f^2, 2k+1)}{2k+1}x^{2k+1}\sum_{d\leq x}\frac{\mu^2(d)f(d)}{d^{k+1}} + 0\left(\sum_{\leq x} d^k A_{2k}\left(\frac{x}{d}\right)\right) =$$

$$=\frac{D(f^2, 2k+1)}{2k+1}x^{2k+1}\sum_{d=1}^{\infty}\frac{\mu^2(d)\, f(d)}{d^{k+1}} + 0\left(x^{2k+1}\sum_{d>}\frac{1}{d^{k+1}}\right) +$$

$$+ 0\left(\sum_{d\leq x} d^k A_{2k}\left(\frac{x}{d}\right)\right), \text{ where } \sum_{d=1}^{\infty}\frac{\mu^2(d)\, f(d)}{d^{k+1}} = D(\mu^2 f, k+1) =$$

$$=\frac{D(f, k+1)}{D(f^2, 2k+2)} \text{ by lemma 2.4.}$$

Using (2.11) the first 0-term is $0\left(x^{2k+1}\frac{1}{x^k}\right)=0(x^{k+1})$ and the second 0-term is for $k>1$ $0\left(\sum_{d\leq x} d^k\left(\frac{x}{d}\right)^{2k}\right)=0(x^{2k})$; for $k=1$ $0\left(\sum_{d\leq x} d\left(\frac{x}{d}\right)^2\right)=0\left(x^2\sum_{d\leq x}\frac{1}{d}\right)=$ $=0\,(x^2\log x)$ by (2.9); in case $k=\frac{1}{2}$ it is $0\left(\sum_{d\leq x} d^{\frac{1}{2}}\frac{x}{d}\log\frac{x}{d}\right)=$ $0\left(x\log x\sum_{d\leq x} d^{-\frac{1}{2}}\right)=0\left(x\log x\cdot x^{\frac{1}{2}}\right)=0\left(x^{\frac{3}{2}}\log x\right)$ using (2.10) and for $k<1$, $k\neq\frac{1}{2}$ the second remain term is $0\left(\sum_{d\leq x} d^k\frac{x}{d}\right)=0\left(x\sum_{d\leq x}\frac{1}{d^{1-k}}\right)=0(x^{k+1})$ or $0\left(\sum_{d\leq x} d^k\left(\frac{x}{d}\right)^{2k}\right)=0\left(x^{2k}\sum_{d\leq x}\frac{1}{d^k}\right)=0(x^{k+1})$ according as $2k<1$ or $2k>1$ by (2.10).

COROLLARY 3.1. $(f=U)$ *For* $k>0$

$$\sum_{n\leq x}\sigma_k(n^2)=\frac{\zeta(2k+1)\,\zeta(k+1)}{(2k+1)\,\zeta(2k+2)}\,x^{2k+1}+0(B_k(x)), \tag{3.2}$$

where $B_k(x)$ *is given in theorem* 3.1.

COROLLARY 3.2. $(f=U,\ k=1)$

$$\sum_{n\leq x}\sigma(n^2)=\alpha x^3+0(x^2\log x), \tag{3.3}$$

where $\alpha=\frac{\zeta(3)\,\zeta(2)}{3\zeta(4)}=\frac{5\zeta(3)}{\pi^2}$.

Now we prove our principal result.

THEOREM 3.2. *If* f *is completely multiplicative and bounded, then for* $k>0$

$$\sum_{n\leq x}(f*E_k)^2(n)=\frac{D(f^2,2k+1)\,D^2(f,k+1)}{(2k+1)\,D(f^2,2k+2)}\,x^{2k+1}+0(C_k(x)) \tag{3.4}$$

where $C_k(x)=x^{2k}$, $x^2\log^2 x$, $x^{\frac{3}{2}}\log^2 x$ or $x^{k+1}\log x$ *according as* $k>1$, $k=1$, $k=\frac{1}{2}$ or $k<1$ *and* $k\neq\frac{1}{2}$.

Proof. We use corollary 2.3. and the above theorem.

$$\sum_{n\leq x}(f*E_k)^2(n)=\sum_{n\leq x}((f*E_k)\circ E_2*fE_k)(n)=$$

$$=\sum_{d\delta=n\leq x}f(d)d^k(f*E_k)(\delta^2)=\sum_{d\leq x}f(d)\,d^k\sum_{\delta\leq\frac{x}{d}}(f*E_k)(\delta^2)=$$

$$= \sum_{d \leqslant x} f(d)\, d^k \left\{ \frac{D(f^2, 2k+1)\, D(f, k+1)}{(2k+1)\, D(f^2, 2k+2)} \left(\frac{x}{d}\right)^{2k+1} + 0\left(B_k\left(\frac{x}{d}\right)\right)\right\} =$$

$$= \frac{D(f^2, 2k+1)\, D(f, k+1)}{(2k+1)\, D(f^2, 2k+2)}\, x^{2k+1} \sum_{d \leqslant x} \frac{f(d)}{d^{k+1}} + 0\left(\sum_{d \leqslant x} d^k B_k\left(\frac{x}{d}\right)\right),$$

where $\sum_{d \leqslant x} \frac{f(d)}{d^{k+1}} = \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + 0\left(\sum_{d > x} \frac{1}{d^{k+1}}\right) = D(f, k+1) + 0\left(\frac{1}{x^k}\right)$

and the remain term is : case $k > 1$ $0\left(\sum_{d \leqslant x} d^k \left(\frac{x}{d}\right)^{2k}\right) = 0(x^{2k})$;

case $k = 1$ $0\left(\sum_{d \leqslant x} d\left(\frac{x}{d}\right)^2 \log\frac{x}{d}\right) = 0\left(x^2 \log x \sum_{d \leqslant x} \frac{1}{d}\right) = 0(x^2 \log^2 x)$;

case $k = \frac{1}{2}$ $0\left(\sum_{d \leqslant x} d^{\frac{1}{2}}\left(\frac{x}{d}\right)^{\frac{3}{2}} \log\frac{x}{d}\right) = 0\left(x^{\frac{3}{2}} \log x \sum_{d \leqslant x} \frac{1}{d}\right) = 0\left(x^{\frac{3}{2}} \log^2 x\right)$;

and in case $k < 1$, $k \neq \frac{1}{2}$ it is $0\left(\sum_{d \leqslant x} d^k \left(\frac{x}{d}\right)^{k+1}\right) = 0(x^{k+1} \log x)$

which proves the theorem.

COROLLARY 3.3. $(f = U)$ *For* $k > 0$

$$\sum_{k \leqslant 1} \sigma_k^2(n) = \frac{\zeta(2k+1)\, \zeta^2(k+1)}{(2k+1)\, \zeta(2k+2)}\, x^{2k+1} + 0(C_k(x)), \tag{3.5}$$

where $C_k(x)$ is given in theorem 3.2.

COROLLARY 3.4. $(f = U,\ k = 1$; Ramanujan)

$$\sum_{n \leqslant x} \sigma^2(n) = \beta x^3 + 0(x^2 \log^2 x), \tag{3.6}$$

where $\beta = \frac{\zeta(3)\, \zeta^2(2)}{3\zeta(4)} = \frac{5}{6}\, \zeta(3)$.

Remark 3.1. In 1970 R. A. Smith [6] improved the error term of Ramanujan's formula (3.6) into

$$0(x^2 \log^{\frac{5}{3}} x),$$

using analytical methods.

REFERENCES

1. T. M. Apostol, *Some properties of completely multiplicative arithmetical functions*, Amer. Math Monthly, *78* (1971) 266–271.
2. K. Chandrasekharan, *Introduction to analytic number theory*, Springer Verlag Berlin-Heidelberg-New York, 1968.
3. E. Cohen, *Arithmetical functions of a greatest common divisor*, Proc. Amer. Math. Soc., *11* (1960) 164–171.
4. J. Lambek, *Arithmetical functions and distributivity*, Amer. Math. Monthly, *73* (1966) 969–973.
5. S. Ramanujan, *Some formulae in the analytic theory of numbers*, Messenger Math., *45* (1916) 81–84; Collected papers 133–135.
6. R. A. Smith, *An error term of Ramanujan*, J. Number Theory, *2* (1970) 91–96.
7. B. M. Wilson, *Proofs of some formulae enunciated by Ramanujan*, Proc. London Math. Soc., *21* (1922) 235–255.
8. *Problem E :* 2235, Amer. Math. Monthly, *78* (1971) 406–407.

UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

NORMAL Π — COMPLEMENTS IN FINITE GROUPS

I. VIRÁG*

Received: September 15, 1986

REZUMAT. — **Π-complement normal în grupuri finite.** În cele patru teoreme din lucrare se dau condiţii necesare şi suficiente pentru existenţa π-complementului normal în grupuri finite.

1. Preliminaires. Let G denote a finite group, π a set of prime numbers and π' the complementary set of π. A normal π—complement in G is a normal Hall π'-subgroup of G.

The purpose of the present paper is to give some necessary and sufficient conditions for the existence of normal π-complements in finite groups.

The notations and the terminology used are largely standard.

If $g \in G$, then g has a unique decomposition in the form $g=g_\pi \cdot g_{\pi'} = g_{\pi'} \cdot g_{\pi'}$, where g_π is a π-element and $g_{\pi'}$, is a π'-element of G. We call g_π, $g_{\pi'}$ respectively, the π-factor, π'-factor of g. If π consists only of one prime p, we write g_p for g_π. Each $g \in G$ is the mutually commuting product of its p-factors $g_p \neq 1$ for the different primes.

Two elements $g, h \in G$ will be said to be π-conjugate in G, if their π-factors are conjugate in G in the ordinary sense. Since the π-factor of an element g is a power of g, conjugate elements are also π-conjugate. The π-conjugacy is an equivalence relation in G. We can thus speak of the π-conjugate classes in G. We use $K_{G,\pi}(g)$ for the π-conjugate class of the element g in G.

It is clear that:

— $K_{G,\pi}(g) = K_{G,\pi}(g_\pi)$ for every $g \in G$.

— $K_{G,\pi}(1)$ is the set of π'-elements of G.

— If G has a normal normal π-complement K, then $K = K_{G,\pi}(1)$ (Hence the normal π-complement is uniquely determined by the set π).

LEMMA 1. (see [4], lemma (20.4), p. 106). *For every π-element $g \in G$, if $C_G(g)$ is the centralizer of g in G, then*

$$|K_{G,\pi}(g)| = |G : C_G(g)| \cdot |K_{C_G(g),\pi}(1)| \tag{1}$$

LEMMA 2. *If H is a Hall π-subgroup of G, then the following conditions are equivalent:*

(a_1) *If two elements of H are conjugate in G, they are conjugate in H.*

(a_2) *For every $h \in H$, $K_{G,\pi}(h) \cap H = K_{H,\pi}(h)$.*

Proof. We observe first that, since H is a π-subgroup, $h = h_\pi$ for every for every $h \in H$. Hence the π-conjugate classes of H ocincides with the conjugate classes of H.

---

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Assume (a_1). It is obvious that $K_{H,\pi}(h) \subseteq K_{G,\pi}(h) \cap H$ for any $h \in H$. Let $k \in K_{G,\pi}(h) \cap H$. It results that $h(=h_\pi)$ and $k(=k_\pi)$ are conjugate in G. Hence from (a_1) h and k are conjugate in H. Thus $k \in K_{H,\pi}(h)$.

Conversely, assume (a_2). If $h, k \in H$ are conjugate in G, then $K_{G,\pi}(h) = K_{G,\pi}(k)$. It follows from (a_2) that $K_{H,\pi}(h) = K_{H,\pi}(k)$.
Hence h and k are conjugate in H.

2. Necessary and sufficient conditions for the existence of the normal π-complements

THEOREM 1. (A reformulation of a theorem of R. Brauer). *The following conditions are necessary and sufficient for the existence of normal π-complement in G:*

(A_1) *There exist a Hall π-subgroup H of G.*

(A_2) *For every $h \in H$, $K_{G,\pi}(h) \cap H = K_{H,\pi}(h)$*

(A_3) *If E is an elementary π-subgroup of G, then E is conjugate in G to a subgroup of H.*

Proof. The statements are immediate consequences of the Brauer's theorem (see [1], Th. 3) and of the LEMMA 2.

Remark 1. ([2], Th. 2) The condition (A_3) in THEOREM 1 can be replaced by the following:

(A_3') *If $1 \neq h \in H$ and P is a Sylow p-subgrup of $C_G(h)$, for some $p \in \pi$ not dividing the order of h, then the elementary π-subgroup $\langle h \rangle \times P$ is conjugate in G to a subgroup of H.*

Remark 2. ([3], Th. 1). The condition (A_3) in THEOREM 1. can be replaced by the statement:

(A_3'') *If $h \in H$ satisfier the condition $C_G(h) \neq G$, then $C_H(h)$ is a Hall π-subgroup of $C_G(h)$ and $C_G(h)$ has normal π-complement.*

THEOREM 2. *The following three conditions are equivalent*

(i) *The finite group G has normal π-complement.*

(ii) (B_1) *There exist a Hall π-subgroup H of G.*
(B_2) *$C_H(h)$ is a Hall π-subgroup of $C_G(h)$ and $|K_{C_G(h),\pi}(1)| = |C_G(h) : C_H(h)|$ for every $h \in H$.*

(iii) (C_1) *There exist a Hall π-subgroup H of G.*
(C_2) *Any π-element of G is conjugate in G to an element of H.*
(C_3) $|K_{C_G(h),\pi}(1)| = |C_G(h)|_{\pi'}$ *for every $h \in H$. ($|C_G(h)|_{\pi'}$ is the largest integer dividing $|C_G(h)|$ al of whose prime factors are in π')*

Proof. (i) implies (ii). Suppose that G has a normal π-complement. The condition (B_1) follows by THEOREM 1 and (B_2) is an immediate consequence of REMARK 2.

(ii) implies (iii). Assume (B_1) and (B_2). It is obious that (C_1) and (C_3) follows from (B_1) and (B_2).

In order to prove (C_2) we use industion on the number of p-factors of π-elements of G. Let g be a π-element of G and $g = g_p h = h g_p$, where $g_p \neq 1$

is the p-factor of g, for some $p \in \pi$. If $h = 1$, then (B_1) and Sylow's theorem shows that g is conjugate in G to an element of H. If $h \neq 1$, then by induction h is conjugate in G to an an element of H. Replacing g by a conjugate, we may suppose that $h \in H$. It follows from (B_2) that $C_H(h)$ contain a Sylow p-subgroup P of $C_G(h)$. Since $g_p \in C_G(h)$, it results that g_p is conjugate in $C_H(h)$ to an element of $P \subseteq C_H(h) \subseteq H$. Hence $g = hg_p$ is conjugate in G to an element of H. Thus (C_2) holds.

(iii) implies (i). Suppose that (C_1), (C_2) and (C_3) hold. We apply THEOREM 1. It is clear that (A_1) holds. It remains to prove (A_2) and (A_3).

Let $K_{H,\pi}(h_i)$ $(i = 1, 2, \ldots, n)$ be the different π-conjugate classes of H. Then

$$H = \bigcup_{i=1}^{n} K_{H,\pi}(h_i) \text{ (disjoint).} \tag{2}$$

Hence

$$|H| = \sum_{i=1}^{n} |K_{H,\pi}(h_i)| \tag{3}$$

By (C_2) any π-element of G are conjugate to an element of H. Hence, for every $g \in G$, there exist an $h_i \in H$ such that $K_{G,\pi}(g) = K_{G,\pi}(h_i)$. It follows that $K_{G,\pi}(h_i)$ $(i = 1, 2, \ldots, n)$ are the all π-conjugate classes of G, i.e.

$$G = \bigcup_{i=1}^{n} K_{G,\pi}(h_i) \tag{4}$$

The equility (1) implies from (C_3) that

$$|K_{G,\pi}(h_i)| = |G : C_G(h_i)| \cdot |C_G(h_i)|_{\pi'}. \tag{5}$$

Since H is a Hall π-subgroup of G, it results that $C_H(h_i)$ is a π—subgroup of $C_G(h_i)$. Hence

$$|C_G(h_i)|_{\pi'} \leqslant |C_G(h_i) : C_H(h_i)|.$$

This inequality implies from (5) that

$$|K_{G,\pi}(h_i)| \leqslant |G : C_G(h_i)| \cdot |C_G(h_i) : C_H(h_i)|. \tag{6}$$

It results that

$$|K_{G,\pi}(h_i)| \leqslant |G : H| \cdot |H : C_H(h_i)| = |G : H| \cdot |K_{H,\pi}(h_i)|$$

Hence by (3) we have

$$\sum_{i=1}^{n} |K_{G,\pi}(h_i)| \leqslant |G : H| \left(\sum_{i=1}^{n} |K_{H,\pi}(h_i)| \right) = |G : H| \cdot |H| = |G|$$

This implies from (4) that

$$G = \bigcup_{i=1}^{n} K_{G,\pi}(h_i) \text{ (disjoint)} \tag{7}$$

It follows that $K_{G,\pi}(h) \cap H = K_{H,\pi}(h)$ for every $h \in H$. Hence (A_2) holds.

It remains to prove (A_3).

We first note, that from (6) and (7) it follows that

$$|K_{G,\pi}(h_i)| = |G : C_G(h_i)| \cdot |C_G(h_i) : C_H(h_i)|$$

Hence from (5) we obtain that

$$|C_G(h_i)|_{\pi'} = |C_G(h_i) : C_H(h_i)| \tag{8}$$

This shows that $C_H(h)$ is a Hall π-subgroup of $C_G(h)$ for every $h \in H$.

Let E be an elementary π-subgroup of G, i.e. E is the direct product of a π-element h and a *p-subgrup* P_0 for some $p \in \pi : E = \langle h \rangle \times P_0$. It follows from (C_2) that h is conjugate in G to an element of H. Replacing E by a conjugate, we may assume that $h \in H$. Since $C_H(h)$ is a Hall π-subgroup of $C_G(h)$, it results that $C_H(h)$ contain a Sylow p-subgroup P of $C_G(h)$. Since P_0 is a p-subgroup of $C_G(h)$, the Sylow's theorem shows that P_0 is conjugate in $C_H(h)$ to a subgroup of $P \subseteq H$. This proves that E is conjugate in G to a subgroup of H. Hence (A_3) holds.

COROLLARY. *If the finite group G has a nilpotent Hall π-subgroup H. then the following conditions are equivalent*

(i) *G possesses normal π-complement*

(ii) $|K_{C_G(h),\pi}(1)| = |C_G(h)|_{\pi'}$ *for every* $h \in H$

Proof. The statements follows from THEOREM 2 and from a Theorem of Wielandt ([6]. The 5.8, p. 285)

As an application of Corollary, we obtain.

THEOREM 3. *Tge following conditions are lquivalent:*

(i) *G is a p-nilpotent group*

(ii) *If P is a Sylow p-subgroup of G, then for every $h \in P$ the number of p'-elements of $C_G(h)$ is $|C_G(h)|_{p'}$.*

Remark 3. It is known (see [5], p. 137) the following conjecture: If n divide the order of a finite group G and the number of solutions of $x^n = 1$ in G is exactly n, then these solutions form a normal subgroup of G.

It $n = |G|_{\pi'}$ then it is obvions that $K_{G,\pi}(1)$ is the set of solutions of $x^n = 1$. We hawe the following reformulation of THEOREM 2.

THEOREM 4. *If $n = |G|_{\pi'}$, then the following three conditions are equivalent:*

(i) *The solutions of $x^n = 1$ in G form a normal subgroup of G.*

(ii)
- (B_1') *There exist a Hall π — subgroup H of G*
- (B_2') *For every $h \in H$, $C_H(h)$ is a Hall π — subgroup of $C_G(h)$ and if $|n_h| = |C_G(h)|_{\pi'}$, then the number of solutions of $x^{n_h} = 1$ in $C_G(h)$ is n_h.*

(iii) $\begin{cases} (C'_1) \text{ There exist a Hall } \pi - \text{subgroup } H \text{ of } G \\ (C'_2) \text{ Any } \pi - \text{element of } G \text{ is conjugate in } G \text{ to an element of } H \\ (C'_3) \text{ For every } h \in H\text{, the number of solutions of } x^{n_h} = 1 \text{ in } C_G(h) \text{ is } \end{cases}$

REFERENCES

1. Brauer, R., *On quotient groups of finite groups*, Math. Zeitschr. 83, (1964), p. 72–84.
2. Dade, E. C., *On normal complement of sections of finite groups*, J. Austral Math. Soc. 19 (Ser. A), (1975) p. 257–262.
3. Suzuki, M., *On the existence of a Hall normal subgroup*, J. Math. Soc. Japan, Vol. 15, No. (1963), p. 387–391.
4. Feit, W., *Characters of finite groups*. W. A. Benjamin, Inc. New York, Amsterdam, 1967.
5. Hall, M., *The theory of groups*. The Macmillan Company New York, 1959.
6. Huppert, B., *Endliche Gruppen I*. Springer-Verlag Berlin Heidelberg New-York, 1967.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

THE DEFINITION OF DISTANCE AND DIAMETER IN FUZZY SET THEORY

GIANGIACOMO GERLA* and RAFFAELE VOLPD*

Received: December 14, 1984

REZUMAT. — Definirea distanţei şi diametrului în teoria mulţimilor fuzzy. În lucrare se defineşte distanţa între două mulţimi nuanţate (fuzzy) şi diametrul unei mulţimi nuanţate. În cazul clasic aceste definiţii se reduc la cele cunoscute.

1. Introduction. In any metric space (S, d) it is possible to define the *distance* between two subsets X and Y of S by setting $\delta(X, Y) = 0$ if $X = \emptyset$ or $Y = \emptyset$ and

$$\delta(X, Y) = \inf \{d(x, y)/x \in X,\ y \in Y\} \text{ otherwise.} \tag{1}$$

The distance between a point x and a set X is defined by setting $\delta(x, X) = \delta(\{x\}, X)$.

This allows, for instance, to characterize the non-empty closed sets as the sets X for which $x \in X$ if and only if $\delta(x, X) = 0$.

Another foundamental concept is that of *diameter* $\Delta(X)$ of a set. One defines it by setting $\Delta(X) = 0$ if $X = \emptyset$ and

$$\Delta(X) = \sup \{d(x, y)/x \in X,\ y \in X\} \text{ otherwise.} \tag{2}$$

In this paper our aim is to define analogue concepts for the fuzzy sets. So we define the *distance* between two fuzzy sets and, hence, between a fuzzy point and a fuzzy set.

We call *closed* a fuzzy set containing all the fuzzy points that have distance from it equals to zero, and we show that the complements of closed sets determine a fuzzy topology, the fuzzy topology of lower semi-continuous functions.

Also, we define the *diameter* of a fuzzy set. This will allow to characterize the fuzzy points as the fuzzy sets with diameter equals to zero.

2. Prerequisites and definitions. Let X a set and $\mathbf{R}$ the set of real numbers. We say *fuzzy subset* of X or, more simply, *fuzzy set* [8] a function $f: X \to [0, 1]$ where $[0, 1]$ denotes the set $\{\alpha \in R/0 \leqslant \alpha \leqslant 1\}$.

We denote by $F(X)$ the class of the fuzzy subsets of X. If $f, g \in F(X)$ then we set $f \leqslant g$ iff $f(x) \leqslant g(x)$ for any $x \in X$. Moreover $-f$, *the complement of* f is the fuzzy subset of X defined by setting $(-f)(x) = 1 - f(x)$ for any $x \in X$. If $(f_i)_{i \in I}$ is a family of fuzzy subsets of X then $\bigvee_{i \in I} f_i$ and $\bigwedge_{i \in I} f_i$ are the fuzzy subsets of X defined by setting

$(\bigvee_{i \in I} f_i)(x) = \sup_{i \in I} \{f_i(x)\}$ and

* Dep. Mat. and Appl. R. Caccioppoli via Mezzocannone 8, 80134 Napoli, Italy

$(\bigwedge_{i\in I} f_i)(x) = \inf_{i\in I}\{f_i(x)\}$ for any $x \in X$.

We denote by f_0 and f_1 the fuzzy sets for which $f_0(x) = 0$ and $f_1(x) = 1$ for any $x \in X$.

Moreover, if $\alpha \in [0, 1]$, we call *α-cut* of a fuzzy set f the subset $C_f^\alpha = \{x \in X / f(x) \geqslant \alpha\}$.

A fuzzy set f is called *crisp* if $f(x) \in \{0, 1\}$ for any $x \in X$. The fuzzy sets crisp can be interpreted as characteristic functions of subsets of X and, hence, they can be identified with these subsets.

For any $a \in X$ and $\alpha \in (0, 1] = \{x \in \mathbf{R}/0 < x \leqslant 1\}$ the fuzzy set f_a^α, defined by setting $f_a^\alpha(x) = 0$ if $x \neq a$ and $f_a^\alpha(x) = \alpha$ if $x = a$, is called *fuzzy point* ([7], [3], [4]).

We say that the fuzzy point f_a^α *belongs* to the fuzzy set f, $f_a^\alpha \in f$, if $f_a^\alpha \leqslant f$ that is if $f(a) \geqslant \alpha$.

We can now define the concept of fuzzy topological space (see references). To this aim we give the following definitions.

DEFINITION 1. *A class τ of fuzzy subsets of X constitutes a fuzzy topology if the following conditions are verified:*

a) $f_0, f_1 \in \tau$

b) *if $f, g \in \tau$ then $f \wedge g \in \tau$*

c) $\bigvee_{i\in I} f_i \in \tau$ *for any family $(f_i)_{i\in I}$ of elements in τ.*

The pair (X, τ) is named *fuzzy topological space*; the elements of τ are named *open*, the complements of these elements are named *closed*.

The following definition is dual of Definition 1.

DEFINITION 2. *A class $C \subset F(x)$ is a system of closed fuzzy subsets of X if the following conditions are verified:*

a) $f_0, f_1 \in C$

b) *if $f, g \in C$ then $f \vee g \in C$*

c) $\bigwedge_{i\in I} f_i \in C$ *for any family $(f_i)_{i\in I}$ of elements of C.*

Obviously, the class of complements of a system of closed fuzzy set is a fuzzy topology and the class of complements of a fuzzy topology constitutes a system of closed fuzzy sets.

3. Distance between two fuzzy sets. Let (S, d) be a metric space. We define a *distance* between two fuzzy subsets f, g of S in the following way:

$$d(f, g) = \int_0^1 \delta(C_f^\alpha, C_g^\alpha)\,d\alpha \qquad (3)$$

Note that if $\beta \geqslant \alpha$ then $C_f^\beta = \{x \in S/f(x) \geqslant \beta\} \subseteq C_f^\alpha = \{x \in S/f(x) \geqslant \alpha\}$ and, hence, $\delta(C_f^\beta, C_g^\beta) \geqslant \delta(C_f^\alpha, C_g^\alpha)$. This proves that $\delta(C_f^\alpha, C_g^\alpha)$ is an increasing function of α and, hence, that the distance between two fuzzy sets is defined for any $f, g \in F(S)$, even if it is finite or infinite. An example of a pair of fuzzy sets with infinite distance is the following.

Let (S, d) be the set of real numbers with the usual distance, and consider f_0^1 and f, where f is the fuzzy set for which $f(x) = x/x + 1$; then $d(f_0^1, f)$ is equal to ∞.

If in f and g there are two crisp points, that is if there exist x, y in S for which f_x^1 and f_y^1 belong respectively to f and g, then, being any contribution $\delta(C_f^\alpha, C_g^\alpha) \leqslant d(x, y)$, the integral in (3) assumes a finite value.

If f and g are the characteristic functions of two subsets X and Y of S then $C_f^\alpha = X$ and $C_g^\alpha = Y$ for every $\alpha > 0$, hence $d(f, g) = \int_0^1 \delta(X, Y)\, d\alpha = 1 \cdot \delta(X, Y) = \delta(X, Y)$. Then (3) generalizes the classical definition of distance between two subsets of a metric space.

Obviously the distance between a fuzzy point f_x^α and a fuzzy set g is $\int_0^\alpha \delta(x, C_g^\beta)\, d\beta$. Moreover the distance between two fuzzy points f_b^β and f_c^γ is equal to $\int_0^{\beta / \gamma} \delta(\{b\}, \{c\})\, d\alpha$ and therefore

$$d(f_b^\beta, f_c^\gamma) = [\gamma \wedge \beta] \cdot d(b, c). \tag{4}$$

This proves that, for the fuzzy points crisp, the distance defined by (3) coincides with the usual one between points.

It is interesting to examine the case that f and g assume values in a finite subset $\{\gamma_0, \ldots, \gamma_n\}$ of $[0, 1]$. Then, if $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_n = 1$ we have

$$d(f, g) = \sum_{i=1}^{n} \delta(C_f^{\gamma_i}, C_g^{\gamma_i}) \cdot (\gamma_i - \gamma_{i-1})\,; \tag{5}$$

if, for $i = 1, \ldots, n$, $\gamma_i - \gamma_{i-1} = 1/n$

$$d(f, g) = 1/n \cdot \left(\sum_{i=1}^{n} \delta(C_f^{\gamma_i}, (C_g^{\gamma_i})\right) \cdot \tag{6}$$

In general, we can also utilize Formulas (5) and (6) to compute a suitable approximation of the distance between two fuzzy subsets.

We can give a definition of closure for fuzzy sets:

DEFINITION 3. *A fuzzy set f is metrically closed if either $f = f_0$ or, for every fuzzy point f_x^α, $f_x^\alpha \in f$ iff $d(f_x^\alpha, f) = 0$. We denote by C the class of the metrically closed fuzzy sets.*

PROPOSITION 1. *The set C is a system of closed fuzzy subsets of X. Equivalently, the set τ of the relative complements defines a fuzzy topology.*

Proof. It is obvious that f_0 and f_1 are elements of C. Let $f \in C$ and $g \in C$, and let f_x^α a fuzzy point. If $f_x^\alpha \in f \vee g$ it is obvious that $d(f_x^\alpha, f \vee g) = 0$. Conversely, suppose that $d(f_x^\alpha, f \vee g) = 0$, then $\delta(x, C_{f \vee g}^\beta) = 0$ for every

$\beta < \alpha$. Suppose, by absurd that $f_x^\alpha \notin f \vee g$, then $f(x) < \alpha$ and $g(x) < \alpha$, i.e. $f_x^\alpha \notin f$ and $f_x^\alpha \notin g$. This implies that $d(f_x^\alpha, f) > 0$ and $d(f_x^\alpha, g) > 0$ and therefore that $\delta(x, C_f^\gamma) > 0$ and $\delta(x, C_g^\gamma) > 0$ for a suitable $\gamma < \alpha$. It follows that $\delta(x, C_f \cup C_g) > 0$ and, since $C_{f\vee g}^\gamma \subseteq C_f^\gamma \cup C_g^\gamma$, $\delta(x, C_{f\vee g}^\gamma) \geqslant \delta(x, C_f^\gamma \cup C_g^\gamma) > 0$, an absurd. This prove that $f_x^\alpha \in f \vee g$ and therefore that $f \vee g \in C$.

Let $(f_i)_{i\in I}$ be a family of elements of C and set $f = \bigwedge_{i\in I} f_i$: we have to prove that $f \in C$. If $f_x^\alpha \in f$ it is obvious that $d(f_x^\alpha, f) = 0$. Assume that $d(f_x^\alpha, f) = 0$ then $\delta(x\, C_f^\beta) = 0$ for every $\beta < \alpha$. If, by absurd, $\alpha > f(x)$, then $\alpha > f(x_j)$, and therefore $f_x^\alpha \notin f_j$, for a suitable $j \in I$. Thus $\int_0^\alpha \delta(x, C_{f_j}^\beta)\, d\beta > 0$ and there exists $\gamma < \alpha$ such that $\delta(x, C_{f_j}) > 0$. Since $C_f^\gamma \subseteq C_{f_j}^\gamma$, we have also that $\delta(x, C_f^\gamma) \geqslant (x, C_{f_j}^\gamma) > 0$, an absurd. Thus we have proved that $\alpha \leqslant f(x)$ and therefore that $f_x^\alpha \in f$. This complete the proof.

Now we show that the above defined fuzzy topology τ coincides with the *natural fuzzy topology* defined in [2].

PROPOSITION 2. *C is the class of the upper semicontinuous functions from S to $[0, 1]$. It follows that τ is the class of the lower semicontinuous functions.*

Proof. Let $f \in C$, then, to prove that f is upper semicontinuous, it suffices to prove that $\{x \in S / f(x) < \alpha\}$ is open for every $\alpha \in [0, 1]$. Equivalently, we can prove that C_f^α is closed. Let $x \in S$ and $\delta(x, C_f^\alpha) = 0$, then $\delta(x, C_f^\beta) = 0$ for every $\beta < \alpha$ and therefore $d(f_x^\alpha, f) = \int_0^\alpha \delta(x, C_f^\beta)\, d\beta = 0$. Thus $f_x^\alpha \in f$ and $x \in C_f^\alpha$. This proves that C_f^α is closed.

Conversely, suppose f upper semicontinuous or, equivalently, that C_f^α is closed for every $\alpha \in [0, 1]$. Moreover, suppose that $d(f_x^\alpha, f) = \int_0^\alpha \delta(x, C_f^\beta)\, d\beta = 0$, then $\delta(x, C_f^\beta) = 0$ for every $\beta < \alpha$. This implies that $x \in C_f^\beta$ and therefore that $f(x) \geqslant \beta$ for every $\beta < \alpha$. In conclusion $f(x) \geqslant \alpha$ and $f_x^\alpha \in f$. This proves that $f \in C$.

4. Diameter of a fuzzy set. Let f be a fuzzy subset of S, then we set

$$\Delta(f) = \sup \{d(x, y) / x \text{ and } y \text{ are fuzzy points of } f\}. \quad (7)$$

The number $\Delta(f)$ may be either finite or infinite, we call it the *diameter of the fuzzy set f*.

If $\Delta(f) < \infty$ then f is called *bounded*.

Being $\delta(f_x^\alpha, f_y^\beta) = d(x, y) \cdot (\alpha \wedge \beta)$, it is obvious that

$$\Delta(f) = \sup \{d(x, y) \cdot [f(x) \wedge f(y)]/x, y \in S\}. \tag{8}$$

PROPOSITION 3. *If f is crisp then the definition of diameter is the classical one. Moreover if $f \leqslant g$ then $\Delta(f) \leqslant \Delta(g)$.*

Proof. If f is the characteristic function of the set X then $\Delta(f) = \sup \{d(x, y) \cdot [f(x) \wedge f(y)]/f(x) \neq 0\ f(y) \neq 0\} = \sup \{d(x, y)/x \in X,\ y \in X\}$.

Suppose that $f \leqslant g$, then $d(x, y) \cdot [f(x) \wedge f(y)] \leqslant d(x, y) \cdot [g(x) \wedge g(y)]$ and $\Delta(f) \leqslant \Delta(g)$.

PROPOSITION 4. *The diameter of a fuzzy set $f \neq f_0$ is equal to zero iff f is a fuzzy point.*

Proof. It is obvious that the diameter of a fuzzy point is zero. Conversely, suppose that f is a fuzzy set for which $\Delta(f) = 0$. Then, by (8), $d(x, y) \cdot [f(x) \wedge f(y)] = 0$ for every $x, y \in S$. By hypothesis, there exists $a \in S$ for which $f(a) \neq 0$ and, if $y \neq a$, since $d(a, y) \neq 0$ then $f(a) \wedge f(y) = 0$. This proves that $f(y) = 0$ for every $y \neq a$ and therefore that f is a fuzzy point.

PROPOSITION 5. *For any $f \in F(S)$* and $\alpha \in (0, 1]$

$$\Delta(C_\alpha^f) \leqslant \Delta(f)/\alpha \tag{9}$$

Then every α-cut of a bounded fuzzy set is bounded while the converse falls,

Proof. If $x, y \in C_f^\alpha$, i.e. $f(x) \geqslant \alpha$, $f(y) \geqslant \alpha$, then $d(x, y) \cdot [f(x) \wedge f(y)] \geqslant d(x, y) \cdot \alpha$. This proves that $\Delta(f) \geqslant \alpha \cdot d(x, y)$ or, equivalently, $d(x, y) \leqslant \Delta(f)/\alpha$.

To prove that there exists a fuzzy set f such that $\Delta(f) = \infty$ and $\Delta(C_f^\alpha) < \infty$ for any $\alpha \in [0, 1]$, let S be the positive real numbers set and define $f: S \to [0, 1]$ by setting $f(x) = 1/(\sqrt{x} + 1)$. Now $\Delta(f) \geqslant d(0, x) \cdot (f(0) \wedge f(x)) = x/(\sqrt{x} + 1)$ for any $x \in S$. Then $\Delta(f) = \infty$ while it is obvious that every cut of f is bounded.

Proposition 5 shows that our definition of bounded fuzzy set is different from Kaufmann's definition [6].

In metric space theory one proves that a subset is bounded if and only if it is contained in a suitable circle. In order to obtain a similar result for fuzzy subsets we give the following definition.

DEFINITION 4. *We call f-circle with center f_c^γ and radius r, the fuzzy set $C(f_c^\gamma, r)$ such that, for any fuzzy point f_b^β, $f_b^\beta \in C(f_c^\gamma, r)$ iff $d(f_b^\beta, f_c^\gamma) \leqslant r$ and $\beta \leqslant \gamma$.*

PROPOSITION 6. *The f-circle $C(f_c^\gamma, r)$ is the fuzzy set defined by*

$$f(z) = \begin{cases} \gamma \text{ if } d(z, c) \leqslant r/\gamma \\ r/d(z, c) \quad \textit{otherwise.} \end{cases} \tag{10}$$

Moreover the diameter of $C(f_c^\gamma, r)$ is not greater than $2r$.

Proof. By definition $f = \bigvee\{f / \beta_x^\beta \leqslant \gamma,\ x \in S \ and\ d(f_x^\beta, f_c^\gamma) \leqslant r\}$, then $f(z) = \bigvee\{f_z^\beta / \beta \leqslant \gamma \ and\ d(f_z^\beta, f_c^\gamma) \leqslant r\} = \bigvee\{\beta/\beta \leqslant \gamma \ and\ \beta \cdot d(x, c) \leqslant r\}$. This proves (10).

To show that $\Delta(f) \leqslant 2r$ observe that, for every pair of fuzzy points $f_b^\beta, f_{b'}^{\beta'}$ with $\beta \leqslant \gamma$ and $\beta' \leqslant \gamma$, the following triangular inequality holds:

$$d(f_b^\beta, f_{b'}^{\beta'}) \leqslant d(f_b^\beta, f_c^\gamma) + d(f_c^\gamma, f_{b'}^{\beta'}). \tag{11}$$

In fact, by $d(b, b') \leqslant d(b, c) + d(c, b')$, we have

$$(\beta \wedge \beta') \cdot d(b, b') \leqslant (\beta \wedge \beta') \cdot d(b, c) + (\beta \wedge \beta') \cdot d(c, b') \leqslant \beta \cdot d(b, c) +$$

$$+ \beta' \cdot d(c, b') = (\beta \wedge \gamma) \cdot d(b, c) + (\beta' \wedge g) \cdot d(c, b') = d(f_b^\beta, f_c^\gamma) + d(f_c^\gamma, f_{b'}^{\beta'}).$$

But $(\beta \wedge \beta') \cdot d(b, b') = d(f_b^\beta, f_{b'}^{\beta'})$ and then (11) is proved.

From this it follows that $\triangle(f) \leqslant 2r$.

PROPOSITION 7. *Let f be a bounded fuzzy set, $\gamma = \sup\{f(x)\}$ and $c \in S$ a point such that $f(c) > 0$. Then f is contained in the f-circle $C(f_c^\gamma, \Delta(f)/f(c))$. It follows that a fuzzy set f is bounded if and only if it is contained in an f-circle.*

Proof. Let $r = \Delta(f)/f(c)$ and denote by g the f-circle $C(f_c^\alpha, r)$. If $d(z, c) \leqslant r/\gamma$ then $g(z) = \gamma = \sup\{f(x)\}$ and therefore $g(z) \geqslant f(z)$. If $d(z, c) > r/\gamma$ then $g(z) = r/d(z, c)$. Since $f(c) \cdot f(z) \leqslant f(c) \wedge f(z)$, we have also that $d(c, z) \cdot f(c) \cdot f(z) \leqslant d(c, z) \cdot (f(c) \wedge f(z)) \leqslant \Delta(f)$. This proves that $f(z) \leqslant r/d(z, c) = g(z)$.

Finally, observe that, if (S, d) is the euclidean plane, then the diameter of an f-circle $C(f_c^\gamma, r)$ is just $2r$. Indeed, let z and z' two points collinear with c such that $d(z, c) = d(z', c) = r/\gamma$. Then $d(z, z') = 2r/\gamma$ and $d(f_z^\gamma, f_{z'}^\gamma) = \gamma \cdot d(z, z') = 2r$. Since f_z^γ and $f_{z'}^\gamma$, are fuzzy points of the f-circle $C(f_c^\gamma, r)$, this proves that the relative diameter is $2r$.

REFERENCES

1. C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. 24 (1968) 182–190.
2. F. Conrad, *Fuzzy topological concepts*, J. Math. Anal. Appl. 74 (1980) 432–440.
3. G. Gerla, *On the concept of fuzzy point*, to appear.
4. G. Gerla, *Generalized fuzzy points*, J. Math. Anal. Appl. (to appear).
5. G. Gerla, *Representation of fuzzy topologies*, Fuzzy Sets and Systems 11 (1983) 103–113.
6. A. Kaufmann, *Introduction à la théorie des sous-ensembles flous*, Ed. Masson (1975).
7. C. K. Wong, *Fuzzy points and local properties of fuzzy topology*, J. Math. Anal. Appl. 46 (197 316–328.
8. L. A. Zadeh, *Fuzzy sets*, Information and control 8 (1965) 338–353.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

PARTICULAR n — α-CLOSE-TO-CONVEX FUNCTIONS

TEODOR BULBOACĂ*

Received: June 15, 1986

REZUMAT. — Funcţii particulare n — α-aproape convexe. În lucrare se dau unele rezultate referitoare la funcţii particulare din clasele de funcţii $AC_n(\delta)$, $AC_n\left(\frac{1}{2}\right)$, $C_n\left(\frac{1}{2}\right)$ introduse de H. S. Al-Amiri în [1].

1. Introduction. Let A be the class of functions $f(z)$, analytic in the unit disc U with $f(0) = f'(0) - 1 = 0$. Like in [2] we denote by $K_{n,\alpha}(\delta)$ the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] > \delta, \quad z \in U$$

where $\alpha \geqslant 0$, $\delta < 1$ and $D^nf(z) = \dfrac{z}{(1-z)^{n-1}} * f(z) = \dfrac{z(z^{n-1}f(z))^{(n)}}{n!}$, where $(*)$ stands for the Hadamard product. Note that the classes $K_{n,\alpha}(\delta)$ and $Z_n(\delta) \equiv \equiv K_{n,0}(\delta)$ are studied in [2] and the classes $K_{n,\alpha}\left(\frac{1}{2}\right)$ and $Z_n\left(\frac{1}{2}\right)$ were introduced by H. S. Al — Amir [1] and S. Ruscheweyh [6] respectively.

Like in [3] we denote by $AC_n(\delta)$ (the class of *n-closeto-convex functions of order* δ) the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)} > \delta, \ z \in U, \text{ where } g \in Z_{n+1}(\delta)$$

and we denote by $C_{n,\alpha}(\delta)$ (the class of *n — α-close-toconvex functions of order* δ) the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}f(z)} + \alpha\frac{D^{n+2}g(z)}{D^{n+2}g(z)}\right] > \delta, \ z \in U \text{ where}$$

$g \in Z_{n+2}(\delta)$. The classes $AC_n\left(\frac{1}{2}\right)$ and $C_{n,\alpha}\left(\frac{1}{2}\right)$ were introduced by H. S. Al — Amiri [1] and some properties of $AC_n(\delta)$ and $C_{n,\alpha}(\delta)$ given in [3] by using sharp subordination results (see [4], [5]).

In this paper we will present some results concerning particular functions of this classes.

2. Preliminaries. Let f and g be regular in U. We say that f is subordinate to g, written $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

* Industrial Lycée, 2900 Arad, Romania

We will need the following lemmas to prove our main results and we denote by $b_\gamma(z) = \sum_{j=1}^{\infty} \frac{\gamma + 1}{\gamma + j} z^j$, $\operatorname{Re} \gamma > -1$.

LEMMA A. [3, Theorem 2]. *Let* $\gamma > -1$ *and*

$$\delta_0 = \max \left\{\frac{n - \gamma}{n + 1}, \frac{2n - \gamma}{2(n + 1)}\right\} \leqslant \delta < 1.$$

If $f \in Z_n(\delta)$ *then* $f * b_\gamma \in Z_n(\tilde{\delta}(n, \gamma, \delta))$, *where*

$$\tilde{\delta}(n, \gamma, \delta) = \frac{1}{n+1}\left(\frac{\gamma + 1}{F\left(1, 2(n + 1)(1 - \delta), \gamma + 2, \frac{1}{2}\right)} - \gamma + n\right)$$

and this result is sharp.

LEMMA B. [4]. *Let* $h, q \in H(U)$ *be univalent in* U *and suppose* $q \in H(U)$. *If* $\psi : C^3 \to C$ *satisfies:*

a) ψ *is analytic in a domain* $D \subset C^3$

b) $(q(0), 0, 0) \in D$ *and* $\psi(q(0), 0, 0) \in h(U)$

c) $\psi(r, s, t) \notin D$ *when* $(r, s, t) \in D$, $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re}(1 + t/s) \geqslant m \operatorname{Re}(1 + \zeta q''(\zeta)/q'(\zeta))$, *where* $|\zeta| = 1$, $m \geqslant 1$, *then for all* $p \in H(U)$ *so that* $(p(z), zp'(z), z^2p''(z)) \in D$, $z \in U$ *we have:*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \ D\ p(z) \prec q(z).$$

3. Main results.

LEMMA 1. Let $\gamma > -1$ *and* $\delta_0 = \max \left\{\frac{n - \gamma}{n + 1}, \frac{2n - \gamma}{2(n + 1)}\right\} \leqslant \delta \leqslant \frac{2n - \gamma + 1}{2(n + 1)}$ *then* (1) $\delta \leqslant \tilde{\delta}(n, \gamma, \delta) = \frac{1}{n+1}\left(\frac{\gamma + 1}{F\left(1, 2(n + 1)(1 - \delta), \gamma + 2; \frac{1}{2}\right)} - \gamma + n\right)$.

Proof. The above inequality is equivalent to

$$F\left(1, 2(n + 1)(1 - \delta), \gamma + 2; \frac{1}{2}\right) \cdot ((n + 1)\delta + \gamma - n) \leqslant \gamma + 1. \quad (2)$$

Because $F\left(1, 2(n + 1)(1 - \delta), \gamma + 2; \frac{1}{2}\right) = 1 + \sum_{k+1}^{\infty} \frac{a_k}{2^k}$ where

$a_k = \frac{b}{c}\frac{b+1}{c+1} \cdot \ldots \cdot \frac{b + k - 1}{c + k - 1}$ and $b = 2(n + 1)(1 - \delta)$, $c = \gamma + 2$ we can easily show that if $\delta_0 \leqslant \delta \leqslant \frac{2n - \gamma + 1}{2(n + 1)}$ then $a_k \leqslant \left(\frac{b}{c - 1}\right)^k$ for all $k \in N$, hence (2) holds.

THEOREM 1. *Let* $\gamma > -1$ *and*

$$\max\left\{\frac{n-\gamma+1}{n+2}, \frac{2n-\gamma+2}{2(n+2)}\right\} \leqslant \delta \leqslant \frac{2n-\gamma+3}{2(n+2)}.$$

If $f \in AC_n(\delta)$ *related to* $g \in Z_{n+1}(\delta)$ *then* $f * b_\gamma \in AC_n(\delta)$ *related to* $g * b_\gamma \in \in Z_{n+1}(\delta)$.

Proof. Because $g \in Z_{n+1}(\delta)$, by using Lemma A we have $G \equiv g * b_\gamma \in \in Z_{n+1}(\bar{\delta}(n+1, \gamma, \delta))$ and from (1) we deduce that $G \in Z_{n+1}(\delta)$. Let $F(z) \equiv \equiv f(z) * b_\gamma(z)$ and $D^{n+1}F(z)/D^{n+1}G(z) = p(z)$, $p(0) = 1$. From the well-known formulas [6]

$$z(D^k f(z))' = (k+1)\, D^{k+1}f(z) - kD^k f(z) \tag{3}$$

$$z(D^k F(z))' = (\gamma+1)D^k f(z) - \gamma D^k f(z),\ \operatorname{Re}\gamma > -1,\ k \in N$$

we obtain

$$\frac{D^{n+1}f(z)}{D^{n+1}g(z)} = p(z) + \alpha(z)\, zp'(z), \text{ where } \alpha(z) = \frac{1}{1+\gamma}\frac{D^{n+1}G(z)}{D^{n+1}g(z)}.$$

Using again (3) and because $G \in Z_{n+1}(\delta)$ we obtain

$$\operatorname{Re}\alpha(z) = \frac{1}{1+\gamma}\left[(n+2)\ \operatorname{Re}\frac{D^{n+2}G(z)}{D^{n+1}G(z)} + \gamma - n - 1\right] > \frac{1}{F\left(1, 2(n+2)(1-\delta), \gamma+2; \frac{1}{2}\right)}$$

hence $\operatorname{Re}\alpha(z) > 0$, $z \in U$. Because $f \in AC_n(\delta)$ related to $g \in Z_{n+1}(\delta)$, then

$$p(z) + \alpha(z)\ zp'(z) \prec h_\delta(z) = \frac{1-(1-2\delta)z}{1+z}.$$

Without loss of generality we can assume that p and h satisfy the conditions of the theorem on the closed disc $\bar{U}$; if not we can replace $p(z)$ by $p_r(z) = = p(rz)$ and $h_\delta(z)$ by $h_{\delta, r}(z) = h_\delta(rz)$, $0 < r < 1$, and these new functions satisfy the conditions of the theorem on $\bar{U}$. We would then prove $p_\delta(z)_{\delta, r} \prec h_\delta(z)$ for all $0 < r < 1$ and by letting $r \to 1^-$ we have $p(z) \prec h_\delta(z)$.

Let $\psi(r, s) = r + \alpha(z)s$ which is analytic in C^2 and $\psi(h_\delta(0), 0) = h_\delta(0) \in \in h_\delta(U)$. A simple calculus shows that

$$\operatorname{Re}\frac{\psi_0 - h_\delta(\zeta_0)}{\zeta_0 h_\delta(\zeta_0)} = m_0\ \operatorname{Re}\alpha(z) > 0,\ z \in U, \text{ where}$$

$$\psi_0 = h_\delta(\zeta_0) + \alpha(z)\, m_0\zeta_0 h'_\delta(\zeta_0),\ m_0 \geqslant 1,\ |\zeta_0| = 1.$$

Using the fact that $\zeta_0 h'_\delta(\zeta_0)$ is an outward normal to the boundary of the convex domain $h_\delta(U)$ we conclude that $\psi_0 \notin h_\delta(U)$ and using Lemma B we have $p(z) \prec h_\delta(z)$ i.e.

$$F \in AC_n(\delta) \text{related to } G \in Z_{n+1}(\delta).$$

Taking $n = 0$ in Theorem 1 we obtain:

COROLLARY 1. *Let* $\gamma > -1$, $\max\left\{\frac{1-\gamma}{2}, \frac{2-\gamma}{4}\right\} \leqslant \delta \leqslant \frac{3-\gamma}{4}$ *and* $f, g \in$ *Then*

$$\operatorname{Re}\frac{f'(z)}{g'(z)} > \delta,\ z \in U \text{ where } \operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > 2\delta - 1,\ z \in U \text{ implies}$$

$$\operatorname{Re}\frac{F'(z)}{G'(z)} > \delta,\ z \in U \text{ where } \operatorname{Re}\left(1 + \frac{zG''(z)}{G'(z)}\right) > 2\delta - 1,\ z \in U \text{ and}$$

$$F = f * b_\gamma,\ G = g * b_\gamma.$$

Remark. Taking $\delta = \frac{1}{2}$ in this corollary we obtain that if $\gamma \in [n, n+1$ and $f \in AC_n\left(\frac{1}{2}\right)$ related to $g \in Z_{n+1}\left(\frac{1}{2}\right)$ then $f * b_\gamma \in AC_n\left(\frac{1}{2}\right)$ related to g $\in Z_{n+1}\left(\frac{1}{2}\right)$. This last result holds for all $\gamma \in C$ with $\operatorname{Re}\gamma \geqslant \frac{n}{2}$, $n \in N$ Theorem 2].

COROLLARY 2. *If* $-1 < \gamma \leqslant 0$ *then* $f \in Z_n\left(\frac{n-\gamma}{n+1}\right)$ *implies that* $f * b_\gamma$ $\in Z_n\left(\tilde{\delta}\left(n, \gamma, \frac{n-\gamma}{n+1}\right)\right)$, *where*

$$\tilde{\delta}\left(n, \gamma, \frac{n-\gamma}{n+1}\right) = \frac{1}{(n+1)\sqrt{\pi}}\frac{\Gamma(\gamma + 3/2)}{\Gamma(\gamma+1)} + \frac{n-\gamma}{n+1} \text{ and this result is sharp.}$$

Proof. If $-1 < \gamma \leqslant 0$ then $\max\left\{\frac{n-\gamma}{n+1}, \frac{2n-\gamma}{2(n+1)}\right\} = \frac{n-\gamma}{n+1}$ and using Lemma A for $\delta = \frac{n-\gamma}{n+1}$ we obtain our result.

Taking $\gamma = 0$ in Lemma A we obtain:

COROLLARY 3. *Let* $\frac{n}{n+1} \leqslant \delta < 1$ *and* $f \in Z_n(\delta)$; *then* $f * b_0 \in Z_n(\tilde{\delta}(n, 0, \delta))$, *where*

$$\tilde{\delta}(n, 0, \delta) = \begin{cases} \frac{1}{n+1}\left[\frac{1 - 2(n+1)(1-\delta)}{2 - 2^{2(n+1)(1-\delta)}} + n\right], & \text{for } \delta \neq \frac{2n+1}{2(n+1)} \\ \frac{1}{n+1}\left[\frac{1}{2\ln 2} + n\right], & \text{for } \delta = \frac{2n+1}{2(n+1)} \end{cases}$$

and this result is sharp.

Taking $n = 0$ in the above corollary we obtain:

COROLLARY 4. *Let* $0 \leqslant \delta < 1$ *and* $f \in A$ *with* $\operatorname{Re}\frac{zf'(z)}{f(z)} > \delta$, $z \in U$. T $\operatorname{Re}\frac{zF'(z)}{F(z)} > \tilde{\delta}$, $z \in U$ *where*

$$\tilde{\delta} = \begin{cases} \dfrac{2\delta - 1}{2 - 2^{2(1-\delta)}}, & \text{for } \delta \neq \dfrac{1}{2} \\ \dfrac{1}{2 \ln 2}, & \text{for } \delta = \dfrac{1}{2} \end{cases}$$

and $F = f * b_0$ *and this result is sharp.*

COROLLARY 5. *If* $\gamma > 0$ *then* $f \in Z_n\left(\dfrac{2n - \gamma}{2(n + 1)}\right)$ *implies that* $f * b_\gamma \in Z_n\left(\dfrac{2n - \gamma + 1}{2(n + 1)}\right)$ *and this result is sharp.*

Proof. If $\gamma > 0$ then $\max\left\{\dfrac{n - \gamma}{n + 1}, \dfrac{2n - \gamma}{2(n + 1)}\right\} = \dfrac{2n - \gamma}{2(n + 1)}$ and using Lemma A for $\delta = \dfrac{2n - \gamma}{2(n + 1)}$ we obtain the above result.

REFERENCES

1. H. S. Al-Amiri, *Certain analogy of the α-convex functions*, Rev. Roum. Math. Pures Appl., XXIII, 10 (1978), 1449–1454.
2. T. Bulboacă, *Applications of the Briot-Bouquet differential subordination*, (to appear).
3. T. Bulboacă, *Classes of* n — *α-close-to-convex functions*, (to appear)
4. S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), 151–171.
5. P. T. Mocanu, D. Ripeanu, I. Şerb, *The order of starlikeness of certain integral operators*, Mathematica (Cluj), 23 (46), No. 2 (1981), 225–230.
6. S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49 (1975), 109–115.

LE CALCUL DE L'INFLUENCE DES PAROIS SUR UN ÉCOULEMENT COMPRESSIBLE ROTATOIRE

SONIA PETRILĂ* et MIHAI BĂRBOSU*

Received: June 10, 1986

ABSTRACT. — Calculation of the Influence of the Walls upon a Compressible Rotating Flow. The problem of the influence of the walls on a fluid flow, produced by a rotational displacement of a thin profile in the fluid mass, is envisaged. Using the Bessel functions and the equations obtained in [2] the authors provide a special technique which allows the computing of the above mentioned influence in the case of a circular wall (of a „channel" type) or of a straight unlimited wall. The flow is considered plane, potential and compressible, the fluid being inviscid.

On sait que le mouvement irrotationnel d'un fluide idéal dû au déplacemen dans la masse fluide d'un corps solide rigide S, de dimensions finies, rapport au repère inertial fixe $0x, y_1z_1$, est régissé par l'équation fondamentale [2]

$$\Delta\varphi - \frac{1}{c^2}\, v \cdot \text{grad}\, w - \frac{1}{c^2}\,\frac{\delta w}{\delta t} = 0$$

où l'on a posé

$$W = \frac{\delta\varphi}{\delta t} + \frac{1}{2}\, v^2$$

Ici on a designé par φ le potentiel des vitesses, par $v = |\vec{v}|$ le module d la vitesse absolute $\vec{v}$, par $\delta/\delta t$ la dérivée partielle par rapport à t dans le sy tème de coordonnées x_1, y_1, z_1, t, et par c la vitesse de propagation du son.

On a supposé aussi que le fluide, au repos à l'infini est assujetti à une l de compressibilité barotropique $\rho = \rho(p)$ — réliant la densité ρ à la pressio p — et que les forces massiques soient négligeables.

Si on écrit l'équation fondamentale dans les variables x, y, z, t liées a corps S en mouvement alors en désignant par $v_r(v_1^{(r)}, v_2^{(r)}, v_3^{(r)})$ la vitesse rela tive au répère $Oxyz$, par $v_e(v_1^{(e)}, v_2^{(e)}, v_3^{(e)})$ la vitesse d'entrainement avec le solid S, par $\omega\,(\omega_1, \omega_2, \omega_3)$ la vitesse de rotation, on obtient avec la convention d sommation par rapport aux indices muets [2]

$$\left(\delta_{ij} - \frac{1}{c^2}\, v_i^{(r)} v_j^{(r)}\right) \frac{\partial^2\varphi}{\partial x_i \partial x_j} + \frac{1}{c^2}\, \varepsilon_{ijk}\, v_i^{(e)}\, \omega_j\, \frac{\partial\varphi}{\partial x_k} =$$

$$= \frac{1}{c^2}\,\frac{\partial^2\varphi}{\partial t^2} + \frac{2}{c^2}\, v_i^{(r)}\, \frac{\partial^2\varphi}{\partial x_i\, \partial t} - \frac{1}{c^2}\,\frac{\partial v_i^{(e)}}{\partial t}\,\frac{\partial\varphi}{\partial x_i}$$

* *For correspondence: University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Nap Romania*

où δ_{ij} sont les composantes du tenseur de Kronecker, ε_{ijk} celles du tenseur de Ricci et $\partial/\partial t$ est l'opérateur de dérivation partielle par rapport à t dans le système de coordonnées x, y, z, t.

Si l'écoulement rélatif au répère $Oxyz$ est stationnaire et si, de plus, le solide S est animé d'une rototranslation uniforme alors, dans le cas particulier du mouvement plan-parallèle, l'équation prend la forme simplifiée [2]

$$\left[1 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial x} + \omega y\right)^2\right]\frac{\partial^2\varphi}{\partial x^2} + \left[1 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial y} - \omega x\right)^2\right]\frac{\partial^2\varphi}{\partial y^2} -$$

$$- \frac{2}{c^2}\left(\frac{\partial\varphi}{\partial x} + \omega y\right)\left(\frac{\partial\varphi}{\partial y} - \omega x\right)\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\omega^2}{c^2}\left(x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y}\right) = 0$$

où $\omega = \omega_3$ est la vitesse angulaire de rotation autour de l'axe $O_1z_1 \equiv Oz$ du solide S qui est maintenant un cylindre à génératrices parallèles à cet axe, les points O et O_1 coincidant.

A cette équation on doit ajouter la condition aux limites

$$\left.\frac{\partial\varphi}{\partial n}\right|_c = -\frac{\omega}{2}\frac{d}{ds}(x^2 + y^2)\Big|_c$$

où C est la frontière de la plaque D d'intersection du solide S avec le plan Oxy du mouvement, le vecteur $\vec{n}$ représente le vecteur unitaire de la normale à C, orientée positivement vers l'exterieur (donc vers le fluide en mouvement) et s désigne l'abscisse curviligne croissante dans le sens direct de parcours de C.

D'autre part si on introduit la fonction de courant ψ du mouvement relatif, celle-ci est reliée au potentiel des vitesses absolues φ, par le système

$$\frac{\partial\varphi}{\partial x} + \omega y = \frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\varphi}{\partial y} - \omega x = -\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x}$$

où u s'éxprime d'une façon non linéaire au moyen des dérivées premières de la fonction φ.

Si on élimine la fonction φ de ce système on aboutit à l'equation aux dérivées partielles que vérifie la fonction de courant ψ du mouvement relatif

$$\left(1 - \frac{u_r^2}{c^2}\right)\frac{\partial^2\psi}{\partial x^2} - \frac{2u_r v_r}{c^2}\frac{\partial^2\psi}{\partial x\partial y} + \left(1 - \frac{v_r^2}{c^2}\right)\frac{\partial^2\psi}{\partial y^2} =$$

$$= 2\omega\left(1 - \frac{u_r^2 + v_r^2}{c^2}\right) + \frac{\omega^2}{c^2}\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right)$$

où $u_r = v_r^{(1)}$ et $v_r = v_r^{(2)}$

équation à laquelle on ajoute la condition aux limites

$$\psi|_c = 0$$

Si le mouvement fluide plan est produit par la rotation autour du poi O d'un profil mince P et peu courbé et si l'incidence par rapport à la vites d'entraînement de rotation est assez petite, l'équation fondamentale en φ pe être linearisée Précisément dans un système de coordonnées polaires — av le pôle en O et l'axe polaire Ox — l'intrados et l'extrados du profil P soie définis respectivement par $r = r_1(\theta)$, $r_2 = r_2(\theta)$, $\theta' < \theta < \theta''$ de sorte que l'o ait $r_2(\theta) \geqslant r_1(\theta)$, $r_2(\theta) \approx R$, $r_1(\theta) \approx R$, où R est une constante positive. L rotation de ce profil mince produit un mouvement fluide absolu de perturb tion, comportant des vitesses assez petites, donc on peut négliger les carr et les produits de $\frac{\partial \varphi}{\partial x}$ et $\frac{\partial \varphi}{\partial y}$ ce qui conduit finalement à l'équation simplifi

$$\left(1 - \frac{\omega^2 y^2}{c_0^2}\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{2\omega^2 xy}{c_0^2} \frac{\partial^2 \varphi}{\partial x \partial y} + \left(1 - \frac{\omega^2 x^2}{c_0^2}\right) \frac{\partial^2 \varphi}{\partial y^2} +$$
$$+ \frac{\omega^2}{c_0^2} \left(x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y}\right) = 0$$

Si on transcrit cette équation en coordonnées polaires elle devient

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r^2} \left(1 - \frac{r^2 \omega^2}{c_0^2}\right) \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} = 0$$

Alors en cherchant ses solutions, sous la forme $\varphi = \Phi(r)\, \psi(\theta)$ on about aux équations differentielles ordinaires suivantes

$$\Phi''(r) + \frac{1}{r} \Phi'(r) + K^2 \left(\frac{\omega^2}{c_0^2} - \frac{1}{r^2}\right) \Phi(r) = 0$$

et

$$\psi''(\theta) + K^2 \psi(\theta) = 0$$

En posant

$$r = \frac{c_0 X}{k\omega} \quad (k \neq 0)$$

on trouve

$$\Phi_k = c_k^1 J_k(X) + c_k^2 Y_k(X) \text{ et}$$
$$\psi_k = \cos(k\theta + k^1)$$

où $J_k(X)$ et $Y_k(X)$ sont les functions de Bessel de première et seconde espè d'indice réel k, c_k^1, c_k^2 et k' étant des constantes.

On trouve ainsi des solutions $\varphi_k = \Phi_k \psi_k$ de l'équation considérée faisa intervenir les fontions de Bessel. Pour avoir la solution du problème mécaniq envisagé il faut essayer d'extraire de ces solutions φ_k celle qui satisfait la co dition simplifiée de glissement sur l'aile mince c'est-à-dire à

$$\frac{\partial \varphi_k}{\partial r}(R, \theta) = -\omega r_j^1(\theta), \; j = 1,2$$

qui vont sur les deux côtés de l'arc circulaire de rayon R représentan squelette du profil pour lequel $\theta' \leqslant \theta \leqslant \theta''$.

Pour résourdre effectivement ce problème on observe d'abord que pour valeurs arbitraires des constantes c_k^1 et c_k^2 le comportement à l'infini — qui exprime le repos du fluide aux grandes distances — est assuré grâce aux suivantes représentations pour les fonctions de Bessel

$$J_k(X) \approx \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(X - \frac{k\pi}{2} - \frac{\pi}{4}\right) \text{ pour } X \to \infty$$

$$Y_k(X) \approx \left(\frac{2}{\pi X}\right)^{1/2} \sin\left(X - \frac{k\pi}{2} - \frac{\pi}{4}\right) \text{ pour } X \to \infty$$

En ce qui concerne la condition de glissement sur l'extrados et l'intrados du profil P en cherchant la solution du problème sous la forme

$$\varphi = \sum_{k=1}^{\infty} [c_k^1 J_k(X) + c_k^2 Y_k(X)] \cos k\theta$$

et en considérant le développement Fourier en sinus de $r_j(\theta)$ c'est-à-dire

$$r_j^1(\theta) = \sum_{k=1}^{\infty} - k a_k^{(j)} \cos k\theta, \qquad j = 1, 2$$

le système algèbrique suivant dans les inconnues c_k^1 et c_k^2 (pour k naturel arbitraire) assurerait la solution complète du problème

$$\left| c_k^1 \frac{\partial J_k(x)}{\partial x} + c_k^2 \frac{\partial Y_k(x)}{\partial x} \right|_{r = R^{(j)}} = c_0 a_k^{(j)}, \quad j = 1, 2$$

Ici on a désigné par $R^{(1)}$ et $R^{(2)}$ la distance minimale, respectivement maximale, entre les points de l'intrados et de l'extrados du profil P et le point fixe O.

Supposons maintenant que l'écoulement fluide produit par la rotation du profil P ait lieu dans un tuyau circulaire fixe dont la séction, dans le plan d'écoulement est donnée par la circonférence $x_1^2 + y_1^2 = R_1^2$. Dans les points de cette circonférence nous aurons la condition de glissement suivante qui remplace, dans ce cas, la condition de comportement à l'infini

$$\vec{v}_a \, \vec{n} \Big|_{r = R_1} = \frac{\partial \varphi}{\partial r}(R_1, \theta) = 0$$

En cherchant de nouveau la solution du problème sous la forme

$$\varphi = \sum_{k=1}^{\infty} [c_k^1 J_k(X) + c_k^2 Y_k(X)] \cos k\theta$$

la condition d'au-dessus s'exprime par une rélation de dépendence, pour quelque k naturel, entre c_k^1 et c_k^2 Si on accepte que R_1 est suffisament **grand** pour que le developpements assymptotiques des $J_k(X)$ et $Y_k(X)$ aient lieu, cet rélation revient à

$$\operatorname{tg}\left(\frac{k\omega R_1}{c_0}-\frac{k\pi}{2}-\frac{\pi}{4}\right)\approx\frac{c_k^2\left(\frac{2c_0}{\pi k\omega R_1}\right)^{1/2}-c_k^1\cdot\frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{c_0}{k\omega R_1}\right)^{3/2}}{c_k^1\left(\frac{2c_0}{\pi k\omega R_1}\right)^{1/2}+c_k^2\cdot\frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{c_0}{k\omega R_1}\right)^{3/2}}.$$

En fin en assimilant l'extrados et l'intrados du profil P avec le bord supérieur, respectivement inférieur, de l'arc $\frac{r_1(\theta)+r_2(\theta)}{2}$, la condition de glissement sur ce profil revient à

$$\left[c_k^1\frac{\partial J_k(X)}{\partial x}+c_k^2\frac{\partial Y_k(X)}{\partial x}\right]_{r=R}=c_0a_k$$

où a_k sont les coefficients du développement Fourier en sinus de la fonction donnée $\frac{r_1(\theta)+r_2(\theta)}{2}$ c'est-à-dire

$$\frac{r_1'(\theta)+r_2'(\theta)}{2}=-\sum_{k=1}^{\infty}-ka_k\cos k\theta$$

La dernière condition, tout ensemble avec celle d'ou-dessus, déterminent univoquement les coefficients c_k^1 et c_k^1 de la solution du problème.

Considerons maintenent le cas quand l'écoulement fluide produit par rotation du profil P ait lieu en présence d'une paroi rectiligne illimitée (supposée parallèle à l'axe fixe Ox_1) d'équation $y_1=-y_0$. En remarquant que l'équation de la paroi peut s'écrire encore, par rapport au repère mobile sous la forme

$$x\sin\alpha+y\cos\alpha+y_0=0\quad\text{ou bien}$$

$$r=-\frac{y_0}{\sin(\theta+\alpha)}\;(\alpha=\widehat{x_1Ox}),$$ la condition de glissement sur lui devient

$$0=\vec{v}_a\,\vec{n}\Big|_\Gamma=\frac{\partial\varphi}{\partial y_1}\Big|_\Gamma=\frac{\partial\varphi}{\partial x}\sin\alpha+\frac{\partial\varphi}{\partial y}\cos\alpha\Big|_\Gamma=$$

$$=\left(\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)\sin\alpha+\left(\sin\theta\frac{\partial\varphi}{\partial r}+\cos\theta\frac{\partial\varphi}{\partial t}\right)\cos\alpha\Big|_\Gamma$$

c'est-à-dire

$$\frac{\partial\varphi}{\partial r}\sin(\theta+\alpha)+\frac{1}{r}\cos(\theta+\alpha)\Big|_{r=-\frac{y_0}{\sin(\theta+\alpha)}}=0$$

Cherchant de nouveau la solution du problème sous la forme

$$\varphi=\sum_{k=1}^{\infty}\left[c_k^1J_k(X)+c_k^2Y_k(X)\right]\cos k\theta$$

cette cdition cononduit à son tour à une rélation de dépendance entre les constantes c_k^1 et c_k^2, plus précisément il faut satisfaire

$$\sum_{k=1}^{\infty} \frac{\omega}{c_0} \left[c_k^1 \frac{\partial J_k(X)}{\partial X} + c_k^2 \frac{\partial Y_k(X)}{\partial X} \right]_{r=-\frac{Y_0}{\sin(\theta+\alpha)}} \cos k\theta =$$

$$= - \sum_{k=1}^{\infty} \frac{Y_0}{\sin(\theta+\alpha)} \left[c_k^1 J_k(x) + c_k^2 Y_k(x) \right]_{r=-\frac{y_0}{\sin(\theta+\alpha)}} \sin k\theta$$

Mais, en utilisant les formules de récurrence pour les dérivées des fonctions de Bessel on obtient

$$\sum_{k=1}^{\infty} \frac{\omega}{2c_0} \{ c_k^1 [J_{k-1}(x) - J_{k+1}(x)] + c_k^2 [Y_{k-1}(x) - Y_{k+1}(x)] \} \cdot \cos k\theta =$$

$$= - \sum_{k=1}^{\infty} \frac{\cos(\theta+\alpha)}{y_0} [c_k^1 J_k(x) + c_k^2 Y_k(x)] \sin k\theta$$

Ou bien

$$\frac{\omega}{2c_0} \left[c_1^1 J_0(x) \cos\theta + c_1^1 \frac{\cos(\theta+\alpha)}{y_0} J_1(x) \sin\theta \right] +$$

$$+ \frac{\omega}{2c_0} \left[c_1^2 Y_0(x) \cos\theta + c_1^2 \frac{\cos(\theta+\alpha)}{y_0} Y_1(x) \sin\theta \right] +$$

$$+ \sum_{k=2}^{\infty} \left\{ J_k(x) \left[(c_{k+1}^1 - c_{k-1}^1) \frac{\omega}{2c_0} \cos k\theta + c_k^1 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta \right] + \right.$$

$$\left. + Y_k(x) \left[(c_{k+1}^2 - c_{k-1}^2) \frac{\omega}{2c_0} \cos k\theta + c_k^2 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta \right]_{r=-\frac{y_0}{\sin(\theta+\alpha)}} \right\}$$

Choisisant alors $c_2^1 = c_1^2 = 0$ et approximant $Y_k(x)$ par $J_k(x) \operatorname{tg}\left(x - \frac{k\pi}{2} - \frac{\pi}{4}\right)$ — forme inspirée par le comportement assymptotique des $J_k(x)$ et $Y_k(x)$ — on aboutit à l'accomplissement tout au long de la paroi $r = \frac{y_0}{\sin(\theta+\alpha)}$, de la rélation

$$(c_{k+1}^1 - c_{k-1}^1) \frac{\omega}{2c_0} \cos k\theta + c_k^1 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta +$$

$$+ \operatorname{tg}\left(-\frac{k\omega}{c_0} \frac{y_0}{\sin(\theta+\alpha)} - \frac{k\pi}{2} - \frac{\pi}{4} \right) \left[(c_{k+1}^2 - c_{k-1}^2) \frac{\omega}{2c_0} \cos k\theta + c_k^2 \frac{\cos(\theta+\alpha)}{y_0} \sin k\theta \right] = 0$$

(pour $k = 2, 3, \ldots$)

Mais parce qu'au voisinage de $\theta = -\frac{\pi}{2}$ la condition de glissement sur la paroi est pratiquement satisfaite (conséquence du caractère rotatoire de l'écoulement) et pour $\theta \notin V\left(-\frac{\theta}{2}\right)$ étant justifiées les approximations $\cos(\theta + \alpha) \approx \cos\alpha$ et $\sin(\theta + \alpha) \approx \sin\alpha$ on est conduit, dans ce cas, à la rélation suivante entre coefficients

$$(c_{k+1}^{1} - c_{k-1}^{1}) + \operatorname{tg}\left(-\frac{k\omega y_0}{c_0 \sin\alpha} - \frac{k\pi}{2} - \frac{\pi}{4}\right)(c_{k+1}^{2} - c_{k-1}^{2}) = 0$$

Evidemment à cette dernière condition pour les coefficients c_k^j on doit ajouter la rélation écrite au—dessus (entre les mêmes coefficients et qui exprime le glissement du fluide sur le contour du profil), ce qui détermine complètement le problème.

BIBLIOGRAPHIE

1. Titus Petrila, *Modele matematice în hidromecanica plană. Cercetări asupra influenței pereților nelimitați*, Editura Academiei R.S.R., 1981.
2. Caius Iacob, *Sur les mouvements rotatoires des fluides compressibles* I (II), *Mécanique appliquée*, t. 26 (27), nr. 3 (4), 1981 (2) pp. 357–369 (437–452).
3. N. N. Lebedev, *Funcții speciale și aplicațiile lor*, Editura Tehnică, 1957.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

ABELIAN GROUPS WITH PSEUDOCOMPLEMENTED LATTICE OF SUBGROUPS

GRIGORE G. CĂLUGĂREANU*

Received: October 13, 1986

ABSTRACT. — In this paper we prove that for the lattice of the subgroups of an abelian group pseudocomplementation and distributivity are equivalent. We also characterize abelian groups which have a Stone lattice or a Heyting algebra of subgroups.

Let L be a lattice with zero and $0 \neq a \in L$. If $C_a = \{x \in L/a \wedge x = 0\}$, the greatest element of C_a (if it exists) is called the pseudocomplement of a in L. (Note that the "pseudocomplement" is differently used for an unspecified maximal element of C_a). If every element in L has a pseudocomplement, L is called a pseudocomplemented lattice.

We first recall the following known facts:

(A) Every distributive compactly generated lattice is pseudocomplemented.

(B) If A is an abelian group, the lattice $L(A)$ of all the subgroups of A is compactly generated.

1. LEMMA. *Let P be an inductive poset. The following conditions are equivalent:* (i) *P has a unique maximal element*; (ii) *P has a greatest element.*

Indeed, if m is the unique maximal element of P and $a \in P$ then $P_a = \{x \in P/a \leqslant x\}$ is inductive and has (by Zorn's lemma) maximal elements which are also maximal in P. So $a \leqslant m$. The converse is obvious.

2. COROLLARY. *Let L be an upper continuous lattice. The following conditions are equivalent:* (i) *C_a has a unique maximal element*; (ii) *C_a has a greatest element.*

Indeed, in an upper continuous lattice, C_a is inductive.

The key result for our paper, from [5] is the following:

(C) Let $B \neq 0$ be a subgroup of an abelian group A. There is a unique B-high subgroup if and only if A/B is a torsion group and for each prime p either $B[p] = A[p]$ or $B[p] = 0$ holds.

3. COROLLARY. *Let $B \neq 0$ be a subgroup of an abelian group A. The following conditions are equivalent:* (i) *B has a pseudocomplement in $L(A)$*; (ii) *there is only one B-high subgroup in A*; (iii) *A/B is a torsion group and for every prime p either $B[p] = A[p]$ or $B[p] = 0$ holds.*

4. PROPOSITION. *For an abelian group A the following conditions are equivalent:* (a) *every nontrivial quotient group of A is a torsion group*; (b) *A is either a torsion group or a torsion-free group of rank* 1.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Proof. Clearly no mixed group has the property (a): if $0 \neq T(A) \neq A$ th
$A/T(A)$ is torsion-free. Obviously every torsion group has the property (
Now, if A is torsion-free of rank $r_0(A) \geqslant 2$ and $0 \neq a \in A$ then $r_0(A/\langle a\rangle) \geqslant$
so that $A/\langle a\rangle$ is not torsion. Finally, if A is torsion-free of rank 1, A h
the property (a), as every rational group has it.

5. PROPOSITION. *For a torsion group A the following conditions are e valent:* (c) *for each subgroup B of A and each p prime either $B[p] = A$ or $B[p] = 0$ holds;* (d) *A is a direct sum of cocyclic groups correspondin different primes.*

Proof. We can obviously reduce our problem to p-groups. But $B[p] = 0$ if and only if $B = 0$ so that only the case $B[p] = A[p]$ needs care. If A a p-group such that $B[p] = A[p]$ holds for each subgroup $B \neq 0$ of A t $A[p] = S(A)$ (the socle) is contained in every nonzero subgroup B of A. this case, having a smallest nonzero subgroup, A is cocyclic. The convers obvious.

6. COROLLARY. *If A is an abelian group, the lattice $L(A)$ is pseudoc plemented if and only if A is either a direct sum of cocyclic groups correspo to different primes or a torsion-free group of rank* 1.

Proof. Using 3, 4 and 5 we only need to observe that (c) is trivially true torsion-free groups.

7. THEOREM. *For an abelian group A the following conditions are equivalent:* (i) *$L(A)$ is a distributive lattice;* (ii) *$L(A)$ is a pseudocomplemented lattice;* (iii) *A is a locally cyclic group;* (iv) $r_0(A) + \max_p r_p(A) \leqslant 1$; (v) *$A$ is either a direct sum of cocyclic groups corresponding to different primes or a torsion-free group of rank* 1.

Proof. One can use [3, p 86, ex. 5] and [2, p 301, T. 78.2]. The rest is done by the previous corollary.

A pseudocomplemented distributive lattice is called a Stone lattice if $a^* \vee a^{**} = 1$, where a^* denotes the pseudocomplement of a in L. If B is a subgroup of A such that A/B is a torsion group, let π be the set of all the primes such that $B[p] = 0$ holds and $B[p] = A[p]$ holds for $p \notin \pi$. Using proposition 2 and 3 from [5] we have $B^* = \bigoplus_{p \in \pi} (T(A))_p$ and $B^{**} = \bigoplus_{p \notin \pi} (T(A))_p$ so that $B^* + B^{**} = T(A)$. Hence only the torsion groups from 7 have Stone lattices of subgroups:

8. PROPOSITION. *For an abelian group A the following conditions are equivalent:* (i) *$L(A)$ is a Stone lattice;* (ii) *A is a direct sum of cocyclic groups corresponding to different primes.*

A lattice with zero is called a Heyting algebra (or a relative pseudocomplemented lattice) if for every $a, b \in L$ the subset $\{x \in L / a \wedge x \leqslant b\}$ has a greatest element denoted $a * b$.

We finally mention the following characterization [1]: (D) A bounded lattice L is a Heyting algebra if and only if L is distributive and for each $b \in L$ the sublattice $1/b = \{x \in L / b \leqslant x\}$ is pseudocomplemented.

The pseudocomplementation and the distributivity of the lattice of all the subgroups of an abelian group being equivalent it immediately follows that

$L(A)$ is a Heyting algebra if and only if $L(A)$ is distributive (any sublattice of a distributive lattice is distributive too).

Remark. The characterization of the class of all abelian groups which have the lattice $L(A)$ a Boole algebra is an easy consequence of 8 (cf. 2, p. 302, Cor. 78.5).

REFERENCES

1. Balbes R., Dwinger P., *Distributive Lattices*, Univ. Missouri Press, Columbia, Miss. 1974.
2. Fuchs L., *Abelian Groups*, Publishing House of the Hungarian Academy of Sciences, Budapest, 1958.
3. Fuchs L., *Infinite Abelian Groups*, vol. 1, Academic Press 1970.
4. Grätzer G., *General Lattice Theory*, Akademie Verlag, Berlin, 1978.
5. Krivonos F. V., *On N-high subgroups of abelian groups*, Vestnik Mosk. Univ., no. 1, 1975, p. 58–64.

CRITICAL RADII AND MAXIMUM MASSES OF RELATIVISTIC STEPENARS

V. URECHE*

Received: October 15, 1986

ABSTRACT. — For the relativistic stepenars, the critical radii and the maximum masses are computed. When the index of the stepenar varies in the range $0 \leqslant \alpha \leqslant 10$, the critical radius and the maximum mass vary, respectively, in the intervals $1.125 \leqslant R_{\min}/R_g \leqslant 4.658$; $0.955 \leqslant M_{axm}/M_{\odot} \leqslant 8.046$, depending on the values assumed for the non-dimensional central pressure. The obtained results are given in tables and plotted on graphs.

1. Introduction. In the newtonian theory of stellar structure, the class of stellar models with the distribution of the density as a power law, having the form

$$\rho = \rho_c(1 - r/R)^{\alpha}, \quad \alpha \geqslant 0, \tag{1}$$

where the notations are the usual ones, was introduced by H u s e y n o v and K a s u m o v (1972). They named these models s t e p e n a r s or p s e u d o - p o l y t r o p e s.

The relativistic stellar models with the distribution of the mass-energy density having the form (1) were firstly studied in our papers (U r e c h e, 1983 a, 1983 b). These models have been named r e l a t i v i s t i c s t e p e n a r s.

2. Main Properties of Relativistic Stepenars. If we introduce the non-dimensional variables (see Ureche, 1980 a), the distribution of the density (1) takes the form

$$\psi = (1 - \eta/\eta_s)^{\alpha}, \quad \alpha \geqslant 0, \tag{2}$$

where η_s is the non-dimensional radius of the star. With the change of variable $\eta = \eta_s y$ and using the non-dimensional form of the equations of relativistic stellar equilibrium from the last cited paper, we obtain the main properties of the relativistic stepenars, namely:

— The non-dimensional mass distribution is given by

$$m(y) = \frac{\eta_s^3}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} f(y), \tag{3}$$

where

$$f(y) = 2 - (1 - y)^{\alpha+1}[(\alpha + 1)(\alpha + 2)y^2 + 2(\alpha \mp 1)y + 2], \tag{4}$$

the total mass of the relativistic stellar configuration having the expression

$$m_s \equiv m(1) = \frac{2\eta_s^3}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} \tag{5}$$

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

— The degree of the concentration of the matter (-energy) towards the centre of the relativistic stellar configuration is given by the ratio between the central density ρ_c and the mean density $\bar{\rho}$, that is

$$\rho_c/\bar{\rho} = (\alpha + 1)(\alpha + 2)(\alpha + 3)/6 \tag{6}$$

— The scale factors from the change of variables are given by

$$a = R\sqrt{\frac{4}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} \cdot \frac{R}{R_g}}, \tag{7}$$

$$M^* = 2M\sqrt{\frac{4}{(\alpha + 1)(\alpha + 2)(\alpha + 3)} \cdot \frac{R^3}{R_g^3}}, \tag{8}$$

where

$$R_g = 2GM/c^2 \tag{9}$$

is the gravitational (Schwarzschild) radius of the relativistic configuration.

— The radius and the mass of the configuration have respectively the expressions

$$R = \frac{(\alpha + 1)(\alpha + 2)(\alpha + 3)}{4\eta_s^2} R_g, \tag{10}$$

$$M = \frac{c^3}{\pi^{1/2}G^{3/2}(\alpha + 1)(\alpha + 2)(\alpha + 3)} \cdot \frac{\eta_s^3}{\rho_c^{1/2}}. \tag{11}$$

3. Critical Radii of Relativistic Stepenars. For the distribution of the mass of stellar model, the exact solution (3) was obtained, while the distribution of the pressure results from the numerical solution of the differential problem

$$\frac{dp}{dy} = -\eta_s^2 \frac{[p + (1 - y)^\alpha]\,[f(y) + (\alpha + 1)(\alpha + 2)(\alpha + 3)y^3p]}{(\alpha + 1)(\alpha + 2)(\alpha + 3)y^2 - 2\eta_s^2 yf(y)} \tag{12}$$

$$p(1) = 0, \quad \eta_s^2 < (\alpha + 1)(\alpha + 2)(\alpha + 3)/4,$$

where the function $f(y)$ is given by the expression (4).

In a previous paper (Ureche, Oproiu, Imbroane, 1985) a numerical analysis of the differential equation (12) was performed. So, for different values of the parameters α (the index of the stepenar) and η_s^2, the tables of the function $p(y)$ were obtained. Here we shall concentrate our attention on those models in which the non-dimensional central pressure takes the values $p_c = 1/3$, $p_c = 1$ and $p_c = \infty$. In Table 1, for the different values of the index of the stepenar α, the values of the parameter η_s^2, corresponding to $p_c = 1/3$, $p_c = 1$ and $p_c = \infty$, are given.

The values of η_s^2 given in Table 1 are the maximum values of this parameter, for $p_c = 1/3$ (classical constraint of GRT), $p_c = 1$ (causal law) and $p_c = \infty$ (absolute limit, which does not depend on the equation of state). Let

Table 1

α	η_s^2		
	$p_c = 1/3$	$p_c = 1$	$p_c = \infty$
0	5/6	9/8	4/3
1	3.2491	4.0289	4.495
2	7.186	8.654	9.46
3	12.57	14.91	16.25
4	19.48	22.89	24.86
5	27.88	32.56	35.27
6	37.75	43.92	47.48
7	49.12	56.97	61.49
8	61.96	71.70	77.31
9	76.29	88.13	95.76
10	92.10	106.25	114.36

η_s^{*2} be one of the values given in Table 1. From (10) for the corresponding minimum radius of the configuration we obtain

$$R_{\min} = \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{4\eta_s^{*2}} R_g. \quad (13)$$

So, using the values in Table 1 with the expression (13) we have computed the minimum radii of relativistic stepenars, that is the critical radii at which the gravitational collapse is inevitable. The obtained results, expressed in terms of the gravitational radius of the configuration, are listed in Table 2.

For the three values of p_c, the quantity $R_{\min}/R_g$ is plotted in Figure 1 as function of α. From Table 2 and Figure 1 one can observe that the critical radii increase with the index of stepenar α. The equation (6) points out the fact that the degree of the concentration of matter towards the centre of configuration also increase with α. Therefore the critical radii increase with the

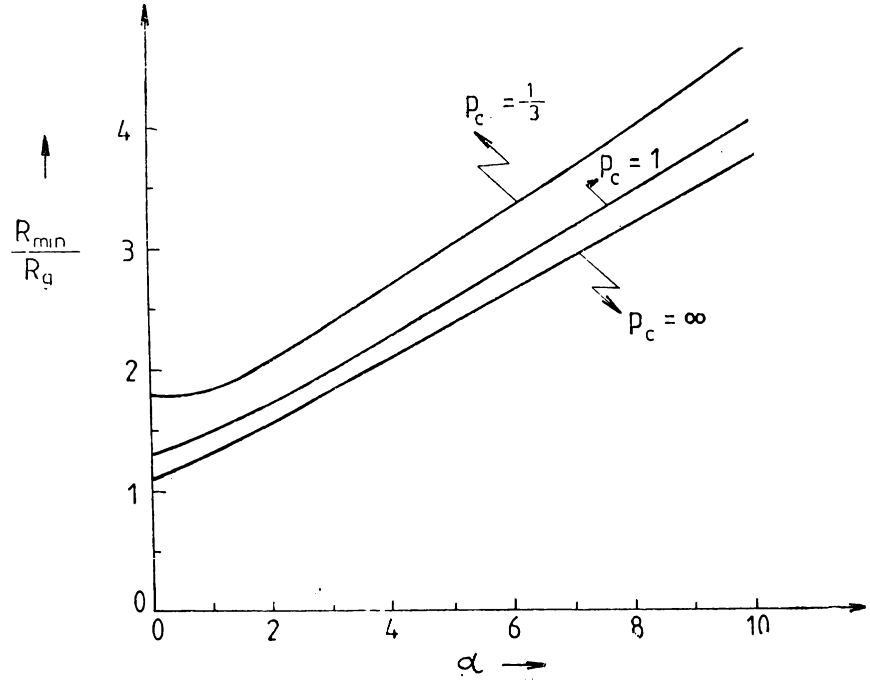

Fig. 1

Table 2

Critical radii of the relativistic stepenars

α	R_{min}/R_g		
	$p_c = 1/3$	$p_c = 1$	$p_c = \infty$
0	1.800	1.333	1.125
1	1.847	1.489	1.335
2	2.087	1.733	1.586
3	2.387	2.012	1.846
4	2.695	2.294	2.112
5	3.013	2.580	2.382
6	3.338	2.869	2.654
7	3.664	3.160	2.927
8	3.995	3.452	3.201
9	4.326	3.744	3.446
10	4.658	4.038	3.751

Table 3

Maximum masses of the relativistic stepenars

α	$M_{max}/M_{\odot}$		
	$p_c = 1/3$	$p_c = 1$	$p_c = \infty$
0	3.976	6.236	8.046
1	3.826	5.283	6.226
2	3.184	4.207	4.809
3	2.604	3.364	3.828
4	2.170	2.764	3.128
5	1.836	2.317	2.612
6	1.574	1.976	2.221
7	1.369	1.710	1.917
8	1.203	1.497	1.676
9	1.067	1.325	1.501
10	0.955	1.183	1.321

increasing of the degree of the concentration of matter (-energy) towards the centre of the relativistic star. An interesting problem would be the study of the asymptotical behaviour of the quantity R_{min}/R_g for $\alpha \to \infty$.

4. Maximum Masses of Relativistic Stepenars. From the equations (6) and (11) for the maximum mass of a relativistic stepenar we obtained the expression:

$$M_{max} = \frac{6^{1/2} c^3}{\pi^{1/2} G^{3/2} (\alpha + 1)^{3/2} (\alpha + 2)^{3/2} (\alpha + 3)^{3/2}} \cdot \frac{\eta_s^{*3}}{\bar{\rho}^{1/2}} \tag{14}$$

With this expression, using the values of the parameter η_s^* from Table 1, we have computed the maximum masses of the relativistic stepenars. For this purpose we took $\bar{\rho} = \rho_{nuc} = 2 \cdot 10^{17}$ kg/m^3 (Brecher, Caporaso, 1977). The computations were performed for the same three values: $p_c = 1/3$, $p_c = 1$ and $p_c = \infty$. So, we obtained the maximum masses of relativistic stepenars. These ones are the limiting masses for the considered models. Over these masses the gravitational collapse is inevitable. The obtained results, expressed in solar masses, are given in Table 3.

The ratio $M_{max}/M_{\odot}$ is plotted in Figure 2, as function of α, for the considered values of p_c. From Table 3 and Figure 2 one can observe that the maximum (critical) masses, called Oppenheimer-Volkoff limiting masses (Zeldovich, Novikov, 1971) decrease with the increasing of the degree of the concentration of matter (-energy) towards the centre of the relativistic star. An interesting problem would also be the study of the asymptotical behaviour of the quantity $M_{max}/M_{\odot}$ for $\alpha \to \infty$.

We note that the results obtained here for $\alpha = 0$ (homogeneous model) and $\alpha = 1$ (linear model) are in agreement with those given in the previous papers (Ureche, 1980 a, b; 1982).

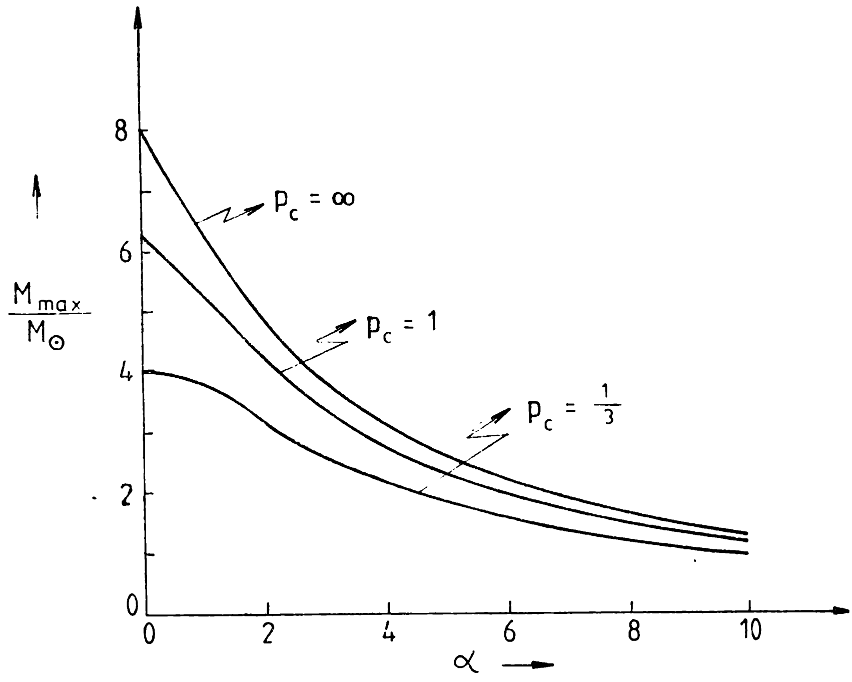

Figure 2

We conclude that the obtained results are equivalent with the following criteria of stability for the relativistic stellar configuration with the power la density distribution (relativistic stepenars)

$$R > R_{min}, \qquad M < M_{max}. \tag{15}$$

REFERENCES

1. Brecher, K., Caporaso, G., 1977, Ann. New York Acad. Sci., 302, 471.
2. Huseynov, O. H., Kasumov, F. R., 1972, *Astrofizika*, 8, 425.
3. Ureche, V., 1980 a, Rev. Roum. Phys., 25, No. 3, 301.
4. URECHE, V., 1980 b, in „Einstein's Cent. in Romania" (Eds. I. Adămuţi et al.), ICPE Bucureşti, 369.
5. Ureche, V., 1982, in „Binary and Multiple Stars as Tracers of Stellar Evolution", Proc. IAU Coll. No. 69 (Eds. Z. Kopal, J. Rahe), Reidel Publ. Comp., Dordrecht, Holland, 73.
6. Ureche, V., 1983 a, in „10-th Int. Conf. on General Relativity and Gravitation" (GR 10, Padova, July 4—9, 1983, Contributed Papers, Vol. 2, Relativistic Astrophysics (Eds. B. Bertotti, S. de Felice, A. Pascolini), Univ. of Padova, 740.
7. Ureche, V., 1983 b, in „Babeş-Bolyai Univ., Fac. of Math., Research Seminaries, Sem. of Stellar Str. and Stellar Evol." Preprint 4, 15 (Proc. of Meet. „Relativistic Objects in Close Binary Systems", Cluj-Napoca, June 8—10, 1982).
8. Ureche, V., Oproiu, T., Imbroane, A. M., 1985, in „Babeş-Bolyai Univ., Fac. of Math., Research Seminaries, Sem. of Stellar Str. and Stellar Evol.", Preprint 2, 4.
9. Zeldovich, Ya. B., Novikov, I. D., 1971, „Stars and Relativity", Univ. Chicago Press Chicago — London.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

CARACTERISATION DES FONCTIONS CONVEXES A L'AIDE DES OPERATEURS CONVOLUTIFS POSITIFS

GR. MOLDOVAN*

Manuscrit reçu le 17 mai 1983

RÉSUMÉ. — Nous allons faire référence a quelque résultats particuliers concernant la caracterisation de fonctions convexes a l'aide de certains opérateurs linéaires et positifs.

DEFINITION — *Soit* $f\colon [a, b] \to \mathbf{R}$. *Si* $\forall x_1, x_2 \in [a, b]$ *et* $\forall \alpha_1 > 0,\ \alpha_2 > 0,$ $\alpha_1 + \alpha_2 = 1$, *on a*

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

alors on dit que fonction f *est convexe.*

Nous allons noter par $[x_1, x_2, x_3; f]$ la différence divisée du deuxiéme ordre de la fontion f.

LEMMA. *Soit* $f \in C[a, b]$. *Une condition nécessaire et suffisante pour que* f *soit convexe est que :*

$$[x_1, x_2, x_3; f] \geqslant 0 \quad \forall x_1, x_2, x_3 \in [a, b]$$

La démonstration de ce lemma on peut voir, par exemple, dans [6].

Considération les opérateurs de Bernstein

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right),\ f \in C[0, 1],\ x \in [0, 1]$$

On a

THÉORÈME. *Soit* $f \in C[0, 1]$. *Une condition nécessaire et suffisante pour que* f *soit convexe est :*

$$B_n\left(f; \frac{i}{n}\right) \geqslant f\left(\frac{i}{n}\right),\ i = 0, 1, \ldots, n;\ n = 1, 2, \ldots \tag{1}$$

DEMONSTRATION. Ce théoréme est donné en [1]. Nous allons indiquer une démonsstration simple du théorème.

On a [4]

$$B_n(f; x) - f(x) = \frac{x(1 - x)}{n} \sum^{n-1} p_{n-1,\, k}(x) \left[x, \frac{k}{n}, \frac{k+1}{n}; f\right], \quad x \in [0, 1] \tag{2}$$

* Université de Cluj-Napoca, Faculté de Mathematique et Physique, 3400 Cluj-Napoca, Roumanie

où

$$p_{n,k}(x) = \binom{k}{n} x^k (1-x)^{n-k}.$$

Si f est convexe, d'après le lemme ci-dessus et le fait que $p_{n-1,k}(x) \geq 0$, $k = 0, 1, \ldots, n$; $n = 1, 2, \ldots$ il résulte la relation (1) du théorème.

Supposons maintenant que (1) soit vérifiée et montrons que f est convexe. Supposons le contraine, donc

$$\exists x_1, x_2, x_3 \in [0, 1] \text{ tels que } [x_1, x_2, x_3; f] = \varsigma < 0.$$

La différence entre

$$\sum_{k=0}^{n-1} p_{n-1,k}(x)\left[x, \frac{k}{n}, \frac{k+1}{n}; f\right] \text{ et } [x_1, x_2, x_3; f]$$

peut être redue aussi petite que l'on veut pour un n suffismment grand dans un point appartenant à [0, 1], correspondant à x_1, x_2, x_3. Il résulte que la somme respective peut être négative ce qui est en contradiction avec (1), donc f est convexe.

Pour les opérateurs de S t a n c u [5]

$$P_n^{[\alpha]}(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \cdot \frac{x(x+\alpha)\ldots(x+\overline{k-1}\alpha)(1-x)(1-x+\alpha)\ldots(1-x+\overline{n-k-1}\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\alpha)} \qquad (3)$$

$$\alpha \in \mathbf{R},\ f \in C[0, 1]$$

on a une formule sémblable à (2) et on peut démontrer de la même façon le théorème suivant :

THÉORÈME. *Soit $f \in C[0, 1]$ et $P_n^{[\alpha]}$, $\alpha > 0$ les opérateurs (3) associe à cette fonction. Une condition nécessaire et suffisante pour que f soit convexe est que :*

$$P_n^{[\alpha]}\left(f; \frac{i}{n}\right) \geq f\left(\frac{i}{n}\right), \quad i = 0, 1, \ldots, n;\ n = 1, 2, \ldots$$

Pour les opérateurs de Bernstein on a aussi :

THÉORÈME. *Une condition nécessaire et suffisante pour qu'une fonction continue f soit convexe sur $[0, 1]$ est que la suite $\{B_n(f; x)\}$, $n = 1, 2, \ldots$; $x \in [0, 1]$ soit non croissante.*

La démonstration de de théorème est donné en [2].

Pour la démonstration on utilise la relation suivante, satisfaite par les polinomes de Bernstein.

$$B_n(f; x) - B_{n+1}(f; x) = \frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}(x)\left[\frac{k}{n}, \frac{k+1}{n+1}; f\right] \qquad (4)$$

Pour les opérateurs $P_n^{[\alpha]}$ on a aussi :

THÉORÈME. *Une condition nécessaire et suffisante pour qu'une fonction continue f soit convexe sur* $[0, 1]$ *est que la suite* $\{P_n^{[\alpha]}(f; x)\}$, $n = 1, 2, \ldots$; $x \in [0, 1]$ *soit non croissante.*

Remarque. On a les mêmes résultats pour les opérateurs $B_n^{[s]}$

$$B_n^{[s]}(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[s(x)]^k\,[1 - s(x)]^{n-k}$$

où $s(x)$ satisfaire des conditions que assurent leur convergence uniforme vers f.

Considérons maintenant les opérateurs convolutifs positifs de type binomial de la forme:

$$L_n(f; x) = A_n(x)\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} P_k(u(x))P_{n-k}(v(x)) \tag{5}$$

$$A_n(x) = [P_n\,(u(x) + v(x))]^{-1} < \infty.$$

Hypothese :

Supposons que les fonctions $a_k : [0, 1] \to \mathbf{R}$

$$b_{n-k} : [0, 1] \to \mathbf{R}$$

$$a_k(x) = \frac{P_n(u(x) + v(x))}{P_{k+1}(u(x) + v(x))}\cdot\frac{P_{k+1}(u(x))}{P_k(u(x))}$$

$$b_{n-k}(x) = \frac{P_n(u(x) + v(x))}{P_{n+1}(u(x) + v(x))}\cdot\frac{P_{n-k+1}(v(x))}{P_{n-k}(v(x))}$$

$$k = 0, 1, \ldots, n;\ n = 1, 2, \ldots$$

satisfassent la relation $1 = a_k(x) + b_{n-k}(x)$

THÉORÈME. a) *Dans l'hypothèse ci-dessus pour les opérateurs convolutifs* (4) *on a :*

$$L_n(f; X) - L_{n+1}(f; x) = \frac{A_{n+1}(x)}{n(n+1)}\sum_{k=0}^{n-1}\binom{n+1}{k} P_{k+1}(u(x)).$$

$$P_{n-k}(v(x))\left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right]$$

$$n = 1, 2, \ldots$$

b) Si $f: [0, 1] \to \mathbf{R}$ *est convexe alors la suite* $\{L_n\}$, $n = 1, 2, \ldots$ *est croissante.*

DEMONSTRATION a) On a :

$$L_{n+1}f; x) = A_{n+1}(x)\sum_{k=1}^{n}\binom{n+1}{k} f\left(\frac{k}{u+1}\right) P_k(u(x))P_{n-k+1}(v(x)) +$$

$$+ A_{n+1}(x)f(0)P_{n+1}(v(x)) + A_{n+1}(x)f(1)P_{n+1}(u(x))$$

En utilisant la relation $1 = a_k(x) + b_{n-k}(x)$ on peut écrire :

$$L_n(f; x) = A_n(x)\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) a_k(x)P_k(u(x))P_{n-k}(v(x)) +$$

$$+ A_n(x)\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) P_k(u(x))P_{n-k}(v(x))b_{n-k}(x) =$$

$$= A_n(x)\sum_{k=1}^{n+1}\binom{n}{k-1} f\left(\frac{k-1}{n}\right) a_{k-1}(x)P_{k-1}(u(x))P_{n-k+1}(v(x)) +$$

$$+ A_n(x)\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) P_k(u(x))P_{n-k}(v(x))b_{n-k}(x)$$

En tenant compte des valeurs fonctions a_k et b_{n-k} données dans l'hypothèse obtient

$$L_{n+1}(f; x) - L_n(f; x) = A_{n+1}(x)\sum_{k=1}^{n}\left[\binom{n+1}{k} f\left(\frac{k}{n+1}\right) - \binom{n}{k-1} f\left(\frac{k-1}{n}\right) - \right.$$

$$\left. - \binom{n}{k} f\left(\frac{k}{n}\right)\right] \cdot P_k(u(x))P_{n-k+1}(v(x))$$

De là il résulte la relation donnée.

b) La dèmonstration rèsulte de a)

EXEMPLES

1° Si $L_n = B_n$ (polynôme Bernstein) alors $u(x) = x$, $v(x) = 1-$ $P_k(u(x)) = x^k$, $A_n(x) = 1$ et l'hypothèse ci-dessus on a $a_k(x) = x$, $b_{n-k}(x)$ $= 1 - x$ et donc évidemmnet $1 = a_k(x) + b_{n-k}(x)$.

La relation du théorème devient alors (4).

2° Si $L_n = P_n^{[\alpha]}$ l'opérateur Stancu alors

$$P_k(u(x)) = x(x + \alpha) \ldots (x - (k - 1)\alpha)$$

$$P_k(u(x) + v(x)) = (1 + \alpha)(1 + 2\alpha) \ldots (1 + (n - 1)\alpha)$$

et on obtient :

$$a_k(x) = \frac{1}{1 + n\alpha}(x + k\alpha), \quad b_{n-k}(x) = \frac{1}{1 + n\alpha}(1 - x + (n - k)\alpha)$$

pour lesquels $1 = a_k(x) + b_{n-k}(x)$.

La relation du théorème devient alors une relation semblable à (4) p les opérateurs $P_n^{[\alpha]}$, [5].

BIBLIOGRAPHIE

1. Karlin S., Ziegler Z. — *Iteration of positive approximation operators*, J. Approx. Theory, 3, 310—338, 1970.
2. Kosmak L., — *Les polynomes de Bernstein des fonctions convexes*, Mathematica, 9 (32), 71—73 1967.
3. Moldovan Gr., — *Discret convolutions and linear positive operators I*, Annales univ. Sct. Budap. de Rolando Eötvös nominates, Section Mathematica, 15, 31—44, 1972.
4. Stancu D. D. — *The remainder of certain linear approximation formulas in two variables*, SIAM J. Numer Anal. Ser. B, 1, 137—162, 1964.
5. Stancu D. D. — *Approximation of functions by a new class of linear polynomials operators*, Rev. Roum. Math. Pures et Appl. 13, 1173—1184, 1968.
6. Roberts A. W., Varberg D. E. — *Convex Functions*, Academic Press, 1973.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

LOOP-EXIT SCHEMES AND GRAMMARS; PROPERTIES, FLOWCHARTABLIES

FLORIAN MIRCEA BOIAN*

Received: March 27, 1985

ABSTRACT. — Some properties for the Loop-Exit grammars and an algorithm for construction of one flowchart for one Loop-Exit scheme are presented in the paper.

1. Introduction. The flowcharts is a traditional tool for the algorithm description. Recently, the Loop-Exit schemes [2, 6] have been are used for the algorithm description, too. Some programming languages such as BLISS [7], Ada [5 and some Pascal implementation [8], used for flowcontrol statements of Loop-Exit type. In this paper, some properties for the Loop-Exit grammars [3, 4 are presented. Also, an elegant algorithm for construction of one flowchart for one Loop-Exit scheme is described.

2. The definition of a Loop-Exit Scheme. Let $\Sigma = \mathbf{AM} \cup \mathbf{TM}$ be a terminal alphabet where **AM** and **TM** are the sets of *assignment and test marks* respectively let

RES = {"+", " –", ";", NULL, IF, THEN, ELSE, ENDIF, LOOP, ENDLOOP, EXIT} be a set of some *reserved symbols* and let $\mathbf{LM} = \{i_1, i_2, \ldots, i_l\}$ be a set of *loop-marked symbols*. Usually, when there is not confusion, we assume that $\mathbf{LM} = \{1, 2, \ldots, l\}$. Suppose that $\mathbf{RES} \cap (\Sigma \cup \mathbf{LM}) = \emptyset$.

Definition 1. A *Loop-Exit-Free Scheme* (LEFS) *over* Σ *is recursively defined as follows:*

a) "NULL; "*is a* LEFS. *For each* $a \in \mathbf{AM}$, "a;" *is a* LEFS.

b) *If* $t \in TM$, α *and* β *are* LEFS *and* i, j, k $\in$ **LM**, *then the followings are* LEFS:

b1) „$\alpha\beta$"

b2) „IF t THEN α[EXIT$_i$;] [ELSE β[EXIT$_j$;]]ENDIF;"

where [δ] *means that* δ *is optional.*

b3) „LOOP$_k$ α ENDLOOP$_k$;"

c) *Each* LEFS *is obtained from a and b rules which satisfies:*

c1) *each two* LOOPs *must have two distinct loop-mark symbols from* **LM**

c2) *for each* LOOP$_k$ α ENDLOOP$_k$; α *has at least an* EXIT$_k$ *in it;*

c3) *for each* EXIT$_k$, *if* LOOP$_k$ α ENDLOOP$_k$; *is in* LEFS, *then*

$\alpha = \alpha'$ EXIT$_k$; α''

Definition 2. *A Loop-Exit Scheme* (LES) *is a* LEFS *such that:*

c3') *for each* EXIT$_k$ *there is* LOOP$_k$ α' EXIT$_k$; α'' ENDLOOP$_k$ *into* LEFS.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Let S be a LES. All the symbols IF of S will be indexed by $1, 2, \ldots$. For each IF_i, the corresponding symbols THEN, ELSE — if this exists — and ENDIF will be indexed by the same index i. In the LES from the examples below we have marked this indexing.

Let $\mathbf{N} = \{I_j \mid \text{if } IF_j \text{ is into } S\} \cup \{L_k, B_k \mid \text{if } LOOP_k \text{ is into } S\}$ be a set of nonterminal symbols where $\mathbf{N} \cap (\Sigma \cup \mathbf{LM}) = \emptyset$.

Definition 3. *Let* α *be a* LEFS. *Through* $\mathfrak{I}(\alpha)$ *we denote the skeleton word over* α, *obtained by the following rules :*

a) *If* $\alpha = \varepsilon$, *then* $\mathfrak{I}(\alpha) = \varepsilon$;

b) *If* α_1 *and* α_2 *are* LEFS, *we have :*

b1) *If* $a \in \mathbf{AM}$ *and* $\alpha = \alpha_1\, a\, ;\, \alpha_2$ *then* $\mathfrak{I}(\alpha) = \mathfrak{I}(\alpha_1) a\, \mathfrak{I}(\alpha_2)$.

b2) *If* $\alpha = \alpha_1\, \text{NULL}\, ;\, \alpha_2$ *then* $\mathfrak{I}(\alpha) = \mathfrak{I}(\alpha_1)\, \mathfrak{I}(\alpha_2)$.

b3) *If* $\alpha = \alpha_1 IF_j\, \beta\, ENDIF_j\, ;\, \alpha_2$ then $\mathfrak{I}(\alpha) = \mathfrak{I}(\alpha_1)\, I_j\, \mathfrak{I}(\alpha_2)$.

b4) If $\alpha = \alpha_1 LOOP_k\, \beta\, ENDLOOP_k\, ;\, \alpha_2$ then $\mathfrak{I}(\alpha) = \mathfrak{I}(\alpha_1)\, L_k\, \mathfrak{I}(\alpha_2)$.

Definition 4. *Let* $X_1\, \alpha\, X_2 \beta Y_2 \delta Y_1$ *be a* LEFS, *where* $X_i = IF_{j_i}\, a\, THEN_{j_i}$ *or* $X_i = IF_{j_i}\, a\, THEN_{j_i}\, \gamma\, ELSE_{j_i}$ *or* $X_i = LOOP_{k_i}$, $i = \overline{1, 2}$ *and according to* X_i *we have* $Y_i = ENDIF_{j_i}$; *or* $Y_i = ENDLOOP_j$; $i = \overline{1, 2}$ *respectively.* Through $\mathfrak{D}(X_1 \alpha X_2)$ *we denote the directly word from* X_1 *to* X_2, *obtained by the ollowing rules :*

a) *If* α *is a* LEFS *then :*

a1) *If* $X_1 = IF_{j_1}\, a\, THEN_{j_1}$ *then* $\mathfrak{D}(X_1 \alpha X_2) = a + \mathfrak{I}(\alpha)$.

a2) *If* $X_1 = IF_{j_1}\, a\, THEN_{j_1}\, \gamma\, ELSE_{j_1}$ *then* $\mathfrak{D}(X_1 \alpha X_2) = a - \mathfrak{I}(\alpha)$.

a3) *If* $X_1 = LOOP_{k_1}$ *then* $\mathfrak{D}(X_1\, \alpha\, X_2) = \mathfrak{I}(\alpha)$

b) *Otherwise :*

b1) *If* $\alpha = \alpha_1\, IF_n\, b\, THEN_n\, \alpha_2$ and $\delta = \delta_2 ENDIF_n\, ;\, \delta_1$ then $\mathfrak{D}(X_1\, \alpha X_2) = \mathfrak{D}(X_1 \alpha_1\, IF_n)\, \mathfrak{D}(IF_n\, \alpha_2 X_2)$.

b2) *If* $\alpha = \alpha_1\, IF_n\, b\, THEN_n\, \gamma_1\, ELSE_n\, \alpha_2$ *and* $\delta = \delta_2\, ENDIF_n\, ;\, \delta_1$ *then* $\mathfrak{D}(X_1\, \alpha X_2) = \mathfrak{D}(X_1 \alpha_1 IF_n)\, \mathfrak{D}(IF_n \alpha_2 X_2)$.

b3) *If* $\alpha = \alpha_1\, LOOP_m\, \alpha_2$ *and* $\delta = \delta_2\, ENDLOOP_m\, ;\, \delta_1$ *then* $\mathfrak{D}(X_1\, \alpha\, X_2) = \mathfrak{D}(X_1\, \alpha_1\, LOOP_m)\, B_m\, \mathfrak{D}(LOOP_m\, \alpha_2 X_2)$.

Definition 5. *Let* S *be a* LES. *The language* $L(S)$ *associated to* S *is generated from the following* CFG :

$$G_S = (\{\nabla\} \cup \{I_j, L_k, B_k, . j \geqslant 0, k \geqslant 0\}, \Sigma \cup \{+, -\}, \mathfrak{L}, \nabla).$$

where „∇" *is a new symbol,* I_j *is a nonterminal for* IF_j — *if this exists* — L_k *and* B_k *are two nonterminals for* $LOOP_k$ — *if this exists — and the set* $\mathfrak{L}$ *of the productions is constructed by the following rules :*

a) $\nabla \to \mathfrak{I}(S)$.

b) *For each* $IF_j\, b\, THEN_j\, \alpha\, ENDIF_j$; *the productions :*

b1) $I_j \to b-$

b2) $I_j \to b + \mathfrak{I}(\alpha)$ *if does not exist* $EXIT_k$ *such that* $\alpha = \alpha'\, EXIT_k$; *are in* $\mathfrak{L}$;

c) *For each* $\mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \alpha\ \mathrm{ELSE}_j\ \beta\ \mathrm{ENDIF}_j$; *the productions:*

c1) $I_j \to b + \mathfrak{I}(\alpha)$ *if* $\alpha \neq \alpha'\ \mathrm{EXIT}_k$;

c2) $I_j \to b - \mathfrak{I}(\beta)$ *if* $\beta \neq \beta'\ \mathrm{EXIT}_k$; *are in* $\mathfrak{L}$;

d) *For each* $\mathrm{LOOP}_k\ \alpha_1\alpha_2\ \delta\ \mathrm{ENDLOOP}_k$; *the productions:*

d1) $L_k \to \mathfrak{I}(\alpha_1\ \alpha_2\ \delta)L_k$ *and* $B_k \to \mathfrak{I}(\alpha_1\alpha_2\ \delta)B_k \mid \varepsilon$

d2) $L_k \to \mathfrak{D}(\mathrm{LOOP}_k\ \alpha_1\ \mathrm{IF}_j)\ b + \mathfrak{I}(\beta)$, if

$\alpha_2 = \mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \beta\ \mathrm{EXIT}_k;\ \mathrm{ENDIF}_j$; *or*

$\alpha_2 = \mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \beta\ \mathrm{EXIT}_k;\ \mathrm{ELSE}_j\ \gamma\ \mathrm{ENDIF}_j$;

d3) $L_k \to \mathfrak{D}(\mathrm{LOOP}_k\ \alpha_1\ \mathrm{IF}_j)\ b - \mathfrak{I}(\beta)$, if

$\alpha_2 = \mathrm{IF}_j\ b\ \mathrm{THEN}_j\ \gamma\ \mathrm{ELSE}_j\ \beta\ \mathrm{EXIT}_k;\ \mathrm{ENDIF}_j$; *are in* $\mathfrak{L}$.

Definition 6. *Let S be a* LES. *The static word associated to S is obtained f*
S by erasing all reserved symbols.

Example 1.

```
LOOP₁
 IF₁ a₁ THEN₁
       LOOP₂
         a₂;
         IF₂ a₃ THEN₂ EXIT₂;
               ELSE₂
                IF₃ a₄ THEN₃ NULL; ELSE₃ EXIT₁; ENDIF₃;
         ENDIF₂;
       ENDLOOP₂;
     ELSE₁
       IF₄ a₅ THEN₄ NULL; ELSE₄ EXIT₁; ENDIF₄;
 ENDIF₁;
ENDLOOP₁;
```

The static word is "$a_1\ a_2\ a_3\ a_4\ a_5$".

Example 2. Let us consider LES from the example 1. The associated
mmar has the follows productions:

$\nabla \to L_1$

$L_1 \to I_1L_1 \mid a_1 + B_2a_2a_3 - a_4 - \mid a_1 - a_5 -$ $\qquad B_1 \to I_1B_1 \mid \varepsilon$

$I_1 \to a_1 + L_2 \mid a_1 - I_4$

$L_2 \to a_2I_2L_2 \mid a_2a_3 +$ $\qquad B_2 \to a_2I_2B_2 \mid \varepsilon$

$I_2 \to a_3 - I_3$

$I_3 \to a_4 +$

$I_4 \to a_5 +$.

3. Orderly properties in the Loop-Exit grammars. *Let S be a LES, let* $G_S = (\{\nabla\} \cup \mathbf{N},\ \Sigma \cup \{+, -\},\ \mathfrak{P},\ \nabla)$ be the attached grammar to S and let $a_1 a_2 \ldots a_n$ be the static word attached to S.

Consider $\alpha \in (\{\nabla\} \cup \mathbf{N} \cup \Sigma \cup \{+, -\})^+$. Similar with [1] we definev:

Definition 7. If $\alpha \neq +\alpha'$ *and* $\alpha \neq -\alpha'$ *then*

$$\text{FIRST}\ (\alpha) = \{a \in \Sigma \mid \alpha \Rightarrow a\beta\}.$$

Suppose that $\alpha = \alpha' +$ *or* $\alpha = \alpha' -$ *if and only* $\alpha = \alpha'' a +$ *or* $\alpha = \alpha'' a -$ *with* $a \in \mathbf{TM}$. *Then*

$$\text{LAST}\ (\alpha) = \{a \mid a \in \mathbf{AM},\ \alpha \overset{*}{\Rightarrow} \beta a\} \cup \{a + \mid a \in \mathbf{TM},\ \alpha \overset{*}{\Rightarrow} \beta a +\} \cup \{a - \mid a \in \mathbf{TM},\ \alpha \overset{*}{\Rightarrow} \beta a -\}.$$

Using the definitions 1—7 we can directly prove the following lemmas:

LEMMA 1. *For each* $A \in \mathbf{N}$, *A being a useful and accesible symbol, there is a symbol* a_i *from the static word so that:*

a) $\text{FIRST}\ (A) = \{a_i\}$;

b) *For each* $A \to \alpha \in \mathfrak{P}$, $\text{FIRST}\ (\alpha) = \{a_i\}$.

LEMMA 2. *If* $a_i \in AM$ *is a symbol from the static word then:*

a) *It does not exist* $w \in L(G_S)$ *so that* $w = xa_i + y$ *or* $w = xa_i - y$

b) *For each* $A \to \alpha a_i \beta \in \mathfrak{P}$, *if* $\varepsilon \neq \beta \neq A$ *then* $\text{FIRST}\ (\beta) = \{a_{i+1}\}$.

LEMMA 3. *If* $a_i \in \mathbf{TM}$ *is a symbol from the static word, then*

a) *Each production from* $\mathfrak{P}$ *which contains* a_i *is either* $A \to \alpha a_i + \beta$ *or* $A \to \alpha a_i - \beta$ *and there are productions of both forms.*

b) *If it exists* $A \to \alpha a_i + \beta$, *so that* $\varepsilon \neq \beta \neq A$ *then it exists* a_j *a symbol from the static word so that* $i < j$ *and for each* $A \to \alpha a_i + \beta \in \mathfrak{P}$. $\varepsilon \neq \beta \neq A$ *it results that* $\text{FIRST}\ (\beta) = \{a_j\}$.

c) *If it exists* $A \to \alpha a_i - \beta$ *so that* $\varepsilon \neq \beta \neq A$ *then it exists* a_k *a symbol from the static word so that* $i < k$ *and for each* $A \to \alpha a_i - \beta$, $\varepsilon \neq \beta \neq A$ *it results that* $\text{FIRST}\ (\beta) = \{a_k\}$.

d) *If the symbol* a_j *verifies b) and* a_k *verifies c) then it results that* $j < k$.

LEMMA 4. *The following properties hold:*

a) $\text{FIRST}\ (a_i) = \{a_i\}$ *for each* $a_i \in \Sigma$

b) $\text{LAST}\ (a_i X) = \{a_i X\}$ *where* $X \in \{\varepsilon, +, -\}$

c) *If* $Y = a_i X$ *where* $X \in \{\varepsilon, +, -\}$ *then*

$$\text{LAST}\ (\alpha Y) = \text{LAST}(Y)$$

d) $\text{LAST}\ (A) = \bigcup \{\text{LAST}(\alpha) \mid A \to \alpha \in \mathfrak{P}\}$

e) *For each two productions* $A \to \alpha$ *and* $A \to \beta$ *we have:*

$$\text{LAST}\ (\alpha) \cap \text{LAST}(\beta) = \emptyset$$

Example 3. Let us consider LES from the example 1, having the associated grammar in the example 2. After eliminating the inaccesible and useless symbols, only the productions $B_1 \rightarrow I_1L_1 | \varepsilon$ must be erased. The FIRST and LA relations are:

FIRST $(a_i) = \{a_i\}, i = \overline{1, 5}$, LAST $(a_2) = \{a_2\}$,

LAST $(a_i +) = \{a_i +\}$ and LAST $(a_i -) = \{a_i -\}$, for $i \neq 2$.

For nonterminals we have:

	Δ	L_1	I_1	L_2	B_2	I_2	I_3	I_4
FIRST	a_1	a_1	a_1	a_2	a_2	a_3	a_4	a_5
LAST	a_4- a_5-	a_4- a_5-	a_3+ a_5+	a_3+	a_4+	a_4+	a_4+	a_5+

4. An algorithm for conversion a LES into a flowchart. Now we give method for conversion to flowchart from LES without inaccessible LEFS [4

ALGORITHM 1.

Input. A LES *A* without inaccesible LEFS.

Output. An equivalent flowchart $(\mathfrak{X}, \mathfrak{U})$ with *S*.

Step 1. Using the definition 6 we'll construct the associated grammar G. Using the algorithms from [1] we eliminate the inaccesible and useless symbol

Step 2. Using the lemmas 1—4 for each symbol *Y* from G_S, the FIRST (Y and LAST (*Y*) relations are found.

Step 3. If $a_1a_2 \ldots a_n$ is the static word associated to *S*, then the set of ve tices $\mathfrak{X}$ is obtained as follows:

— for each symbol a_i from the static word, if $a_i \in$ **AM** then A_i : "[a_i]" is a vertex else, (if $a_i \in$ **TM**) A_i: "◇a_i" is a vertex;

— for each $a_i X \in$ LAST (∇) we have one stop vertex „Δ";

— the start vertex "∇" is added to $\mathfrak{X}$.

Step 4.

Let $\mathfrak{U} := \{(\nabla, \text{FIRST}(\nabla)\} \cup \{(A_i, \Delta)$ marked $X | a_i X \in \text{LAST}(\nabla), X \in \{\varepsilon, +, -\}$

Step 5. For each production $A \rightarrow \alpha\beta$ from G_S, $\alpha \neq \varepsilon \neq \beta$, add $\{(A_i, A$ marked $X | \{a_k\} = \text{FIRST}(\beta), a_i X \in \text{LAST}(\alpha), X \in \{\varepsilon, +, -\}\}$ to the set

Example 4. Let us consider LES from the example 1, having the gramm in the example 2 and FIRST and LAST relations from the example 3. Afte applying the step 4, we have:

$\mathfrak{U} = \{(\nabla, A_1)$ unmarked (marked with ε), (A_4, Δ) and (A_5, Δ), both marke "—"}.

When we apply the step 5 to $L_1 \rightarrow I_1L_1$ with $\alpha = I_1$ and $\beta = L_1$, we obta the edges (A_3, A_1) and (A_5, A_1), both marked "+". When we apply the step 5 t $L_2 \rightarrow a_2I_2L_2$ with $\alpha = a_2$ and $\beta = I_2L_2$, we obtain the edge (A_2A_3) unmarked.

After applying the algorithm 1, we obtain the flowchart from fig. 1.

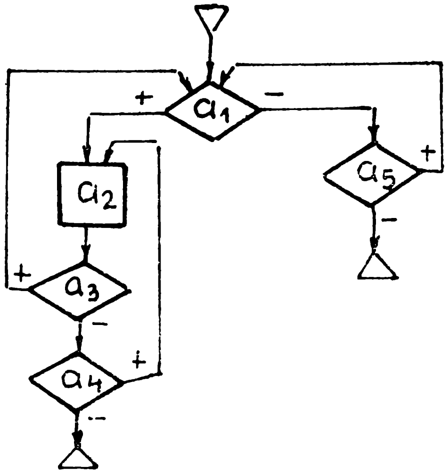

Fig. 1

THEOREM 1. *For each* LES *without inaccessible* LEFS [4] *using the algorithm* 1, *a flowchart* (X, U) *equivalent with* LES *is obtained.*
Proof this theorem was presented in [3].

REFERENCES

1. Aho A. V., J. D. Ullman, *The Theory of Parsing, Translation and Compilig.* Prentice Hall, Englewood Cliffs, New Jersey, vol. I, 1972.
2. Arsac J. J., *Syntactic Source to Source Transformation an Program Manipulation.* Comm. ACM, 22, 1, (1979) pp. 43–53.
3. Boian F. M., *Syntactic Equivalence between Marked Graph Schemes and Loop-Exit Schemes.* „Babes-Bolyai" University, Faculty of Mathematics, Seminar on Computer Science, Preprint no 4, 1984, pp. 34–56.
4. Boian F. M., *Reducing the Loop-Exit Schemes.* Mathematica 28 (51). 1, 1986 pp. 1–7
5. Gehani N., *Ada, An Advanced Introduction.* Prentice Hall, Englewood Cliffs, New Jersey, 1983.
6. Kosaraju S. R., *Analysis of Structured programms.* Journal of Comp. Syst. Sci. 9 (1973), pp. 232–255.
7. Wulf W. A. et al., BLISS. *A Language for Systems Programming.* Comm. ACM. 14, 12 (1971 pp. 780–790.
8. * * * *Pascal for PDP* 11 *under RSX/IAS.*

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI 3, 1986

CONTINUOUS SELECTIONS OF MULTI-VALUED MAPS WITH NONCONVEX RICHT-HAND SIDES AND THE PICARD PROBLEM FOR THE MULTI-VALUED HYPERBOLIC EQUATION $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$

GEORGETA TEODORU*

Received: October 9, 1986

ABSTRACT. — The Picard problem is considered for the multivolued hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, where F is a continuous multi-valued map defined on $A \subset R^{n+2}$ with compact values, but nonconvex in R^n. An existence theorem of a continuous selection is proved for $F(x, y, z)$, with $z \in K$, where K is a compact, convex, set of absolutely continuous functions, submitted to certain conditions. An operator is then defined by means of this selection, for which one applies the Schander Fixed Point Theorem — the fixed point being just the solution of the Picard probleme.

1. Introduction. In this paper we are concerned with the Picard proble[...] for the multi-valued hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, where F is a mult[...] valued map, defined in a suitable subset of $\mathbf{R}^{n+2}$, with values that are nonemp[...] compact but not necessarily convex subsets of $\mathbf{R}^n$. The Picard problem is defin[...] by analogy with the Picard problem for quasilinear hyperbolic equations [1] in [2], [3], where F is a multi-valued map defined on a subset of $\mathbf{R}^{n+2}$ and valu[...] in the set of compact convex nonempty subsets of $\mathbf{R}^n$, satisfying conditions [...] Caracthéodory type. Using the Fixed Point Theorem of Kakutani-Ky Fan o[...] proves that the problem above has at least a solution.

In this note one proves an existence theorem of a continuous selection i[...] each of the maps $(x, y) \to F(x, y, z(x, y))$ relative to a given family of continuo[...] maps $(x, y) \to z(x, y)$, as in [4] — [8], and using the Schauder Fixed Po[...] Theorem one obtains the existence of a solution of the Picard problem.

2. Continuous approximate selections. Let be the multivalued map $F: A \to$ comp X, where $A \subset \mathbf{R}^{n+2}$, $A = D \times B$, $D = [0, a] \times [0, b] \subset \mathbf{R}^2$, $B \subset \mathbf{R}^n$ the closed ball centered in origin with radius $c = M_1 + Mab$, M_1 given by (3[...] M given by (4), $X \subset \mathbf{R}^n$ is the closed ball centered in origin with radius M,[...] being a compact metric space with the metric d induced on X by the norm defin[...] on $\mathbf{R}^n$.

Let H be the Hausdorff-Pompeiu metric [9] on comp X induced by d. Th[...] comp X is a compact metric space with respect to the metric H.

---

* *Polytechnic Institute of Jassy, Department of Mathematics, 6600 Iaşi, Romania*

Let $C(D; \mathbf{R}^n)$ be the Banach space of continuous functions defined on D and valued in $\mathbf{R}^n$ and $L^1(D; \mathbf{R}^n)$ the Banach space of equivalence classes of Lebesgue integrable functions defined on D and valued in $\mathbf{R}^n$.

Let the following hypotheses be satisfied:

(H_0) The curve $\gamma : x = \psi(y)$, $0 \leqslant y \leqslant b$, is defined by the function $\psi \in C^1([0, b]; \mathbf{R})$, satisfying the conditions

$$\psi(0) = 0, 0 \leqslant \psi(y) \leqslant a, \; o \leqslant y \leqslant b, \tag{1}$$

(H_1) The functions $P \in AC([0, a]; \mathbf{R}^n)$, $Q \in AC([0, b]; \mathbf{R}^n)$, where $AC([\alpha_1, \alpha_2]; \mathbf{R}^n)$ is the space of obsolutely continuous functions $f : [\alpha_1, \alpha_2] \to \mathbf{R}^n$, endowed with the norm

$$|f| = \sup_{t \leqslant [\alpha_1, \alpha_2]} ||f(t)|| + \int_{\alpha_1}^{\alpha_2} ||f'(t)|| dt,$$

satisfy the condition $P(0) = Q(0)$,

(H_2) The function $\alpha : D \to \mathbf{R}^n$ defined by

$$\alpha(x, y) = P(x) + Q(y) - P(\psi(y)), \; (x, y) \in D, \tag{2}$$

is bounded and therefore, the is $M_1 > 0$ such that

$$||\alpha(x, y)|| \leqslant M_1, \; (x, y) \in D. \tag{3}$$

It follows that α is absolutely continuous;

$$\alpha \in C^*(D; \mathbf{R}^n), \; [10], \; §§ \; 565-568.$$

Let K be the set of absolutely continuous functions $z : D \to \mathbf{R}^n$, $z \in C^*(D; \mathbf{R}^n)$, [10], satisfying the conditions (3), (4), (5), where

$$\left\| \frac{\partial^2 z(x, y)}{\partial x \, \partial y} \right\| \leqslant M, \text{ a.e. } (x, y) \in D, \tag{4}$$

and

$$\begin{cases} z(x, 0) = P(x), \; 0 \leqslant x \leqslant a. \\ z(\psi(y), y) = Q(y), \; 0 \leqslant y \leqslant b. \end{cases} \tag{5}$$

Then, the following two propositions hold:

Proposition 1. *K is a nonempty convex and compact subset of the Banach space* $C(D; \mathbf{R}^n)$.

Proof. The relation $z \in K$ implies $z \in C(D; \mathbf{R}^n)$. We observe that $\frac{\partial^2 z}{\partial x \partial y}$ exists a.e. $(x, y) \in D$, as $z \in C^*(D; \mathbf{R}^n)$, [10].

Let $M(x, y)$ be any point of D. Consider the parallel to x-axis, that intersects the curve γ in the point $N(\psi(y), y)$. Let $M_0(x, 0)$ and $N_0(\psi(y. 0)$ be the ortogonal projections of M and N on the x-axis. Denote $D_0(x, y)$ the rectangle MNN_0M_0, given by

$$D_0(x, y) = \{\psi(y) \leqslant u \leqslant x, 0 \leqslant v \leqslant y\}.$$

Integrating $\frac{\partial^2 z(x, y)}{\partial x \partial y}$ over $D_0(x, y)$, one obtains

$$\iint\limits_{D_0(x, y)} \frac{\partial^2 z(u, v)}{\partial u \partial v} du dv = \int_0^y dv \int_{\psi(y)}^x \frac{\partial^2 z(u, v)}{\partial u \partial v} du = \int_0^y dv \frac{\partial z}{\partial v}(u, v)\Big|_{u=\psi(y)}^{u=x} =$$

$$= \int_0^y \frac{\partial z}{\partial v}(x, v)\, dv - \int_0^y \frac{\partial z}{\partial v}(\psi(y), v) dv = z(x, y) - z(x, 0(\psi(y), y) +$$

$$+ z(\psi(y), 0) = z(x, y) - P(x) - Q(y) + P(\psi(y)), \ (x, y) \in D.$$

Using (2), it follows

$$z(x, y) = P(x) + Q(y) - P(\psi(y)) + \iint\limits_{D_0(x, y)} \frac{\partial^2 z(u, v)}{\partial u \partial v} du dv =$$

$$= \alpha(x, y) + \iint\limits_{D_0(x, y)} \frac{\partial^2 z(u, v)}{\partial u \partial v} du dv, \ (x, y) \in D, \tag{6}$$

or

$$z(x, y) = P(x) + Q(y) - P(\psi(y)) + \int_0^y dv \int_{\psi(y)}^x \frac{\partial^2 z(u, v)}{\partial u \partial v} du, \ (x, y) \in D. \tag{6'}$$

The compactness of K follows using (6) or (6') and the Arzelà-Ascoli Theorem. The convexity of K is obvious.

Remark. The relation $z \in K$ implies $(x, y, z(x, y)) \in A$ for each $(x, y \in D$. Because each $z \in K$ generate a multi-valued map $(x, y) \to F(x, y, z(x, y))$ from D to comp X, we shall denote this map by $G(z)$,

$$G(z)(x, y) = F(x, y, z(x, y)), \ (x, y) \in D. \tag{7}$$

Proposition 2. *Let* $F : A \to$ comp X *be a multi-valued continuous map. Then, for each* $\varepsilon > 0$, *there exists a continuous function* $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ *such that for each* $z \in K$ *we have*

$$d(g(z)(x, y), G(z)(x, y)) < \varepsilon, \ a.e. \ (x, y) \in D. \tag{8}$$

Proof. The proof is analogous to that given in [4]—[8] and is based on the construction of the function g by means of the continuous partition of the unity. Let $\varepsilon > 0$ be given. In view of the fact that F is continuous on A and A is compact, F is uniformly continuous on A and there is $\Delta > 0$ such that

$$H(F(x, y, z), F(\xi, \eta, \bar{z})) < \varepsilon,$$

for any two points (x, y, z), $(\xi, \eta, \bar{z})$ in A with $\|(x, y) - (\xi, \eta)\| < \Delta$, $\|z - \bar{z}\| < \Delta$.

Let $\{\mathcal{U}_k\}_{1\leqslant k\leqslant N}$ be a finite open cover of K such that diam $\mathcal{U}_k < \Delta$ for any $k = \overline{1, N}$. Let $\{p_k\}_{1\leqslant k\leqslant N}$ be the continuous partition of unity subordinate to $\{\mathcal{U}_k\}_{1\leqslant k\leqslant N}$; select for each k a point $z_k \in \mathcal{U}_k$ and let $\{v_k\}_{1\leqslant k\leqslant N}$ be a sequence of Lebesgue measurable functions $v_k : D \to \mathbf{R}^n$ such that, for every k, $v_k(x, y) \in G(z_k)(x, y)$ a.e. $(x, y) \in D$. Such function v_k exists because each $G(z_k)$ is continuous and measurable in D, [11]; $v_k \in \mathfrak{L}^1(D; \mathbf{R}^n)$ for every k. We can take $N = N_1 N_2$. Denote $\mathcal{U}_k = \mathcal{U}_{ij}$, $v_k = v_{ij}$, $z_k = z_{ij} \in \mathcal{U}_{ij}$, $p_k(z) = p_{ij}(z)$ and suppose

$$p_{ij}(z) = q_i(z) r_j(z), \; i = \overline{1, N_1}, \; j = \overline{1, N_2}.$$

The functions $p_{ij} : K \to \mathbf{R}$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$ satisfy the properties:

a) $0 \leqslant p_{ji}(z) \leqslant 1$, for $z \in K$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$,

b) $p_{ij}(z) = 0$ if $z \notin \mathcal{U}_{ij}$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$,

c) $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} p_{ij}(z) = \sum_{i=1}^{N_1} \sum_{i=1}^{N_2} q_i(z) r_j(z) = 1$, for $z \in K$.

For each $z \in K$ define the continuous functions $\lambda_{ij} : K \to \mathbf{R}$

$$\lambda_{ij}(z) = x_i(z) y_j(z), \; i = \overline{1, N_1}, \; j = \overline{1, N_2},$$

where

$$\begin{cases} x_0(z) = 0 \\ x_i(z) = x_{i-1}(z) + a q_i(z) \sum_{j=1}^{N_2} r_j(z), \; i = \overline{1, N_1}, \end{cases}$$

and

$$\begin{cases} y_0(z) = 0 \\ y_j(z) = y_{j-1}(z) + b r_j(z) \sum_{i=1}^{N_1} q_i(z), \; j = \overline{1, N_2}. \end{cases}$$

For each $z \in K$ define the rectangles

$$D_{ij}(z) = [x_{i-1}(z), x_i(z)) \times [y_{j-1}(z), y(z)), \; i = \overline{1, N_1}, \; j = \overline{1, N_2},$$

which constitute a partition of D excepting lines $x = a$ and $y = b$.

We construct the desired function $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$,

$$\begin{cases} g(z)(x, y) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \chi[D_{ij}(z)](x, y)\, v_{ij}(x, y), \; 0 \leqslant x < a, \; 0 \leqslant y < b, \\ g(z)(a, y) = v_{l, N_2}(a, y), \; l = \min\{i \geqslant 1 ; x(z) = a\}, \\ g(z)(x, b) = v_{N_1, p}(x, b), \; p = \min\{j \geqslant 1 ; y(z) = b\}. \end{cases} \tag{9}$$

where $\chi[D_{ij}(z)]$ is the characteristic function of $D_{ij}(z)$.

Obviously, g maps K into $\mathfrak{L}^1(D; \mathbf{R}^n)$. Moreover, for a given $z \in K$ and any fixed $(x, y) \in D$, there exists a unique (i, j) such that $(x, y) \in D_{ij}(z)$ and this implies $z \in \mathcal{U}_{ij}$. Thus, $g(z)\,(x, y) = v_{ij}(x, y)$ and $||z(x, y) - z_{ij}(x, y)|| < \Delta$ so that

$$d(g(z)\,(x, y), G(z)(x, y)) \leqslant d\,(v_{ij}(x, y),\ G(z_{ij})(x, y)) + H\,(G(z_{ij})\,(x, y);$$
$$G(z)\,(x, y)) < d(v_{ij}(x, y), G(z_{ij})\,(x, y)) + \varepsilon. \tag{10}$$

It follows that, for each $z \in K$, $d(g(z)\,(x, y), G(z)\,(x, y)) < \varepsilon$ a.e. $(x, y) \in D$, therefore (8) holds. We show that g is continuous on K. Then, for any points z, w in K and any $(x, y) \in D$, $0 \in x < a$, $0 \leqslant y < b$,

$$||g(z)\,(x, y) - g(w)\,(x, y)|| \leqslant \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \chi\,[D_{ij}(z)\,\Delta D_{ij}(w)]\,(x, y)\,||v_{ij}(x, y)|| \tag{11}$$

where $D_{ij}(z)\,\Delta D_{ij}(w) = (D_{ij}(z) - D_{ij}(w)) \cup (D_{ij}(w) - D_{ji}(z))$.

Since K is compact, $\{\lambda_{ij}\}_{i=\overline{1, N_1},\ j=\overline{1, N_2}}$ is a uniformly equicontinuous family of real valued functions. Thus, for every $\eta > 0$, there exists a $\delta > 0$ such that, for any $z \in K$, $w \in K$ satisfying $||z(x, y) - w(x, y)|| < \delta$ at every $(x, y) \in D$,

$$|\lambda_{ij}(z) - \lambda_{ij}(w)| < \varepsilon/2MN,$$

and hence $\mu(D_{ij}(z)\,\Delta D_{ij}(w)) < \eta/MN$,

so that (11) implies

$$|g(z) - g(w)|\,|_{\mathfrak{L}^1} = \iint_D ||g(z)\,(x, y) - g(w)\,(x, y)||\,dxdy < \eta. \tag{12}$$

Therefore $g : K \rightarrow \mathfrak{L}^1(D\,; \mathbf{R}^n)$ is uniformly continuous.

3. Continuous selections. On analogy of [4]—[8] one proves the following existence theorem of a continuous selection for multi-valued map $G(z)$.

Theorem 1. *If $F : A \rightarrow$ comp X is a multi-valued continuous map, then there exists a continuous function $g : K \rightarrow \mathfrak{L}^1(D\,; \mathbf{R})$ such that, for any $z \in K$, $g(z)(x,y) \in G(z)(x, y)$, a.e. $(x, y) \in D$, that is $g(z)$ is a continuous selection for $G(z)$, given by* (7).

Proof. Define a sequence of continuous functions $g^n : K \rightarrow \mathfrak{L}^1(D\,; \mathbf{R}^n)$, $n \in N$, submitted to the following conditions:

1) $d(g^n(z)\,(x, y), G(z)\,(x, y)) < \frac{1}{2^{n+1}}$, a.e. $(x, y) \in D$,

2) $\mu\left(\left\{(x, y) \in D \mid \ ||g^n(z)\,(x, y) - g^{n-1}(z)\,(x, y)|| \geqslant \frac{1}{2^{n-1}}\right\}\right) < \frac{1}{2^n}$.

The condition 2) shows that for each $z \in K$, the sequence $\{g^n(z)\}_{n \in \mathbf{N}}$ converges, in the norm of $\mathfrak{L}^1(D\,; \mathbf{R}^n)$, to an element $g(z)$ and the convergence is uniform on K, because the condition 2) is satisfied uniformly for any $z \in K$. Using the Lebesgue Dominated Convergence Theorem it follows that $g(z) \in \mathfrak{L}^1(D\,; \mathbf{R}^n)$ for each $z \in K$. By continuity of the functions g^n, $n \in \mathbf{N}$, it follows that $g : K \rightarrow \mathfrak{L}^1(D\,; \mathbf{R}^n)$ is continuous. Therefore, for any $z \in K$, there exists a measurable function $g(z) : D \rightarrow X$ such that the sequence $\{g^n(z)\}_{n \in \mathbf{N}}$ converges to $g(z)$ a.e. in measure, and a subsequence of $\{g^n(z)\}_{n \in \mathbf{N}}$ that converges to $g(z)$ a.e. on D. Then, from the condition 1), follows that for each $z \in K$ we have $g(z(x, y)) \in G(z)(x, y)$, a.e. $(x, y) \in D$, because $G(z)(x, y)$ is closed for any $z \in K$.

The sequence $\{g^n(z)\}_{n \leq N}$ is obtained by induction. From Proposition 2 it follows that there exists a continuous function $g^0 : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ such that for any $z \in K$

$$d(g^0(z)(x, y), G(z)(x, y)) < \frac{1}{2}, \text{ a.e. } (x, y) \in D. \tag{13}$$

Also, from Proposition 2 and the continuity of F on $A = D \times B$ there exists $\Delta_1 > 0$ such that

$$H(F(x, y, z), F(\xi, \eta, \tilde{z})) < \frac{1}{4} \tag{14}$$

for each (x, y, z), $(\xi, \eta, \tilde{z})$ in A with $||(x, y) - (\xi, \eta)|| < \Delta_1$, $||z - \tilde{z}|| < \Delta_1$ and

$$\mu(\{(x, y) \in D \mid ||g^0(z)(x, y) - g^0(\tilde{z})(x, y)|| > 0\}) < \frac{1}{2} \tag{15}$$

for each $z \in K$, $\tilde{z} \in K$ satisfying $||z(x, y) - \tilde{z}(x, y)|| < \Delta_1$ for any $(x, y) \in D$.

By analogy with the Proposition 2, let $\{\mathfrak{U}_k^1\}_{1 \leq k \leq N(1)}$ be an open finite cover of K, such that diam $\mathfrak{U}_k^1 < \Delta_1$, for any k; let $\{p_k^1\}_{1 \leq k \leq N(1)}$ be the continuous partition of unity subordinate to $\{\mathfrak{U}_k^1\}_{1 \leq k \leq N(1)}$; we select for each k a point $z_k^1 \in \mathfrak{U}_k^1$ and a Lebesgue measurable function $v_k^1 : D \to \mathbf{R}^n$ such that

$$v_k^1(x, y) \in G(z_k^1)(x, y), \text{ a.e. } (x, y) \in D,$$

and

$$||v_k^1(x, y) - g^0(z_k^1)(x, y)|| = d(g^0(z_k^1)(x, y), G(z_k^1)(x, y)). \tag{16}$$

It follows from the continuity of each $G(z_k^1)$, that are measurable on D, [12]. By analogy with the Proposition 2, consider $N(1) = N_1(1) N_2(1)$ and denote $\mathfrak{U}_k^1 = \mathfrak{U}_{ij}^1$, $v_k^1 = v_{ij}^1$, $z_k^1 = z_{ij}^1 \in \mathfrak{U}_{ij}^1$ and $p_k^1(z) = p_{ij}^1(z) = q_i^1(z)\, r_j^1(z)$, $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$.

The continuous partition of unity, $\{p(z_{ij}^1)\}$, $p_{ij}^1 : K \to \mathbf{R}$ satisfies:

a) $0 \leq p_{ij}^1(z) \leq 1$ for all $z \in K$, $i = 1, N_1(1)$, $j = 1, N_2(1)$,

b) $p_{ij}^1(z) = 0$ if $z \notin \mathfrak{U}_{ij}^1$,

c) $\sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} p_{ij}^1(z) = \sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} q_i^1(z) r_j^1(z) = 1$ for all $z \in K$.

For each $z \in K$, define the continuous functions $\lambda^1_{ij} : K \to \mathbf{R}$

$$\lambda^1_{ij}(z) = x^1_i(z)\, y^1_j(z), \; i = \overline{1, N_1(1)} \; j = \overline{1, N_2(1)}$$

with

$$\begin{cases} x^1_0(z) = 0 \\ x^1_i(z) = x^1_{i-1}(z) + aq^1_i(z) \sum_{j=1}^{N_2(1)} r^1_j(z), \; i = \overline{1, N_1(1)}, \end{cases}$$

and

$$\begin{cases} y^1_0(z) = 0 \\ y^1_i(z) = y^1_{j=1}(z) + br^1_j(z) \sum_{i=1}^{N_1(1)} q^1_i(z), \; j = \overline{1, N_2(1)}. \end{cases}$$

For each $z \in K$ consider the rectangles

$$D^1_{ij}(z) = [x^1_{i-1}(z), x^1_i(z)) \times [y^1_{j-1}(z), y^1_j(z)), \; i = \overline{1, N_1(1)}, \; j = \overline{1, N_2(1)},$$

establishing a partition of D, except for the lines $x = a$ and $y = b$.),

Define the function $g^1 : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$

$$\begin{cases} g^1(z)\,(x, y) = \sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} \chi[D^1_{ij}(z)](x, y) v^1_{ij}(x, y), \; 0 \leqslant x < a, \; 0 \leqslant y < b, \\ g^1(z)(a, y) = v^1_{l,\, N_2(1)}(a, y), \quad l = \min\{i \geqslant 1 \,;\, x^1_i(z) = a\}, \\ g^1(z)\,(x, b) = v_{N_1(1),\, p}(x, b), \quad p = \min\{j \geqslant 1 \,;\, y^1_i(z) = b\}. \end{cases} \tag{17}$$

The function g^1 is continuous (see the proof of continuity of g in the Proposition 2). To verify the conditions 1), 2) suppose that $z \in K$ is given, and $(x, y) \in D$ fixed ($0 \leqslant x < a$, $0 \leqslant y < b$).

Then $(x, y) \in D^1_{ij}(z)$ for a unique pair of indices (i, j), therefore $p^1_{ij}(z) < \vec{0}$. Then

$$g^1(z)(x, y) = v^1_{ij}(x, y) \tag{18}$$

and $||z(x, y) - z^1_{ij}(x, y)|| < \Delta_1$ such that

$$d(g^1(z)\,(x, y), G(z)\,(x, y)) = d(v^1_{ij}(x, y), G(z)\,(x, y)) \leqslant \tag{19}$$
$$\leqslant d(v^1_{ij}(x, y), G(z^1_{ij})(x, y)) + H(G(z^1_{ij})(x, y), G(z)\,(x, y)) \leqslant \frac{1}{2^1}$$

a.e. $(x, y) \in D$, that is the condition 1) holds for $n = 1$.

Moreover, using (13), (16) and (18), it follows

$$||g^1(z)(x, y) - g^0(z)\,(x, y)|| \leqslant ||v^1_{ij}(x, y) - g^0(z^1_{ij})(x, y)|| +$$
$$+ ||g^0(z^1_{ij})(x, y) - g^0(z)(x, y)|| = d(g^0(z^1_{ij})(x, y), G(z^1_{ij})(x, y)) + \tag{20}$$
$$+ ||g^0(z^1_{ij})(x, y) - g^0(z)(x, y)|| < \frac{1}{2} + ||g^0(z^1_{ij})(x, y) - g^0(z)\,(x, y)||$$

that implies

$$\mu(\{(x,y) \in D \mid \|g^1(z)(x,y) - g^0(z)(x,y)\| \geqslant 1\}) \leqslant$$

$$\leqslant \mu\left(\left\{(x,y) \in D \mid \|g^0(z^1_{ij})(x,y) - g^0(z)(x,y)\| \geqslant \frac{1}{2}\right\}\right) < \frac{1}{2}, \tag{21}$$

that is the condition 2) holds for $n = 1$.

Obviously, a similar construction can be used for $n > 1$, and the theorem is proved.

4. *The Picard problem.*

Consider the multi-valued equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z),\ (x, y) \in D,\ z \in B, \tag{22}$$

where $F : D \times B \to \text{comp}\, X$.

The Picard problem is defined in [2], [3] and consists in determination of a solution of the equation (22) satisfying the conditions (5) in the hypotheses (H_0), (H_1), (H_2). We state the following theorem.

Theorem 2. Let $F : D \times B \to \text{comp}\, X$ *be a multi-valued map satisfying the hypothesis*

(H_3) *F is continuous on $D \times B$.*

If the hypotheses $(H_0)-(H_3)$ *is fulfilled, the Picard problem* (22) + (5) *has at least an absolutely continuous solution* $\bar{z} : D \to \mathbf{R}^n$, $\bar{z} \in C.(D; \mathbf{R}^n)$.

Proof. Using the Theorem 1 it follows that there exists a continuous selection $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ for $G(z)$ given by (7). Define, for each $z \in K$, the function $h(z) : D \to \mathbf{R}^n$ by

$$h(z)(x,y) = \alpha(x,y) + \iint_{D_3(x,y)} g(z)(u,v)\, du dv =$$

$$= P(x) + Q(y) - P(\psi)(y)) + \int_0^y dv \int_{\psi(y)}^x g(z)(u,v) du,\ (x,y) \in D. \tag{23}$$

Then, $h(z) \in C^*(D; \mathbf{R}\ )$ for each $z \in K$, [10]. One obtains $h(K) \subset K$. Using the Schauder Fixed Point Theorem, it follows that there exists $\bar{z} \in K$ such that $h(\bar{z}) = \bar{z}$, that is $h(\bar{z})(x,y) = \bar{z}(x,y)$, $(x,y) \in D$.

That implies from (23) $\bar{z}(x, 0) = P(x)$, $0 \leqslant x \leqslant a$, $\bar{z}(\psi(y), y) = Q(y)$, $0 \leqslant \leqslant y \leqslant b$, therefore (5) and (22) hold for $\bar{z}$, consequently z is a solution of the Picard problem (22)+(5), a.e. $(x,y) \in D$.

REFERENCES

1. Cinquini Cibrario M., Cinquini S., *Equazioni a derivate parziali di tipo iperbolic* Edizioni Cremonese, Roma, 1964,
2. Teodoru G., *Le problème de Picard pour une équation aux dérivées partielles multivoque*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, „Babeş-Bolyai" University, Faculty of Mathematics, Research Seminaries, Preprint nr. 6, 193–1?? (1984).
3. Teodoru G., *Studiul soluţiilor ecuaţiilor de forma* $\frac{\partial z^2}{\partial x \partial y} \in F(x, y, z)$, Teză de doctorat, Universitatea ,,Al. I. Cuza" Iaşi, Facultatea de Matematică, 1984.
4. Teodoru G., *Continuous Selections for multi-valued functions*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, ,,Babeş-Bolyai" University, Faculty of Mathematics, Research Seminaries, Preprint nr. 6, 273–279, (1986).
5. Teodoru G., *Absolutely continuous Solutions for Cauchy problem for a multi-valued hyperbolic equation*, Buletin Inst. Politehnic Iaşi, s. Matematică-Mecanică (to appear).
6. Teodoru G., *Continuous Selections of multi-valued maps with non-convex right-hand side or the Goursat problem for the multi-valued hyperbolic equation* $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, Mathematica (to appear).
7. Antosiewicz H. A., and Cellina A., *Continuous Selections and Differential Relations*, J. Diff. Eq., T. 19, 386–398 (1975).
8. Bogatîrev A. V., *Continuous branches of multi-valued mappings with non-convex right-hand side* (Russian), Math. Sb., T. 120 (162), 3, 344–353 (1983).
9. Rus I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 197?
10. Carathéodory C., *Vorlesungen über Reelle Funktionen*, Chelsea Publishing Company, 19??
11. Castaing Ch., *Quelques problèmes de mésurabilité liés à la théorie de la commande*, C. R. Acad. Sci., Paris, T. 262, 409–414 (1966).
12. Filippov A. F., Classical solutions of differential equations with multi-valued right-hand side, SIAM J. Control 5 (1967), 609–621.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

OBSERVATORUL ASTRONOMIC AL UNIVERSITĂŢII

ÁRPÁD PÁL*

Intrat în redacţie: 10 noiembrie 1986

ABSTRACT. — Astronomical Observatory of the University. The paper deals with the development of the modern astrononomical research in Cluj. The founding, endowing and activity of the Astronomical Observatory of the University are presented, as well as the difficult work of its managers along the time. The modern residence of the Observatory and the rich scientific activity within the framework of this institution are also pointed out.

Cercetările moderne de astronomie au început la Cluj odată cu înfiinţarea Universităţii româneşti (1919), avînd drept iniţiatori pe *profesorul Gheorghe Bratu* (1881—1941) şi *profesorul Gheorghe Demetrescu* (1885—1969), care au elaborat planurile celui dintîi observator modern înzestrat la Cluj şi au format primii astronomi ce urmau să ducă mai departe creaţia lor**.

Observatorul astronomic al Universităţii din Cluj a fost construit şi dotat între anii 1920—1934, în partea de sud a oraşului, unde a avut multă vreme un cîmp larg de vizibilitate. La stăruinţele profesorului Gheorghe Bratu (directorul Observatorului între anii 1919—1923 şi 1928—1941), se fac primele comenzi de aparate şi cărţi, iar profesorul Gheorghe Demetrescu (directorul Observatorului între anii 1923—1928) completează aceste planuri, care au fost realizate astfel: în anul 1924 se obţine terenul, iar în 1927 se construieşte sala meridiană, în care se montează o lunetă de treceri, transformată dintr-un teodolit vechi, şi încep lucrările practice de astronomie. După mari greutăţi materiale, legate de asigurarea fondurilor necesare, cînd sursa principală de venituri o constituiau taxele studenţeşti, între anii 1928—1931 este construită clădirea ecuatorială cu o cupolă mobilă avînd diametrul de 5 m (construită şi montată de casa Gillon din Paris). Aici au fost instalate în următorii doi ani: ecuatorialul, Prin avînd un telescop Newton (cu oglindă parabolică, D = 50 cm, F = 250 cm) şi o lunetă cu obiectiv Zeiss (D = 20 cm, F = 300 cm), ambele instrumente fiind montate de inginerul Nicolae Bratu, fiul prof. Gh. Bratu. Alte instrumente mai mici, o lunetă de treceri, două sextante, două teodolite, cronometre şi pendule (de timp mediu şi sideral) au completat înzestrarea Observatorului. În anul 1934 este terminat pavilionul central pentru bibliotecă şi laboratoare.

Rolul şi meritele profesorului Gheorghe Bratu în domeniul astronomiei sînt pregnant înfăţişate în raportul Facultăţii de Ştiinţe a Universităţii din Cluj privind transferul său de la Catedra de analiză matematică la Catedra de astronomie. Cităm:

* *Universitatea din Cluj-Napoca, Facultatea de Matematică şi Fizică, 3400 Cluj-Napoca, România*

** *Dar preocupările de astronomie pe meleagurile transilvănene sînt foarte vechi, contopindu-se cu începuturile culturii şi civilizaţiei. Istoria astronomiei consemnează cunoştinţe şi cercetări astronomice remarcabile ale geto-dacilor, observatoare astronomice medievale înfiinţate în focarele de cultură, datînd încă din veacul al XV-lea, precum şi lucrări astronomice scrise, de mare valoare — mergînd pînă la elaborarea unor sisteme ale lumii — ce se păstrează şi astăzi în muzeele din Alba Iulia, Cluj-Napoca, Oradea şi alte oraşe (a se vedea bibliografia de la sfîrşitul articolului).*

„Dl. Profesor Gh. Bratu, de la începutul carierei sale ştiinţifice, şi-a manifestat înclinarea spre Astronomie. În adevăr, bursier prin concurs al Academiei Române în 1909, Dsa a fost trimis la Paris cu specială destinaţie de a face studii şi lucrări practice de Astronomie.

În această calitate, Dsa a obţinut diploma de Astronomia aprofundată la Facultatea de Ştiinţe din Paris, fiind clasificat în fruntea candidaţilor.

Ca „Astronome adjoint" la Observatorul din Paris a făcut timp de trei ani (1909—1912) lucrări practice de Astronomie, trecînd succesiv cele trei servicii fundamentale, după cum se vede din certificatul eliberat la 19 Iulie 1912 de Dl. Baillaud, Directorul acelui Observator. Dsa a lucrat 1) *La serviciul meridian* (...); 2) *La serviciul equatorial* (...); 3) *La serviciul fotografic al cerului*.

Certificatul Observatorului din Paris se termină cu următoarele aprecieri: „Rezultă din aceste expuneri că Dl. Bratu, cu activitatea căruia Observatorul nostru se poate lăuda, a făcut la acest Observator un ansamblu de lucrări foarte complete".

Paralel cu aceste studii şi lucrări de astronomie, Dl. Bratu a trecut şi Doctoratul în Matematici la Sorbona, în iunie 1914. Subiectul tezei sale „*Asupra echilibrului firelor*" e o problemă de Mecanică Analitică în strînsă legătură cu problema de Mecanică cerească a echilibrului maselor fluide.

Întors în ţară, Dl. Bratu şi-a continuat activitatea paralel în cele două direcţii: de Astronomie şi de Matematică pură.

În 1919 fiind numit profesor de Analiză Matematică la Facultatea de Ştiinţe din Cluj, i s-a încredinţat şi suplinirea catedrei de Astronomie, iar de la 1 Octombrie 1920 atît Direcţia cît şi organizarea Observatorului Astronomic din Cluj.

În această privinţă situaţia era cu deosebire grea (...). Crearea unui Observator Astronomic la Cluj, mai ales în condiţiile financiare de după război, era o problemă deosebit de anevoioasă şi dacă azi putem spune că sîntem aproape de realizarea ei completă, aceasta se datoreşte marilor calităţi de organizator precum şi tenacităţii, abnegaţiei şi muncii neobosite a Dlui Profesor Gh. Bratu. De altfel toţi Colegii noştri s-au putut convinge de aceste calităţi în numeroşii ani în care Dsa a făcut administraţia facultăţii noastre, fie ca decan fie ca prodecan.

Remarcăm de asemenea că Dl. Profesor Bratu a profitat de cercetările ştiinţifice făcute în străinătate, publicînd pe lîngă vreo 30 de Memorii de Matematică pură şi următoarele lucrări de Astronomie: 1) Efemeridele planetei 498 Tokio; 2) Efemeridele planetei 537 Pauly (...); 3) Despre planeta Marte (...).

De la 1923 la 1928 catedra de Astronomie la Facultatea de Ştiinţe din Cluj a fost ocupată de Dl. Profesor G. Demetrescu. De la plecarea Dlui Demetrescu la Bucureşti în Martie 1928 şi pînă astăzi, Dl. Profesor G. Bratu a reluat Direcţia Observatorului precum şi catedra de Astronomie în suplinire. De atunci Dsa face neîntrerupt cursul de Astronomie la Facultatea noastră. (...)

Deoarece, prin lipsa unui Observator Astronomic, orice cercetare astronomică era imposibilă la Cluj, Dl. Prof. Bratu şi-a pus ca prim scop al activităţii Dsale realizarea acestui Observator şi de numele său va rămîne legată această creaţie.

Putem spune azi, cu legitimă mîndrie şi mulţumire, că: Sala meridiană e complet instalată şi serviciul regulat al orei este asigurat. Sala ecuatorială e clădită; marea cupolă de 6 m diametru e montată; luneta ecuatorială e complet construită şi achitată la Paris şi urmează să fie adusă la Cluj în cursul lunii Aprilie 1931. Telescopul aferent e şi el gata şi ţinem să remarcăm că graţie relaţiilor şi intervenţiilor Dlui Bratu, oglinda parabolică de 0,50 m diametru a fost construită chiar în atelierele Observatorului Astronomic din Paris. Este pentru prima dată cînd acest Observator consimte să lucreze pentru un alt Observator din lume. În fine, clădirea pavilionului central cuprinde deja subsolul şi parterul.

Putem spune cu drept cuvînt că activitatea Dlui Prof. Bratu s-a identificat cu crearea acestui aşezămînt de cercetări ştiinţifice care, sîntem siguri, va fi o podoabă a facultăţii noastre.

Ca atare, socotim în unanimitate că locul Dlui Prof. Bratu este la Direcţia acestui Observator pentru ca, odată instalat, să poată culege roadele ştiinţifice ale strădaniei sale neobosite de 11 ani (...)". (Semnează membrii Consiliului Facultăţii de Ştiinţe: Th. Angheluţă — decan, N. Abramescu, A. Maior, E. Racoviţă, P. Sergescu, D. Pompeiu, Gh. Spacu ş.a.; actul de arhivă nr. 1054—1930/31.)

P r o f e s o r u l G h e o r g h e D e m e t r e s c u, fiind numit la Universitatea din Cluj la 1 iunie 1923, dar păstrînd un contact permanent cu Observatorul din Bucureşti — unde a revenit definitiv la 1 martie 1928 ca prim-astronom şi vice-director, a desfăşurat, de asemnea, o activitate remarcabilă, atît la catedră, cît şi la Observatorul astronomic, contribuind temeinic la formarea primelor promoții de absolvenți în matematică ai Universității, dintre care s-au afirmat ca astronomi valoroși Ioan Armeanca și Ioan Curea.

Activitatea științifică propriu-zisă în cadrul Observatorului din Cluj începe din 1933, cînd profesorul Gh. Bratu angajează Observatorul într-o colaborare cu Observatorul din Paris, la „*Catalogul hărții fotografice a Cerului*", lucrare de colaborare mondială, condusă de acesta din urmă, din care primului îi revenea reducerea clișeelor fotografice pentru zona de +20° (între anii 1933—1947). Această lucrare — „operă monumentală și istorică", după caracterizarea prof. Gh. Bratu — a avut drept scop eternizarea Cerului secolului XX, imprimîndu-l pe plăci de cupru, dar, evident, înregistrarea în cataloagele cerești era suficientă pentru știință.

În cadrul unei conferințe ținute la Universitatea din Cluj, profesorul Gh. Bratu spunea despre această lucrare:

„Pentru a se putea studia schimbările ce se produc în poziția și în strălucirea stelelor în timp de veacuri, schimbări ce nu pot fi observate în viața unui om, e absolut necesar ca pozițiile și strălucirile actuale ale stelelor să fie *înscrise în cataloage cerești*, spre a se păstra și spre a se putea *compara* situația cerului de azi cu cea a cerului de peste 100, 200, 1000 de ani.

Cum facerea acestor cataloage cere un timp îndelungat și o muncă uriașă, la 1889, la un congres internațional ținut la Paris s-a decis să se *fotografieze* bucată cu bucată toată bolta cerească, făcîndu-se poze de pătrate de pe sfera cerească de 2° lungime pe 2° lățime. La această operă internațională, pusă sub direcția Observatorului din Paris, s-au angajat 24 observatoare din lume (...).

Observatorul din Paris își luase pe seama lui studierea și fotografierea a 4 zone cerești. După cererea mea, la lucrarea unei zone, zona +20°, colaborează și Observatorul din Cluj.

Observatorul din Paris ne procură clișeele și datele necesare și noi la Cluj facem calcule pentru zona +20° — ceea ce reprezintă o muncă de mai mulți ani, pentru a determina cu cea mai mare exactitate *coordonatele cerești* ale stelelor cuprinse în zona dintre paralelele de 19° și 21° latitudine cerească. Cînd lucrarea va fi terminată, rezultatele vor fi publicate la Paris într-un volum special.

Numai cu aceste sacrificii imense și cu această muncă uriașă geniile veacurilor viitoare vor putea descoperi *legi noi* în Știința încă puțin cunoscută, numită *Astrofizica* sau *Astronomia stelară*".

Observatorul astronomic din Cluj era terminat în 1934, dar îi mai lipseau accesoriile. În următorii patru ani, tinerii colaboratori ai Observatorului realizează completările necesare: *astronomul Ioan Armeanca* (1900—1954) pune în funcțiune laboratorul de fotometrie fotografică și fotoelectrică (printre primele din lume) cu un fotometru Guthnick cu electrometru Lindemann cu cadrane; iar *astronomul Ioan Curea* (1901—1977) reinstalează vechea stație seismică, cu seismografe Mainka.

În 1938, Observatorul din Cluj, condus de Prof. Bratu, este angajat ca unitate de cercetare deja pe *trei direcții fundamentale*:

a) colaborează la „Catalogul hărții fotografice a Cerului" în continuare (prin prof. Gh. Bratu, în colaborare cu I. Armeanca, Gheorghe Chiș și Ștefan Radu);

b) studiul fotoelectric al stelelor variabile (prin astronomul I. Armeanca);

c) studiul seismelor din Transilvania (prin astronomul și seismologul I. Curea).

Profesorul I. Armeanca, socotit primul astrofizician român în sensul strict al cuvîntului, și-a început munca de specializare în domeniul astrofizicii la Göttingen, unde se dedică fotometriei stelare. Timp de trei ani aici, apoi la Observatorul din Kiel, sub conducerea profesorilor Rosenberg și Stobbe, își încheie teza sa de doctorat cu titlul „*Străluciri fotografice și fotovizuale ale stelelor* *din vecinătatea polului"* (1933), care constituie o remarcabilă contribuţie la extinderea fotometriei fotografice, fiind citată în toate lucrările de bază de fotometrie. I. Armeanca a extins secvenţa polară Nord la toate stelele dintr-o regiune de 100′ × 100′, stabilind cu multă precizie strălucirile fotografice a 260 de stele pînă la magnitudinea de 16,25 şi strălucirile fotovizuale a 220 de stele pînă la magnitudinea de 14,71, utilizînd metoda diferenţială. A făcut un studiu comparativ al fotometrului termoelectric Zeiss cu cel fotoelectric Rosenberg şi a stabilit ecuaţiile de culoare şi cele de distanţă ale obiectivelor. Pe baza secvenţei polare a lui I. Armeanca s-a obţinut o creştere a preciziei fotometriei fotografice şi vizuale şi a sporit posibilitatea de utilizare a ei.

Fotometrul achiziţionat şi asamblat în anii 1936—1938 este instalat în anii 1939—1940 la telescopul Newton al Observatorului din Cluj, dînd rezultate foarte bune.

Profesorul I. Curea, pasionat astronom şi seismolog, realizatorul de mai tîrziu — în calitatea sa de rector — al Universităţii şi Observatorului astronomic din Timişoara, ca şi al staţiilor seismice din Banat, şi-a adus o contribuţie importantă la consolidarea direcţiilor de cercetare astronomică din Cluj. În teza sa de doctorat, referitoare la determinarea polului ceresc pe cale fotografică, dă o metodă proprie, care a fost utilizată şi peste hotare în lucrările de astronomie.

O bruscă scădere a activităţii Observatorului astronomic din Cluj a avut loc odată cu declanşarea războiului şi, ca urmare a Dictatului de la Viena, Observatorul, împreună cu Facultatea de Ştiinţe căreia îi aparţinea, este mutat la Timişoara cu întreaga-i zestre, mai puţin cupola şi celelalte clădiri. Luneta ecuatorială este reinstalată provizoriu în Grădina horticolă din centrul oraşului, într-o clădire de lemn, unde, la adăpostul camuflajului impus de rigorile războiului, astronomul I. Armeanca reia observaţiile fotoelectrice. Dar în urma unui bombardament este distrus întreg echipamentul fotoelectric, încît după 1945, cînd Observatorul revine la vechea matcă din Cluj, pot fi continuate doar lucrările de fotometrie fotografică.

Profesorul Gheorghe Bratu, greu lovit de evacuarea şi greutăţile de reinstalare a Observatorului la Timişoara, moare fulgerător, la 1 septembrie 1941, în deplină capacitate creatoare.

Între anii 1941—1945, directorul Observatorului din Cluj-Timişoara a fost *profesorul Constantin Pîrvulescu* (1890—1945), profesor de astronomie la Facultatea de Ştiinţe a Universităţii refugiate, care — după cum se ştie — a deschis cercetării româneşti drumul astronomiei galactice (studiul stelelor duble, al roiurilor stelare, al rotaţiei Galaxiei) şi al celei extragalactice.

În *perioada 1945—1954*, începută prin repunerea instrumentelor în stare de funcţionare, activitatea Observatorului din Cluj a fost condusă cu multă competenţă şi autoritate de *profesorul Ioan Armeanca*. Se continuă tradiţionala problemă a studiului stelelor variabile pe calea fotometriei vizuale şi fotografice şi se încheie lucrarea de colaborare cu Observatorul din Paris. Rezultatele acestei colaborări sînt inserate în lucrarea *„Catalogue de 11.755 étoiles de la zone +17° à +25° et de magnitudes 9,5 à 10,5", Publications de l'Observatoire de Paris, Ed. Gauthier-Villars, 1950.*

În anul 1951, Observatorul este transferat de la Universitate la Filiala din Cluj a Academiei R. P. Române, organizîndu-se ca unitate de cercetare. În *perioada* 1951—1961, Observatorul a primit un sprijin substanţial de la Academie, atît pentru dezvoltarea planului tematic, cît şi pentru creşterea bazei materiale şi a numărului de cercetători, căpătînd şi personal tehnic-administrativ. Noile condiţii de lucru, precum şi posibilităţile create pentru noi colaborări internaţionale — în special cu unele observatoare sovietice (Moscova, Odesa) — deschid o nouă perspectivă cercetării astronomice clujene. În afară de I. Armeanca, Gh. Chiş şi Şt. Radu, care se aflau la Observator, în perioada menţionată au venit la această instituţie succesiv: *Ioan Todoran* (1 decembrie 1951), *Elvira Botez* (1 decembrie 1951, pînă în anul 1962), Árpád Pál (1 mai 1957, după efectuarea stagiului de doctorat la Universitatea „M. V. Lomonosov" din Moscova, Institutul Astronomic „P. K. Şternberg", Catedra de Mecanică cerească).

Din 1961, Observatorul astronomic trece în cadrul Universităţii „Babeş—Bolyai" din Cluj, păstrîndu-şi structura de unitate de cercetare.

După moartea prematură a prof. I. Armeanca, conducerea Observatorului din Cluj este preluată de *profesorul Gheorghe Chiş*, elev al prof. Gh. Bratu şi colaborator al prof. I. Armeanca; timp de 23 de ani (1954—1977) el va conduce această instituţie cu acelaşi devotament ca şi predecesorii săi.

Profesorul Gh. Chiş şi-a început activitatea la Observatorul din Cluj la 1 februarie 1936, fiind numit în postul de preparator. A fost trecut asistent, în cadrul acestui Observator, la 1 februarie 1943, şi şef de lucrări în acelaşi an, la 1 decembrie. La 1 octombrie 1950 devine conferenţiar de matematici generale, iar la 1 octombie 1954 trece la specialitatea sa, astronomie şi astrofizică. În data de 1 ianuarie 1960 ocupă postul de profesor titular în această specialitate, pe care o păstrează pînă la 1 iulie 1977, cînd devine profesor consultant prin ieşirea la pensie. Între anii 1962—1968 a fost decanul Facultăţii de Matematică-mecanică a Universităţii din Cluj.

Cercetările ştiinţifice ale prof. Gh. Chiş se referă la următoarele patru domenii: a) probleme de astromertie prin: participarea la „*Catalogul hărţii fotografice a cerului*" (participare amintită mai sus), determinări de coordonate geografice, determinări de poziţii de comete, planete mici şi sateliţi artificiali, cercetări cu caracter astronomic asupra calendarului geto-dacic din vestigiile sanctuarului de la Sarmizegetusa; b) probleme de stele variabile prin studii fotometrice — fotovizuale, fotografice şi fotoelectrice — ale stelelor binare fotometrice şi de tip RR Lyrae, reintroducînd metoda fotometriei fotoelectrice la Observatorul din Cluj; c) probleme de mecanică cerească prin determinări de orbite de comete şi de sateliţi artificiali ai Pămîntului; d) probleme de cercetări spaţiale, prin înfiinţarea în cadrul Observatorului din Cluj a *Staţiei de observare a sateliţilor artificiali (cod COSPAR: 1132)* şi participarea la programele de colaborare internaţională INTEROBS, INTERKOSMOS, EUROBS, vizînd folosirea observaţiilor sateliţilor artificali ai Pămîntului (de poziţie şi fotometrice) la studiul variaţiilor parametrilor structurali ai atmosferei înalte a Pămîntului, în corelaţie cu variaţiile indicilor activităţii solare şi geomagnetice.

Dintre toate domeniile pe care prof. Gh. Chiş le-a îmbrăţişat, astrofizica a rămas domeniul său de predilecţie. Dovadă a importanţei lucrărilor sale şi ale colaboratorilor săi din acest domeniu, în 1974, prof. Gh. Chiş a fost ales preşedinte al *Subcomisiei nr. 5 (Stele duble)*, în cadrul Comisiei de colaborare internaţională între academiile de ştiinţe din ţări socialiste, în problema „*Fizica şi* *evoluția stelelor*", precum și vicepreședinte al Comitetului Național Român de Astronomie (pînă în 1980).

În anul 1976, extinderea orașului obligă Observatorul la *strămutarea instrumentelor de observații* (achiziționate de Gh. Bratu) pe dealul Feleacului, *în zona Făget* (8 km sud de Cluj-Napoca, 750 m altitudine), existînd aici condiții de astroclimat adecvate unor observații astrofizice. Conform hotărîrii Senatului Universității, clădirea și cupola Stației de observare din Făget au fost construite — după concepția profesorului Gh. Chiș, executantă fiind Întreprinderea „Electrometal" Cluj-Napoca — din resurse interne și cu forțe locale, alocînd în acest scop circa 1 000 000 lei (Fig. 1). Din anul 1977, Stația de observare din Făget trece în nomenclatorul Centrului de Astronomie și Științe Spațiale București, care în noua organizare, încadrează toate cadrele de cercetare astronomică din țară, care lucrează pentru această unitate de cercetare.

Rămînîd la catedră și la Observator și după pensionare, profesorul Gh. Chiș a condus Seminarul de cercetare „Structura și evoluția stelelor", precum și activitatea unor doctoranzi în specialitatea „Astronomie și astrofizică", pînă în ultimele zile ale vieții sale. În urma unei boli necruțătoare, el s-a stins din viață la 19 mai 1981.

Din 1977, ca director al Observatorului astronomic a fost numit *profesorul Árpád Pál*, decanul de atunci al Facultății de Matematică a Universității (1976—1984). Colectivul de astronomi clujeni, la începutul noii perioade, a avut de depășit nenumărate dificultăți și piedici, care păreau uneori de neînvins pentru activitatea astronomică: mutarea provizorie a zestrei și personalului Observatorului, inclusiv a cercetătorilor C.A.S.S., în clădirea Institutului de Matematică de pe lîngă Facultatea de Matematică a Universității (str. Republicii nr. 37), cu excepție cupolei vechi, care a fost adăpostită în Parcul Sportiv al Universității; demolarea vechilor clădiri (din str. Republicii nr. 109), în 1978 — în locul lor fiind construită moderna Întreprindere de Electronică Industrială și Automatizări; demersuri și stăruințe pentru obținerea aprobărilor și fondurilor necesare reconstruirii Observatorului pe un nou amplasament, în valoare totală de 2 000 000 lei; și mai apoi, coordonarea lucrărilor de construcție și montaj (proiectantul fiind ELECTROUZINPROIECT București, iar executantul — S.C.P.C. Cluj-Napoca) și mutarea în noul edificiu.

Era pentru noi un motiv de legitimă satisfacție să consemnăm încheierea lucrărilor și darea în folosință, în vara anului 1982, a unui *modern și impunător pavilion al Observatorului astronomic*, în extremitatea sudică a Grădinii Botanice a Universității (str. Cireșilor 19), a cărei vegetație bogată condiționează aerul din jurul Observatorului (Fig. 2). Noua clădire asigură condiții superioare de desfășurare a activităților didactice și de cercetare științifică, în ea fiind amplasate: laboratoare didactice și de cercetare — printre care sala cupolei (vechi), adăpostind noul refractor Coudé (D = 150 mm, F = 2250 mm), achiziționat de la firma Zeiss din R.D.G., în 1980 (Fig. 3), sala meridiană și Stația de observare a sateliților artificiali ai Pămîntului (pe terasa clădirii), biblioteca și sala de lectură, cabinetul de astronomie, atelierele de mecanică și electronică, centrala termică și celelalte utilități.

De un deosebit sprijin am beneficiat în realizarea acestui obiectiv din partea organelor județene și municipale de partid și de stat, din partea Ministerului Educației și Învățămîntului, precum și din partea proiectanților și

Fig. 1. Stația de observare din Făget.

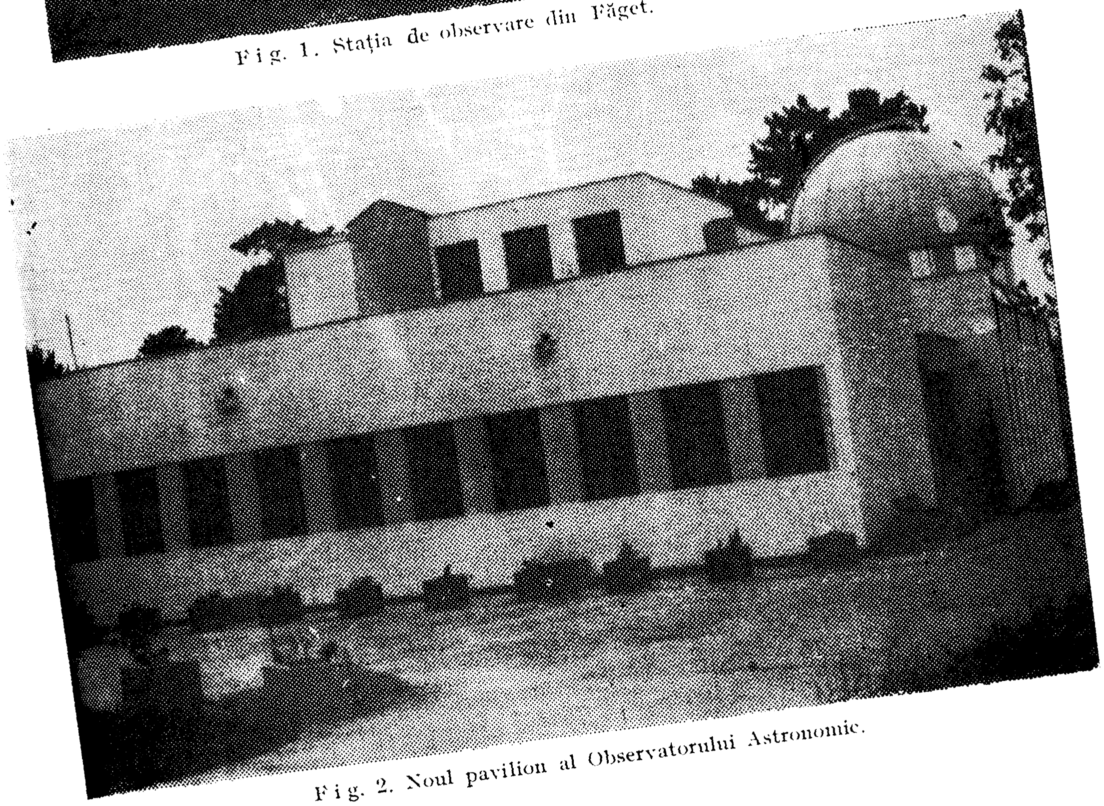

Fig. 2. Noul pavilion al Observatorului Astronomic.

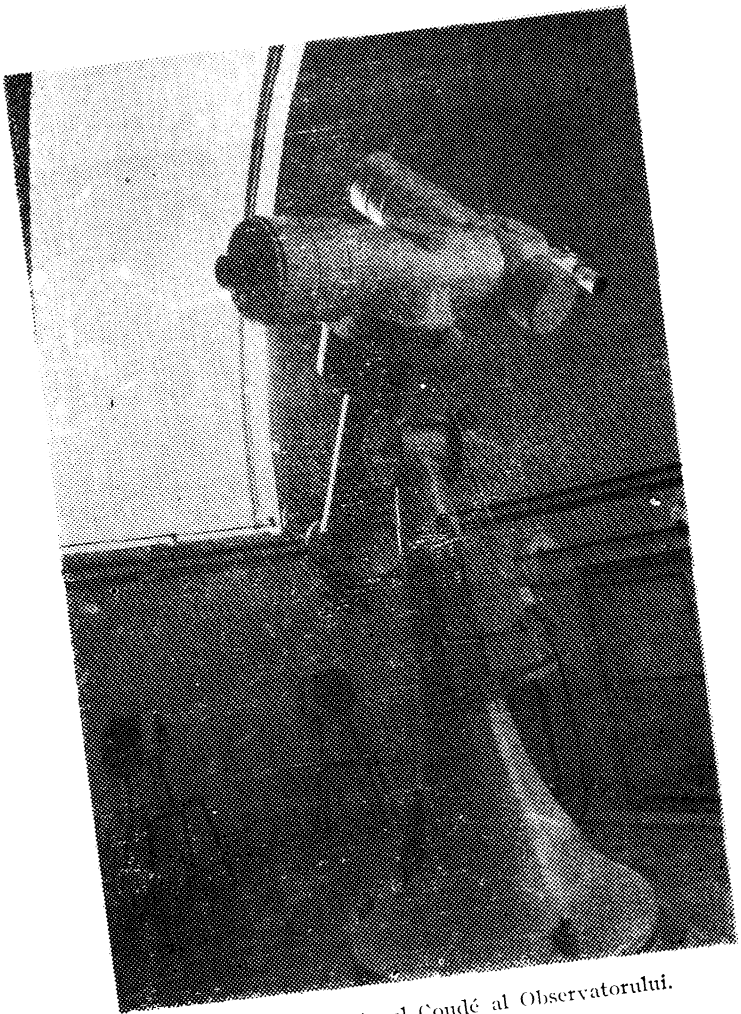

Fig. 3. Refractorul Coudé al Observatorului.

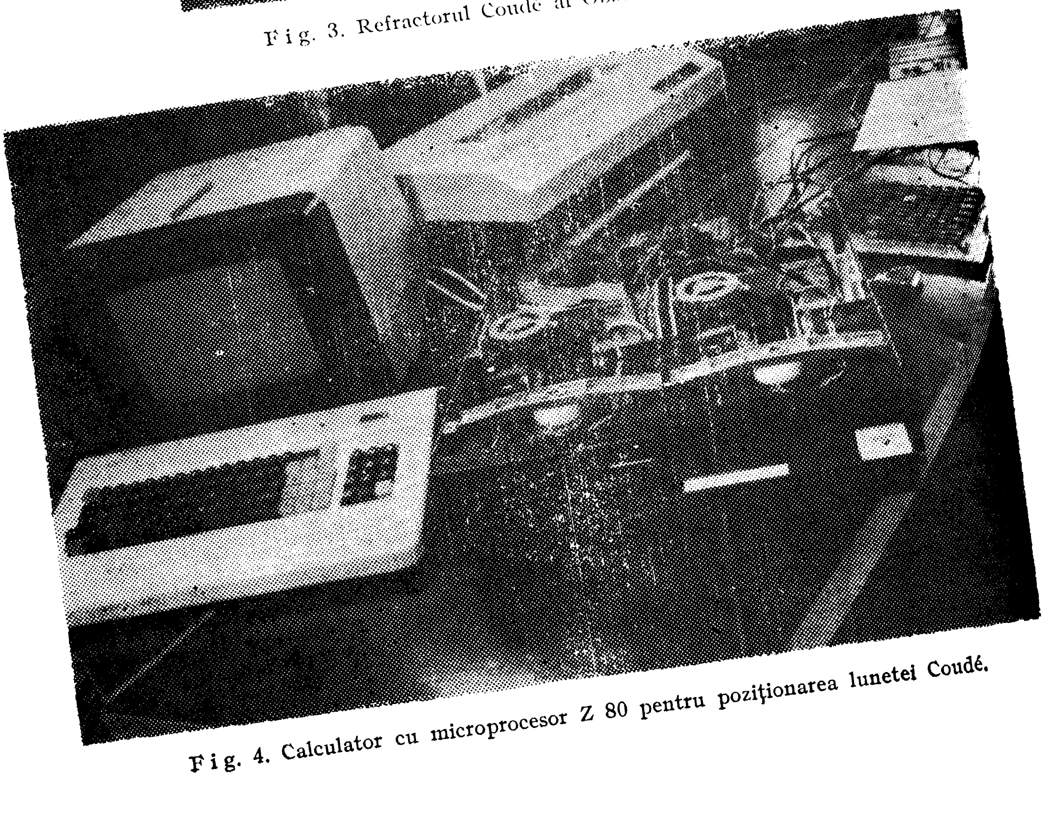

Fig. 4. Calculator cu microprocesor Z 80 pentru poziţionarea lunetei Coudé.

constructorilor întreprinderilor amintite. Tuturor le exprimăm, şi pe această cale, cele mai vii mulţumiri şi profunda noastră recunoştinţă!

În ultima vreme, în *dotarea Observatorului astronmoic* au intrat mai multe aparate noi: un complex de aparate pentru măsurarea timpului, aparate de măsurare a coordonatelor cereşti (teodolit, sextant, stereocomparator), calculatoare electronice, instalaţie pentru poziţionarea automată (pe bază de microprocesor) a lunetei Coudé (Fig. 4) ş.a., care contribuie atît la îmbunătăţirea cercetării ştiinţifice, cît şi a procesului didactic.

S-a mărit, de asemenea, fondul de cărţi şi publicaţii periodice, Observatorul astronomic dispunînd la ora actuală de o zestre de peste 16000 de volume (cărţi şi reviste), precum şi de alte materilale documentare de mare valoare culturală.

Cercetarea ştiinţifică în cadrul Observatorului nostru se realizează *în prezent* în cadrul a *două seminarii de cercetare*, ale căror tematici le vom schiţa în cele ce urmează.

Preocupări în cadrul *Seminarului „Mecanică cerească şi cercetări spaţiale"* (înfiinţat în anul 1972; conducător: dr. Á. Pál): 1. *Probleme de mecanică cerească:* varietăţi diferenţiabile şi topologice cu aplicaţii în mecanica cerească; studiul metodei medierii şi al aplicaţiilor ei în diferite probleme de mecanică cerească (orbite intermediare ale asteroizilor, cometelor, sateliţilor artificiali ş.a.); studiul problemei restrînse a celor trei corpuri şi elaborarea de modele matematice pentru cazul eliptic al acestei probleme; aplicarea metodei transformărilor Lie la studiul mişcării perturbate a corpurilor cereşti; aplicarea unor metode topologice la studiul problemei celor două şi trei corpuri; studiul soluţiilor cu ciocniri în problema a două şi n ($\geqslant 3$) corpuri. 2. *Teoria mişcării sateliţilor artificiali ai Pămîntului (SAP):* studiul mişcării perturbate a SAP sub influenţa diferiţilor factori gravitaţionali şi negravitaţionali; studiul metodelor de integrare numerică a ecuaţiilor mişcării perturbate a SAP; elaborarea de modele matematice ale mişcării SAP, ca şi de algoritmi şi programe de calcul în vederea rezolvării acestor modele; determinări şi ameliorări de orbite ale SAP; determinări de efemeride; studii asupra mişcării sateliţilor geostaţionari; studiul mişcării de rotaţie a SAP în jurul centrului propriu de masă. 3. *Probleme privind structura atmosferei terestre în cadrul relaţiilor Soare-Pămînt:* studiul evoluţiei parametrilor de stare ai atmosferei înalte pe baza datelor observaţionale asupra frînării orbitele a SAP în atmosferă; elaborarea de formule (legi empirice) noi pentru aproximarea parametrilor de stare ai atmosferei înalte; studiul corelaţiei între evoluţia parametrilor de stare ai atmosferei înalte şi activitatea solară şi geomagnetică; elaborarea de algoritmi şi programe de calcul pentru determinarea de valori ale densităţii şi ale altor parametri de stare ai atmosferei înalte. 4. *Probleme privind observarea sateliţilor artificiali:* studiul vizibilităţii sateliţilor artificiali (condiţii de vizibilitate, vizibilitatea satelit-satelit, cazuri particulare); elaborarea de algoritmi şi programe de calcul pentru determinarea de efemeride ale SAP; studii asupra metodelor de reducere a observaţiilor de SAP; probleme privind identificarea SAP (metode, criterii).

O bună parte a rezultatelor obţinute au apărut în următoarele *publicaţii ale Seminarului: Visual Observations of Artificial Earth Satellites. Supplement Dedicated to the Twentieth Anniversary of the First Artificial Earth Satellite Launch.*

University Babeş—Bolyai Cluj-Napoca, Astronomical Observatory, Satellite Tracking Station No. 1132, Cluj-Napoca, 1977; *„Babeş—Bolyai" University, Faculty of Mathematics, Research Seminaries, Seminar of Celestial Mechanics and Space Research, Preprints*: 2/1980, 3/1982, 2/1984, 10/1985.

Preocupări în cadrul *Seminarului „Structura şi evoluţia stelelor"* (înfiinţat în anul 1947, actuala denumire datînd din 1977; conducător: dr. V. Ureche): *1. Interpretarea curbelor de lumină la sisteme stelare binare strînse*: modele de interpretare a curbelor de lumină; studiul efectului de reradiaţie (reflexie); studiul efectului de elipticitate; determinarea elementelor absolute ale componentelor. *2. Studiul variaţiei perioadei la binare strînse*: deplasarea liniei apsidale; efectul relativist în mişcarea liniei apsidale; prezenţa celui de-al treilea corp; evoluţia sistemelor binare în faza transferului de masă; construirea de suprafeţe echipotenţiale Roche şi studiul stabilităţii. *3. Studiul fotometric al unor variabile de tip RR Lyrae*: construirea de curbe de lumină din observaţii; determinări de perioade multiple; studiul efectului Blajko; ajustarea curbelor observate cu funcţii spline; determinarea parametrilor fizici. 4. *Studiul stabilităţii pulsaţionale a stelelor*: efectul rotaţiei asupra stabilităţii; efectul mareic asupra stabilităţii. 5. *Studiul structurii şi stabilităţii stelelor relativiste*: modele analitice şi semianalitice de structură internă: omogen, liniar, politropic, de tip „stepenar"; criterii de stabilitate a stelelor relativiste; razele critice şi masele maxime ale stelelor neutronice; geometria continuumului spaţiu-timp în interiorul şi în vecinătatea obiectelor relativiste; diagrame de imersiune; energia gravitaţională a stelelor relativiste.

Mare parte a rezultatelor obţinute au apărut în următoarele *publicaţii ale Seminarului*: *Contributions of the Astronomical Observatory, Univ. „Babeş—Bolyai"*, Cluj-Napoca, 1976; *Contributions of the Astronomical Observatory. Proceedings of the Colloquium of Astronomy, Section Astrophysics, Cluj-Napoca, November 1977*, Cluj-Napoca, 1978; *„Babes—Bolyai" University, Faculty of Mathematics, Research Seminaries, Seminar of Stellar Structure and Stellar Evolution, Preprints*: 4/1983, 2/1985, 6/1986.

În virtutea acestor preocupări, pe care îi este axată activitatea, Observatorul clujean participă la mai multe *colaborări internaţionale*, sub egida Uniunii Astronomice Internaţionale (IAU = International Astronomical Union) şi a Comitetului de Cercetări Spaţiale (COSPAR = Committee on Space Research), precum şi în cadrul a două cooperări între academiile de ştiinţe din ţări socialiste: „Fizica şi evoluţia stelelor" — în care Observatorului nostru îi revine coordonarea Subcomisiei (Subproiectului) nr. 5 „Stele duble" şi „Fizica cosmică", în cadrul temei „Cercetări şi experimente comune cu ajutorul observaţiilor sateliţilor artificiali în scopuri astronomice, geofizice şi geodezice". Ţara noastră a găzduit ultimele reuniuni ale acestor colaborări în 1982 şi, respectiv, în 1983.

Pentru dezvoltarea relaţiilor internaţionale ale cercetării astronomice româneşti, în 1930 s-a înfiinţat Comitetul Naţional Român de Astronomie, care a devenit membru al U.A.I., unde rezultatele noastre au primit o binemeritată apreciere, oglindită şi în faptul că numai dintre astronomii clujeni cinci au fost aleşi membri individuali ai acestui for internaţional.

Observatorul astronomic are şi menirea de a contribui la pregătirea viitorilor profesori de matematică şi de fizică, în cadrul *cursurilor şi seminariilor* cuprinse în planurile de învăţămînt ale secţiilor respective, *lucrările practice de obser-*

vații fiind singurele activități care pot asigura o bună înțelegere, aprofundare și asimilare a cunoștințelor teoretice predate la cursuri și exersate în seminarii. La această activitate cu studenții participă, deopotrivă, cadrele didactice și cercetătorii Observatorului. Cursurile, culegerile de probleme, programe de calcul și lucrări de laborator, publicate în edituri sau litografiate pe plan local de către colectivul nostru reprezintă și ele tot atîtea ajutoare în pregătirea de specialitate a studenților și în inițierea lor în cercetarea științifică.

Ca lăcaș de știință și cultură, destinat studiului cerului înstelat, Observatorul atrage o frecvență abundentă de vizitatori, în special clase de elevi, terasa lui spațioasă de observații — unde se pot așeza mai multe intrumente astronomice portabile de amatori — fiind adecvată organizării unor ședințe demonstrative ce contribuie, începînd cu „trăirea emoțională a lui Galilei", însoțită de explicații ale specialistului, la formarea unei concepții juste, materislist-științifice despre Univers.

BIBLIOGRAFIE

1. I. Breahnă, *Observatorul Astronomic*, Buletinul Informativ al Universității „Al. I. Cuza" Iași, ianuarie—iunie 1985.
2. G. Chiș, Á. Pál, *Les débuts et le développement de l'astronomie à Cluj*, „Proceedings of the 16-th International Congress of the History of Science", A. Scientific Sessions, Bucharest, August 26—September 3, 1981.
3. I. Curea, *Observatorul Astronomic*, Tipografia Universității Timișoara, 1969.
4. C. Drâmbă, *75 de ani de la înființarea Observatorului Astronomic din București*, Anuarul Astronomic 1983, Ed. Acad. R.S.R., București, 1982.
5. L. Heinrich, *Az elsö kolozsvári csillagda* (*Primul observator astronomic din Cluj*), Ed. Kriterion, București, 1978.
6. *Istoria științelor în România. Matematica, Mecanica, Astronomia*, Ed. Acad. R.S.R., București, 1981.
7. *Lucrările simpozionului Profesor Gheorghe Bratu*, organizat cu prilejul împlinirii a 100 de ani de la naștere, Litografia Universității „Babeș-Bolyai" Cluj-Napoca, 1981.
8. V. Marian, I. Józsa, *Vechile observatoare astronomice din Transilvania*, București, 1957.
9. *Progrese în astronomie*, Sesiune științifică organizată cu ocazia împlinirii a 100 de ani de la nașterea astronomului Gheorghe Demetrescu, București, 25—26 octombrie 1985.
10. I. M. Ștefan, V. Ionescu-Vlăsceanu, *Momente și figuri din istoria astronomiei românești*, Ed. științifică, București, 1968.
11. I. M. Ștefan, E. Nicolau, *Scurtă istorie a creației științifice și tehnice românești*, Ed. Albatros, București, 1981.
12. N. Teodorescu, G. Chiș, *Cerul — o taină descifrată* (*Astronomia în viața societății*), Ed. Albatros, București, 1982.

RECENZII

1) E d w a r d W. S t r e d u l i n s k y, **Weighted Inequalities and Degenerate Elliptic Partial Differential Equations,** Lecture Notes in Mathematics. Vol. 1074, 143 pages, 1984.

The main purpose of this book is the investigation of various weighted spaces and weighted inequalities which are relevant to the study of solvability of the problems concerning linear or non-linear partial differential equations and in the analysis of the properties of the solutions. The usefulness of these results is illustrated in the latter part of the book where they are used to establish continuity for weak solutions of degenerate elliptic equations.

The book may be used by the specialists who work in the domain of the theory of partial differential equations and by the students who are specializing in this domain.

S. SZILÁGYI

K. J a r o s z, **Perturbations of Banach Algebras.** Lecture Notes in Mathematics 1120, Springer Verlag 1985, 117 pp.

The book is dealing with three kinds of small perturbations for Banach algebras: ε-perturbations of the multiplication, ε-isomorphism and ε-isometries. The author proves stability results under small perturbations for various classes of Banach algebras. As the theory is only at the initial stage the author states many problems — the book ends with a list of 20 open problems.

S. COBZAŞ

K. S u n d a r e s a n, S. S w a m i n a t h a n, **Geometry and Nonlinear Analysis in Banach Spaces,** Lecture Notes in Mathematics 1131, Springer Verlag 1985, 115 pp.

The book is concerned with differential nonlinear analysis in infinite dimensional Banach spaces. The rich and elegant finite dimensional theory do not extend automatically to the infinite dimensional case. The authors treat topics as: Smoothness classification of B-space, Smooth partitions of the unity, Smoothness and approximation, Infinite dimensional differentiable manifolds. The book is clearly written, collects together many topics scattered in various journals and will be useful to a large class of mathematicians (especially analysts).

S. COBZAŞ

P a l l e T. E. J o r g e n s e n and Robert T. M o o r e, **Operator Commutation Relations,** Mathematics and its Applications, D. Reidel Publishing Co. 1984, 493 pp.

The authors consider infinitesimal and global commutation relations for operators showing that apparently distinct topics can be unified, in an unexpected way, through a certain analysis of operator commutation relations, leading as well to new results in diverse areas of mathematics and its applications. The book is of interest for mathematicians (both pure and applied) and for researchers in mathematical physics and quantum chemistry.

S. COBZAŞ

J e r r o l d M a r s d e n, A l a n W e i n s t e i n, **Calculus I, II and III,** Springer Verlag New York, Berlin, Heidelberg, Tokio 1985.

This three-volume book represents a very good introduction to real differential and integral calculus. In a didactic and rigorous manner the authors present the basic notions and results including many geometric and physical aspects of calculus with a wealth of excellent applications. Each volume contains many solved and proposed exercices and problems. The book presents interest and is useful for the students in mathematics.

D. ANDRICA

Banach Center Publications vol. 14, **Mathematical Control Theory,** Edited by C. Olech, B. Jakubczyk and J. Zabczyk, P.W.N. Warszawa 1985, 643 pp.

These are the Proceedings of the XVI-th semester of the Banach International Mathematical Center (September—December 1980). The book contains forty papers covering various topics in optical control theory, written by eminent specialists in the field as A. Bensoussan, L. D. Berkovitz, F. H. Clarke, R. Gabasov, J-L. Lions, P-L. Lions, F. Mignot, S. Rolewicz et al.

S. COBZAȘ

Jindrich Nečas, **Introduction to the Theory of Nonlinear Elliptic Equations,** Teubner-Texte für Mathematik, Band 52, Leipzig, 1983.

În prezenta carte autorul studiază probleme la limită pentru ecuații cu derivate parțiale de ordinul al doilea de tip eliptic. Se studiază probleme ca: spații Sobolev și Morrey-Camponate, soluții slabe, metode aproximative, regularitatea soluțiilor și aplicații în teoria elasticității. Cartea profesorului J. Nečas reprezintă o foarte bună introducere în teoria problemelor la limită relative la ecuații cu derivate parțiale de tip eliptic neliniare.

I. A. RUS

Lars Hörmander: **The Analysis of Linear Partial Differential Operators;** Vol. 1: *Distribution Theory and Fourier Analysis;* Vol. 2: *Differential Operators with Constant Coefficients,* Springer-Verlag, Berlin, 1983.

The volumes I and II are a systematic study of distribution theory and of partial differential operators with constant coefficients, Basic properties of distributions, Convolutions, Fourier transformation, Spectral analysis of of singularities, Hyperfunctions, Existence and approximation of solution of differential equations, Differential operators of constant strenghth, Scattering theory, Analytic function theory and differential equations, Convolution equations. These two volumes are part of a remarkable book of highest quality and of greatest importance for research workers and graduate students in mathematics.

I. A. RUS

Hideyuki Majima, **Asymptotic Analysis for Integrable Connections with Irregular Singular Points,** Lect. Notes in Math., 1075, Springer-Verlag (1984).

The book is an excellent research monograph. Using strongly asymptotic expansions of functions of several variables, the author proves existence theorems of asymptotic solutions to integrable systems of partial differential equations under certain general conditions. Other topics in this book: Riemann—Hilbert—Birkhoff problem, Poincaré's lemma and de Rham cohomology theorem.

I. A. RUS

E. Zeidler, **Nonlinear Functional Analysis and Its Applications. III.** *Variational Methods and Optimization,* Springer Verlag 1985, 662 pp.

The book is a considerably expanded version of the book of the author, „Vorlesungen über nichtlineare Funktionalanalysis III". *Variationsmethoden und Optiemierung,* Teubner Texte zur Mathematik Leipzig 1977, 239 pp., and belongs to a cycle of five books on nonlinear functional analysis: I Fixed point teohrems, II Monotone Operators, IV—V Applications to mathematical physics, published originaly in German as Teubner Texte and translated (and expanded) in English and published by Springer Verlag. This is a comprehensive monograph on optimization and variational problems. The book is very well organized and very clear written. Each chapter (and there are 57 chapters) ends with a set of problems and bibliographical comments. The bibliography is very extensive (30 pages). The book ends with a list of symbols, a list of theorems and an index of notions. The book is a valuable contribution to optimization theory and related topics.

S. COBZAȘ

Jean Paul Gauthier. **Structure des systemes non-lineaires.** Éditions du CNRS, Paris, 1984, 307 p.

Dans l'Introduction du livre on présente les idées genérales, les sources et les buts du travail. Les rappels néccessaires de géométrie différentielle et de Topologie, ainsi que la théorie du controle des systèmes non-linéaires avec ses applications sont développés d'une manièer attractive et accesible d'après le schéma suivant: I. Variétés différentiables. II. Gouvernabilité. III. Observabilité et Observateurs. IV. Stabilisation. V. Découplage. VI. Bibliographie. Le livre est destiné aux étudiants qui débutent dans la recherche, autant que aux spécialistes en Automatique.

M. ȚARINĂ

Graphentheorie: eine Entwicklung aus dem 4-Farben Problem, von Martin Aigner, Stuttgart: Teubner 1984 (Teubner-Studienbücher: Mathematik) ISBN 3–519–02068–8.

Das vorliegende Buch eines bekannten Autors enthält eine sehr gute Einführung in die Graphentheorie mit nahezu allen wichtigen Begriffen und Resultate. Es wird dabei insbesondere die wichtige Rolle geschildert die das 4-Farben Problem in der Entwicklung der Graphentheorie spielte: sein Ursprung, die ersten Versuche zur Lösung des Problems mit all seinen Sackgassen und schliesslich seine ungewöhnliche Lösung mit Hilfe des Computers.

H. KRAMER

Global Analysis — Studies and Applications I, (Edited by Yu. G. Borisovich and Yu. E. Gliklikh), Lectures Notes in Mathematics vol. 1108, Springer Verlag 1984, 301 pp.

The volume contains the translations of the Voronezh University Press series „Novoe v global'nom analyze" for the years: 1982 — Equations on manifolds; 1983 — Topological and geometrical methods in mathematical physics; 1984 — Geometry and topology in global nonlinear problems. The aim of the series is to publish survey (expository) papers and a small number of short communications. Among the members of the editorial board and contributors there are well known specialists as A. T. Fomenko, A. S. Mishchenko, S. P. Novikov, M. M. Postnikov, A. M. Vershik et al. The translation and publication in Lectures Note Series make these important contributions to global analysis accessible to a larger set of readers.

S. COBZAȘ

Nonlinear Analysis and Optimization, Bologna 1982, Edited by C. Vinti, Lecture Notes in Mathematics vol. 1107, Springer Verlag 1984, 214 pp.

These are the Proceedings of a meeting organized in Bologna in Honour of Professor Lamberto Cesari (a similar meeting took place in 1980 at the University of Texas at Arlington). The book begins with a paper of D. Graff on L. Cesari scientific activity and a paper of J. Serrin, Applied mathematics and scientific thought. There are also ten contributed papers by eminent specialists in the field: L. Cesari himself, A. Bensoussan, J. Frehse, J. P. Gossez, P. Hess, R. Kannan, J. Mawhin et al.

S. COBZAȘ

Y. Okuyama, **Absolute Summability of Fourier Series and Orthogonal Series,** Lecture Notes in Mathematics vol. 1067, Springer Verlag 1984, 117 pp.

The absolute summability of a series is a generalization of the concept of absolute convergence just as the summability is an extension of the concept of convergence. The absolute summability methods for non-absolute convergent series (Nörlund — and Riesz — absolute summability) are given both for trigonometric series and for the Walsh orthogonal system. The book will be useful to all interested in harmonic analysis.

S. COBZAȘ

Raghavan Narasimhan, **Complex Analysis in One Variable,** Birkhäuser Verlag 1985, 266 pp.

The aim of this book is to present, from a modern point of view, the theory of functions of one complex variable, relating the subject to other branches of mathematics, especially several complex variables (a field which owes much to the author of this book). The author achieves masterly this end and the result is an excelent monograph in complex function theory. The book also contains a chapter (Chapter 8) on several complex variables but, as the author points out in the preface, as a whole, the book is about one variable.

S. COBZAȘ

P. Schapira, **Microdifferential Systems in the Complex Domain,** Grundlehren der mathematischen Wissenchaften vol. 269, Springer Verlag 1985, 214 pp.

The subject of this book involves several branches of mathematics as: microlocal analysis, linear partial differential equations, algebra and complex analysis. Its aim is to present, at an accessible level, to the analyst the algebraic methods used in this field and to the algebraist some topics from partial differential equations. The book is a very good introduction to this difficult and very active domain of research.

S. COBZAŞ

H. Schlichtkrull, **Hyperfunctions and Harmonic Analysis on Symmetric Spaces,** Progress in Mathematics vol. 49, Birkhäuser Verlag 1984, 1985 pp.

The book is divided in two parts. The first one (Chapters 1 and 2) is expository (few proofs are given) and gives an introduction to microlocal analysis and hyperfunctions. In the second part, containing also some original contributions of the author, these results are applied to symmetric spaces. The book is an outhgrowth of an essay which received a gold medal from the University of Copenhagen.

S. COBZAŞ

Albrecht Fröhlich, **Classgroups and Hermitian Modules,** Progress in Mathematics, Birkhäuser Verlag 1984, Boston — Basel — Stuttgart.

Cartea conține o expunere sistematică și detailată a abordării cu ajutorul omorfismului Galois a diferitelor clasgrupuri atașate ordinelor și în particular inelelor grupale, abordare care se dovedește fundamentală în cercetări recente. Cartea este utilă în cercetări de teoria numerelor algebrice, K-teorie, forme patratice și Hermitiene și teoria modulară.

GR. CĂLUGĂREANU

H. Junek, **Locally Convex Spaces and Operator Ideals,** Teubner Texte zur Mathematik, Band 56, Leipzig, 1983, 180 pp.

A. Pietsch was the first who studied operator ideals in Banach spaces and applied them to nuclear spaces. The author gives in this book a systematic exposition of the theory of ideals of operators ranging in locally convex spaces (LCS), showing that many properties of several classes of LCS are just consequences of some stability properties of operator ideals acting on them. A special attention is paid to $F-$, $DF-$ spaces, to spaces of differentiable and holomorphic functions and to spaces of unbounded operators.

S. COBZAŞ

Recent Trends in Mathematics, Reinhardsbrun 1982, Teubner Texte zur Mathematik, Band 50, Leipzig 1983, 329 pp.

These are the Proceedings of a Conference held in Reinhardsbrunn RDG, from October 11 to October 13, 1982, editde by H. Kurke, J. Mecke, H. Triebel and R. Thiele. The conference was attended by 62 mathematicians working in various branches of mathematics (S. V. Bochkariev, L. D. Kudryavtsev, Z. Cieselski, W. Dickmeis, R. J. Nessel, K-H. Elster, A. Göpfert, J. Necas et al.). The book contains 40 of the contributed papers, the programme of the conferenceand the list of participants.

S. COBZAŞ

Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Leipzig 1983, Teubner Texte zur Mathematik, Band 67, Leipzig 1984, 234 pp.

The book contains the contributions of the participants at this conference grouped in three sections: A. Topological algebras and their representations; B. Operator ideals and geometry of Banach spaces, and C. Algebraic approach to quantum field theory and statistical physics. Sections A + C contain 18 papers and Section B, 12 papers. The book contains valuable contributions to these fields and it is of interest for a large class of mathematicians and physicists.

S. COBZAŞ

Thomas Zink, **Cartiertheorie kommutativer formaler Gruppen,** B. G. Teubner, Leipzig, 1984.

The theory of commutative formal groups has a special importance in algebraic number theory and in algebraic geometry over a field of characteristic p. The french mathematician

P. Cartier found a new approach to this theory which is simpler and more general than others and which has interesting applications to abelian manifolds. The book of Th. Zink is for students and mathematicians interested in algebraic geometry or number theory and familiar with commutative algebra. It presents the theory in a new way based on concepts of deformation theory. During the six chapters of the book, besides the main theorems of the theory, basic facts on isogenies, deformations of p-divisible formal groups and Dieudonne's classification are treated.

RODICA COVACI

L. Lovász, M. D. Plummer: **Matching Theory**, Akadémiai Kiadó, Budapest, 1986, 544 + XXXIII pp.

This book deals with the matchings (sets of edges without common points) in graphs. In the theory of matchings a lot of applied problems can be modelled, from which the entire theory was really borne.

A complete treatment of this and related ubjects is divided into twelve chapters. These chapters are the followings: 1. Matchings in bipartite graphs, 2. Flow theory, 3. Size and structure of maximum matchings, 4. Bipartite graphs with perfect matchings, 5. General graphs with perfect matchings, 6. Some graph-theoretical problems related to matchings, 7. Matching and linear programming, 8. Determinants and matchings, 9. Matching algorithms, 10. The f-factor problem, 11. Matroid matching, 12. Vertex packing and covering, and References with an impressive number of titles. Algorithmical aspects are also considered.

This well-written book is recommanded to all, who are interested in matching problems.

Z. KÁSA

ÎNTREPRINDEREA POLIGRAFICĂ CLUJ,
Municipiul Cluj-Napoca, Cd. nr. 579/1986

Revista ştiinţifică a Universităţii din Cluj-Napoca, **STUDIA UNIVERSITATIS BABEŞ-BOLYAI,** apare începînd cu anul 1986 în următoarele condiţii:

matematică — trimestrial
fizică — semestrial
chimie — semestrial
geologie-geografie — semestrial pentru geologie şi anual pentru geografie
biologie — semestrial
filosofie — semestrial
ştiinţe economice — semestrial
ştiinţe juridice — semestrial
istorie — semestrial
filologie — semestrial

STUDIA UNIVERSITATIS BABEŞ-BOLYAI, the scientific journal of the University of Cluj-Napoca, starting with 1986 is issued as follows:

mathematics: quarterly
physics: biannually
chemistry: biannually
geology-geography: biannualy on geology and yearly on geography
biology: biannually
philosophy: biannually
economic sciences: biannually
juridical sciences: biannually
history: biannually
philology: biannually

43 875

Abonamentele se fac la oficiile poştale, prin factorii poştali şi prin difuzorii de presă, iar pentru străinătate prin „ROMPRESFILATELIA", sectorul export-import presă, P. O. Box 12—201, telex. 10376 prsfir, Bucureşti Calea Griviţei nr. 64—66.

STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

4

1986

CLUJ-NAPOCA

ANUL XXXI 1986

STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

4

Redacția: 3400 CLUJ-NAPOCA, str. M. Kogălniceanu, 1 ● Telefon 1 61 01

SUMAR — CONTENTS — SOMMAIRE

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXI, 4, 1986

CLASSES OF n-α-CLOSE-TO-CONVEX FUNCTIONS

TEODOR BULBOACĂ*

Received: June 15, 1986

REZUMAT. — Clase de funcţii n-α-aproape convexe. Se introduc clase noi de funcţii univalente, care generalizează unele clase definite de H. S. Al-Amiri [1] şi S. Ruscheweyh [6] şi se stabilesc unele proprietăţi de incluziune între aceste clase.

1. **Introduction** Let A be the class of functions $f(z)$, analytic in the unit disc U with $f(0) = f'(0) - 1 = 0$. As in [2] we denote by $K_{n,\alpha}(\delta)$ the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] > \delta, \quad z \in U,$$

where $\alpha \geqslant 0$, $\delta < 1$ and $D^n f(z) = \dfrac{z}{(1-z)^{n-1}} * f(z) = \dfrac{z(z^{n-1}f(z))^{(n)}}{n!}$, where $(*)$ stands for the Hadamard product. Some results concerning the classes $K_{n,\alpha}(\delta)$ and $Z_n(\delta) \equiv K_{n,0}(\delta)$ are presented in [2]; note that the classes $K_{n,\alpha}\left(\frac{1}{2}\right)$ and $Z_n\left(\frac{1}{2}\right)$ were introduced by *H. S. Al-Amiri* [1] and *S. Ruscheweyh* [6] respectively.

Let $AC_n(\delta)$ be the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)} > \delta, \quad z \in U, \quad \text{where } g \in Z_{n+1}(\delta),$$

and we call this class the class of *n-close-to-convex functions of order* δ.

Let $C_{n,\alpha}(\delta)$ be the class of functions $f \in A$ which satisfy

$$\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}g(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right] > \delta, \ z \in U, \text{ where}$$

$g \in Z_{n+2}(\delta)$, and we call this class the class of *n-α-close-to-convex functions of order* δ.

Note that $C_{n,1}(\delta) = AC_{n+1}(\delta)$, $C_{n,0}(\delta) = AC_n(\delta)$, $Z_{n+1}(\delta) \subset AC_n(\delta)$ and $Z_{n+2}(\delta) \cup C_{n,\alpha}(\delta)$; the classes $C_{n,\alpha}\left(\frac{1}{2}\right) = C_n(\alpha)$, $AC_n\left(\frac{1}{2}\right) = C_n$ were introduced by H. S. Al-Amri [1], who proved that $C_n(\alpha) \subset C_n$, $\alpha \geqslant 0$ and $C_n(\alpha) \subset C_n(\beta)$, $\alpha > \beta \geqslant 0$. In this paper we shall study some properties of these classes and several particular results will also be given.

* *Industrial Lycée, 2900 Arad, Romania*

2. **Preliminaries.** Let f and g be regular in U. We say that f is subordinate to g, written $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We will need the following theorems to prove our main results.

THEOREM A. Let $\frac{1}{2} \leqslant \delta < 1$ *and* $n \in N$; *then* $Z_{n+1}(\delta) \subset Z_n(\delta_n^*)$ *where* $\delta_n^* = 1/F\left(1,\ 2(n+2)(1-\delta),\ n+2;\ \frac{1}{2}\right)$, *and this result is sharp.*

This theorem is a particular case of Theorem 3[2], when $\alpha = 1$.

THEOREM B. [5]. *Let* $\beta > 0$, $\beta + \gamma > 0$ *and* $-\frac{\gamma}{\beta} \leqslant \delta < 1$. *Then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1-(1-2\delta)z}{1+z} \equiv h_\delta(z),\quad q(0) = 1$$

has a univalent solution is U, *given by*

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}$$

where $Q(z) = \int\limits_0^1 \left(\frac{1-z}{1-rz}\right)^{2\beta(1-\delta)} t^{\beta+\gamma-1}\ dt$, $z \in U$, *and* $q(z) \prec \frac{1-(1-2\delta)z}{1+z}$.

If $p(z) = 1 + p_1 z + \ldots$ *is regular in* U *and satisfies the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1-(1-2\delta)z}{1+z}$$

then $p(z) \prec q(z)$ *and this subordination is sharp.*

THEOREM C. [3]. *Let* $d\mu(t)$ *be a positive measure on* $[0, 1]$ *and let* $Q(z, t)$ *be a complex-valued function defined on* $U \times [0, 1]$, *such that* $Q(z, \cdot)$ *is integrable on* $[0, 1]$ *for each* $z \in U$. *Suppose that* Re $Q(z, t) > 0$ *in* U, $Q(-r, t)$ *is real and*

$$\text{Re}\ \frac{1}{Q(z,r)} \geqslant \frac{1}{Q(-r,\ t)} \quad for\ |z| \leqslant r < 1$$

and $t \in [0, 1]$.

$$If\ Q(z) = \int\limits_0^1 Q(z,\ t)\ d\mu(t),\ then\ \ \text{Re}\ \frac{1}{Q(z)} \geqslant \frac{1}{Q(-r)}\ for\ |z| \leqslant r.$$

THEOREM D. [4]. *Let* $h,\ q \in H(U)$ *be univalent in* U *and suppose* $q \in H(\overline{U})$. *If* $\psi : \mathbb{C}^3 \to \mathbb{C}$ *satisfies:*

a) ψ *is analytic in a domain* $D \subset \mathbb{C}^3$,

b) $(q(0),\ 0,\ 0) \in D$ *and* $\psi(q(0),\ 0,\ 0) \in h(U)$,

c) $\psi(r,\ s,\ t) \notin D$ *when* $(r,\ s,\ t) \in D$, $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$\operatorname{Re}(1 + t/s) \geqslant m \operatorname{Re}(1 + \zeta q''(\zeta)/q'(\zeta))$, *where* $|\zeta| = 1$, $m \geqslant 1$, *then for all* $p \in H(U)$ *so that* $(p(z), zp'(z), z^2p''(z)) \in D$, $z \in U$ *we have:*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \Rightarrow p(z) \prec q(z).$$

3. **Main results.** *THEOREM* 1. *Let* $\alpha \geqslant 0$ *and* $\frac{1}{2} \leqslant \delta < 1$; *if* $f \in C_{n,\alpha}(\delta)$ *related to* $g \in Z_{n+2}(\delta)$ *then* $f \in AC_n(\delta^*_{n+1})$ *related to* $g \in Z_{n+1}(\delta^*_{n+1})$, *where* $\delta^*_{n+1} = 1/F\left(1,\ 2(n+3)(1-\delta),\ n+3;\ \frac{1}{2}\right)$.

Proof. Using Theorem A we obtain $g \in Z_{n+1}(\delta^*_{n+1})$ i.e.

$$\frac{D^{n+2}g(z)}{D^{n+1}g(z)} \prec h_{\delta_{n+1}}(z).$$

Let $p(z) = D^{n+1}f(z)/D^{n+1}g(z)$; then $p(0) = 1$ and using

$z(D^nf(z))' = (n+1)D^{n+1}f(z) - nD^nf(z)$, $n \in \mathbf{N}$ we obtain that $f \in C_{n,\alpha}(\delta)$ is equivalent to $p(z) + \alpha(z)\, zp'(z) \prec h_\delta(z)$, where $\alpha(z) = \frac{\alpha}{n+2}\frac{D^{n+1}g(z)}{D^{n+2}g(z)}$.
Without loss of generality we can assume that p and h_δ satisfy the conditions of the theorem on the closed disc $\bar{U}$; if not we can replace $p(z)$ by $p_r(z) = p(rz)$ and $h_r(z)$ by $h_{\delta,r}(z) = h_\delta(rz)$, $0 < r < 1$, and these new functions satisfy the conditions of the theorem on $\bar{U}$. We would then prove $p_r(z) \prec h_{\delta,r}(z)$ for all $0 < r < 1$ and by letting $r \to 1^-$, we have $p(z) \prec h_\delta(z)$.

Because $g \in Z_{n+1}(\delta^*_{n+1})$, for $\alpha > 0$ we have $\operatorname{Re}\alpha(z) > 0$, $z \in U$. Let $\psi(r, s) = r + \alpha(z)s$ which is analytic in C^2 and $\psi(h_\delta(0), 0) = h_\delta(0) \in h_\delta(U)$. A simple calculus shows that

$$\operatorname{Re}\frac{\psi_0 - h_\delta(\zeta_0)}{\zeta_0 h'_\delta(\zeta_0)} = m_0 \operatorname{Re}\alpha(z) > 0, \text{ where}$$

$$\psi_0 = h_\delta(\zeta_0) + \alpha(z)m_0\zeta_0 h'_\delta(\zeta_0),\ m_0 \geqslant 1,\ |\zeta_0| = 1.$$

Using this fact together with the fact that $\zeta_0 h'_\delta(\zeta_0)$ is an outward normal to the boundary of the convex domain $h_\delta(U)$ we conclude that $\psi_0 \notin h_\delta(U)$ and using Theorem D we have $p(z) \prec h_\delta(z)$. A simple calculus shows that $\delta^*_{n+1} \leqslant \delta$ hence $p(z) \prec h_{\delta^*_{n+1}}(z)$ and the proof of the theorem is complete.

Remarks. 1°. For $\delta = \frac{1}{2}$ we obtain $\delta^*_{n+1} = \frac{1}{2}$ and the above result becomes Theorem 1 [1].

2° Theorem 1 shows that if $\frac{1}{2} \leqslant \delta < 1$ and $\alpha \geqslant 0$, the $C_{n,\alpha}(\delta) \subset AC_n(\delta^*_{n+1})$. Taking $\alpha = 1$ we obtain $AC_{n+1}(\delta) \subset AC_n(\delta^*_{n+1})$.

COROLLARY 1. *Let* $\frac{1}{2} \leqslant \delta < 1$ *and* $\alpha > \beta \geqslant 0$. *Then* $C_{n,\alpha}(\delta) \subset C_{n,\beta}(\delta^*_{n+1})$ *where* $\delta^*_{n+1} = 1/F\left(1,\ 2(n+3)(1-\delta),\ n+3;\ \frac{1}{2}\right)$.

Proof. If $\beta = 0$, using Theorem 1 we have

$$C_{n,\alpha}(\delta) \subset AC_n(\delta^*_{n+1}) = C_{n,0}(\delta^*_{n+1}).$$

If $\beta \neq 0$, using Theorem 1 and $\delta^*_{n+1} \leqslant \delta$ we obtain that if

$$f \in C_{n,\alpha}(\delta) \text{ then } \operatorname{Re}\left[(1-\beta)\frac{D^{n+1}f(z)}{D^{n+1}g(z)} + \beta\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right] =$$

$$= \frac{\beta}{\alpha}\left\{\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}g(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right] + \left(\frac{\alpha}{\beta} - 1\right)\operatorname{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)}\right\} >$$

$$> \frac{\beta}{\alpha}\left(\delta + \left(\frac{\alpha}{\beta} - 1\right)\delta^*_{n+1}\right) \geqslant \delta^*_{n+1}.$$

Let $\gamma \in C$ with $\operatorname{Re}\gamma > -1$ and $b_\gamma(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j$.

In [6], S. Ruscheweyh showed that if $\operatorname{Re}\gamma \geqslant \frac{n-1}{2}$ and $f \in Z_n\left(\frac{1}{2}\right)$ then $f * b_\gamma \in Z_n\left(\frac{1}{2}\right)$. Our next theorem presents a result concerning this function.

THEOREM 2. *Let* $\gamma > -1$ *and* $\delta_0 = \max\left\{\frac{\alpha-\gamma}{n+1},\ \frac{2n-\gamma}{2(n+1)}\right\} \leqslant \delta < 1$. *If* $f \in Z_n(\delta)$ *then* $f * b_\gamma \in Z_n(\tilde{\delta}(n,\gamma,\delta))$, *where*

$$\tilde{\delta}(n,\gamma,\delta) = \frac{1}{n+1}\left(\frac{\gamma+1}{F\left(1,\ 2(n+1)(1-\delta),\ \gamma+2;\ \frac{1}{2}\right)} - \gamma + n\right)$$

and this result is sharp.

Proof. Let $F(z) = f(z) * b_\gamma(z)$; using the well-known formulas [6]:

$$z(D^k f(z))' = (k+1)D^{k+1}f(z) - kD^k(z),\ k \in \mathbf{N}$$

$$z(D^k F(z))' = (\gamma+1)D^k f(z) - \gamma D^k f(z),\ \operatorname{Re}\gamma > -1,\ k \in \mathbf{N}$$

we obtain that $f \in Z_n(\delta)$ is equivalent to

$$p(z) + \frac{zp'(z)}{(n+1)p(z) + \gamma - n} \prec h_\delta(z), \text{ where } p(z) = \frac{D^{n+1}F(z)}{D^n F(z)}.$$

Considering the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma'}=h_{\delta}(z) \text{ where } \beta=n+1,\ \gamma'=\gamma-n$$

and using Theorem B we deduce that this equation has the univalent solution

$$q(z)=\frac{1}{\beta Q(z)}-\frac{\gamma'}{\beta}, \text{ where } Q(z)=\int_0^1\left(\frac{1-z}{1-}\right)^{2\beta(1-\delta)}t^{\beta+\gamma'-1}dt.$$

We also have $p(z)\prec q(z)$ and $q(z)$ is the best dominant.
Using a method similar to that of P. T. Mocanu, D. Ripeanu and I. Şerb [5] we show tha $\inf\{\operatorname{Re} q(z): z\in U\}=q(-1)$.

We use the following well-known formulas:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b,c;z), \text{ with } c>b>0$$

$$F(a,b,c;z)=F(b,a,c;z)$$

$$F(a,b,c;z)=(1-z)^{-a}F\left(a,\ c-b,\ c;\frac{z}{z-1}\right) \text{ which hold for all}$$

$z\in \mathbf{C}\setminus(1,+\infty)$.

If $\delta_0<\delta<1$, we denote $a=2\beta(1-\delta)$, $b=\beta+\gamma'$, $c-b=1$ and using the above relations we deduce

$$Q(z)=\frac{1}{\beta+\nu'}F\left(1,a,c;\frac{z}{z-1}\right).$$

Since $c>a>0$ we obtain that $Q(z)=\int_0^1\frac{1-z}{1-(1-t)z}d\mu(t)$, $z\in U$ where

$$d\mu(t)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)}t^{a-1}(1-t)^{c-a-1}\,dt \text{ is a positive measure.}$$

If we let $Q(z,t)=\dfrac{1-z}{1-(1-t)z}$, then $\operatorname{Re} Q(z,t)>0$,

$$Q(-r,t)\in\mathbf{R} \text{ for } 0\leqslant r<1,\ t\in[0,1] \text{ and } \operatorname{Re}\frac{1}{Q(z,t)}\geqslant\frac{1}{Q(-r,t)}$$

for $|z| \leqslant r < 1$, $t \in [0, 1]$. By using Theorem C we deduce

$$\operatorname{Re}\frac{1}{Q(z)} \geqslant \frac{1}{Q(-r)}, \quad |z| \leqslant r < 1$$

and by letting $r \to 1^-$ we have $\operatorname{Re}\frac{1}{Q(z)} > \frac{1}{Q(-1)}$, $z \in U$.

Then, by letting $\delta \to \delta_0^+$ we obtain our result.

Now we will prove that our result is sharp. If we let $\frac{D^{n+1}F(z)}{D^n F(z)} = q(z)$ we obtain $\frac{z(D^n F(z))'}{D^n F(z)} = (n+1)q(z) - n \equiv \tilde{q}(z)$, and letting $D^n F(z) = \varphi(z)$, $\varphi(0) = = \varphi'(0) - 1 = 0$ we deduce $\frac{z\varphi'(z)}{\varphi(z)} = \tilde{q}(z)$, $\tilde{q}(0) = 1$. This last differential equation has the regular solution $\varphi(z) = z \exp \int_0^z \frac{\tilde{q}(t) - 1}{t}\, dt$, hence $D^n F(z) = = z \exp (n+1) \int_0^z \frac{q(t) - 1}{t}\, dt \equiv G_n(z)$. Because $z(D^{n-1}F(z))' + (n-1)D^{n-1}F(z) = nD^nF(z)$ we deduce $D^{n-1}F(z) + \frac{1}{n-1} z(D^{n-1}F(z))' = \frac{n}{n-1} G_n(z)$, $G_n(0) = 0$, $n > 1$ hence $D^{n-1}F(z) = \frac{n}{z^{n-1}} \int_0^z G_n(t)t^{n-2}\, dt \equiv G_{n-1}(z)$. A simple calculus shows that

$$D^{n-2}F(z) = \frac{n-1}{z^{n-2}} \int_0^z G_{n-1}(t)\, t^{n-3}dt \equiv G_{n-2}(z)$$

$$D^1F(z) = \frac{2}{z} \int_0^z G_2(t)dt \equiv G_1(z),$$

and $F(z) = \int_0^z \frac{G_1(t)}{t}\, dt$; since $zF'(z) = (1+\gamma)f(z) - \gamma F(z)$ we conclude that $f(z) = \frac{1}{1+\gamma}(\gamma F(z) + zF'(z))$ is the extremal function and this completes the proof of our theorem.

COROLLARY 2. *If* $\gamma \geqslant \max\left\{\frac{n-1}{2},\ n-1\right\}$, *then* $f \in Z_n\left(\frac{1}{2}\right)$ *implies that* $f * b_\gamma \in Z_n\left(\tilde{\delta}\left(n, \gamma; \frac{1}{2}\right)\right)$, *where* $\tilde{\delta}\left(n, \gamma, \frac{1}{2}\right) = \frac{1}{n+1}\left[\frac{\gamma+1}{F\left(1, n+1, \gamma+2; \frac{1}{2}\right)} - \gamma + n\right] > \frac{1}{2}$, *and this result is sharp.*

Proof. Taking $\delta = \frac{1}{2}$ in Theorem 2 we obtain the first part of this corollary; the relation $\tilde{\delta}\left(n, \gamma, \frac{1}{2} > \frac{1}{2}\right)$ is equivalent to $F\left(1, n+1, \gamma+2; \frac{1}{2}\right) < \frac{2(\gamma+1)}{2\gamma-n+1}$ and a simple calculus shows that the last inequality holds.

Taking $n = 0$ and $n = 1$ in Theorem 2 we obtain respectively:

COROLLARY 3. *If* $\gamma > -1$, $\max\left\{-\gamma, -\frac{\gamma}{2}\right\} \leqslant \delta < 1$ *and* $f \in A$, *then* $\operatorname{Re}\frac{zf'(z)}{f(z)} > \delta$, $z \in U$ *implies* $\operatorname{Re}\frac{zF'(z)}{F(z)} > \tilde{\delta}(0, \gamma, \delta)$, $z \in U$, *where* $F(z) = f(z) * b_\gamma(z)$ *and* $\tilde{\delta}(0, \gamma, \delta) = \frac{\gamma+1}{F\left(1, 2(1-\delta), \gamma+2; \frac{1}{2}\right)} - \gamma.$

COROLLARY 4. *If* $\gamma > -1$, $\max\left\{\frac{1-\gamma}{2}, \frac{2-\gamma}{4}\right\} \leqslant \delta < 1$ *and* $f \in A$, *then* $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 2\delta - 1$, $z \in U$ *implies* $\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) > 2\tilde{\delta}(1, \gamma, \delta) - 1$, $z \in U$ *where* $F(z) = f(z) * b_\gamma(z)$ *and*

$$\tilde{\delta}(1, \gamma, \delta) = \frac{1}{2}\left[\frac{\gamma+1}{F\left(1, 4(1-\delta), \gamma+2; \frac{1}{2}\right)} - \gamma + 1\right].$$

REFERENCES

1. H. S. Al-Amiri, *Certain analogy of the α-convex functions*, Rev. Roum. Math. Pures Appl. **XXIII'** 10(1978), 1944–1454.
2. T. Bulboacă, *Applications of the Briot-Bouquet differential subordination*, Mathematica (Cluj), [to appear].
3. J. Feng, D. R. Wilken, *A remark on convex and starlike functions*, J. London Math. Soc., (2), **21** (1980), 287–290.
4. S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math., J., **28** (1981), 151–171.
P. T. Mocanu, D. Ripeanu, I. Serb, *The order of starlikeness of certain integral operators*, Mathematica (Cluj), **23** (46), Nr. 2 (1981), 225–230.
6. S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), 109–115.

STUDIA UNIV. BABEŞ BOLYAI, MATHEMATICA, XXXI, 4, 1986

STUDIUL RESTULUI ÎNTR-O FORMULĂ DE APROXIMARE A FUNCŢIILOR DE DOUĂ VARIABILE CU AJUTORUL UNUI OPERATOR DE TIP FAVARD-SZÁSZ

ALEXANDRU LEONTE* şi ION VÎRTOPEANU*

Intrat în redacţie: 10 iulie 1983

ABSTRACT. — Study on the Rest in an Approximation Formula of the Two-Variable Functions by means of a Favard-Szász-Type Operator. One class of positive, linear operators is constructed on the multitude of the functions $f: D \to R \to$, where $D = \{(x, y) : x \geqslant 0, y \geqslant 0\}$, by relation (2). An analysis is further made on the rest in an approximation formula of functions by means of these operators.

1. In [1] D. D. Stancu dă o metodă de construcţie a unei clase de operatori liniari pozitivi depinzînd de un parametru real α, definind pentru orice funcţie $f: I \to \mathbf{R}$, I fiind un interval al axei reale, aplicaţia

$$(L_m^{\langle\alpha\rangle} f)(x) = \frac{1}{\varphi_m^{\langle\alpha\rangle}(0)} \sum_{k=0}^{\infty} (-1)^k \frac{x^{[k,-\alpha]}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) f(x_{m,k}), \tag{1}$$

unde $x \in I = [0, a]$, $a > 0$, $x^{[k,-\alpha]} = x(x+\alpha)\dots(x+(k-1)\alpha)$, $x_{m,k} \in \in JCI$, $(\varphi_m^{\langle\alpha\rangle})$ fiind un şir de funcţii depinzînd de α, analitice într-un domeniu D care conţine discul $|z - a| \leqslant a$ şi care pot fi dezvoltate în serie Newton convergente pe D, D_α^k fiind operatori diferenţe Nörbund

$$D_\alpha^k g(x) = D_\alpha(D^{k-1} g(x)),\ D_\alpha g(x) = \frac{g(x+\alpha) - g(x)}{\alpha},\ D_\alpha^0 g(x) = g(x).$$

Luînd în particular

$$\varphi_m^{\langle\alpha\rangle}(x) = (1 + \alpha m)^{-\frac{x}{\alpha}},\ x_{m,k} = \frac{}{m}$$

pentru care

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^k \left(\frac{m}{1+\alpha m}\right)^k (1+\alpha m)^{-\frac{x}{\alpha}}$$

se obţine operatorul de tip Favard-Szász

$$(L_m^{\langle\alpha\rangle} f)(x) = \sum_{k=0}^{\infty} w_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}\right),$$

unde $w_{m,k}^{\langle\alpha\rangle}(x) = (1+\alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1+\alpha m}\right)^k \frac{x^{[x,-\alpha]}}{k!}$.

* *Universitatea din Craiova, Facultatea de Ştiinţe ale Naturii, 1100 Craiova, România*

2. În această lucrare construim o clasă de operatori liniari pozitivi pe mulţimea funcţiilor

$$f : D \to \mathbf{R}, \text{ unde } D = \{(x, y) : x \geqslant 0,\ y \geqslant 0\}$$

şi facem o analiză a restului într-o formulă de aproximare a funcţiilor prin aceşti operatori.

Fie α, β doi parametri reali, m, n numere naturale.

Definim operatorul $L_{m,n}^{\langle\alpha,\beta\rangle}$ prin

$$(L_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle}(y) f\left(\frac{k}{m}, \frac{j}{n}\right), \tag{2}$$

unde

$$v_{m,k}^{\langle\alpha\rangle}(x) = (1 + \alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1 + \alpha m}\right)^{k} \cdot \frac{x^{[k,-\alpha]}}{k!}, \tag{3}$$

$$w_{n,j}^{\langle\beta\rangle}(y) = (1 + \beta_n)^{-\frac{y}{\beta}} \left(\frac{n}{1 + \beta n}\right)^{j} \cdot \frac{y^{[j,-\beta]}}{j!} \tag{3'}$$

Este evident că operatorul $L_{m,n}^{\langle\alpha,\beta\rangle}$ este un operator liniar pozitiv pentru $\alpha > 0$, $\beta > 0$.

Notînd $$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = f(x, y) - (L_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y),$$

avem $$f(x, y) = (L_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) + (R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) \tag{4}$$

şi scopul este de a obţine forme lucrative ale restului $(R_{m,n}^{\langle\alpha,\beta\rangle} f)$ (x, y) în această formulă de aproximare.

3. Înainte de toate, fie funcţiile $l_{ij}(n, y) = x^i y^j$, $i = 0, 1, 2$; $j = 0, 1, 2$. Avem

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{00})(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle}(y) =$$

$$= \left[\sum_{k=0}^{\infty} (1 + \alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1 + \alpha n}\right)^{k} \cdot \frac{x(x + \alpha) \dots (x + (k - 1)\alpha)}{k!}\right] \times$$

$$\times \left[\sum_{j=0}^{\infty} (1 + \beta_n)^{-\frac{y}{\beta}} \left(\frac{n}{1 + \beta n}\right)^{j} \cdot \frac{y(y + \beta) \dots (y + (j - 1)\beta)}{j!}\right] = 1.$$

Un calcul similar, arată

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{10})(x, y) = x, \ (L_{m,n}^{\langle\alpha,\beta\rangle} l_{01}(x, y) = y,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{11})(x, y) = xy,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{02})(x, y) = y^2 + \frac{1 + \beta n}{n} y,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{20})(x, y) = x^2 + \frac{1 + \alpha n}{m} x,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{12})(x, y) = xy^2 + \frac{1 + \beta n}{n} x \cdot y, \ (L_{m,n}^{\langle\alpha,\beta\rangle} l_{21})(x, y) = x^2 y + \frac{1 + \alpha m}{m} xy,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{22})(x, y) = x^2 y^2 + \left(\frac{1 + \alpha m}{m} + \frac{1 + \beta n}{n}\right) xy(x + y) + \frac{(1 + \alpha m)(1 + \beta n)}{mn} x$$

Prin urmare

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{ij})(x, y) = 0, \text{ pentru } (i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\},$$

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{12})(x, y) = -\frac{1 + \beta n}{n} xy,$$

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{21})(x, y) = -\frac{1 + \alpha m}{m} xy,$$

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{22})(x, y) = -\left(\frac{1 + \alpha m}{m} + \frac{1 + \beta n}{n}\right) xy(x + y) - \frac{(1 + \alpha m)(1 + \beta n)}{mn} xy.$$

Din aceasta, rezultă că

$$\lim_{m,n \to \infty} (L_{m,n}^{\langle\alpha,\beta\rangle} l_{ij})(x, y) = l_{ij}(x, y), \text{ pentru}$$

$$i, j \in \{0, 1, 2\}.$$

4. Fie Y fixat. Folosind notațiile și formele (19) din [1], obținem

$$f(x, y) = (L_m^{\langle\alpha\rangle} f(\cdot, y))(x) + (R_m^{\langle\alpha\rangle} f(\cdot, y))(x),$$

cu

$$(L_m^{\langle\alpha\rangle} f(\cdot, y))(x) = \sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}, y\right) \text{ și}$$

$$(R_m^{\langle\alpha\rangle} f(\cdot, y))(x) = -\sum_{k=0}^{\infty} \frac{x + k\alpha}{m} v_{m,k}^{\langle\alpha\rangle}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right] =$$

$$= -\frac{1 + \alpha m}{m} x \sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x + \alpha) \left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right].$$

Dezvoltînd pe $f\left(\frac{k}{m}, y\right)$ după aceeași formulă, avem

$$f\left(\frac{k}{m}, y\right) = \sum_{j=0}^{\infty} w_{n,j}^{\langle\beta\rangle}(y) f\left(\frac{k}{m}, \frac{j}{n}\right) + \left(R_n^{\langle\beta\rangle} f\left(\frac{k}{m}, \cdot\right)\right)(y),$$

unde

$$\left(R_n^{\langle\beta\rangle} f\left(\frac{k}{m}, \cdot\right)\right)(y) = -\sum_{j=0}^{\infty} \frac{j + j\beta}{n} w_{n,j}^{\langle\beta\rangle}(y) \left[y, \frac{j}{n}, \frac{j+1}{n}; f\left(\frac{k}{m}, \cdot\right)\right].$$

Rezultă

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) \cdot w_{n,j}^{\langle\beta\rangle}\left(\frac{k}{m}, \frac{j}{n}\right) -$$

$$-\sum_{k=0}^{\infty} \frac{x + k\alpha}{m} v_{n,k}^{\langle\beta\rangle}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right] -$$

$$-\sum_{j=0}^{\infty} \frac{y + j\beta}{n} w_{n,j}^{\langle\beta\rangle}(y) \cdot \left[y, \frac{j}{n}, \frac{j+1}{n}: f(x, \cdot)\right] -$$

$$-\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(x + k\alpha)(y + j\beta)}{mn} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle}(y) \begin{bmatrix} x, & \frac{k}{m}, & \frac{k+1}{n} & \\ & & & ; f \\ y, & \frac{j}{n}, & \frac{j+1}{n} & \end{bmatrix}$$

și deci

$$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = -\sum_{k=0}^{\infty} \frac{x + k\alpha}{m} v_{m,k}^{\langle\alpha\rangle}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right] -$$

$$-\sum_{j=0}^{\infty} \frac{y + j\beta}{n} w_{n,j}^{\langle\beta\rangle}(y) \left[y, \frac{j}{n}, \frac{j+1}{n}; f(x, \cdot)\right] -$$

$$-\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(x + k\alpha)(y + j\beta)}{mn} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle} \cdot \begin{bmatrix} x, & \frac{k}{m}, & \frac{k+1}{m} & \\ & & & ; f \\ y, & \frac{j}{n}, & \frac{j+1}{n} & \end{bmatrix} \qquad (5)$$

unde $\begin{bmatrix} x, & \frac{k}{m}, & \frac{k+1}{m} & \\ & & & ; f \\ y, & \frac{j}{n}, & \frac{j+1}{n} & \end{bmatrix}$ reprezintă diferența divizată bidimensională a lui f pe sistemul de puncte indicate.

Presupunînd că diferențele divizate de ordinul al doilea ale lui f în raport cu x, pentru fiecare y fixat, și în raport cu y, pentru fiecare x fixat, și diferențele bidimensionale sînt egal mărginite de o aceeași constantă M, rezultă

$$|(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y)| \leqslant M\left(\frac{1+\alpha m}{m} x + \frac{1+\beta n}{n} y + \frac{1+\alpha n+\beta n+\alpha\beta mn}{mn}\right).$$

5. Să presupunem că funcția f are derivate parțiale continue pînă la ordinul al patrulea pe $(0, \infty) \times (0, \infty)$.

Atunci, conform [3 formula (11)], avem

$$(R^{\langle\alpha\rangle} f(\cdot, y))(x) = -\frac{1+\alpha m}{2m} x \frac{\partial^2 f}{\partial x^2}(\xi, y).$$

Reluînd calculul prin care am dedus formula (5), obținem:

$$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = -\frac{1+\alpha m}{2m} x \frac{\partial^2 f}{\partial x^2}(\xi, y) - \frac{1+\beta n}{2n} y \frac{\partial^2 f}{\partial y^2}(x, \eta) -$$
$$-\frac{(1+\alpha n)(1+\beta n)}{4mn} xy \frac{\partial^4 f}{\partial x^2 \partial y^2}(\xi, \eta). \quad (6)$$

6. În [1], D. D. Stancu arată că dacă funcția f: $[0, \infty) \to \mathbf{R}$ are derivate pînă la ordinul al doilea pe $[0, \infty)$, atunci restul $(R_m^{\langle\alpha\rangle} f)(x)$ din formula

$$f(x) = (L_m^{\langle\alpha\rangle} f)(x) + (R_m^{\langle\alpha\rangle} f)(x) \quad (7)$$

poate fi pus sub forma

$$(R_m^{\langle\alpha\rangle} f)(x) = \int_0^\infty G_m^{\langle\alpha\rangle}(t, x) f''(t) dt, \quad (8)$$

unde

$$G_m^{\langle\alpha\rangle}(t, x) = (R_m^{\langle\alpha\rangle} \psi_x)(t), \ \psi_x(t) = (x-t)_+.$$

$R_m^{\langle\alpha\rangle}\psi_x$ fiind restul relativ la variabila x, t fiind fixat.

Vom nota însă, prin analogie.

$$H_n^{\langle\beta\rangle}(\tau, \eta) = (R_n^{\langle\beta\rangle}\psi)(\tau).$$

Vom presupune că funcția f are derivate parțiale pînă la ordinul IV inclusiv. Aplicînd formulele (7), (8), pentru Y fixat, avem

$$f(x, y) = \sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}, y\right) + \int_0^\infty G_m^{\langle\alpha\rangle}(t, x) \frac{\partial^2 f}{\partial t^2}(t, y) dt. \quad (9)$$

Exprimînd cu această formulă $f\left(\frac{k}{m}, y\right)$, schimbînd rolul variabilelor, obţinem

$$f\left(\frac{k}{m}, y\right) = \sum_{j=0}^{\infty} w_{n,j}^{\langle\beta\rangle}(y) f\left(\frac{k}{m}, \frac{j}{n}\right) + \int_0^{\infty} H_n^{\langle\beta\rangle}(\tau, y) \frac{\partial^2 f}{\partial\tau^2}\left(\frac{k}{m}, \tau\right) d\tau \tag{10}$$

Formula (9), aplicată funcţiei $\frac{\partial^2 f}{\partial y^2}$, ne conduce la

$$\sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) \frac{\partial^2 f}{\partial\tau^2}\left(\frac{k}{m}, \tau\right) = \frac{\partial^2 f}{\partial\tau^2}(x, \tau) - \int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) \frac{\partial^{(IV)} f}{\partial t^2 \partial\tau^2}(t, \tau) d\tau \tag{11}$$

Inlocuind (10) şi (11) în (9), obţinem

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) \cdot w_{n,j}^{\langle\beta\rangle}(y) \cdot f\left(\frac{k}{m}, \frac{j}{n}\right) + \int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) \frac{\partial^2 f}{\partial t^2}(t, y) dt +$$

$$+ \int_0^{\infty} H_n^{\langle\beta\rangle}(\tau, y) \frac{\partial^2 f}{\partial\tau^2}(x, \tau) d\tau - \int_0^{\infty}\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) H_n^{\langle\beta\rangle}(\tau, y) \cdot \frac{\partial^{(IV)} f}{\partial t^2 \partial\tau^2}(t, \tau) dt d\tau,$$

ceea ce conduce la expresia integrală a restului din formula (4) :

$$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = \int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) \cdot \frac{\partial^2 f}{\partial t^2}(t, y) dt + \int_0^{\infty} H_n^{\langle\beta\rangle}(\tau, y) \cdot \frac{\partial^2 f}{\partial\tau^2}(x, \tau) d\tau -$$

$$- \int_0^{\infty}\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) H_n^{\langle\beta\rangle}(\tau, y) \cdot \frac{\partial^{(IV)}}{\partial t^2 \partial\tau^2}(t, \tau) dt d\tau \tag{12}$$

Sîntem recunoscători prof. D.D. Stancu, care ne-a îndemnat să întreprindem acest studiu şi ale cărui indicaţii ne-am înlesnit realizarea lui.

BIBLIOGRAFIE

1. Stancu, D. D., *A study of the remainder in an approximation formula using a Favard-Szász type operator*, Studia Univ. Babeş—Bolyai, Mathematica, XXV, 4, (1980), 70—76.
2. Stancu, D. D., *On the remainder of approximation of functions by means of a parameter-dependent linear polynomial operator*, Studia Univ. Babeş—Bolyai, Ser. Math., 1962, (1971), 59—66.
3. Stancu, F., *Asupra restului în formulele de aproximare prin operatorii lui Mirakyan de una şi două variabile*, Analele Şt. Univ. A. I. Cuza, Iaşi, Matem., 14 (1968), 415—422.
4. Favard, J., *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl., 23 (1944), 219—247.
5. Szász, O., *Generalization of S. Bernstein's polinomialis to the infinit interval*, J. Res. Nat. Bur. Standards, 45 (1950), 239—245.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

ON STRONGLY-STARLIKE AND STRONGLY-CONVEX FUNCTIONS

PETRU T. MOCANU*

Received: October, 8, 1986

REZUMAT. — Asupra funcţiilor tare stelate şi tare convexe. Rezultatul principal al lucrării este conţinut în următoarea teoremă.

TEOREMA 1. Dacă $0 < \alpha \leqslant 2$ şi f este o funcţie olomorfă în discul unitate U, $f(0) = f'(0) - 1 = 0$, care satisface condiţia (2), atunci

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha \frac{\pi}{2} \quad z \in U.$$

1. Introduction. Let A be the class of analytic functions f in the unit disc $U = \{z;\ |z| < 1\}$, which are normalized by $f(0) = f'(0) - 1 = 0$. A function $f \in A$ is called stronglystarlike (strongly-convex) of order α, $0 < \alpha \leqslant 1$, if $|\arg\ [zf'(z)/f(z)]\,| < \alpha\pi/2$ $(|arg[1 + zf''(z)/f'(z)]\,| < \alpha\pi/2)$, for $z \in U$. If $\alpha = 1$ these concepts reduce to the well-known concepts of starlikeness and convexity, respectively.

For $0 < \alpha \leqslant 1$ it is easy to show that each stronglyconvex function of order α is strongly — starlike of order α, i.e. the following implication

$$f \in A \text{ and} \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \alpha\frac{\pi}{2} \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2}$$

holds. In terms of subordination this implication can be written as follows

$$f \in A \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \Rightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (1)$$

where $0 < \alpha \leqslant 1$. This result fails if $\alpha > 1$.

In the case $\alpha = 1$, we improved (1) by the following „open door' theorem [3].

THEOREM A. *If* $f \in A$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}.$$

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Geometrically, Theorem A shows that if $1 + zf''(z)/f'(z)$ lies in the complex plane slit along the half-lines $u = 0,\ v \geq \sqrt{3}$ and $u = 0,\ v \leq -\sqrt{3}$, then $zf'(z)/f(z)$ lies in the right half-plane, i.e. the function f is starlike.

In this paper we improve (1) by the following result, which holds for all $\alpha \in (0, 2]$.

THEOREM 1. *If* $0 < \alpha \leq 2$ *and* $f \in A$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \tag{2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

2. **Preliminaries.** If F and G are analytic functions in U, then F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$.

We will need the following two lemmas to prove Theorem 1.

LEMMA 1 [1, p. 128]. *Let* q *be analytic and injective on* $U \setminus E(q)$, *where* $E(q) \subset \partial U$ *consists of a finite number of isolated singularities. Let* p *be analytic in* U, *with* $p(0) = q(0)$. *If there exist points* $z_0 \in U$ *and* $\zeta_0 \in \partial U \setminus E(q)$, *such that* $q'(\zeta_0) \neq 0$, $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(U)$, *then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

where $m \geq 1$.

LEMMA 2 *Let* P *be an analytic function in* U *such that*

$$P(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \equiv h(z), \quad 0 < \alpha \leq 2. \tag{3}$$

If p *is analytic in* U, $p(0) = 1$ *and satisfies the differential equation*

$$zp'(z) + P(z)p(z) = 1, \tag{4}$$

then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Proof. If we let $q(z) = [(1 - z)/(1 + z)]^{\alpha}$, then

$$h(z) = \frac{1}{q(z)} - \frac{zq'(z)}{q(z)}.$$

The domain $h(U)$ is symmetric with respect to the real axis. Therefore, if $z = e^{i\theta}$, then in order to obtain the boundary of $h(U)$ it is sufficient to suppose $0 \leq \theta \leq \pi$.

Letting ctg $(\theta/2) = t$ and $h(e^{i\theta}) = u + iv$, we find

$$\begin{cases} u = u(t) = At^{\alpha} \\ v = v(t) = Bt^{\alpha} + \dfrac{\alpha(1+t^2)}{2t}, \quad t \geqslant 0. \end{cases} \tag{5}$$

where $A = \cos(\alpha\pi/2)$ and $B = \sin(\alpha\pi/2)$.

If $\alpha = 1$, then $u = 0$ and $v \geqslant \sqrt{3}$ and we find that $h(U)$ is the complex plane slit along the half-lines $u = 0$, $v \geqslant \sqrt{3}$ and $u = 0$, $v \leqslant -\sqrt{3}$.

We note that $A > 0$ (i.e. $u > 0$), for $0 < \alpha < 1$ and $A < 0$ (i.e. $u < 0$), for $1 < \alpha \leqslant 2$. In the last two cases it is possible to eliminate the parameter t in (5) and to express v as a function of u. Actually we find

$$v = \frac{B}{A}u + \frac{\alpha}{2}\left[\left(\frac{u}{A}\right)^{1/\alpha} + \left(\frac{u}{A}\right)^{-1/\alpha}\right], \quad \begin{cases} u > 0, \text{ for } 0 < \alpha < 1 \\ u < 0, \text{ for } 1 < \alpha \leqslant 2 \end{cases} \tag{6}$$

We also have $v(0) = v(\infty) = \infty$.

In all cases we deduce that h is univalent in U.

Now we suppose that $0 < \alpha < 2$ and the solution p of (3) is not subordinate to q. Then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(U)$. Since $q(-1) = \infty$, it is clear that $\zeta_0 \neq -1$. Suppose $\zeta_0 = 1$. Then $p(z_0) = 0$ is the corner of the sector $q(U)$. If $p'(z_0) = 0$, by letting $z = z_0$ in (4), we obtain a contradiction. If $p'(z_0) \neq 0$ and $0 < \alpha < 1$, then the image by p of the circle $|z| = |z_0|$ cannot pass through the corner $w = 0$ without itself having a corner, which contradicts $p'(z_0) \neq 0$. If $p'(z_0) \neq 0$ and $1 \leqslant \alpha < 2$, then is it easy to show that

$$(3 - \alpha)\frac{\pi}{2} \leqslant \arg[z_0 p'(z_0)] \leqslant (1 + \alpha)\frac{\pi}{2},$$

which shows that Re $[z_0 p'(z_0)] \leqslant 0$. Hence, if we let $z = z_0$ in (4), we again obtain a contradiction. All the above contradictions show that $\zeta_0 \neq 1$. Therefore we can apply Lemma 1 to obtain $z_0 p'(z_0) = m\,\zeta_0 q'(\zeta_0)$ and from (4) we deduce

$$P(z_0) = \frac{1}{q(\zeta_0)} - \frac{m\,\zeta_0 q'(\zeta_0)}{q(\zeta_0)} \equiv Q(\zeta_0, m). \tag{7}$$

If we let $\zeta_0 = e^{i\theta}$, we can suppose $0 \leqslant \theta \leqslant \pi$. Letting ctg $(\theta/2) = t$, we obtain

$$Q(\zeta_0, m) = u(t) + iV(t),$$

where

$$V(t) = v(t) + \frac{(m-1)\,\alpha\,(1+t^2)}{2t}, \quad t \geqslant 0,$$

and $u(t)$, $v(t)$ are given by (5).

Since $m \geq 1$, we deduce $V(t) \geq v(t)$, which shows that $P(z_0) = Q(\zeta_0, m) \notin h(U)$. This contradicts the condition (3).
Therefore we must have $p \prec q$.

In the case $\alpha = 2$, the result can be obtained by a limiting procedure.

3. **Proof of Theorem 1.** Let $f \in A$ satisfy (2) and let $g(z) = zf'(z)$. From (2) we deduce $g(z)/z \neq 0$, for $z \in U$. Therefore the functions $p(z) = f(z)/g(z)$ and $P(z) = zg'(z)/g(z)$ are analytic in U. Moreover p satisfies the differential equation (4). Since the condition (2) is equivalent to (3), by Lemma 2 we deduce $p(z) \prec [(1-z)/(1+z)]^{\alpha}$, which is equivalent to $[zf'(z)/f(z)] \prec [(1+z)/(1-z)]^{\alpha}$. This completes the proof of Theorem 1.

The above proof shows that Theorem 1 can be stated in the following equivalent form.

THEOREM 2. *Let $0 < \alpha \leq 2$ and let $g \in A$ satisfy*

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2}.$$

If

$$f(z) = \int_0^z \frac{g(t)}{t}\, dt$$

then $f \in A$, $f(z)/z \neq 0$ and

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

If we integrate the differential equation

$$1 + \frac{zf''(z)}{f'(z)} = P(z),$$

we easily obtain another equivalent form of Theorem 1.

THEOREM 3. *If the analytic function P satisfies in U the condition* (3), *then*

$$\left|\arg \int_0^1 \left(\exp \int_z^{tz} \frac{P(w)-1}{w}\, dw\right) dt\right| < \alpha\frac{\pi}{2}, \text{ for } |z| < 1.$$

4. **Particular cases.**

a) For $\alpha = 1/2$ the equation (6) becomes

$$v = u + \frac{u^2}{2} + \frac{1}{8u^2}, \qquad u > 0.$$

It is easy to show that $v > 1$. Hence from Theorem 1 we deduce the following result.

COROLLARY 1.1. *If $f \in A$ and*

$$\left| \operatorname{Im} \frac{zf''(z)}{f'(z)} \right| \leqslant 1, \quad z \in U,$$

then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{4}, \quad z \in U.$$

For example, if we take $f(z) = e^z - 1$, we obtain

$$\left| \arg \frac{z}{e^z - 1} \right| < \frac{\pi}{4}, \text{ for } |z| < 1.$$

Using Theorem 3, we obtain the following equivalent form of Corollary 1.1.

COROLLARY 3.1. *If Q is analytic in* U, $Q(0) = 0$ *and* $|\operatorname{Im} Q(z)| \leqslant 1$ *in* U, *then*

$$\left| \arg \int_0^1 \left(\exp \frac{2}{\pi} \int_z^{tz} \frac{Q(w)}{w} dw \right) dt \right| < \frac{\pi}{4}, \text{ for } |z| < 1.$$

For example, if we take

$$Q(z) = \frac{2}{\pi} \log \frac{1+z}{1-z},$$

we obtain

$$\left| \arg \int_0^1 \left(\exp \frac{2}{\pi} \int_z^{tz} \frac{1}{w} \log \frac{1+w}{1-w} dw \right) dt \right| < \frac{\pi}{4}, \text{ for } |z| < 1.$$

b) For $\alpha = 1$ Theorem 1 reduces to Theorem A, In this case the equations (5) become

$$u = 0 \text{ and } v = t + \frac{1}{2}\left(t + \frac{1}{t}\right), \quad t > 0.$$

c) For $\alpha = 2$ the equation (6) becomes

$$v = \sqrt{-u} + \frac{1}{\sqrt{-u}}, \quad u < 0.$$

Since $v \geqslant 2$, from Theorem 1 we deduce the following result, COROLLARY 1.2. *If $f \in A$ and*

$$\left| \operatorname{Im} \frac{zf''(z)}{f'(z)} \right| < 2, \qquad z \in U,$$

then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \pi, \qquad z \in U.$$

Using Theorem 3, we obtain the following equivalent form of Corollary 1.2.

COROLLARY 3.2. *If Q is analytic in U, $Q(0) = 0$ and $|\operatorname{Im} Q(z)| < 2$ in U, then*

$$\left| \arg \int_0^1 \left\{ \exp \int_z^{tz} \frac{Q(w)}{w} dw \right\} dt \right| < \pi, \qquad \text{for } |z| < 1.$$

5. **Remark.** If $0 < \alpha \leqslant 1$, Theorem 1 is a particular case of a more general result recently obtained in [2], by using a „subordination chain" technique. The present proof is elementary; moreover for $1 < \alpha \leqslant 2$ Theorem 1 cannot be deduced from the result in [2], since the subordination chain condition is not satisfied in this case.

REFERENCES

1. S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28** (1981), 157–171.
2. S. S. Miller and P. T. Mocanu, *Marx-Strohhäcker differential subordination systems*, Proc. Amer. Math. Soc. (to appear).
3. P. T. Mocanu, *Some integral operators and starlike functions*, Rev. Roumaine Math. Pures Appl., **31**, 3 (1986), 231–235.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXI, 4, 1986

SIMPLEX-LIKE METHOD FOR THE PARETO MINIMUM SOLUTIONS OF AN INCONSISTENT SYSTEM

I. MARUSCIAC*

Received: October 12, 1986

REZUMAT. Metoda simplex pentru soluţii minime Pareto ale unui sistem inconsistent. În 1977 s-a definit o clasă de funcţii convexe în medie de ordinul α (αm-convexe) [2] şi s-a arătat că orice funcţie αm-convexă este pseudo convexă. În această notă se arată că în anumite condiţii orice funcţie pseudo convexă este αm-convexă cu un număr α determinat.

1. **Introduction.** Recently [6] we have defined Pareto minimum solutions of an inconsistent system and we have shown that there is a strong connection between these extremal approximate solutions of a system and a multicriterion optimization problem. In [6] we gave some properties of the Pareto minimum solutions of an inconsistent semi-infinite linear system.

In this paper we are going to present a simplex-like technique to generate extreme Pareto minimum solutions of an inconsistent linear system.

2. **Pareto minimum solutions of a system.** Let I be an index set and $f_i : \mathbf{R}^n \to \mathbf{R}$, $i \in I$. Consider the system

$$f_i(x) = 0, \; i \in I. \tag{1}$$

If $I = \{1, 2, \ldots, m\}$ and

$$f_i(x) = \sum_{j=1} a_{ij} x_j - b_i, \quad i \in I \tag{2}$$

then system (1) becomes a finite linear system:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i \in I \tag{3}$$

or

$$Ax = b, \tag{3}$$

where $A = (a_{ij})$ is a $m \times n$ real matrix and $b = (b_i)$ — a column matrix of the type $m \times 1$

If $I = \mathbf{N}$ and f_i are given in (2), the system (3) is a semiinfinit linear system.

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

DEFINITION 1. *Vector $x \in \mathbf{R}^n$ is called Pareto minimum solution (or Pareto minimum point) of the system* (1) *if there is no $y \in \mathbf{R}^n$ such that*

(i) $$\forall i \in I \Rightarrow |f_i(y)| \leqslant |f_i(x)|$$

(ii) $$\exists i_0 \in I, \qquad |f_{i_0}(y)| < |f_{i_0}(x)|$$

DEFINITION 2. *Vector $x \in \mathbf{R}^n$ is called weak Pareto minimum solution of the system* (1) *if there is no $y \in \mathbf{R}^n$ such that*

$$\forall i \in I \Rightarrow |f_i(y)| < |f_i(x)|$$

If we denote by $P(f, I)$, $P_w(f, I)$ the set of all Pareto minimum and weak Pareto minimum solutions of the system (1) respectively, then obviously

$$P(f, I) \subset P_w(f, I)$$

The converse inclusion generally does not hold (see [6]).

3. **Equivalent multicriterion optimizatuin problem.** Consider $X \subseteq \mathbf{R}^n$, $f: X \to \mathbf{R}^m$, $g: X \to \mathbf{R}^p$. Assume that

$$S = \{x \in X : g(x) \leqslant 0\} \neq \emptyset.$$

We remind that $x^0 \in S$ is called Pareto minimum solution (efficient solution) or Pareto minim point on S of the vectorminimization problem:

$$f(x) \to \min$$

subject to

$$g(x) \leqslant 0, \; x \in X$$

if there is no $x \in S$ such that

$$f(x) \leqslant f(x^0), \; f(x) \neq f(x^0).$$

Assume that $I = \{1, 2, \ldots, m\}$ and consider the following vector-minimization problem (V.P):

$$(u_1, u_2, \ldots, u_m) \to \min \tag{4}$$

subject to

$$|f_i(x)| \leqslant u_i, \; i \in I, \; x \in \mathbf{R}^n, \; u \in \mathbf{R}^+_m$$

THEOREM 1. ([6, *Theorem* **1**]). *$x_0 \in P(f, I)$ if and only if (x^0, u^0) is a Pareto minimum solution to the problem $(V.P)$, where*

$$u_0 = (|f_1(x_0)|, \ldots, |f_m(x^0)|)$$

In what follows we shall deal with the case when

$$f_i(x) = \sum_{j=0}^{n} a_{ij} x_j - b_i, \; i \in I = \{1, 2, \ldots, m\}$$

i.e. we shall consider the Pareto minimum solutions of the system

$$Ax = b, \tag{6}$$

where $A \in \mathcal{M}_{m\times n}(\mathbf{R})$ and $b \in \mathcal{M}_{m\times 1}(\mathbf{R})$.

We denote by $P(A, b)$ the set of all Pareto minimum solutions of the system (6).

In the present case the problem (V.P) becomes a Pareto linear program (PLP) : minimize $u \in \mathbf{R}^m$ subject to

$$|Ax - b| \leqslant u,\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^m_+$$

From Theorem 1 it follows

Corollary 1 shows that to find $x^0 \in P(A, b)$ is equivalent to find a Pareto minimum solution to the vector minimization problem (PLP).

DEFINITION 3. *An approximate solution* $x^* \in \mathbf{R}^n$ *of the system* (*1*) *is called the least squares solution of* (*1*) *if*

$$\sum_{i=1}^{m} f_i^2(x^*) = \inf_{x\in\mathbf{R}^n} \sum_{i=1}^{m} f_i^2(x)$$

THEOREM 2. *Problem* (PLP) *is always consistent and* $P(A, b) \neq \emptyset$ *for each matricsis* A *and* b.

Proof. Consider $x^* \in \mathbf{R}^n$ the least squares solution to (6), i.e. a solution of the system

$$\sum_{i=1}^{m} f_i(x)a_{ij} = 0,\ j = 1, 2, \ldots, n$$

which always exists. Then

$$(x^*, u_1^*, u_2^*, \ldots, u_m^*) \in \mathbf{R}^{n+m},$$

where

$$u_i^* = |f_i(x^*)|,\ i = 1, 2, \ldots, m$$

is a feasible solution to the problem (PLP), since $u_i^* \geqslant 0,\ i = 1, 2, \ldots, m$.

But if x^* is the least squares solution to (6), then $x^* \in P(A, b)$. Indeed, if $x^* \notin P(A, b)$, then there is $x \in \mathbf{R}^n$ such that (i)—(ii) are satisfied. Then

$$\sum_{i=1}^{m} f_i^2(x) < \sum_{i=1}^{m} f_i^2(x^*),$$

which contradicts the fact that x^* is the least squares solution to the system (6).

4. **Optimality criteria.** To solve the problem (PLP) it is convenient to express our multiobjective programming problem in th following simplex-like tableau

$$\begin{array}{r|cc|c|} & -x & -u & 1 \\ \hline y = & A & -E_m & b \\ \bar{y} = & -A & -E_m & -b \\ \hline f = & 0 & -E_m & 0 \\ \hline \end{array} \tag{7}$$

where $E_m \in \mathcal{M}_{m\times m}$ (**R**) is the unit matrix and $\bar{y} = (y_{m+1} \ldots y_{2m})^T$.

Without loss of generality, we assume that

$$\operatorname{rank} A = n,$$

otherwise system (6) is generally consistent and every its solution is always a Pareto minimum solution (see [6, Theorem 4])

Then, after n Jordan elimination steps (J.e.s.) we can eliminate variables x. Assume that $x_1, x_2, \ldots, x_n$ were eliminated from the first n lines. Then we get the tableau

$$\begin{array}{r|cc|c|} & -y_1 \ldots -y_n & -u_1 \ldots -u_m & 1 \\ \hline y_{n+1} = & & & \\ \vdots & & & \\ y_m = & & A_1 & b_1 \\ y_{m+1} = & & & \\ \vdots & & & \\ y_{2m} & & & \\ \hline f = & 0 & -E_m & 0 \\ \hline \end{array}$$

in which we have omitted the first n lines corresponding to the variables x, writing separately

$$x = B^{-1}b' - B^{-1}y', \tag{8}$$

where

$$B = (a_{ij})_{i,j=1}^{n}, \quad b' = (b_1 \ldots b_n)^T, \quad y' = (y_1 \ldots y_n)^T$$

Continuing with the first stage of the simple algorithm, to determine a basic feasible solution (b.f.s.) to the problem (PLP) (that in view of Theorem 2 exists) we get the tableau

$$\begin{array}{r|c|c|} & -z & 1 \\ \hline v = & D & d \\ \hline f = & C & c \\ \end{array} \tag{9}$$

corresponding to the canonical vector-minimization problem (CPLP):

$$f(z) = -Cz + c \to \min$$

$$Dz \leqslant d,\ z \geqslant 0$$

where $D \in \mathcal{M}_{(2m-n)\times(n+m)}$, $C \in \mathcal{M}_{m\times(n+m)}$ and $d \in \mathbf{R}^{2m-n}$, $c \in \mathbf{R}^m$.

To simplify the notation, we have denoted by $z \in \mathbf{R}_+^{n+m}$ and $v \in \mathbf{R}_+^{2m-n}$ the nonbasic and basic variables of the problem (CPLP) respectively.

THEOREM 3. [4, *Theorem 1*]. *Let* $(0, d)$ $(d \geqslant 0)$ *be a b.f.s. given in* (9) *and let* $Q = \{i : d_i = 0\}$. *Then* $(0, d)$ *is Pareto minimum solution to* (CPLP) *if and only if*

$$\begin{aligned} &Cu \geqslant 0,\ Cu \neq 0 \\ &D_Q u \leqslant 0,\ u \geqslant 0 \end{aligned} \tag{10}$$

is inconsistent, where

$$D_Q = (d_{ij}),\ i \in Q,\ j = 1, 2, \ldots, n + m$$

Remark 1. If $(0, d)$ is a non-degenerate b.f.s., then $Q = \emptyset$, and (10) becomes

$$Cu \geqslant 0,\ Cu \neq 0,\ u \geqslant 0 \tag{10'}$$

We denote by

$$a^{i\cdot} = (a_{i1}, a_{i2}, \ldots, a_{in}),\ a^{\cdot j} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$$

the row-vector and column-vector of the matrix $A = (a_{ij}) \in \mathcal{M}_{m\times m}$ respectively.

Now, let $i_1, i_2, \ldots, i_k$ $(1 \leqslant k \leqslant m)$ be distinct numbers of I such that

$c_{i_1 j} \leqslant 0,\ j \in J_0 = \{1, 2, \ldots, n + m\},\ J_1 = \{j \in J_0 : c_{i_1 j} = 0\}$

$c_{i_2 j} \leqslant 0,\ j \in J_1,\ J_2 = \{j \in J_1 : c_{i_2 j} = 0\}$

. . . .

$c_{i_k j} \leqslant 0,\ j \in J_{k-1}, J_k = \{j \in J_{k-1} : c_{i_k j} = 0\}$.

Obviously

$$J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots \supseteq J_k$$

THEOREM 4. *If* $J_0, J_1, \ldots, J_{k-1}$ *are nonempty and* $J_k = \emptyset$, *then* $(0, d)$ *is a Pareto minimum solution to* (*CPLP*).

Proof. If $(0, d)$ is not a Pareto minimum solution to (CPLP) then there is $(\bar{z}, \bar{v})$ such that

$$C\bar{z} \geqslant 0,\ C\bar{z} \neq 0. \tag{11}$$

Let $H = \{j \in J_0 : \bar{z}_j > 0\}$. From (11) it is clear that $\emptyset \neq H \subseteq J_1$. Indeed, if there is $h \in H \setminus J_1$, then from $c_{i_1 h} < 0$ it follows

$$\sum_{j \in H} c_{i_1 j} \bar{z}_j < 0$$

contradicting (11). Therefore $H \subseteq J_1$.

Denote by

$$s = \max \{h \leqslant \{1, 2, \ldots, k\} : H \subseteq J_h\}$$

Then we have $1 \leqslant s \leqslant k$, $J_{s+1} \neq J_s$.

Since

$$c_{i_{s+1} j} < 0, \ j \in J_s \setminus J_{s+1}$$

it follows

$$c^{i_{s+1}} \cdot \bar{z} = \sum_{j \in J_s} c_{i_{s+1}} \bar{z}_j < 0$$

which again contradicts (11).

From the proof of Theorem 4 it follows

COROLLARY 1. *If* $J_0, J_1, \ldots, J_{m-1}$ *are non empty, then* $(0, d)$ *is a Pareto minimum solution to* $(CPLP)$.

COROLLARY 2. *Let* $c^i \leqslant 0$ *and* $J_1 = \{j \leqslant J_0 : c_{ij} = 0\}$.
If there is $s \in I \setminus \{i\}$ *such that*

$$\forall j \in J_1 \Rightarrow c_{sj} < 0$$

then $(0, d)$ *is a Pareto minimum solution to* $(CPLP)$.

5. **Description of the algorithm.** The general outline of the algorithm following from Theorems 3 and 4 is as follows.

Step 0. Starting from (7) eliminate variables x and construct Tableau (9).

Step 1. Starting from Tableau (9) proceed to a b.f.s.

Step 2. Set $i := 1$, $i_i := i_1$, $J_i := J_0 = \{1, 2, \ldots, n + m\}$

Step 3. Minimize

$$f_{i_i}(z) = - \sum_{j \in J_i} c_{ij} z_j \text{ on } S = \{z : Dz \leqslant d, \ z \geqslant 0\}$$

Step 4. Set $i := i + 1$; $i_i := i_{i+1}$, $J_i := J_{i+1} = \{j \in J_i : c_{ij} = 0\}$.

Step 5. If $J_i = \emptyset$ or $i + 1 \geqslant m + 1$, then go to Step 6, else go to Step 3.

Step 6. Calculate x^* from (8) for $(v, z) = (0, d)$ and terminate.

Remark 2. To minimize $f_i(z)$ at the Step 3 we can use the simplex algorithm corresponding to the first $2m - n$ rows, objective function f_i in the Tableau (9) and column $j \in J_i$.

Remark 3. To compute another Pareto minimum solution to the system (6) we have to itrate the algorithm by changing the initial index i_1 in the Step 2.

Remark 4. To generate all extreme Pareto minimum solutions to (6) we can apply the method given in [3], by taking, for instance, supercriterion

$$F(x, u) = \sum_{i=1}^{m} u_i.$$

6. **A numerical example.** Consider the following inconeistent linear system

$$\begin{aligned} x_1 \quad\; &= 0 \\ x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The initial simplex tableau is the following

	$-x_1$	$-x_2$	$-u_1$	$-u_2$	$-u_3$	1
$y_1 =$	1	0	−1	0	0	0
$y_2 =$	0	1	0	−1	0	0
$y_3 =$	1	1	0	0	−1	1
$y_4 =$	−1	0	−1	0	0	0
$y_5 =$	0	−1	0	−1	0	0
$y_6 =$	−1	−1	0	0	−1	−1
$f_1 =$	0	0	−1	0	0	0
$f_2 =$	0	0	0	−1	0	0
$f_3 =$	0	0	0	0	−1	0

Step 0. Eliminating x_1 and x_2, after two Jordan steps (J.S) we get the tableau

	$-y_1$	$-y_2$	$-u_1$	$-u_2$	$-u_3$	1
$y_3 =$	−1	−1	1	1	−1	1
$y_4 =$	1	0	−2	0	0	0
$y_5 =$	0	1	0	−2	0	0
$y_6 =$	1	1	−1	−1	−1	−1
$f_1 =$	0	0	−1	0	0	0
$f_2 =$	0	0	0	−1	0	0
$f_3 =$	0	0	0	0	−1	0

$$x_1 = -y_1 + u_1$$
$$x_2 = -y_2 + u_2$$

Step 1. After one J.s. we get the tableau

	$-y_1$	$-y_2$	$-u_1$	$-y_6$	$-u_3$	1
$y_3 =$	0	0	0	1	−2	0
$y_4 =$	1	0	−2	0	0	0
$y_5 =$	−2	−1	2	−2	2	2
$u_2 =$	−1	−1	1	−1	1	1
$f_1 =$	0	0	−1	0	0	0
$f_2 =$	−1	−1	1	−1	1	1
$f_3 =$	0	0	0	0	−1	0

which gives a b.f.s. $y_1 = y_2 = u_1 = y_6 = u_3 = 0$; $y_3 = y_4 = 0$, $y_5 = 2$, $u_2 = 1$.

Step 2. Set $i_1 := 1$, $J_1 = \{1, 2, 4, 5\}$.

Step 3. Minimize

$$y_1 + y_2 + y_6 - u_3 \qquad (f_2(z) \to \min)$$

After one J.s. we get the tableau

	$-y_1$	$-y_2$	$-u_1$	$-y_6$	$-u_2$	1
$y_3 =$	−2	−2	2	−1	2	2
$y_4 =$	1	0	−2	0	0	0
$y_5 =$	0	1	0	0	−2	0
$u_3 =$	−1	−1	1	−1	1	1
$f_1 =$	0	0	−1	0	0	0
$f_2 =$	0	0	0	0	−1	0
$f_3 =$	−1	−1	1	−1	1	1

Step 4. $i_2 := 2$, $J_2 = \{1, 2, 4\}$

Step 5. $J_3 = \emptyset$.

Step 6. A Pareto minimum solution is given by the b.f.s.

$$y_1^0 = y_0^2 = u_1^0 = y_6^0 = u_2^0 = 0$$

$$y_3^0 = 2,\ y_4^0 = 0,\ y_5^0 = 0,\ u_3^0 = 1$$

i.e.

$$x_1^0 = -y_1^0 + u_1^0$$

$$x_2^0 = -y_2^0 + u_2^0$$

Therefore $x^0 = (0, 0)$ is the first extreme Pareto minimum solution of the given system.

Iterating the algorithm by taking $i_1 := 2$, after one J.s. we get another extreme Pareto minimum solution $x^1 = (1, 0)$, which is written from the tableau

	$-y_1$	$-y_2$	$-u_3$	$-y_6$	$-u_2$	1
$y_3 =$	0	0	-2	1	0	0
$y_4 =$	-1	-2	2	-2	2	2
$y_5 =$	0	1	0	0	-2	0
$u_1 =$	-1	-1	1	-1	1	1
$f_1 =$	-1	-1	1	-1	1	1
$f_2 =$	0	0	0	0	-1	0
$f_3 =$	0	0	-1	0	0	0

in wich $i_2 := 3$ and $J_1 = 1,2,3,4$, $J_2 = \emptyset$.

The last iteration of the algorithm, starting with $i_1 := 3$ and taking $i_2 := 1$, $i_3 := 2$, gives the last extreme Pareto minimum solution $x^2 = (0,1)$.

REFERENCES

1. J. P. Evans and R. E. Steuer, *A revised simplex method for linear multiobjective programs*. Mathematical Programming 5 (1973), 54–72.
2. I. Maruşciac and M. Rădulescu, *Un problème général de la programmation linéaire à plusieurs fonctions économiques*, Studia Univ. Babeş–Bolyai, ser. Math. Mech. 2 (1970), 55–65.
3. I. Maruşciac, *On the hierarchy of the efficient extreme points in multiobjective programming*. Studia Univ. Babeş–Bolyai, Mathematica, 1, (1977), 53–60.
4. I. Maruşciac, *An efficient realization of an algorithm to compute the first efficient point of a linear multiobjective programm*. Studia Univ. Babeş–Bolyai, Mathematica, (1978), 66–72.
5. I. Maruşciac, *Infrapolynomials and Pareto optimization*, Mathematica 22 (45) (1980), no. 2, 297–307.
6. I. Maruşciac, *On infrasolutions and Pareto minimum solutions of an inconsistent system*. Babeş–Bolyai Univ. Fac. of Mathem. Research Seminaries, Optimization Theory, no. 5 (1985), 1–22.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

NUMERICAL METHODS IN FUZZY HIERARCHICAL PATTERN RECOGNITION

I. Cluster Substructure of a Fuzzy Class

D. DUMITRESCU*

Received: October 15 1986

ABSTRACT. — In Part I, a multilevel fuzzy classification is introduced. The cluster substructure of a fuzzy class is described by a fuzzy partition of this class. A refinement relation between fuzzy partitions is defined. Some convexity properties for fuzzy partitions are given. A generalization of the Fuzzy ISODATA clustering algorithm is developed. A stratified classification may be obtained using this algorithm.

In Part II, a fuzzy hierarchy is defined. A divisive clustering algorithm to obtain a binary fuzzy hierarchy is given. The algorithm represents an effective technique for identifying the cluster structure of a data set.

1. **Introduction.** This paper presents a fuzzy hierarchical approach of the pattern recognition problem. The main task in pattern recognition is the class identification. The most real-world classes are fuzzy in nature. The classical sets are therefore not appropriate to describe such classes. A class of patterns may be conceived as a fuzzy set. The cluster structure of a collection X of patterns will be given by a set of disjoint fuzzy sets which form a fuzzy partition of X. The cluster substructure of a fuzzy class C may be described by a fuzzy partition of C. The hierarchical structure of X is given by a chain of fuzzy partitions ordered by the refinement relation. This chain generates a binary fuzzy hierarchy. Hierarchies may be obtained agglomeratively or divisively. In this paper a fuzzy divisive procedure to build a fuzzy hierarchy is developed. The classes are subdivided as long as is necessary to produce the final objective classification.

A decomposition criterion which permits to retain in the hierarchy only „real" clusters is used. In this way no *a priori* knowledge concerning the optimal number of clusters is required. The method gives therefore a solution of the cluster validity problem.

The method is more powerful than the one-level classification methods because it permits a more intimate exploration of the cluster substructure. No evaluation of a validity functional [1, 4, 14, 15] is needed. Our approach is essentially different from the hierarchial clustering methods based on fuzzy relations. In the method of Bezdek and Harris [2], for example, fuzzy relations are used to obtain classical hierarchies. As the author knows, the present procedure is the unique which produces a hierarchy of fuzzy classes.

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

2. **Prerequisites.** Let $X = \{x_1, \ldots, x_p\}$ be a set of patterns. Every pattern x_i is specified by the values of d features. $x_{ij} \in \mathbf{R}$ represents the value of the j-th feature with respect to $x_i \cdot x_i$ may be thus considered as a vector (or point) in $\mathbf{R}^d$.

A fuzzy set on X is a function $A : X \to [0, 1]$. We denote by $L(X)$ the class of all fuzzy sets on X. The set operations of fuzzy sets are defined using the triangular norms (t-norms) and t-conorms (see for instance) [11]). In this paper we consider the t-norm $T(x, y) = \max(x + y - 1, 0)$ and the t-conorm $S(x, y) = \min(x + y, 1)$.

Let A and B be two fuzzy sets from $L(X)$. The reunion $A \cup B$ is defined by

$$(A \cup B)(x) = \min(A(x) + B(x), 1), \qquad \forall x \in X.$$

The intersection $A \cap B$ is defined by

$$A \cap B)(x) = \max(A(x) + B(x) - 1, 0), \qquad \forall x \in X$$

The inclusion on $L(X)$ is defined as usually

$$A \subseteq B \text{ if } A(x) \leqslant B(x), \qquad \forall x \in X.$$

The family $A_1, \ldots, A_n$, $n \geqslant 2$, of fuzzy sets is called disjoint [5] iff

$$\left(\bigcup_{i=1}^{j} A_i\right) \cap A_{j+1} = \emptyset, \qquad j = 1, \ldots, n-1,$$

where $\emptyset(x) = 0, \qquad \forall x \in X.$

The family $A_1, \ldots, A_n$ of fuzzy sets is said to be a fuzzy partition of the fuzzy set C iff it is disjoint and its reunion is just C. A_i is an atom or member of the partition. It is not difficult to prove [5] that the family $A_1 \ldots, A_n$ is a fuzzy partition of C iff $\sum_i A_i(x) = C(x)$, $\forall x \in X$. For $C = X$ this equality is just Ruspini's definition of a fuzzy partition. We denote by $F(C)$ ($F_n(C)$) the class of all fuzzy partitions of C (having n atoms).

Let P, Q be from $F(C)$. Q is said to be a *refinement* of P, $P \prec Q$, iff every atom of P is a reunion of some atoms of Q.

It is easy to see that if $P = \{A_1, \ldots, A_n\}$, $P \in F_n(C)$ and $Q_i \in F(A_i)$, then $\{Q_1, \ldots, Q_n\} = Q \in F(C)$ and $P \prec Q$. The refinement relation is an order relation on $F(C)$ [11].

3. **Convexity propeties.** Let M_{np} be the linear space of real $(m \times p)$ matrices. Any fuzzy partition $P = \{A_1, \ldots, A_n\}$, $P \in F_n(C)$ may be characterized by matrices in M_{np}. Let m_{ij} be the ij-th element of the matrix M and define

$$U_n(C) = \left\{M \in M_{np} \,|\, m_{ij} \in [0, 1], \sum_{j=1}^{n} m_{ij} = C(x_j),\ \forall j\right\}.$$

There is an isomorphism $f: F_n(C) \to U_n(C)$, defined by $f(P) = M$ where $m_{ij} = A_i(x_j)$. Throughout this section we identify a fuzzy partitions with the matrix associated to it by this isomorphism.

A fuzzy partition is called non-degenerate iff none of its atoms is empty, i.e. $\sum_j A_i(x_j) > 0$, for every i. P is degenerate iff $\sum_j A_i(x_j) \geqslant 0$, for each i. Let $F_{n0}(C)$ be the set of all degenerate fuzzy partitions of C having n atoms. We denote by by $F_{nh}(X)$ $(F_{nh0}(X))$ the set of all non-degenerate (degenerate) classical or hard partitions of X. Using the established isomorphism we may speak about the convex combination of fuzzy partitions. We are now able to state the next convexity property.

PROPOSITION 1. *The sets $F_n(C)$ and $F_{n0}(C)$, where $C \in L(X)$, $C \neq \emptyset$, are convex.*

Proof. Let us consider $P_j = \{A_1^j, \ldots, A_n^j\}$, $P_j \in F_n(C)$, and $a_1, \ldots, a_k \geqslant 0$, $\sum_{j=1}^{k} a_j = 1$. We define the convex combination $B_i(x) = \sum_{j=1}^{k} a_j A_i^j(x)$ and denote $Q = \{B_1, \ldots, B_n\}$. We have thus $\sum_{i=1}^{n} B_i(x) = \sum_{i=1}^{n} \sum_{j=1}^{k} a_j A_i^j(x) = \sum_{j=1}^{k} a_j \sum_{i=1}^{n} A_i^j(x) = \sum_{j=1}^{k} a_i \cdot C(x) = C(x)$, for every x from X. It follows thus that Q is in $F_n(C)$.

It is not difficult to see that the convex hull of $F_{nh}(X)$ is a subset of $F_n(X)$. The inclusion conv $F_{nh}(X) \subset F_n(X)$ is strict [3]. The next proposition proves this affirmation. It gives a necessary and sufficient condition that a fuzzy partitions from $F_n(X)$ admits a convex decomposition with non-degenerate hard partitions.

PROPOSITION 2. *Let $P = \{A_1, \ldots, A_n\}$ be from $F_n(X)$. P is in* conv $F_{nh}(X$ *if and only if* $\sum_{x \in X} A_i(x) \geqslant 1$, $i = 1, \ldots, n$.

Proof. Necessity. If $P \in \text{conv} F_{nh}(X)$ then there exist $a_1, \ldots, a_k \geqslant 0$, $\sum_{j=1}^{k} a_j = 1$ and $Q_j = \{B_1^j, \ldots, B_n^j\}$, Q_j from $F_{nh}(X)$, $j = 1, \ldots, k$, such that $A_i(x) = \sum_{j=1}^{k} a_j B_i^j(x)$, for every x from X. Q_i is non-degenerate and thus $\sum_x B_i^j(x) \geqslant 1$, for every i, j. We may write

$$\sum_x A_i(x) = \sum_{j=1}^{k} a_j \sum_x B_i^j(x) \geqslant \sum_{j=1}^{k} a_j = 1.$$

For sufficiency, an algorithm for the convex decomposition of every fuzzy partitions P with $\sum_{i=1}^{n} A_i(x) \geqslant 1$ has been elaborated. Every partition in the obtained convex decomposition is non-degenerate. Because of its technical character this algorithm is omitted here. It will be presented in a further paper.

Remark. The theorem gives „the additional property P in $F_n(X)$ needs to distinguish it as a member of conv $F_{nh}(X)$" required in [3]. It may play a central role in clustering by convex decomposition.

Example. Let us consider $X = \{x_1, x_2, x_3\}$ and $P = \{A_1, A_2\}$ the fuzzy partition of X given by $A_1(x_j) = \lambda$, $\forall j$, $A_2(x_j) = 1 - \lambda$, $\forall j$. The associated matrix is

$$f(P) = \begin{pmatrix} \lambda & \lambda & \lambda \\ 1-\lambda & 1-\lambda & 1-\lambda \end{pmatrix}.$$

According with Proposizion 2, P admits a convex decomposition with non-degenerate hard partitions iff $3\lambda \geqslant 1$ and $3(1-\lambda) \geqslant 1$, i.e. $\lambda \in [1/3, 2/3]$.

For $\lambda = 1/2$ the decomposition is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} +$$

$$+ \frac{1}{6}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This result contradicts the affirmation made in [3] that P (or $f(P)$) is not in conv $F_{2h}(X)$.

4. Multilevel classification. The cluster structure of the set of patterns $X = \{x_1, \ldots, x_p\}$, $x_i \in \mathbf{R}^d$, may be described by a fuzzy partition of X. A class of patterns (or a cluster) corresponds to an atom A_i of a fuzzy partition P of X. In the following sections, we'll refer to the atom A_i as fuzzy class or the cluster A_i.

In our two-level classification model the cluster substructure of the fuzzy class A_i is given by a fuzzy partition of A_i. We may consider a multilevel fuzzy classification in which the cluster substructure of a fuzzy class C from a level l is described by a fuzzy partition P of C. The atoms of P belong to the level $l + 1$.

Let $P = \{A_1, \ldots, A_n\}$ be a fuzzy partition of the fuzzy class C. A fuzzy class A_i is represented by a prototype $L_i \in \mathbf{R}^d$. $D(x_j, L_i)$ denotes a dissimilarity index measuring the degree in which x_j differs from the prototype L_i. D is a function $D : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ such that

(i) $$D(x, y) \geqslant 0, \quad D(x, x) = 0, \qquad \forall x, y.$$

(ii) $$D(x, y) = D(y, x). \qquad , \qquad \forall x, y.$$

D may be a squared distance on $\mathbf{R}^d$.

In order to define a local dissimilarity D_i which depends on A we consider a distance d on $\mathbf{R}$ given by

$$d_i(x, y) = \begin{cases} \min\,(A_i(x), \quad A_i(y))\, d(x, y), & \text{if } x, y \in X \\ A_i(x)\, d(x, y) & \text{if } x \in X,\ y \neq X \\ d(x, y) & \text{if } x, y \notin X \end{cases}$$

The local dissimilarity is thus

$$D_i(x, y) = d_i^2(x, y), \qquad \forall x, y.$$

If $L_i \in X$ then $A_i(L_i) = \max\limits_{x \in X} A_i(x)$ and therefore we may write

$$D_i(x_j, L_i) = d_i^2(x_j, L_i) = (A_i(x_j))^2 d^2(x_j, L_i).$$

The inadequacy $I(A_i, L_i)$ between the fuzzy class A_i and its prototype L_i may be defined as

$$I(A_i, L_i) = \sum_{j=1}^{p} D_i(x_j, L_i) = \sum_{j=1}^{p} (A_i(x_j))^2 d^2(x_j, L_i).$$

The inadequacy $J(P, L)$ between the partition P and its representation $L = (L_1, \ldots, L_n)$ is given by

$$J(P, L) = \sum_{i=1}^{n} I(A_i, L_i) = \sum_{i=1}^{n} \sum_{j=1}^{p} (A_i(x_j))^2\, d^2(x_j, L_i),$$

where J is a function $J : F_n(C) \times \mathbf{R}^{dn} \rightarrow \mathbf{R}$.

The detection of the cluster substructure of the fuzzy class C reduces to search for the partition $P \in F_n(C)$ and its representation $L \in \mathbf{R}^{dn}$ which minimize $J(P, L)$. The optimal fuzzy partition is thus obtained as the solution of the minimization problem:

$$\begin{cases} \text{minimize } J(P, L) \\ P \in F_n(C) \\ L \in \mathbf{R}^{dn} \end{cases} \tag{1}$$

The next proposition gives a local solution of this problem:

PROPOSITION 3. i) $P \in F_n(C)$ *is a minimum of the function* $J(\cdot, L)$ *if and only if*

$$A_i(x_j) = \frac{C(x_j)}{\sum\limits_{k=1}^{n} \frac{d^2(x_j, L_i)}{d^2(x_j, L_k)}}, \qquad \forall i, j. \tag{2}$$

ii) *$L \in \mathbf{R}^{dn}$ is a minimum of the function $J(P,\cdot)$ if and only if*

$$L_i = \frac{\sum_{j=1}^{p} (A_i(x_j))^2 \cdot x_j}{\sum_{j=1}^{p} (A_i(x_j))^2}, \qquad \forall i. \tag{3}$$

Proof. For necessity the Lagrange multipliers method is used. For sufficiency one shows that the Hessians associated with $J(\cdot,L)$ and $J(P,\cdot)$ are positive definite.

Remark. The Picard iteration with (2) and (3) is used to obtain a local solution of the problem (1). The starating point in the Picard iteration may be an arbitrary choice for P or an arbitrary choice for L. For $C = X$ this procedure reduces to the well-known Fuzzy ISODATA algoritm [1, 12]. For this reason we will call it Generalized Fuzzy ISODATA (GFI) algorithm. Using the GFI algoritm a stratified classification of the pattern set may be obtained. In the second part of this paper a hierarchical divisive procedure to detect the optimal cluster structure in the data set X will be given.

REFERENCES

1. Bezdek J., *Cluster validity with fuzzy sets*. J. Cybernetics, **3** (1974), 58–72.
2. Bezdek J., Harris J., *Fuzzy partitions and relations; an axiomatic basis for clustering*, Fuzzy Sets and Systems, **1** (1978), 111–127.
3. Bezdek J., Harris J., *Convex decomposition of fuzzy partitions*, J. Math. Anal. Appl. **67** (1979), 490–512.
4. Bezdek J., Windham M., Ehrlich, R., *Statistical parameters of cluster validity functionals*, Intern. J. of Computer and Information Sciences, **9** (1980), 323–336.
5. Butnariu D., *Additive fuzzy measures and integrals*, I, J. Math. Anal. Appl **93** (1983), 436–452.
6. Dumitrescu D., *On fuzzy systems in pattern recognition*, Proceedings of the 2 nd National Symposium on System Theory, Craiova 1982, **2**. pp. 255–261.
7. Dumitrescu D., *On fuzzy partitions*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1983, pp. 57–60.
8. Dumitrescu D., *On fuzzy partitions in cluster analysis*, Seminar Angheluţă, Cluj-Napoca, 1983, p. 105–108.
9. Dumitrescu, D., *Hierarchical classification*, Proc. of the 4-th Conference on Informatics Iaşi, 1983, pp. 349–355.
10. Dumitrescu, D., *Pattern recognition in system control*, Proc. of the 3-d National Sympo, sium on System Theory, Craiova, 1984, vol. 2, pp. 368–372.
11. Dumitrescu D., *Hierarchical classification with fuzzy sets*, University of Cluj-Napoca, Fac. Math. Res. Seminars 4 (1984), 36–55.
12. Dunn J., *A fuzzy relative of the ISODATA process and its use in detecting compact, well-separated clusters*, J. Cybern **3** (1974), 319–350.
13. RUSPINI E., *Numerical methods for fuzzy clustering*, Inf. Sci, **2** (1970), 319–350.
14. Windham M., *Cluster validity for the fuzzy c-means algorithm*, IEEE Trans. on Pattern Anal. and Mach. Intell., **4** (1982), 357–363.
15. Windham M., *Cluster validity for fuzzy clustering algorithm*, Fuzzy Sets and Systems, **5** (1981), 177–185.

ON A GENERAL TYPE OF CONVEXITY

GH. TOADER*

Received: October 17, 1986

REZUMAT. — Asupra unui tip general de convexitate. În lucrare se dă o caracterizare operatorilor integrali (2) ce conservă funcţiile S-convexe definite in [5].

In their book [5], A. W. Roberts and D. E. Varberg have proposed, for an independent study project, the following general notion of convexity. Let S be a subset of $I \times I$ (where $I = [0, 1]$) and $D = [0, b]$. The function $f: D \to \mathbf{R}$ is said to be S-convex it it verifies the relation:

$$f(sx + ty) \leqslant s \cdot f(x) + t \cdot f(y) \tag{1}$$

for any $(s, t) \in S$ and any $x, y \in D$.

The set of all S-convex functions defined on D is denoted by $K(S)$. Theoretically S can be a subset of $\mathbf{R}^2$ and a S-convex function can be defined on some subsets of a linear space. But even in the case given before can appear some complications. For example, from (1) we can see that $s + t \leqslant 1$ for any $(s, t) \in \in S$. Otherwise b must be infinite because $(s \cdot t) \cdot x \in D$ for $x \in D$.

Apart from the well known examples of S-convexity given in [5], let us to mention here another one, given by us in [7]. For a given $m \in I$, we say that the function $f: D \to \mathbf{R}$ is m-convex if:

$$f(sx + m(1 - s)y) \leqslant s \cdot f(x) + m(1 - s) \cdot f(y)$$

for any $x, y \in D$ and any $s \in I$. A function is m-convex if and only if it is S_m- convex, where:

$$S_m = \{(s, t) : s \in I,\ t = m(1 - s)\}.$$

As follows from Lemma 2, m-convexity is a notion intermediate to convexity ($m = 1$) and starshapendness ($m = 0$). So it may be considered similar to a notion given for complex functions by P. T. Mocanu in [4].

For $s = t = 0$, from (1) we have $f(0) \leqslant 0$, that we suppose to be valid for any function which appears in what follows.

To answer to some questions from [5], we consider the following relation between sets: $S < S'$ if for any $(s, t) \in S$ there is an $(s, t') \in S'$ such that $t \leqslant t'$. We put $0 < S$ for $I \times \{0\} < S$.

LEMMA 1. *If $0 < S$, any S-convex function f is starshaped.*

* *Polytechnical Institute, Department of Mathematics, 3400 Cluj-Napoca, Romania*

Proof. For any $s \in I$, ther is a $t \geqslant 0$ such that $(s, t) \in S$. So, for any $x \in D$, we have:

$$f(sx) = f(sx + t \cdot 0) \leqslant s \cdot f(x) + t \cdot f(0) \leqslant s \cdot f(x).$$

LEMMA 2. *If* $0 < S < S'$, then $K(S) \supset K(S')$.

Proof. Let f be in $K(S')$ and x, y in D. For any $(s, t) \in S$ there is a $(s, t') \in S$ such that $t' \geqslant t$. Hence:

$$f(sx + ty) = f(sx + t'(t/t')y) \leqslant sf(x) + t'f((t/t')y) \leqslant sf(x) + tf(y).$$

Remark 1. As $s + t \leqslant 1$ for $(s, t) \in S$, we deduce that the usual convexity is the most restrictive.

COROLLARY 1. *If* $0 < S$ *and* $G \subset S$, *where*:

$$G = \{(s, t_s) : s \in I,\ t_s = \inf \{t : (s, t) \in S\}\},$$

then $K(S) = K(G)$.

Remark 2. This property gives an answer, at least partial, to the question on the minimality of the set S which determines a class $K(S)$.

But our central objective in this note is related to another problem. In [2] A. M. Bruckner and E. Ostrow have proved that the integral mean:

$$F(f)(x) = \frac{1}{x} \int_0^x f(v)\, \mathrm{d}v$$

preserves the convexity, the starshapendness and the superadditivity of the function f. In [3] it is considered a more general mean:

$$F_g(f)(x) = \frac{1}{g(x)} \int_0^x g'(v)\, f(v)\, dv. \tag{2}$$

In [6] we have obtained a characterization of the weight-functions g which give integral means F_g that preserve the above properties. We want to extend now this characterization to the case of S-convexity.

THEOREM. *The function* $F_g(f)$ *is* S*-convex for any* S*-convex function* f *if and only if the function* g *is of the form*:

$$g(x) = k \cdot x^a,\ k \in \mathbf{R},\ a > 0. \tag{3}$$

Proof. The function $f_0(x) = cx$ is S-convex for any real c. Hence so must be also the function:

$$F_0(x) = F_g(f_0)(x) = \frac{c}{g(x)} \int_0^x g'(v) \cdot v dv.$$

But, c being of arbitrary sign, this happens if and only if, for $c = 1$:

$$F_0(sx + t \cdot y) = s \cdot F_0(x) + t \cdot F_0(y)$$

for $(s, t) \in S$; $x, y \in D$. Thus (see [11]) $F_0(x) = bx$ and so g must be of the form (3). If $a > 0$, (2) is not defined for $f(x) = c$.

Conversely, if g is given by (3), then (2) becomes:

$$F_a(f)(x) = \frac{a}{x^a} \int_0^x v^{a-1} \cdot f(v)\, dv. \tag{4}$$

aking the substitution (given in [3]): $v = x \cdot w^{1/a}$, from (4) we get:

$$F_a(f)(x) = \int_0^1 f(x \cdot w^{1/a}) dw.$$

If f is in $K(S)$, for any $(s, t) \in S$ and any $x, y \in D$, we have:

$$F_a(f)(sx + ty) = \int_0^1 f((sx + ty)w^{1/a}) dw \leqslant s \int_0^1 f(xw^{1/a}) dw +$$

$$+ t \int_0^1 f(yw^{1/a}) dw = s \cdot F_a(f)(x) + t \cdot F_a(f)(y)$$

that is $F_a(f)$ is also in $K(S)$.

If we denote:

$$M^a K(S) = \{f : F_a(f) \in K(S)\}$$

we have thus the following:

COROLLARY 2. *If* $0 < S < S'$ *and* $a > 0$, *then*:

$$\begin{array}{ccc} K(S') & \subset & K(S) \\ \cap & & \cap \\ M^a K(S') & \subset & M^a K(S). \end{array}$$

REFERENCES

1. **Aczél, J.**, *Lectures on functional equations and their applications*, Academic press, New York – London, 1966.
2. **Bruckner, A. M.** and **Ostrow E.**, *Some function classes related to the class of convex functions*, Pacific J. Math. **12** (1962), 1203–1215.
3. **Mocanu, C.**, *Monotony of weight-means of higher order*, Analyse Numér. Théor. Approx. **11** (1982), 115–127.
4. **Mocanu, P. T.** *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica **11** (34) (1969), 127–133.
5. **Roberts, A. W.** and **Varberg, D. E.**, *Convex functions*, Academic Press, New York, 1973.
6. **Toader, Gh.**, *An integral mean that preserves some function classes*, Bulet. Inst. Politehn. Cluj-Napoca, Ser. Mat.–Mec. **27** (1984), 17–20.
7. **Toader, Gh.**, *Some generalizations of the convexity*, Proc. Colloq. Approx. Optim., Cluj-Napoca, 1984, 329–338.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

FURTHER REMARKS ON THE FIXED POINT STRUCTURES

IOAN A. RUS*

Intrat în redacție: 18 october 1986

REZUMAT. — Alte observații asupra structurilor de punct fix. În [5] am realizat o teorie a structurilor de punct fix în spații metrice. În prezenta lucrare se extind aceste rezultate în cazul unei mulțimi oarecare.

1. **Introduction.** The purpose of this paper is to improve the results given in [5].

We follow terminologies and notations in [4] and [5].

2. **Fixed point structures.** Let X be a nonempty set and $Y \in P(X)$. We denote by $\mathbf{M}(Y)$ the set of all mapping $f: Y \to Y$.

DEFINITION 2.1. A triple (X, S, M) is a fixed point structure if

(i) $S \subset P(X)$ is a nonempty subset of $P(X)$,

(ii) $M: P(X) \to \bigcup_{Y \in P(X)} \mathbf{M}(Y)$, $Y \mapsto M(Y) \subset \mathbf{M}(Y)$, is a mapping such that, if $Z \subset Y$, then

$$M(Z) \supset \{f|_Z : f \in M(Y) \text{ and } f(Z) \subset Z\},$$

(iii) Every $Y \in S$ has the fixed point property with respect to $M(Y)$.

Now, let us consider some simple examples.

Example 2.1. X is a nonempty set, $S = \{\{x\} \mid x \in X\}$, and $M(Y) = \mathbf{M}(Y)$

Example 2.2. (Bourbaki-Birkhoff). $(X, \leqslant)$ is an ordered set, $S = \{Y \in P(X) \mid (Y, \leqslant)$ has a maximal element$\}$ and $M(Y) = \{f: Y \to Y \mid x \leqslant f(x)$, for all $x \in Y\}$.

Example 2.3. (Knaster, Tarski, Birkhoff). $(X, \leqslant)$ is a complete lattice, $S = \{Y \in P(X) \mid (Y, \leqslant)$ is a complete sublattice of $X\}$ and $M(Y) = \{f: Y \to Y \mid f$ is order-preserving mapping$\}$.

Example 2.4. (Banach, Caccioppoli). (X, d) is a complete metric space, $S = P_{cl}$ and $M(Y) = \{f: Y \to Y \mid f$ is a contraction$\}$.

Example 2.5. (Niemytzki, Edelstein). (X, d) is a complete metric space, $S = P_{cp}(X)$ and $M(Y) = \{f: Y \to Y \mid f$ is a contractive mapping$\}$.

Example 2.6. (Schauder) X is a Banach space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$.

Example 2.7. (Dotson). X is a Banach space, $S = P_{cp,st}(X)$ and $M(Y) = \{f: Y \to Y \mid f$ is a nonexpansive mapping$\}$.

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

Example 2.8. (Browder). X is a Hilbert space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{f : Y \to Y \mid f$ is a nonexpansive mapping$\}$.

Example 2.9. (Schauder). X is a Banach space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{f : Y \to Y \mid f$ is completely continuous$\}$.

Example 2.10. (Tychonov). X is a locally convex space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$.

3. **Mappings with the intersection property.** Now, let us introduce

DEFINITION 3.1. Let X be a nonempty set, $Z \subset P(X)$ and $Z \neq \emptyset$. A mapping $\theta : Z \to \mathbf{R}_+$ has the intersection property if $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} \theta(Y_n) = 0$ implies $Y_\infty := \bigcap\limits_{n\in\mathbf{N}} Y_n \neq \emptyset$ and $\theta(Y_\infty) = 0$.

For some examples of mappings with the intersection property see [5]. Consider, however

Example 3.1. Let (X, d) be a complete metric space. If $x_1, x_2, x_3 \in X$, then we denote by $\delta_2(x_1, x_2, x_3)$ the area of the triangle $\triangle(x_1, x_2, x_3)$. For $Y \in P_b(X)$ let

$$\delta_2(Y) := \sup \{\delta_2(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in Y\}.$$

If Z is the set of all connected and bounded subset of X, then $\delta_2 : Z \to \mathbf{R}_+$, $Y \mapsto \delta_2(Y)$, has the intersection property.

For some properties of the mappings with the intersection property see [5].

4. **Compatibility with the fixed point structures.** Let us give

DEFINITION 4.1. Let (X, S, M) be a fixed point structure, $\theta : Z \to \mathbf{R}_+$ $(S \subset Z \subset P(X))$, $\eta : Z \to Z$. The pair (θ, η) is compatible with (X, S, M) if

(i) there exists Z_1, $S \subset Z_1 \subset Z$, such that $\theta|_{Z_1}$ has the intersection property,

(ii) η is a closure operator,

(iii) $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$,

(iv) $F_\eta \cap Z_0 \subset S$.

Now we illustrate this definition by some examples.

Example 4.1. Let X be a Banach space, $S = P_{cp,cv}(X)$, $M(Y) = C(Y, Y)$, $Z = P_b(X)$, $Z_1 = P_{b,cl}(X)$, $\theta = \alpha_k$, and $\eta(A) = \overline{co}A$, $A \in Z$.

Example 4.2. Let (X, S, M) be as in Example 4.1. $\theta = \gamma$ (measure of non compact-convexity), $\eta(A) = \bar{A}$, $A \in P_b(X)$.

5. (θ, φ)**-contractions**

DEFINITION 5.1. Let X be a set, $Z \subset P(X)$, $Z \neq \emptyset$, $\theta : Z \to \mathbf{R}_+$ and $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ a comparison function. A mapping $f : Y \to X$ is a (θ, φ) — contraction $(Y \subset X)$ if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $\theta(f(A)) \leqslant \varphi(\theta(A))$ for all $A \in Z \cap I(f)$.

Now we have.

THEOREM 5.1. *Let* (θ, η) *be a compatible pair with the fixed point structure* (X, S, M). *Let* $Y \in F_{\eta/Z_1}$ *and* $f \in M(Y)$. *If* f *is a* (θ, φ) *— contraction, then* $F_f \neq \emptyset$ *and* $\theta(F_f) = 0$.

Proof. Let $Y_1 = \eta(f(Y))$. Since $Y \in F_{\eta/Z_1}$, we have $Y_1 \subset Y$. Let $Y_2 = \eta(f(Y_1)), \ldots, Y_n = \eta(f(Y_n)), \ldots$. We denote $A_\infty := \bigcap_{n \in \mathbb{N}} Y_n$. From the Definition 4.1. we have $Y_\infty \neq \emptyset$, $\theta(Y_\infty) = 0$ and $Y_\infty \in F_\eta$. On the other hand $Y_n \in I(f)$ and $Y_\infty \in I(f)$. These imply $Y_\infty \in S$. Since $f \in M(Y)$ and (X, S, M) is a fixed point structure we have $F_f \neq \infty$.

From $f(F_f) = F_f$ and the condition (ii) in Definition 5.1. we have $\theta(F_f) = 0$.

For some consequences of this general result see [5].

6. θ-condensing mappings

DEFINITION 6.1. Let X be a set, $Z \subset P(X)$, $Z \neq \emptyset$, and $\theta : Z \to \mathbf{R}_+$. A mapping $f : Y \to X$ is a θ-condensing mapping if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $A \in I(f)$, $\theta(A) \neq 0$ implies $\theta(f(A)) < \theta(A)$.

We have

THEOREM 6.1. *Let* (θ, η) *be a compatible pair with the fixed point structure* (X, S, M). *Let* $Y \in F_{\eta/Z_1}$ *and* $f \in M(Y)$. *If*

(i) $A \in Z$, $x \in Y$ *imply* $A \cup \{x\} \in Z$,

(ii) $\theta(A \cup \{x\}) = \theta(A)$ *for all* $A \in Z$, $x \in Y$,

(iii) f *is* θ-*condensing*.

Then $F_f \neq \emptyset$ *and* $\theta(F_f) = 0$.

Proof. The proof is the same as the proof of the Theorem 5.1. in [5]

For some consequences of this theoreme see [5].

REFERENCES

1. J. Dugundji and A. Granas, *Fixed point theory*, P.W.N., Warszawa, 1982.
2. M. A. Krasnoselskii, P. P. Zabreiko, *Geometrical methods of nonlinear analysis*, Springer-Verlag, Berlin, 1984.
3. I. A. Rus, *Principii si aplicatii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.
4. I. A. Rus, *Generalized contractions*, Seminar on fixed point theory (Cluj-Napoca), Preprint nr. 3 (1983), 1–130.
5. I. A. Rus, *Fixed point structures*, Mathematica, **28** (1986), 59–64.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

NOTE ON A CONJECTURE IN PRIME NUMBER THEORY

DORIN ANDRICA*

Received: October 22, 1986

ABSTRACT. — Let p_n be the n-th natural prime number and let $\alpha_n = \sqrt{p_{n+1}} - \sqrt{p_n}$, $n \geq 1$. One presents some asymptotic estimates for $(\alpha_n)_{n\geq 1}$. Relations among the diverses conjectures in the prime number theory are also considered.

In what follows we shall denote by p_n the n^{th} prime number. Consider the sequences $(d_n)_{n\geq 1}$, $(\alpha_n)_{n\geq 1}$ defined by $d_n = p_{n+1} - p_n$ and $\alpha_n = \sqrt{p_{n+1}} - \sqrt{p_n}$. It is well-known that $\limsup\limits_{n\to\infty} d_n = +\infty$ (see [1], [7]). From this point of view the sequence $(\alpha_n)_{n\geq 1}$ has a different behaviour than $(d_n)_{n\geq 1}$. In this sense we begin with the following conjecture:

CONJECTURE 1. *The following inequality holds*

$$\alpha_n < 1 \tag{1}$$

for every natural numbers n.

The inequality (1) hs verified on a computer Felix C-256 with a programm in FΦRTRAN IV for all prime numbers $\leq 10^6 + 3$, so for the first 78.500 primes. The numerical tests and the programm was accomplished by Dan Grecu from Politehnic Institute of Cluj-Napoca.

Our next theorem contains some remarks abouth the lim inf and lim sup of some sequences which contain the difference $\alpha_n = \sqrt{p_{n+1}} - \sqrt{p_n}$.

THEOREM 1. *If $\beta \in [0, 1/2)$ then*

$$\liminf_{n\to\infty} (n \ln n)^\beta \cdot \alpha_n = 0 \tag{2}$$

$$\limsup_{n\to\infty} (n/\ln n)^{1/2} \cdot \alpha_n = \infty \tag{3}$$

Proof. To prove the relation (2), following the method of [8], we consider the function $f\colon [1, \infty) \to [1, \infty)$, $f(x) = x^\alpha$, $\alpha \in (0, 1)$. Denote $\lambda_n = \sqrt{p_n}$. It is clear that $\lambda_1 < \lambda_2 < \ldots$ and using the inequality $tf(t) \leq \lambda_{n+1} f(\lambda_{n+1})$ for $t \in [\lambda_n, \lambda_{n+1}]$ we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_{n+1})} \leq \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} \frac{dt}{tf(t)} = \int_{\lambda_1}^{\infty} \frac{dt}{t^{1+\alpha}} < +\infty \tag{4}$$

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

But, ob the other hand, we have:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_n)} - \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_{n+1})} = \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)\left(\frac{1}{f(\lambda_n)} - \frac{1}{f(\lambda_{n+1})}\right) <$$

$$< \sum_{n=1}^{\infty} \left(\frac{1}{f(\lambda_n)} - \frac{1}{f(\lambda_{n+1})}\right) = \frac{1}{f(\lambda_1)} \tag{5}$$

and one obtains

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_n)} < +\infty \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{p_{n+1}}(\sqrt{p_n})^{\alpha}} < +\infty \tag{7}$$

In the following, using the divergence of the series $\sum_{n=1}^{\infty} 1/p_n$ ([1], [3] p. 135, [8]) we get easy

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{p_n p_{n+1}}} = +\infty. \tag{8}$$

From (7) and (8) it follows

$$\liminf_{n\to\infty} \left(\frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{p_{n+1}}(\sqrt{p_n})^{\alpha}}\right) \Big/ \frac{1}{\sqrt{p_n p_{n+1}}} = 0$$

This is equivalent with $\liminf_{n\to\infty} p_n^{\frac{1-\alpha}{2}} \cdot \alpha_n = 0$. Taking into account that $p_n \sim n \ln n$ ([10] p. 153) we get the relation (2) where $\beta = \frac{1-\alpha}{2} \in [0, 1/2)$.

To prove the relation (3) we use the inequality of Rankin ([2] p. 355, [6] p. 99):

$$p_{n+1} - p_n > C \ln p_n \frac{\ln\ln p_n \cdot \ln\ln\ln\ln p_n}{(\ln\ln\ln p_n)^2} \tag{9}$$

which is true for an infinity of natural numbers n.

We have successively

$$\alpha_n = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} > \frac{p_{n+1} - p_n}{2\sqrt{p_{n+1}}} > C \frac{\ln p_n}{\sqrt{p_{n+1}}} \cdot \frac{\ln\ln p_n \cdot \ln\ln\ln\ln p_n}{(\ln\ln\ln p_n)^2}$$

for an infinity of natural indices. It follows

$$\frac{\sqrt{p_{n+1}}}{\ln p_n} \cdot \alpha_n > C \cdot \frac{\ln\ln p_n \cdot \ln\ln\ln\ln p_n}{(\ln\ln\ln p_n)^2} \tag{10}$$

so that

$$\limsup_{n\to\infty} \frac{\sqrt{p_{n+1}}}{\ln p_n} \cdot \alpha_n = +\infty \tag{11}$$

Using again the asymptotic relation $p_n \sim n \ln n$ we get immediately

$$\limsup_{n\to\infty} \frac{n \ln n}{\ln (n \ln n)} \cdot \alpha_n = +\infty \tag{12}$$

and after an elementary calculation we obtain the relation (3).

Remarks. 1. Fro the relation (2) we give a proof similar to that given for the relation (3) using instead of (9) the inequality of B o m b i e r i [4]

$$p_{n+1} - p_n < (0{,}46\ldots) \ln p_n \tag{13}$$

which is true for an infinity of natural numbers n. We have

$$\alpha_n = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} < \frac{p_{n+1} - p_n}{2\sqrt{p_n}} < K \frac{\ln p_n}{\sqrt{p_n}}, \quad K = 0{,}46\ldots$$

for an infinity of n.

It follows

$$p_n^{\beta} \cdot \alpha_n < K \frac{\ln p_n}{p_n^{1/2-\beta}} \tag{14}$$

from where one obtains

$$\liminf_{n\to\infty} p_n^{\beta} \cdot \alpha_n = 0 \tag{15}$$

and, consequently, using the relation $p_n \sim n \ln n$ we have (2). This method has the disadvantage that it uses the strong inequality (13) and this inequality has a very difficult proof.

2. One sets, in a natural way, the question if the relation (2) remains true in the limit case $\beta = 1/2$. It is very surprising the fact that in this case we have:

$$\liminf_{n\to\infty} \sqrt{p_n} \cdot \alpha_n \geqslant 1 \tag{16}$$

If $\liminf_{n\to\infty} \sqrt{p_n} \cdot \alpha_n < 1$, then there is a positive number such that for an infinity of natural numbers n one has

$$\sqrt{p_n} \cdot \alpha_n \leqslant 1 - \varepsilon \tag{17}$$

The inequality (17) is equivalent with

$$\frac{p_{n+1} - p_n}{1 - \varepsilon} \leqslant 1 + \sqrt{p_{n+1}/p_n}$$

But $p_{n+1} - p_n \geqslant 2$ and it follows

$$(1 + \varepsilon)/(1 - \varepsilon) \leqslant \sqrt{p_{n+1}/p_n} \tag{18}$$

But this relation is in contradiction with the known fact that $\lim\limits_{n\to\infty} p_{n+1}/p_n = 1$.

3. From the relation (2) one obtains $\liminf\limits_{n\to\infty} \alpha_n = 0$, a result proved by L. Panaitopol [7].

Let us recall other three conjectures in prime number theory.

CONJECTURE 2 ([11]). *One has the equality*

$$\lim_{n\to\infty} \alpha_n = 0 \tag{19}$$

CONJECTURE 3 (A. Schinzel [9]). *For $x \geqslant 8$ between x and $x + (\ln x)^2$ there is a prime number.*

CONJECTURE 4 ([3] p. 73). *For $n \geqslant 1$ the interval $[n^2, (n+1)^2]$ contains a prime number.*

In connection with Conjecture 4 the best known result is due to M. N. Huxley (see [5]) and states that there is a prime number between n^2 and $n^2 + n\theta$ for every $\theta > 7/6$ and $n \geqslant n_0(\theta)$, where $n_0(\theta)$ is a sufficiently great natural number.

The following theorem establishes the connections of our Conjecture 1 with the above mentioned conjectures.

THEOREM 2. *The following implications hold:*

$$\begin{array}{ccc} & C\,3 & \Rightarrow C\,2 \\ & \Downarrow & \\ C\,1 \Rightarrow & C\,4 & \end{array}$$

C 2 $\Rightarrow$ C 1 *when n is sufficiently great.*

C 2 $\Rightarrow$ C 4 *when n is sufficiently great.*

Proof. „C 3 $\Rightarrow$ C 2". Supposing that C 3 is true, it follows $p_n < p_{n+1} < p_n + (\ln p_n)^2$, so that

$$\alpha_n < \sqrt{p_n + (\ln p_n)^2} - \sqrt{p_n} = \frac{(\ln p_n)^2}{\sqrt{p_n + (\ln p_n)^2} + \sqrt{p_n}} < \frac{(\ln p_n)^2}{\sqrt{p_n}}$$

which implies C 2.

„C 3 $\Rightarrow$ C 4". By the Conjecture 3, in the interval $[n^2, (n+1)^2]$ there is a prime number. But we have the inequality $n^2 + 4(\ln n)^2 < n^2 + 2n + 1$, for every natural number n, which implies C 4.

„C 1 $\Rightarrow$ C 4". If C 4 is not true then there is a natural number n such that $p_k < n^2 < (n+1)^2 < p_{k+1}$. Then an elementary calculation shows that $\sqrt{p_{k+1}} - \sqrt{p_k} = \alpha_k > 1$, which is a contradiction.

„C 2 $\Rightarrow$ C 1 for sufficiently great n" is clear.

„C 2 ⇒ C 4 for sufficiently great n". We obtain easily this implication taking into account that C 1 ⇒ C 4.

In conection with Conjecture 1 it presents also interest the proof of the weaker statement that the sequence $(\alpha_n)_{n\geq 1}$ is bounded.

The author is greatfully indebted to Şerban Buzeţeanu, from University of Bucharest, for the idea of the elementary proof of the relation (2). Also the author thanks to Cătălin Badea for the interesting remark 2.

REFERENCES

1. D. Andrica, *Asupra şirului* $(p_{n+1} - p_n)_{n<1}$ G. M. No. 7 (1978), 277–279.
2. A. A. Buhčtab, *Teoria numerelor* (Russian), Moscova, 1966.
3. I. Creangă, *Introducere în teoria numerelor*, Ed. Didactică şi Pedagogică, Bucureşti, 1965.
4. H. Davenport, *Teoria multiplicativă a numerelor* (Russian), Moscova, 1971.
5. H. Halberstam, *Some unsolved problems in higher arithmetic*, in *The Encyclopedia of Ignorance*, edited by R. Duncan and M. Weston-Smith, Pergamon Press, 1982.
6. D. S. Mitrinović, M. S. Papadić, *Inequalities in the number theory*, Nis, 1978.
7. L. Panaitopol, *Problem* 141, G. M. Ser. A 79 (1974), No. 2, p. 72.
8. I. Sandor, *Despre şiruri, serii şi aplicaţii în teoria numerelor prime*, G. M. Ser. A (2), Vol. 6 (1985), No. 1–2, p. 38–48.
9. A. Schinzel, *Remarks on the paper „Sur certaines hypothèses concernant les nombers premiers"*, Acta Arith. 7 (1961), p. 1–8.
10. W. Sierpinski, *Elementary theory of numbers*, Warszawa, 1964.

STUDIA UNIV. BABEŞ–BOLYAI, MATHEMATICA, XXXI, 4, 1986

ON GENERALIZED MEASURES OF THE AMOUNT OF INFORMATION BASED ON THE STRATIFIED RANDOM SAMPLE

IOAN MIHOC*

Received: Octovber 22, 1986

ABSTRACT. – In this paper we define a new measure of the amount of information associated with a generalized random variable as well as the measures based on the stratified random sample.

1. **Generalized measures of the amount of information.** Let $\{\Omega, \mathfrak{B}, P\}$ be a probability space, that is, Ω an arbitrary nonempty set, called the set of elementary events; $\mathfrak{B}$ a σ — algebra of subsets of Ω, containing Ω itself, the elements of $\mathfrak{B}$ being called events; and P a probability measure, that is, a nonnegative and additive set function, defined on $\mathfrak{B}$, for which $P(\Omega) = 1$.

Let

$$\Delta_N^* = \left\{\mathfrak{P} = (p_1, p_2, \ldots, p_N);\ p_i > 0,\ i = \overline{1,N}, \sum_{i=1}^{N} p_i = 1\right\}, \tag{1.1}$$

be the set of all probability distributions associated with a discrete finite random variable X.

S h a n n o n [8] introduced a measure of information by the quantity

$$H(\mathfrak{P}) = H(X) = -\sum_{i=1}^{N} p_i \log_2 p_i, \tag{1.2}$$

called entropy of the distribution $\mathfrak{P}$ (or, entropy of the random variable X).

Measure (1.2) satisfies the additivity

$$H(\mathfrak{P} * \mathfrak{Q}) = H(\mathfrak{P}) + H(\mathfrak{Q}), \tag{1.3}$$

where

$$\mathfrak{P} * \mathfrak{Q} = (p_1q_1, \ldots, p_1q_N, \ldots, p_Nq_1, \ldots, p_Nq_N) \in \Delta_{NN}^* \tag{1.4}$$

is direct product of the distributions $\mathfrak{P}$ and $\mathfrak{Q}$, $\mathfrak{P}, \mathfrak{Q} \in \Delta_N^*$.

R é n y i [7] introduced a generalization of the notion of a random variable.

DEFINITION 1. An incomplete random variable X, is a function $\xi = \xi(\omega)$ measurable with respect to the measure on $\mathfrak{B}$ and defined on a subset Ω_1 of Ω, where $\Omega_1 \in \mathfrak{B}$ and $P(\Omega_1) > 0$.

* *University of Cluj-Napoca, Faculty of Economic Sciences, 3400 Cluj-Napoca, Romania*

The only difference between an ordinary random variable (ξ is an ordinary or complete random variable if $P(\Omega_1) = 1$) and an incomplete random variable is thus that the latter is not necessarily defined for every $\omega \in \Omega$. Therefore, for a incomplete random variable we have $0 < P(\Omega_1) < 1$.

DEFINITION 2. If $0 < P(\Omega_1) \leqslant 1$, then random variable ξ, defined on the Ω_1, is a generalized random variable. The distribution of a generalized random variable X will called a generalized probability distribution.

In this sense, the ordinary distributions can be considered as a particular case of a latter.

We denote by

$$w(\mathfrak{P}) = \sum_{i=1}^{N} p_i, \tag{1.5}$$

the weight of the distribution $\mathfrak{P}$.

Using the above definitions it follows that:

— if $w(\mathfrak{P}) = 1$, then $\mathfrak{P}$ is an ordinary distribution;

— if $0 < w(\mathfrak{P}) < 1$, then $\mathfrak{P}$ is an incomplete distribution;

— if $0 < w(\mathfrak{P}) \leqslant 1$, then $\mathfrak{P}$ is a generalized probability distribution

Also, we denote by

$$\Delta_N = \{\mathfrak{P} = (p_1, p_2, \ldots, p_N);\ p_i > 0,\ i = \overline{1,N},\ 0 < w(\mathfrak{P}) \leqslant 1\}, \tag{1.6}$$

the set of all finite discrete generalized probability distributions.

DEFINITION 3. [5] The measure of the amount of information, associated to a generalized random variable X, have the form

$$H_{\alpha^*}(\mathfrak{P}) = H_{\alpha^*}(X) = -\frac{1}{\alpha^*}\log_2\left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right), \tag{1.7}$$

where

$$q_i = \frac{p_i^{\beta+a_i}}{\sum\limits_{j=1}^{N} p_j^{\beta+a_j}},\ i = \overline{1,N},\ \sum_{i=1}^{N} q_i = 1, \tag{1.8}$$

$$\alpha^* = \frac{\alpha - n}{n},\ \alpha^* \in (-1, 0) \cup (0, \infty);\ \alpha > 0,\ \alpha \neq n,\ n \geqslant 1, \tag{1.9}$$

$$\beta + a_i \geqslant 1,\ i = \overline{1,N},\ \mathfrak{P} \in \Delta_N. \tag{1.10}$$

This measure can be called the measure of the information of order α/n and of type $\{\beta + a_i\}$, associated to the generalized propability distribution $\mathfrak{P}$, [4].

Remark 1. The measure (1.7) is a generalized measure of the amount of information in the Daroczy's sense [1] that is

$$H_{\alpha^*}(\mathfrak{P}) = -\log_2 M_{\alpha^*}(\mathfrak{P}) = H_{\varphi}(\mathfrak{P})_f, \tag{1.11}$$

where

$$M_\varphi(\mathfrak{L})_f = M_{\alpha^*}(\mathfrak{L}) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right)^{1/\alpha^*} = \left(\frac{\sum_{i=1}^{N} p_i^{\beta+a_i} \cdot p_i^{\alpha^*}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}\right)^{1/\alpha^*}, \tag{1.12}$$

represents the weighted mean associated to the generalized probability distribution $\mathfrak{L}$, and f, φ represent the weight function, respectively, the representation function, namely

$$f(t) = t^{\beta+a_i}, \quad \varphi(t) = t^{\alpha^*}, \quad t \in (0, 1]. \tag{1.13}$$

In the paper [6] to prove a theorem whence follows the form (1.11) of the measure (1.7), the additivity property (1.3), as well as the properties of the functions f and φ which were considered by Daróczy.

2. **Measures of the amount of informazion based on the stratified random sample.** Let C be a population (collectivity) and X a common property of hers elements. We want to study this collectivity relativ to this common property (characteristic).

Because, in general, the population C is heterogeneous in comparison with the characteristic X, we consider a stratification of the population C so that to obtain an homogeneity in each strata (subpopulation).

We assume that the population C is divided into N mutually exclusive subpopulations $C_1, C_2, \ldots, C_N$,

$$C = C_1 \cup C_2 \cup \ldots \cup C_N. \tag{2.1}$$

We denote by $m = M(X) = M(X \mid C)$, $\sigma^2 = D^2(X) = D^2(X \mid C)$ the expectation and the variance of the random variable (of the characteristic) X relating to whole population C. Also, we denote by

$$m_i = M(X \mid C_i), \quad \sigma_i^2 = D^2(X \mid C_i), \quad i = \overline{1,N}, \tag{2.2}$$

the expectations and variances of the same characteristic X but relating to the subpopulation (strata) C_i.

Let

$$p_i = P(X = x \mid x \in C_i), \quad i = \overline{1,N},$$

be the probability (the proportion) that a certain element x of the population C to belong to the strata C_i. We assume that all these probabilities are known.

A sample S from C obtained by taking random samples of size n_1 from C_1, of size n_2 from $C_2, \ldots,$ of size n_N from C_N is called a stratified sample of total size

$$n = n_1 + n_2 + \ldots + n_N. \tag{2.3}$$

In this paper, we discuss the stratified random sampling for estimation of the population expectation m if we assume that σ^2, m_i, σ_i^2 and p_i, $i = \overline{1,N}$, are known. Alsom, using these specifications as well as the Definition 3, we shall define measures of the amount of information based on the stratified random sample.

DEFINITION 4. If the size of sample, n, satisfies the relations

$$n = \frac{n_1}{p_1} = \frac{n_2}{p_2} = \ldots = \frac{n_N}{p_N}, \tag{2.4}$$

then the sample S is called a represenative sample. Also, if n from Definition 3 has just this semnification, then about the measure (1.7) we shall say that is *representative*.

DEFINITION 5. If the sizes of sample n_i, $i = \overline{1,N}$, to determine from the condition that the sample mean

$$\overline{X} = \sum_{i=1}^{N} p_i \overline{X}_i, \tag{2.5}$$

where

$$\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}, \quad i = \overline{1,N}, \tag{2.5'}$$

(x_{ij} are elements from the strata C_i), to be an efficient estimator for unknown expectation m, that is,

$$n_i = \frac{n p_i \cdot \sigma_i}{\sum_{j=1}^{N} p_j \cdot \sigma_j}, \quad i = \overline{1,N}, \tag{2.6}$$

then about the measure (1.7) we shall say that is *optimum*.

Therefore, the representative information of sample, $H_S^R(X)$, may be written in the form

$$H_S^R(X) = -\log_2 M_{\alpha_R^*}(X), \tag{2.7}$$

where

$$M_{\alpha_R^*}(X) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha_R^*}\right)^{1/\alpha_R^*}, \tag{1.8}$$

and

$$\alpha_R^* = \frac{\alpha \cdot p_i}{n_i} - 1, \quad \forall i, \ i = \overline{1,N}. \tag{2.9}$$

In the same way, the optimum information of sample, $H_S^0(X)$, has the form

$$H_R^0(X) = -\log_2 M_{\alpha_0^*}(X), \tag{2.10}$$

where

$$M_{\alpha_0^*}(X) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha_0^*}\right)^{1/\alpha_0^*}, \tag{2.11}$$

and

$$\alpha_0^* = \frac{p_i \cdot \sigma_i}{n_i \sum_{j=1}^{N} p_j \cdot \sigma_j} - 1, \quad \forall\, i,\ i = \overline{1, N}. \tag{2.12}$$

Now, we shall compare, between them, the measures (2.7) and (2.10). But, to compare these measures means, in fact, to compare the generalized means corresponding to (2.8) and (2.11).

Remark 2. Because the parameters (2.9) and (2.12) are independent by the index i, then when $i = \overline{1, N}$, it follows that among the standard deviations σ_1, $\sigma_2, \ldots, \sigma_N$, exist at least an index i so that

$$\sigma_i > \bar{\sigma} = \sum_{j=1}^{N} p_j\ \sigma_j, \tag{2.13}$$

where $\bar{\sigma}$ represents just the mean value of the standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_N$.

If the inequality (2.13) is realized, then

$$\alpha_0^* - \alpha_R^* = \frac{\alpha \cdot p_i}{n_i}\left(\frac{\sigma_i}{\sigma} - 1\right) > 0, \tag{2.14}$$

and hence

$$\alpha_R^* < \alpha_0^*. \tag{2.15}$$

THEOREM. *If the parameters* α_R^* *and* α_0^* *are in the relation* (2.15), *then*

$$M_{\alpha_R^*}(X) < M_{\alpha_0^*}(X) \tag{2.16}$$

and

$$H_S^0(X) < H_S^R(X). \tag{2.17}$$

Proof. Because the parameters α_R^* and α_0^* belong to the set $A = (-1, 0) \cup \cup (0, \infty)$ we shall distinguish two cases.

Case 1. $0 < \alpha_R^* < \alpha_0^*$ and $\alpha_R^* = \gamma \cdot \alpha_0^*$, $0 < \gamma < 1$.

If we denote

$$u_i = q_i \cdot p_i^{\alpha_0^*}, \quad v_i = q_i. \quad i = \overline{1, N}, \tag{2.18}$$

then

$$q_i \cdot p_i^{\alpha_R^*} = u_i^{\gamma} \cdot v_i^{1-\gamma} \tag{2.19}$$

and therefore

$$\sum_{i=1}^{N} q_i \cdot p_i^{\alpha_R^*} = \sum_{i=1}^{N} u_i^{\gamma} \cdot v_i^{1-\gamma}. \tag{2.20}$$

Using the inequality [2]

$$b_1^{p_1} \cdot b_2^{p_2} \cdot \ldots \cdot b_N^{p_N} < p_1 b_1 + b_2 p_2 + \ldots + p_N n_N, \tag{2.21}$$

which is a generalization of the inequality between the arithmetic and geometric means of N nonnegative numbers, we obtain

$$\frac{u_i^{\gamma}}{\left(\sum_{j=1}^{N} u_j\right)^{\gamma}} \cdot \frac{v_i^{1-\gamma}}{\left(\sum_{j=1}^{N} v_i\right)^{1-\gamma}} < \gamma \cdot \frac{u_i}{\sum_{j=1}^{N} u_j} + (1-\gamma) \cdot \frac{v_j}{\sum_{j=1}^{N} v_j} \cdot \tag{2.22}$$

Summing this inequality over i, we obtain

$$\frac{\sum_{i=1}^{N} u_i^{\gamma} \cdot v_i^{1-\gamma}}{\left(\sum_{j=1}^{N} u_j\right)^{\gamma} \cdot \left(\sum_{j=1}^{N} v_j\right)^{1-\gamma}} < \sum_{i=1}^{N} \left[\gamma \cdot \frac{u_i}{\sum_{j=1}^{N} u_j} + (1-\gamma) \cdot \frac{v_i}{\sum_{j=1}^{N} v_j}\right] = \gamma + (1-\gamma) = 1, \tag{2.23}$$

and hence the inequality

$$\sum_{i=1}^{N} u_i^{\gamma} \cdot v_i^{1-\gamma} < \left(\sum_{i=1}^{N} u_i\right)^{\gamma} \cdot \left(\sum_{i=1}^{N} v_i\right)^{1-\gamma} \cdot \tag{2.24}$$

In view of (2.18) and if we effectuate the calculations we obtain just the inequality (2.16) and hence the inequality (2.17).

Case 2: $-1 < \alpha_R^* < \alpha_0^* < 0$.

Using the relations [3]

$$M_{-\alpha_R^*}(X) = \frac{1}{\left(\sum_{i=1}^{N} q_i \left(\frac{1}{p_i}\right)^{\alpha_R^*}\right)^{1/\alpha_R^*}} \tag{2.25}$$

$$M_{-\alpha_0^*}(X) = \frac{1}{\left(\sum_{i=1}^{N} q_i \left(\frac{1}{p_i}\right)^{\alpha_0^*}\right)^{1/\alpha_0^*}}, \tag{2.26}$$

the proof of this case is similar to the preceding case.

Remark 3. Using the Remark 2, that is, the facts that parameters α_R^* and α_0^* are independent by the index i, is possible, also, to find an index s so that

$$\sigma_s < \sigma = \sum_{j=1}^{N} p_j \cdot \sigma_j, \tag{2.27}$$

or, an index r se that

$$\sigma_r = \bar{\sigma} = \sum_{j=1}^{N} p_j \cdot \sigma_j. \tag{2.28}$$

And in these cases the Theorem is also true, only that the inequalities (2.16) and (2.17) will be

$$M_{\alpha_R^*}(X) > M_{\alpha_0^*}(X),\ H_S^0(X) > H_S^R(X), \tag{2.29}$$

respectively,

$$M_{\alpha_R^*}(X) = M_{\alpha_0^*}(X),\ H_S^0(X) = H_S^R(X). \tag{2.30}$$

Remark 4. If the characteristic X follows a uniform distribution relating to each strata C_i, $i = \overline{1, N}$, namely $p_i = \frac{1}{N}$, $i = \overline{1, N}$, and $\sigma_1 = \sigma_2 = \ldots = \sigma_N$, then

$$H_S^0(X) = H_S^R(X) = \log_2 N, \tag{2.31}$$

where $\log_2 N$ is hist the Shannon's information associated with a uniform distribution.

REFERENCES

1. D a r ó c z y, Z., *Einige Ungleichungen über die mit Gewichtsfunktionen gebildeten Mittelwerte*, Monatsh. Math., **68** (1968), 102–112.
2. D a y k i n, D. E., E l i e z e r, C. J., *Generalizations of the A. M. and G. M. inequality*, Mathematics Magazine, Tom **40.** 5 (1967), 247–250.
3. H a r d y, G. H., L i t t l e w o o d, J. E., P o l y a, G., *Inequalities*, 1967.
4. K a p u r, J. N., *Generalized entropy of order α and type β*, Math. Seminar, 4 (1967), 78–94.
5. M i h o c, I., *Conjugate measures of the amount of information*, Mathematica, Cluj-Napoca, **2 (20)**. (1978), 171–176.
6. M i h o c, I., *Măsura cantității de informație privită ca valoare medie generalizată* (to appear).
7. R é n y i, A., *On measures of entropy and information*, Proc. 4th Berk. Symp. Math. Statist. and Prob., Tom I, (1961), 547–561.
8. S h a n n o n, C. E., *A mathematical theory of communication*, Bell System Techn. J., **27** (1948), 379–423; 623–656.

STUDIA UNIV. BABEŞ–BOLYAI, MATHEMATICA, XXXI, 4, 1986

ON A CLASS OF MULTIVARIATE LINEAR POSITIVE APPROXIMATING OPERATORS

D. D. STANCU*

Received: October 25, 1986

REZUMAT. — Asupra unei clase de operatori de aproximare liniari pozitivi multidimensionali. În această lucrare se prezintă o extindere multidimensională a unui operator liniar pozitiv, de tip Bernstein, introdus şi studiat, în anul 1983, în lucrarea [5], în cazul unidimensional. Se dau evaluări ale restului formulei de aproximare corespunzătoare şi se estimează ordinul de aproximare, folosind modulul de continuitate multidimensional.

1. In our earlier paper [5] we have introduced and investigated the approximation properties of a linear positive operator L_m^r, of Bernstein type, depending on a non-negative integer parameter r, m being a natural number such that $m > 2r$. This operator, which maps into itself the Banach space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$, is defined explicitly by

$$(L_m^r f)(x) := \sum_{k=0}^{m} w_{m,k}^r(x) f\left(\frac{k}{m}\right), \tag{1}$$

where

$$w_{m,k}^r(x) := \begin{cases} \binom{m-r}{k} x^k (1-x)^{m-r-k+1} & \text{if } 0 \leqq k < r \\ \binom{m-r}{k} x^k (1-x)^{m-r-k+1} & \text{if } r \leqslant k \leqslant m-r \\ \binom{m-r}{k-r} x^{k-r+1} (1-x)^{m-k} & \\ \binom{m-r}{k-r} x^{k-r+1} (1-x)^{m-k} & \text{if } m-r < k \leqslant m. \end{cases} \tag{2}$$

It is easy to see that if $r = 0$ or $r = 1$ then this operator reduces to the Bernstein operator B_m, defined by

$$(B_m f)(x) := \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{3}$$

where

$$p_{m,k}(x) := \binom{m}{k} x^k (1-x)^{m-k}. \tag{4}$$

---

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

As we have shown, one can express $(L_m^r f)(x)$ in terms of the Bernstein fundamental polynomials (4) in the following form

$$(L_m^r f)(x) = \sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)f\left(\frac{k}{m}\right) + xf\left(\frac{k+r}{m}\right)\right]. \tag{5}$$

In this paper we present a multivariate extension of the operator L_m^r and investigate how it can be used in the multivariate constructive approximation theory.

In the Euclidian space $E_s \equiv \mathbf{R}^s$ of all s-tuples of real numbers $(x_1, x_2 \dots, x_s)$ we consider the s-dimensional unit cube

$$\Omega_s = \{(x_1, x_2, \dots, x_s) \in \mathbf{R}^s \mid 0 \leqslant x_i \leqslant 1, \quad i = 1(1)s\}.$$

To any real-valued function f defined on Ω_s and arbitrary vector of non-negative integer components $(r_1, r_2, \dots, r_s)$ $(m_s > 2r_i.\ i = 1(1)s)$, we define the s-dimensional linear positive operator

$$L_{m_1,\dots,m_s}^{r_1,\dots,r_s}$$

by the following formula

$$\left(L_{m_1,\dots,m_s}^{r_1,\dots,r_s} f\right)(x_1, \dots, x_s) := \tag{6}$$

$$= \sum_{k_1=0}^{m_1-r_1} \dots \sum_{k_s=0}^{m_s-r_s} p_{m_1-r_1,k_1}(x_1) \dots p_{m_s-r_s,k_s}(x_s) \cdot \left(Q_{m_1,\dots,m_s,k_1,\dots,k_s}^{r_1,\dots,r_s} f\right)(x_1, \dots, x_s),$$

where

$$\left(Q_{m_1,\dots,m_s,k_1,\dots,k_s}^{r_1,\dots,r_s} f\right)(x_1, \dots, x_s) :$$

$$= (1-x_1)\dots(1-x_s)f\left(\frac{k_1}{m_1}, \dots, \frac{k_s}{m_s}\right) +$$

$$+ x_1(1-x_2)\dots(1-x_s)f\left(\frac{k_1+r_1}{m_1}, \frac{k_2}{m_2}, \dots, \frac{k}{m_s}\right) +$$

$$+ \dots + (1-x_1)\dots(1-x_{s-1})x_s f\left(\frac{k_1}{m_1}, \dots, \frac{k_{s-2}}{m_{s-2}}, \frac{k_{s-1}+r_{s-1}}{m_{s-1}}, \frac{k_s}{m_s}\right) +$$

$$+ x_1x_2(1-x_3)\dots(1-x_s)f\left(\frac{k_1+r_1}{m_1}, \frac{k_2+r_2}{m_2}, \frac{k_3}{m_3}, \dots, \frac{k_s}{m_s}\right) + \dots +$$

$$+ (1-x_1)\dots(1-x_{s-2})x_{s-1}x_s f\left(\frac{k_1}{m_1}, \dots \frac{k_{s-2}}{m_{s-2}}, \frac{k_{s-1}+r_{s-1}}{m_{s-1}}, \frac{k_s+r_s}{m_s}\right) +$$

$$+ \ldots + (1 - x_1)x_2 \ldots x_s f\left(\frac{k_1}{m_1}, \frac{k_2 + r_2}{m_2}, \ldots, \frac{k_s + r_s}{m_s}\right) +$$

$$+ x_1(1 - x_2)x_3 \ldots x_s f\left(\frac{k_1 + r_1}{m_1}, \frac{k_2}{m_2}, \frac{k_3 + r_3}{m_3}, \ldots, \frac{k_s + m_s}{m_s}\right) +$$

$$+ \ldots + x_1 \ldots x_{s-1}(1 - x_s)f\left(\frac{k_1 + r_1}{m_1}, \ldots, \frac{k_{s-1} + r_{s-1}}{m_{s-1}}, \frac{k_s}{m_s}\right).$$

One observes that for $r_1 = \ldots = r_s = 0$ this operator reduces to the s-dimensio nalBernstein operator $B_{m_1, \ldots, m_s}$, defined by

$$(B_{m_1, \ldots, m_s} f)(x_1, \ldots, x_s) : =$$

$$\sum_{k_1=0}^{m_1} \ldots \sum_{k_s=0}^{m_s} p_{m_1, k_1}(x_1) \ldots p_{m_s, k_s}(x_s) f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right).$$

It should be noticed that the higher-dimensional analogous of the operator L_m^r from (1) can also be expressed under the following form

$$\left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} f\right)(x_1, \ldots, x_s) =$$

$$= \sum_{k_1=0}^{m_n} \ldots \sum_{k_s=0}^{m_s} w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s}(x_1, \ldots, x_s) f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right),$$

where

$$w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s}(x_1, \ldots, x_s) = \prod_{i=1}^{s} w_{m_i, k_i}^{r_i}(x_i).$$

3. By using formula (6) we can find easily the values of the operator $L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s}$ applied to the test functions $e_{i_1, \ldots, i_s}$, defined — for any point $(x_1, \ldots, x_s) \in \Omega_s$ — by

$$e_{i_1, \ldots, i_s}(x_1, \ldots, x_r) = x_1^{i_1} \ldots x_s^{i_s} \; (0 \leqslant i_1 + \ldots + i_s \leqslant 2).$$

We have

$$\left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} e_{i_1 \ldots, i_s}\right)(x_1, \ldots, x_s) = x_1^{i_1} \ldots . x_s^{i_s}(i_p = 0,1 \,;\, p = 1(1)s), \tag{7}$$

and

$$\left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} e_{i_1}, \ldots, i_s\right)(x_1, \ldots, x_s) = x_p^2 + \left[1 + \frac{r_p(r_p - 1)}{m_p}\right] \frac{x_p(1 - x_p)}{m}, \tag{8}$$

for

$$i_p = 2,\ i_1 = \ldots = i_{p-1} = i_{p+1} = \ldots = i_s = 0,\ p = 1(1)\, s.$$

Appealing to the known Bohman-Korovkin-Volkov uniform convergence criterion, we may formulate the following convergence theorem.

THEOREM 1. *If f $C(\Omega_s)$, then we have*

$$\lim_{m_1, \ldots, m_s \to \infty} L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f = f,$$

uniformly on the unit s-cube Ω_s.

4. We now proceed to determine the expression of the remainder of the approximation formula of a function $f \in C(\Omega_s)$ by means of the s-dimensional Bernstein type operator introduced in this paper:

$$f(x_1, \ldots, x_s) = \left(L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) + \left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s). \quad (9)$$

One observes first that the polynomial $L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f$, which — according to (7) — reproduces the linear functions, interpolates the function f at the vertices of the s-cube Ω. This is the reason that formula (9) has the degree of exactness $(\mathbf{1}, \ldots, \mathbf{1})$, as can be easily seen from the equalities (7) and (8).

Now refering to an expression of the remainder given in the one-dimensional case in our paper [5] and to a generalization of the formula (7.2) for the remainders, presented in our earlier paper [3], in the bivariate linear approximation formulas, we can formulate the following.

THEOREM 2. *If $f \in C(\Omega_s)$, then for any point $(x_1, \ldots, x_s) \in \Omega_s$, the remainder of the approximation formula* (9) *can be expressed, by means of one and two-dimensional divided differences, in the following form*

$$\left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) = \quad [10)$$

$$= \sum_{i=1}^{s} \left[1 + \frac{r_i(r_i - 1)}{m_i}\right] \cdot \frac{x_i(1 - x_i)}{m_i} \left[\xi^{(i)}_{m_i},\ \eta^{(i)}_{m_i},\ \zeta^{(i)}_{m_i};\ f(x_n, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_s)\right]_{t_i} -$$

$$- \sum_{\substack{i, j=1 \\ (i<j)}}^{s} \left[1 + \frac{r_i(r_i - 1)}{m_i}\right]\left[1 + \frac{r_j(r_j - 1)}{m_j}\right] \frac{x_i(1 - x_i)}{m_i} \cdot \frac{x_j(1 - x_j)}{m_j} \Big) \cdot$$

$$\cdot \left[\begin{matrix} \xi^{(i)}_{m_i}, & \eta^{(i)}_{m_i}, & \zeta^{(i)}_{m_i} \\ \xi^{(i)}_{m_j}, & \eta^{(j)}_{m_j}, & \zeta^{(j)}_{m_j} \end{matrix};\ f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_s)\right]_{t_i, t_i}$$

$$-\left[1+\frac{r_1(r_1-1)}{m_1}\right]\dots\left[1+\frac{r_s(r_s-1)}{m_s}\right]\frac{x_1(1-x_1)}{m_1}\dots\frac{x_s(1-x_s)}{m_s}\cdot$$

$$\begin{bmatrix}\xi_{m_1}^{(1)}, & \eta_{m_1}^{(1)}, & \zeta_{m_1}^{(1)} & \\ \cdot & \cdot & \cdot & ;\ f(t_1,\dots,t_s)\\ \xi_{m_s}^{(s)}, & \eta_{m_s}^{(s)}, & \zeta_{m_s}^{(s)} & \end{bmatrix}$$

where $\xi_{m_i}^{(j)}$, $\eta_{m_i}^{(j)}$, $\zeta_{m_i}^{(j)}$ $(i, j = 1(1)s)$ *are certain points in* $(0,1)$.

If we take into account formula (6.8) from our paper [3], which corresponds to the extension to several variables of a Peano-Milne integral representation formula of a linear functional having a certain degree of exactness, we can state the following theorem.

THEOREM 3. *If the function f has continuous partial derivatives of second order on Ω_s, then the remainder of the approximation formula* (9) *can be represented, for any point of Ω_s, under the following form*

$$\left(R_{m_1,\dots,m_s}^{r_1,\dots,r_s} f\right)(x_1,\dots,x_s) =$$

$$= \sum_{i=1}^{s}\int_0^1 G_{m_i}^{r_i}(t_i\,;\,x_i) f_{x_i^2}\,dt_i - \sum_{\substack{i,j=1\\(i<j)}}^{s}\int_0^1\int_0^1 G_{m_i}^{r_i}(t_i\,;\,x_i)G_{m_j}^{r_j}(t_j\,;\,x_j) f_{x_i^2x_j^2}\cdot dt_i\,dt_j +$$

$$+ \dots\dots\dots\dots\dots\dots +$$

$$+ (-1)^{s-1}\int_0^1\dots\int_0^1 G_{m_1}^{r_1}(t_1\,;\,x_1)\dots G_{m_s}^{r_s}(t_s\,;\,x_s) f_{x_1^2\dots x_s^2}\cdot dt_1\dots dt_s,$$

where

$$G_{m_i}^{r_i}(t_i\,;\,x_i) = (R_{m_i}^{r_i}\varphi_{x_i})(t_i),$$

$$\varphi_{x_i}(t_i\,;\,x_i) = (x_i - t_i)_+ = \frac{1}{2}\left[x_i - t_i + |x_i - t_i|\right]$$

and $R_{m_i}^{r_i}$ represents the remainder in the approximation formula of $f(x_1,\dots,x_s)$ by means of $L_{m_i}^{r_i} f$, $L_{m_i}^{r_i}$ being the one-diensional linear positive operator corresponding to the argument x_i.

If we use the explicit expressions given in [5] for these Peano kernels one can see that their values on Ω are nonpositive on [0,1], so that appealing to the mean value theorem of the integral calculus, or using the Cauchy mean value theorem for the divided differences occuring in formula (10), we can state.

THEOREM 4. *If the function f is twice continuous differentiable in Ω_s, with respect to each of its arguments, then for any point $(x_1, \ldots, x_s) \in \Omega$ there exists a point $(\xi_1, \xi_2, \ldots, \xi_s)$ in this domain such that*

$$\left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) = -\sum_{i=1}^{s}\left[1+\frac{r_i(r_i-1)}{m_i}\right]\frac{x_i(1-x_i)}{2m_i} f_{\xi_i^2} -$$

$$-\sum_{\substack{i, j=1 \\ (i<j)}}^{s}\left[1+\frac{r_i(r_i-1)}{m_i}\right]\left[1+\frac{r_j(r_j-1)}{m_j}\right]\frac{x_i(1-x_i)}{2m_i}\cdot\frac{x_j(1-x_j)}{2m_j} f_{\xi_i^2\xi_j^2} -$$

$$-\left[1+\frac{r_1(r_1-1)}{m_1}\right]\cdots\left[1+\frac{r_s(r_s-1)}{m_s}\right]\frac{x_1(1-x_1)}{2m_1}\cdots\frac{x_s(1-x_s)}{2m_s} f_{\xi_1^2\cdots\xi_s^2}.$$

We note that in the special case s = 2 this result has been given in our recent paper [6], while in the case $s = 2$, $r_1 = 0$, $r_2 = 0$ or $r_1 = 1$, $r = 1$ all these results were found in our earlier paper [4].

5. We can give also an asymptotic estimate of the remainder in the approximation formula (9), which corresponds to a result of Voronovskaja in the case of the Bernstein operator.

THEOREM 5. *If for the function $f: \Omega \to \mathbf{R}$, at an interior point $(x_1, \ldots, x_s)$ of Ω_s the second total differential $d^2 f(x_1, \ldots, x_s)$ exists, then we have the asymptotic formula*

$$\left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) =$$

$$= \sum_{i=1}^{s}\left[1+\frac{r_i(r_i-1)}{m}\right]\frac{x_i(1-x_i)}{2m} f_{x_i^2}(x_1, \ldots, x_s) + \sum_{p=1}^{s}\frac{1}{m_p}\cdot\varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s},$$

where $\varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}$ thend to 0 as $m_1, \ldots, m_s$ thend to ∞.

Proof. Let $(t_1, \ldots, t_s) \in \Omega_s$. It is known that because of our hypotheses on f, there exists a function $g: \Omega_s \to \mathbf{R}$ such that we have $g(t_1, \ldots, t_s) \to 0$ as $t_1 \to x_1, \ldots, t_s \to x_s$, while $f(t_1, \ldots, t_s)$ may be expanded, according to Taylor's formula, in the following form

$$f(t_1, \ldots, t_s) = f(x_1, \ldots, x_s) + \sum_{i=1}^{s}(t_i - x_i) f_{x_i}(x_1, \ldots, x_s) +$$

$$+\frac{1}{2}\sum_{i=1}^{s}(t_i - x_i)^2 f_{x_i^2}(x_1, \ldots, x_s) + \sum_{i, j=1}^{s}(t_i - x_i)(t_j - x_j) f_{x_i x_j}(x_1, \ldots, x_s) +$$

$$+\left[\sum_{i=1}^{s}(t_i - x_i)^2\right](t_1, \ldots, t_s).$$

If we insert here $t_i = k_i/m_i$, multiply by the fundamental polynomial

$$w^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, k_1, \ldots, k_s}(x_1, \ldots, x_s),$$

and take into consideration the equalities (7), (8), we obtain

$$\left(L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s)$$

$$f(x_1, \ldots, x_s) + \sum_{i=1}^{s} \left[1 + \frac{r_i(r_i - 1)}{m_i}\right] \frac{x_i(1 - x_i)}{2m_i} f_{x_i^2}(x_1, \ldots, x_s) +$$

$$+ \rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}(x_1, \ldots, x_s),$$

where

$$\rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}(x_1, \ldots, x_s) =$$

$$= \sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} w^{r_1, \ldots, r_s}_{m_1, \ldots, m_s k_1, \ldots, k_s}(x_1, \ldots, x_s) \left[\sum_{p=1}^{s} \left(\frac{i_p}{m_p} - x_p\right)^2\right] \cdot g\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right).$$

Since $g(t_1, \ldots, t_s) \to 0$ as $t_1 \to x_1, \ldots, t \to x_s$, it follows that for any ε positive we can choose the positive numbers $\delta_1, \ldots, \delta_s$ such that $|g(t_1, \ldots, t_s)| < \varepsilon$ whenever $|t_i - x_i| \leq \delta_i$, $i = 1(1)s$. By replacing $t_i = k_i/m_i$, $i = 1(1)s$, we can write

$$\left|g\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right| < \varepsilon \text{ when } \left|\frac{k_i}{m_i} - x_i\right| \leq \delta_i, \quad i = 1(1)s.$$

Since

$$\left|\rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}\right|(x_1, \ldots, x_s) \leq \sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} w^{r_1, \ldots, r_s}_{m_1, \ldots, m_s k_1, \ldots, k_s}(x_1, \ldots, x_s) \cdot$$

$$\cdot \left[\sum_{p=1}^{s} \left(\frac{i_p}{m_p} - x_p\right)^2\right] \cdot \left|g\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right|,$$

we may proceed further in like manner as in the case of one variable [1]. One concludes that there exist

$$\varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, p} = \varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, p}(x_1, \ldots, x_s)$$

tending to 0 as $m_1, \ldots, m_s$ tend to ∞, so that we can write

$$\rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}(x_1, \ldots, x_s) \sum_{p=1}^{s} \frac{1}{m_p} \cdot \varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, p}.$$

6. We now discuss the estimation of the order of approximation of a function $f \in C(\Omega_s)$ by means of the operator considered in this paper, in order to see the speed of convergence of these operators.

We shall make use of the modulus of continuity ω on Ω_s, defined by

$$\omega(f;\, \delta_1, \ldots, \delta_s) = \max |f(x_1'', \ldots, x_s'') - f(x_1', \ldots, x_s')|,$$

where $\delta_1 > 0, \ldots, \delta_p > 0$, while $(x_1', \ldots, x_s')$ and $(x_1'', \ldots, x_s'')$ are points from Ω_s, so that

$$|x_i'' - x_i'| \leqq \delta_i, \quad i = 1(1)s.$$

We shall now establish

THEOREM 6. *If $f \in C(\Omega_s)$, then we have*

$$\left|f(x_1, \ldots, x_s) - \left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} f\right)(x_1, \ldots, x_s)\right| \leqq$$
$$\leqq \left[1 + \sum_{i=1}^{s} \frac{1}{\alpha_i} \sqrt{1 + \frac{r_i(r_i - 1)}{m_i}}\right] \omega\left(f;\, \alpha_1 \sqrt{\frac{x_1(1 - x_1)}{m_1}}, \ldots, \alpha_s \sqrt{\frac{x_s(1 - x_s)}{m_s}}\right) \tag{11}$$

where $\alpha_1, \ldots, \alpha_s$ are any positive constants.

Proof. Because we have on Ω_s:

$$w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s} \geqq 0$$

and

$$L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} e_{0, \ldots 0} = e_0,$$

we can write

$$f(x_1, \ldots, x_s) - \left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s}\right)(x_1, \ldots, x_s)\Big| \leqq$$
$$\leqq \sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s}(x_1, \ldots, x_s)\left|f(x_1, \ldots, x_s) - f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right|.$$

We shall use now the following two properties of the modulus of continuity

$$|f(x_1'', \ldots, x_s'') - f(x_1', \ldots, x_s')| \leqq \omega(f;\, |x_1'' - x_1'|, \ldots, |x_s'' - x_s'|),$$
$$\omega(f;\, \lambda_1\delta_1, \ldots, \lambda_s \delta_s) \simeq (1 + \lambda_1 + \ldots + \lambda_s)\omega(f;\, \delta_1, \ldots, \delta_s).$$

Since

$$\left|f(x_1, \ldots, x_s) - f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right| \leqq \omega\left(f;\, \frac{1}{\delta_1}\left|x_1 - \frac{k_1}{m_1}\right|\delta_1, \ldots, \delta_s\left|x_s - \frac{k_s}{m_s}\right|\delta_s\right) \leqslant$$
$$\leqslant \left(1 + \sum_{i=1}^{s} \frac{1}{\delta_i}\left|x_i - \frac{k_i}{m_i}\right|\right)\omega(f;\, \delta_1, \ldots, \delta_s),$$

it follows that we can write

$$\left|f(x_1, \ldots, x_s) - \left(L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s)\right| \leqslant$$

$$\leqq \left(1 + \sum_{i=1}^{s} \sum_{k_i=0}^{m_i} \frac{1}{\delta_i} \left| x_i - \frac{k_i}{m_i} \right| w^{r_i}_{m_i, k_i}(x_i)\right) \omega(f; \delta_1, \ldots, \delta_s).$$

In accordance with the Cauchy-Schwarz inequality and with the identities (7) and (8), we have

$$\sum_{k_i=0}^{m_i} w^{r_i}_{m_i, k_i}(x_i) \left| x_i - \frac{k_i}{m_i} \right| \leqq \left[\sum_{k_i=0}^{m_i} w^{r_i}_{m_i, k_i}(x_i) \left(x_i - \frac{k_i}{m_i}\right)^2\right]^{1/2} \leqq$$

$$\leqslant \sqrt{\left[1 + \frac{r_i(r_i - 1)}{m_i}\right] \frac{x_i(1 - x_i)}{m_i}} .$$

By using these inequalities and selecting

$$\delta_p = \alpha_p \sqrt{\frac{x_p(1 - x_p)}{m_p}}, \qquad p = 1(1)s,$$

$\alpha_1, \ldots, \alpha_s$ being any positive constants, we finally get the inequality (11).

Now, since for any $(x_1, \ldots, x_s) \in \Omega_s$ we have $x_p(1 - x_p) \leqq 1/4$, we can select $\alpha_1 = \ldots = \alpha_s = 2$ and we obtain the following result.

THEOREM 7. *If $f \in C(\Omega_s)$ in the maximum norm over Ω_s we have*

$$\left\| f - L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f \right\|$$

$$\leqq \left(1 + \frac{1}{2}\sum_{p=1}^{s} \sqrt{1 + \frac{r_p(r_p - 1)}{m_p}}\right) \omega\left(f; \frac{1}{\sqrt{m_1}}, \ldots, \frac{1}{\sqrt{m_s}}\right).$$

In the particular case $s = 1$, $r_1 = 0$ or $r_1 = 1$, this inequality reduces to the inequality given in 1935 by T. Popoviciu[2] for the Bernstein polynomials.

REFERENCES

1. Lorentz, G. G., *Bernstein polynomials*. University of Toronto Press. Toronto, 1953.
2. Popoviciu, T., *Sur l'approximation des fonctions convexes d'ordre supérieur*. Mathematica **10** (1935), 49–54.
3. Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*. SIAM Journ. Numer. Anal. (B) **1** (1964), 137–163.
4. Stancu, D. D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*. Mathematics of Computation **17** (1963), 270–278.
5. Stancu, D. D., *Approximation of functions by means of a new generalized Bernstein operator*. Calcolo **20** (1983), 211–229.
6. Stancu, D. D., *Bivariate approximation by some Bernstein type operators*. Proceedings of the Colloquim on Approximation and Optimization. Cluj-Napoca, Oct. 25–27, 1984.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

ON A PROBLEM OF AREOLAR MECHANICS

CONSTANTIN TUDOSIE*

Received: March 27, 1985

ABSTRACT. — In this paper a method is given for determining, in polar coordinates, the linear accelerations on plane curves, considering the functions r, θ as zero order accelerations, and the derivatives $\dot{r}$, $\dot{\theta}$ as first order accelerations. At the same time the areolar accelerations of the mobile body are also determined. The differential equation solution is obtained by introducing some unknown functions, of the t-time variable, called „direct connexion functions".

1. Introduction. The real development of complex phenomena cannot be comprised in simple differential equations. The simplicity disappears when the progression of the phenomenon, in all its extent, is slow or fast. In this case, the easy determination of these out of common equations' solutions disappears, and new and pretentious methods must often be resorted to.

In this paper a method is given for determining linear accelerations on plane curves, considering functions r, θ as zero order accelerations, and the derivatives $\dot{r}$, $\dot{\theta}$ as first order accelerations. At the same time, the areolar accelerations of the mobile body in curvilinear motion are also determined.

2. Description of the method. Let be the areolar differential of the motion of a mobile body on a plane curvilinear trajectory

$$a_2(t)\ddot{A} + a_1(t)\dot{A} + a_0(t)A = f(t), \tag{1}$$

with the given initial conditions $\overset{(i)}{A}(0) = \overset{(i)}{A}_0$, $(i = 0, 1)$, where $A(t)$ is an area.

By denoting with $R = R(\theta)$ the polar equation of the plane trajectory, the elementary area dA has the expression

$$dA = \frac{1}{2} R^2(\theta) d\theta,$$

or

$$dA = \frac{1}{2} r^2(t)\dot{\theta}(t) dt, \tag{2}$$

where $r(t) = R[\theta(t)]$ and $\theta(t)$ are the polar coordinates of the mobile body instantaneous position.

* Polytechnical Institute, Department of Mathematica, 3400 Cluj-Napoca, Romania

By integrating (2), it follows

$$A(t) = A_0 + \frac{1}{2}\int_0^t r^2(s)\dot{\theta}(s)ds, \tag{3}$$

and the derivatives of A are

$$\dot{A}(t) = \frac{1}{2}r^2(t)\dot{\theta}(t), \ \ddot{A}(t) = \frac{1}{2}(2\dot{r}r\dot{\theta} + r^2\ddot{\theta}). \tag{4}$$

By introducing the „direct connexion functions" [3]

$$\omega_{i+1,\,i}(t), \quad \bar{\omega}_{i+1,\,i}(t), \quad \omega_{2,\,0}(t); \quad (i = 0,\ 1),$$

we have the relations

$$\dot{r} = \bar{\omega}_{0,\,1}r, \ \ddot{r} = \bar{\omega}_{2,\,1}\dot{r}, \tag{5}$$

$$\dot{\theta} = \omega_{1,\,0}\theta, \quad \ddot{\theta} = \omega_{2,\,1}\dot{\theta}, \quad \ddot{\theta} = \omega_{2,\,1}\omega_{1,\,0}\theta. \tag{6}$$

By integrating the first relation (5) and the second one (6), one obtains

$$r(t) = r_0 \exp\left[\int_0^t \bar{\omega}_{1,\,0}(s)ds\right], \tag{7}$$

$$\dot{\theta}(t) = \dot{\theta}_0 \exp\left[\int_0^t \omega_{2,\,1}(s)ds\right]. \tag{8}$$

By observing (5) and (6), relations (4) become

$$\dot{A} = \frac{1}{2}r^2\dot{\theta}, \tag{9}$$

$$\ddot{A} = \frac{1}{2}r^2\dot{\theta}(2\bar{\omega}_{1,\,0} + \omega_{2,\,1}). \tag{10}$$

By substituting (8) in (3), (9), and (10), it follows

$$A(t) = A_0 + \frac{1}{2}\dot{\theta}_0\int_0^t r^2(s)\exp\left[\int_0^s \omega_{2,\,1}(\sigma)d\sigma\right]ds, \tag{11}$$

$$\dot{A}(t) = \frac{1}{2}\dot{\theta}_0 r^2(t)\exp\left[\int_0^t \omega_{2,\,1}(s)ds\right], \tag{12}$$

$$\ddot{A}(t) = \frac{1}{2}\dot{\theta}_0(2\bar{\omega}_{1,\,0} + \omega_{2,\,1})r^2(t)\exp\left[\int_0^t \omega_{2,\,1}(s)ds\right]. \tag{13}$$

By observing (11), (12), and (13), equation (1) becomes

$$\frac{1}{2}\,\dot{\theta}_0 r^2(t)\,\exp\left[\int_0^t \omega_{2,1}(s)ds\right]\{a_2(t)\,[2\bar{\omega}_{1,0}(t)+\omega_{2,1}(t)]+a_1(t)\}+$$

$$+\,a_0(t)\left\{A_0+\frac{1}{2}\,\dot{\theta}_0\int_0^s r^2(s)\,\exp\left[\int_0^t \omega_{2,1}(\sigma)d\sigma\right]ds\right\}=f(t). \tag{14}$$

By integrating expression (8), one obtains

$$\theta(t)=\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds. \tag{15}$$

By substituting (7) in the first relation (5), we have

$$\dot{r}(t)=r_0\bar{\omega}_{1,0}(t)\,\exp\left[\int_0^t \bar{\omega}_{1,0}(s)ds\right]. \tag{16}$$

By observing (8), the second relation (6) becomes

$$\ddot{\theta}(t)=\dot{\theta}_0\omega_{2,1}(t)\,\exp\left[\int_0^t \omega_{2,1}(s)ds\right]. \tag{17}$$

From the third relation (6) and from

$$\ddot{\theta}=\omega_{2,0}\theta, \tag{18}$$

it follows

$$\omega_{2,0}(t)=\omega_{2,1}(t)\omega_{1,0}(t). \tag{19}$$

By substituting (15) in the first relation (6), and from (18), we have

$$\dot{\theta}(t)=\omega_{1,0}(t)\left\{\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\}, \tag{20}$$

$$\ddot{\theta}(t)=\omega_{2,0}(t)\left\{\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\}. \tag{21}$$

By equalizing (17) with (21), one obtains

$$\dot{\theta}_0\omega_{2,1}(t)\,\exp\left[\int_0^t \omega_{2,1}(s)ds\right]-\omega_{2,0}(t)\left\{\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\}=0. \tag{22}$$

From (8) and (20) it results

$$\dot{\theta}_0 \exp\left[\int_0^t \omega_{2,1}(s)ds\right] - \omega_{1,0}(t)\left\{\theta_0 + \dot{\theta}_0 \int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\} = 0. \quad (23)$$

By substituting (16) in the second relation (5), it is obtained

$$\ddot{r}(t) = r_0\bar{\omega}_{2,1}(t)\bar{\omega}_{1,0}(t) \exp\left[\int_0^t \bar{\omega}_{1,0}(s)ds\right]. \quad (24)$$

By integrating the second relation (5), it follows

$$\dot{r}(t) = \dot{r}_0 \exp\left[\int_0^t \bar{\omega}_{2,1}(s)ds\right]. \quad (25)$$

From (16) and (25), one obtains

$$r_0\bar{\omega}_{1,0}(t) \exp\left[\int_0^t \bar{\omega}_{1,0}(s)ds\right] - \dot{r}_0 \exp\left[\int_0^t \bar{\omega}_{2,1}(s)ds\right] = 0. \quad (26)$$

The expressions (7), (11), (12), (13), (14), (15), (16), (19), (20), (21), (22), (23), (24) and (26) make up a system (S) of 14 equations with 14 unknown quantities

$$\overset{(i)}{r},\ \overset{(i)}{\theta},\ \overset{(i)}{A},\ (i = 0, 1, 2),\ \omega_{i+1,i},\ \bar{\omega}_{i+1,i},\ (i = 0, 1),\ \omega_{2,0}.$$

For determining the solution, the initial conditions $\overset{(i)}{r}(0) = \overset{(i)}{r}_0$, $(i = 0, 1, 2)$, $\theta(0) = \theta_0$ are also given.

The constant $\ddot{A}_0$ results from (1), for $t = 0$.

The constant $\dot{\theta}_0$ has the value

$$\dot{\theta}_0 = 2\dot{A}_0 r_0^{-2}.$$

The value $\ddot{\theta}_0$ is obtained from the second relation (4), for $t = 0$. From (5) and (6), for $t = 0$, we have

$$\bar{\omega}_{1,0}(0) = \dot{r}_0(r_0)^{-1}, \quad \bar{\omega}_{2,1}(0) = \ddot{r}_0(\dot{r}_0)^{-1}$$

$$\omega_{1,0}(0) = \dot{\theta}_0(\theta_0)^{-1}, \quad \omega_{2,1}(0) = \ddot{\theta}_0(\dot{\theta}_0)^{-1}.$$

For $t = 0$, from (19) it follows

$$\omega_{2,0}(0) = \omega_{2,1}(0)\omega_{1,0}(0).$$

Knowing the functions $\overset{(i)}{r}(t)$ and $\overset{(i)}{\theta}(t)$, $(i = 0, 1, 2)$, the module of the speed and acceleration of the mobile body is determined

$$|\bar{v}| = [\dot{r}^2 + (r\dot{\theta})^2]^{\frac{1}{2}},$$

$$|\bar{a}| = [(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2]^{\frac{1}{2}}.$$

The functions $r = r(t)$ and $\theta = \theta(t)$ are parametrical equations, in polar coordinates, ale the mobile body trajectory, which lead to the polar equation of the plane curve, by eliminating parameter t.

REFERENCES

1. Démidovitsch, B., Maron, I. *Éléments de calcul numérique*, Éditions Mir, Moscou, 1973.
2. Iacob, C. *Mecanică teoretică*. Editura didactică și pedagogică, București, 1971.
3. Tudosie, C. *Deduction of higher order accelerations by the method of associated angular velocity*. Strojnícky Časopis, 34, č. 3, pp. 337–341, 1983.
4. Tudosie, C. *Determination of higher order accelerations by a functional method*. Acta Technica, ČSAV, No. 2, pp. 218–224, 1983.
5. Tudosie, C. *A method for calculating the higher order accelerations*. Mathematica, Tome 25 (48), No. 1, pp. 69–74, 1983.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

ON SATURATED Π-FORMATIONS

RODICA COVACI*

Received: December 21, 1985

ABSTRACT. — A theorem giving necessary and sufficient conditions for a π-formation to be saturated is proved in the paper.

1. **Preliminaries.** It is the aim of this note to prove a theorem which gives necessary and sufficient conditions for a π-formation (i.e. a π-closed formation) to be saturated.

All groups considered are finite. Let π be a set of primes, π′ the complement to π in the set of all primes and $O_{\pi'}(G)$ the largest normal π′-subgroup of a group G.

We first give some useful definitions.

DEFINITION 1.1. ([4], [5], [7]) a) *A class* $\mathfrak{X}$ *of groups is a homomorph if* $\mathfrak{X}$ *is epimorphically closed, i.e. if* $G \in \mathfrak{X}$ *and* N *is a normal subgroup of* G, *then* $G/N \in \mathfrak{X}$.

b) *A homomorph* $\mathfrak{F}$ *is a formation if* $G/N_1 \in \mathfrak{F}$, $G/N_2 \in \mathfrak{F}$ *implies* $G/N_1 \cap N_2 \in \mathfrak{F}$.

c) *A formation* $\mathfrak{F}$ *is saturated if* $\mathfrak{F}$ *is Frattini closed, i.e.* $G/\Phi(G) \in \mathfrak{F}$ *implies* $G \in \mathfrak{F}$, *where* $\Phi(G)$ *denotes the Frattini subgroup of* G.

d) *A group* G *is primitive if* G *has a maximal subgroup* H *with* $\text{core}_G H = 1$, *where* $\text{core}_G H = \bigcap \{H^g / g \in G\}$.

e) *A homomorph* $\mathfrak{F}$ *is a Schunck class if* $\mathfrak{F}$ *is primitively closed, i.e. if any group* G, *all of whose primitive factor groups are in* $\mathfrak{F}$, *is itself in* $\mathfrak{F}$.

DEFINITION 1.2. ([4]) *Let* $\mathfrak{F}$ *be a class of groups,* G *a group and* H *a subgroup of* G. H *is an* $\mathfrak{F}$*-covering subgroup of* G *if*: (i) $H \in \mathfrak{F}$; (ii) $H \leqslant K \leqslant G$, $K_0 \trianglelefteq K$, $K/K_0 \in \mathfrak{F}$ *imply* $K = HK_0$.

DEFINITION 1.3. a) ([3]) *A group is* π *— solvable if every chief factor is either a solvable* π *— group or a* π' *— group. For* π *the set of all primes, we obtain the notion of solvable group.*

b) *A class* $\mathfrak{F}$ *of groups is* π *— closed if*

$$G/O_{\pi'}(G) \in \mathfrak{F} \Rightarrow G \in \mathfrak{F}.$$

A π-closed homomorph, formation, respectively Schunck class is called π — *homomorph*, π — *formation*, respectively π — *Schunck class*.

In the proof of the main theorem we need the following results:

LEMMA 1.4. ([4]) *If* $\mathfrak{X}$ *is a homomorph,* G *a group,* N *a normal subgroup of* G, K/N *an* $\mathfrak{X}$*-covering subgroup of* G/N *and* H *an* $\mathfrak{X}$ *— covering subgroup of* K, *then* H *is an* $\mathfrak{X}$ *— covering subgroup of* G.

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

LEMMA 1.5. ([1]) *A solvable minimal normal subgroup of a group is abelian.*

LEMMA 1.6. ([1]) *If S is a maximal subgroup of G with* $\text{core}_G\, S = 1$ *and N is a minimal normal subgroup of G, then $G = SN$ and $S \cap N = 1$.*

LEMMA 1.7. ([5]) *Let $\mathfrak{F}$ be a class of groups. $\mathfrak{F}$ is a saturated formation if and only if $\mathfrak{F}$ is a Schunck class and a formation.*

LEMMA 1.8. ([2]) *Let $\mathfrak{F}$ be a π* — homomorph. Then $\mathfrak{F}$ is a Schunck class if and only if any π — solvable group has $\mathfrak{F}$ — covering subgroups.

2. The main result

THEOREM 2.1. *Let $\mathfrak{F}$ be a π formation. The following conditions are equivalent:*

(1) *$\mathfrak{F}$ is saturated;*

(2) *if G is a π — solvable group and $G \notin \mathfrak{F}$, but for the minimal normal subgroup N of G we have $G/N \in \mathfrak{F}$, then N has a complement in G;*

(3) *any π — solvable group G has $\mathfrak{F}$ — covering subgroups.*

Proof.

(1) $\Rightarrow$ (2). $\mathfrak{F}$ being a saturated π — formation, $\mathfrak{F}$ is, by 1.7., a π — Schunck class. Hence, applying 1.8., G has an $\mathfrak{F}$ — covering subgroup H. We shall prove that H is a complement of N in G. Indeed, $HN = G$, because of 1.2. (ii) used for $H \leqslant G = G$, $N \trianglelefteq G$, $G/N \in \mathfrak{F}$. Further we have $H \cap N = 1$, as the following shows. Since G is π — solvable, N is either a solvable π — group or a π' — group. Let us suppose that N is a π' — group. It follows that $N \leqslant O_{\pi'}(G)$ and we have

$$G/O_{\pi'}(G) \simeq (G/N)/(O_{\pi'}(G)/N).$$

But $G/N \in \mathfrak{F}$, hence $G/O_{\pi'}(G) \in \mathfrak{F}$, which implies, by the π — closure of $\mathfrak{F}$, the contradiction $G \in \mathfrak{F}$. So N is a solvable π — group. By 1.5., N is abelian. It follows that $H \cap N \trianglelefteq G$. Since $H \cap N \neq N$ ($H \cap N = N$ leads to $N \subseteq H$, hence $G = HN = H$, in contradiction with $G \notin$ F and $H \in \mathfrak{F}$), we have $H \cap N = 1$.

(2) $\Rightarrow$ (3). By induction on $|G|$. Two cases are possible:

1) $G \in \mathfrak{F}$. Then G is its own $\mathfrak{F}$ — covering subgroup.

2) $G \notin \mathfrak{F}$. Let N be a minimal normal subgroup of G. By the induction, G/N has an $\mathfrak{F}$ — covering subgroup E/N. We can have two possibilities:

2a) $G/N \in \mathfrak{F}$. Then $E/N = G/N$. By (2), N has a complement V in G. Again two cases are possible:

2a$_1$) $\text{core}_G V \neq 1$. The induction shows that $G/\text{core}_G V$ has an $\mathfrak{F}$ — covering subgroup $H/\text{core}_G V$. Let us suppose that $H = G$. Then $G/\text{core}_G V \in \mathfrak{F}$. Hence $G/N \cap \text{core}_G V \in \mathfrak{F}$, because $\mathfrak{F}$ is a formation. But V being a complement of N in G, we have $N \cap \text{core}_G V = 1$. It follows the contradiction $G \in \mathfrak{F}$. So $H < G$. By the induction, H has an $\mathfrak{F}$ — covering subgroup $\bar{H}$. By 1.4., $\bar{H}$ is an $\mathfrak{F}$ — covering subgroup of G.

2a$_2$) $\mathrm{core}_G V = 1$. We shall prove that V is an $\mathfrak{F}$ — covering subgroup of G. Since

$$V \simeq V/V \cap N \simeq VN/N = G/N,$$

we have $V \in \mathfrak{F}$. Let now

$$V \leqslant K \leqslant G, \; K_0 \trianglelefteq K, \; K/K_0 \in \mathfrak{F}.$$

We shall prove that $K = VK_0$. It is easy to see that V is a maximal subgroup in G. Indeed, $V < G$, for $V \in \mathfrak{F}$ but $G \notin \mathfrak{F}$. Further, if $V \leqslant V^* < G$, supposing $V < V^*$, it follows that there is an element $v^* \in V^* \setminus V$; but $G = VN$ implies $v^* = vn$, with $v \in V$ and $n \in N$. We obtain that $n = v^{-1}v^* \in V \cap N$. Since $V \cap N = 1$, $n = 1$. So $v^* = v \in V$, a contradiction. Hence $V = V^*$ and V is maximal in G. It follows that we have either $K = V$ or $K = G$. In the first alternative, $K = KK_0 = VK_0$. If $K = G$, we notice that $K_0 \neq 1$, for if $K_0 = 1$ it follows the contradiction $G \in \mathfrak{F}$. Let M be a minimal normal subgroup of G with $M \subseteq K_0$. We are in the hypotheses of 1.6.: V is a maximal subgroup of G with $\mathrm{core}_G V = 1$ and M is a minimal normal subgroup of G. Hence $G = VM$. It follows that $K = G = VM = VK_0$.

2b) $G/N \notin \mathfrak{F}$. Then $E/N < G/N$, hence $E < G$. By the induction, E has an $\mathfrak{F}$ — covering subgroup $\overline{E}$. Applying 1.4, $\overline{E}$ is an $\mathfrak{F}$ — covering subgroup of G.

(3) $\Rightarrow$ (1). By 1.8., $\mathfrak{F}$ is a Schunck class. Since $\mathfrak{F}$ is a formation, 1.7. implies that $\mathfrak{F}$ is saturated.

REFERENCES

1. B a e r, R., *Classes of finite groups and their properties*, Illinois J. Math., 1, 2, 1957, 115–187.
2. C o v a c i, R., *Projectors in finite π-solvable groups*, Studia Univ. Babeş-Bolyai, Math., XXII, 1, 1977, 3–5.
3. Č u n i h i n, S. A., *O teoremah tipa Sylowa*, Dokl. Akad. Nauk SSSR, 66, 2, 1949, 165–168.
4. G a s c h ü t z, W., *Zur Theorie der endlichen auflösbaren Gruppen*, Math. Z., 80, 4, 1963, 300–305.
5. G a s c h ü t z, W., *Selected topics in the theory of soluble groups*, Australian National University, Canberra, 1969.
6. H u p p e r t, B., *Endliche Gruppen* I, Berlin-New York, Springer-Verlag, 1967.
7. S c h u n c k, H., H — *Untergruppen in endlichen auflösbaren Gruppen*, Math. Z., 97, 4, 1967, 326–330.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 4, 1986

NUMERICAL RESULTS FOR THE FREE CONVECTION FLOW FROM A VERTICAL PLATE WITH GENERALISED WALL TEMPERATURE DISTRIBUTION

H. S. TAKHAR* and I. POP**

Received: October 27, 1986

ABSTRACT. — The problem of natural convection over a semi-infinite vertical flat plate with non-uniform wall temperature is studied by using a numerical method. The wall derivates of the universal functions for the Prandtl numbers 0.733 and 7 are tabulated. Such tabulations serve to calculate the heat transfer and skin friction from the plate.

Introduction. As is well known, the problem of natural convection boundary layer flow over a semi-infinite vertical flat plate is one of the most basic problems in the study of heat transfer over external surfaces and numerous papers dealing with various physical or mathematical aspects of this problem have been published. An excellent review article concerning this problem is given by Jaluria [1].

Recently Kundu [4] has considered a special form of the problem of free convection flow over a vertical semi-infinite flat plate, viz., that of a wall with a temperature distribution of the form

$$T_w - T_\infty = \bar{x}^n \sum_{i=1}^{r} A_i \bar{x}^i \tag{1}$$

where A_i are constants, T_w is the wall temperature, T_∞ is the ambient temperature and $\bar{x}$ measures the distance along the plate from the leading edge. However, the derived differential equations have not been analytically or numerically solved in Kundu's paper. It is, therefore, there aim of his Research Note to complete Kundu's problem by giving a numerical solution shooting techniques employing the fourth order Runge-Kutta routines as outlined by Soundalgekar, Takhar and Singh [3]. At the same time, we shall correct some misprints in his derived equations. The first and second-order wall derivatives of the universal functions are given in a table. It is worth mentioning that having a numerical solution is very helpful in evaluation of both data and approximate methods in design, and in other further calculations, such as those related to instability.

Basic ecuations. The present problem is formulated on the basis of a semi-infinite vertical surface with the origin at the leading edge. The x-axis is vertically upward and y is perpendicular to the plate. Employing the Boussinesq approximation and neglecting the viscous term in the energy equation, the governing differential equations for the solution of natural convection

* *University of Manchester, Department of Engineering, Manchester M13 9PL, England*
** *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

flow past a semiinfinite vertical flat plate with variable wall temperature can be written, in terms of dimensionless quantities, as

$$\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} = 0 \tag{2a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \theta + \frac{\partial^2 u}{\partial y^2} \tag{2b}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2}\,. \tag{2c}$$

The boundary conditions of the problem are

$$\begin{aligned} &u = v = 0, \quad \theta = \theta_w(x) \quad &\text{at} \quad y = 0 \\ &u, \quad \theta \to 0 &\text{as} \quad y \to \infty \end{aligned} \tag{2d}$$

Here u, v are the velocity components along x, y-axes ; θ is the temperature and Pr is the Prandtl number. The dimensionless quantities in equations (2) are related to their corresponding dimensional variables through the following definitions :

$$\begin{gathered} x = \bar{x}/L, \; y = \bar{y}/L, \; u = \bar{u}L/\nu, \; v = \bar{v}L/\nu \\ \theta = (g\beta L^3/\nu^2)(T - T_\infty) \end{gathered} \tag{3}$$

where L is there reference length and other physical quantities have their usual meaning.

Next, to reduce equations (2) to ordinary differential ones, we introduce the following variables, after Kundu [2]:

$$\psi = 4(a_0/4)^{1/4} x^{(n+3)/4} \sum_{i=0}^{r} x^i f_i(\eta) \tag{4a}$$

$$\theta = x^n \sum_{i=0}^{r} a_i \, x^i \, \theta_i \, (\eta) \tag{4b}$$

where the stream function ψ is defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{5}$$

and

$$\eta = (a_0/4)^{1/4} \; x^{(n-1)/4} \; y \tag{6}$$

is an independent variable.

Substitution of equations (4) into (2) then yields the following system of ordinary differential equations:

$$a_0 \sum_{i=0}^{r} (2n + 2 + 4i) f_i' f_{r-i}' - a_0 \sum_{i=0}^{r} (n + 3 + 4i) f_i f_{r-i}'' = a_0 f''' + a_r \theta_r \quad (7a)$$

$$4 \sum_{i=0}^{r} (n + i) a_i \theta_i f_{r-i}' - \sum_{i=0}^{r} (n + 3 + 4i) a_{r-i} f_i \theta_{r-i}' = a_s \theta_r''/\mathrm{Pr}$$

subject to the boundary conditions

$$f_r = f_r' = 0, \quad \theta = 1 \qquad \text{at} \qquad \eta = 0$$

$$f_r', \ \theta \to 0 \qquad \text{as} \qquad \eta \to \infty$$

In the above equations primes denote differentiation with respect to η.

Analysis and results. To shorten the paper, we give in the Table 1 the wall temperature distributions, the stream function transformations and the first 24 differential equations of the problem. The universal functions and their surface derivatives needed for the evaluations of flow and heat transfer parameters are computed for Prandtl numbers of 0.733 and 7 respectively when $n = 0$.

Table 1. Functions and differential equations for different r

r	No. of equations to be solved	Functions and differential equations
0	2	$\theta_w = a_0 x^n, \ f_0, \theta_0$ $(2n + 2) f_0'^2 - (n + 3) f_0 f'' = f_0''' + \theta_0$ $\theta'' + \mathrm{Pr}[(n + 3) f_0 \theta' - 4n f_0' \theta_0] = 0$
1	$2 + 2 = 4$	$\theta_w = x^n(a_0 + a_1 x), \ f_1 = (a_1/a_0) F_{11}, \ \theta_1 = \Phi_{11}$ $(4n + 8) f_0' F_{11}' - (n + 3) f_0 F_{11}'' - (n + 7) f_0'' F_{11} =$ $= F_{11}''' + \Phi_{11}$ $4[n\theta_0 F_{11}' + (n + 1) \Phi_{11} f_0'] - [(n + 3) f_0 \Phi_{11}' + (n +$ $+ 7) F_{11} \theta_0'] = \Phi_{11}''/ \ \mathrm{Pr}$
2	$4 + 4 = 8$	$\theta_w = x^n(a_0 + a_1 x + a_2 x^2), \ f_2 = (a_1/a_0)^2 F_{21} + (a_2/a_0) F_{22}$ $\theta_2 = (a_1/a_2 a_0) \Phi_{21} + \Phi_{22}$ $(4n + 12)(f_2' F_{91}' + F_{11}'^2/2) - (n + 3) f_0 F_{21}'' - (n + 7) F_{11} F_{11}'' -$ $- (n + 11) f'' F_{21} = F_{21}''' + \Phi_{21}$ $(4n + 12) f_0' F_{22}' - (n + 3) f_0 F_{22}'' - (n + 11) f_0'' F_{22} =$ $= F_{22}''' \Phi_{22}$ $4[n\theta_0 F_{21}' + (n + 1)\Phi_{11} F_{11}' + (n + 2) \Phi_{21} f_0'] - [(n + 3)$ $f_0 \Phi_{21}' + (n + 7) F_{11} \Phi_{11}' + (n + 11) F_{21} \theta_0'] = \Phi_{21}''/\mathrm{Pr}$ $4[n\theta_0 F_{22}' + (n + 2) \Phi''_{21} f_0'] - [(n + 3) f_0 \Phi_{22}' + (n + 11)$ $F_{22} \Phi_0'] = \Phi_{22}''/\mathrm{Pr}$

3 $\quad 8 + 6 = 14$

$$\theta_w = x^n(a_0 + a_1 x + a_2 x^2 + a_3 x^3), \qquad f_1 = (a_1/a_0)^3 F_{31} + (a_2 a_1/a_0^2) F_{32} + (a_3/a_0) F_{33},$$

$$\theta_3 = (a_1^3/a_3 a_0^2)\, \Phi_{31} + a_2 a_1/a_3 a_0)\Phi_{32} + \Phi_{33}$$

$$(4n + 16)(f_0' F_{31}' + F_{11}' F_{21}') - (n + 3) f_0 F_{32}'' - (n+7) F_{11} F_{22}'' - (n + 11) F_{21} F_{11}'' - (n + 15) F_{31} f_0'' = F_{31}''' + \Phi_{31}$$

$$(4n + 16)(f_0' F_{32}' + F_{11}' F_{22}') - (n + 3) f_0 F_{32}'' - (n+7) F_{11} F_{22}'' - (n + 11) F_{22} F_{11}'' - (n + 15) F_{32} f_0'' = F_{32}''' + \Phi_{32}$$

$$(4n + 16) f_0' F_{33}' - (n + 3) f_0 F_{33}'' - (n + 15) F_{33} f_0'' = F_{33}''' + \Phi_{33}$$

$$4[n\theta_0 F_{31}' + (n+1)\,\Phi_{11} F_{21}' + (n + 2)\Phi_{21} F_{11}' + (n + 3)\Phi_{31} f_0'] - ([n + 3)\,\Phi_{31}' f_0 + (n + 7)\,\Phi_{21}' F_{11} + (n + 11)\,\Phi_{11}' F_{21} + (n + 15)\,\theta_0' F_{31}] = \Phi_{31}''/\mathrm{Pr}$$

$$4[n\theta_0 F_{32}' + (n+11)\,\Phi_{11} F_{22}' + (n+2)\,\Phi_{22} F_{11}' + (n+3)\Phi_{32} f_0'] - [(n + 3)\,\Phi_{32}' f_0 + (n + 7)\,\Phi_{22}' F_{11} + (n + 11)\,\Phi_{11}' F_{22} + (n + 15)\,\theta_0' F_{32}] = \Phi_{32}''/\mathrm{Pr}$$

$$4[n\theta_0 F_{33}' + (n+3)\,\Phi_{33} f_0'] - [(n+3)\,\Phi_{33}' f_0 + (n + 15)\,\theta_0' F_{33}] = \Phi_{33}''/\mathrm{Pr}$$

4 $\quad 14 + 10 = 24$

$$\theta_w = x^n(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4), \quad f_4 = (a_1/a_0) F_{41} + (a_1^2 a_2/a_0^3) F_{40} + (a_1 a_3/a_0^2) F_{43} + (a_4/a_0) F_{44} + (a_2^2/a_0^2) F_{45},$$

$$\theta_4 = (a_1^4/a_4 a_3^3)\, \Phi_{41} + (a_1 a_3/a_4 a_0)\, \Phi_{43} + \Phi_{44} + (a_2^2/a_4 a_0)\, \Phi_{45}$$

$$(4n + 20)(f_0' F_{41}' + F_{11}' F_{31}' + F'^2/2) - (n + 3) f_0 F_{41}'' - (n + 7) F_{11} F_{31}'' - (n+11) F_{21}'' F_{21} - (n + 15) F_{31} F_{11}'' - (n + 19) F_{41} f_0'' = F_{41}''' + \Phi_{41}$$

$$(4n + 20)(f_0' F_{42}' + F_{11}' F_{32}' + F_{21}' F_{22}') - (n + 3) f_0 F_{42}'' - (n + 7)\, F_{11} F_{32}'' - (n + 11)\, (F_{22}'' F_{21} + F_{22} F_{21}'') - (n + 15) F_{11}'' F_{32} - (n + 19) f_0'' F_{42} = F_{42}''' + \Phi_{42}$$

$$(4n + 20)(f_0' F_{43}' + F_{11}' F_{33}') - (n+3)(f_0 F_{43}'' - (n + 7) F_{11} F_{33}'' - (n + 15) F_{11}'' F_{32} - (n + 19) f_0' F_{42} = F_{42}''' + \Phi_{42}$$

$$(4n + 20) f_0' F_{44}' - (n + 3) f_0 F_{44}'' - (n + 19) f_0'' F_{44} = F_{44}''' + \Phi_{44}$$

$$(4n + 20)(f_0' F_{45}' + F_{22}'^2/2) - (n + 3) f_0 F_{45}'' - (n + 11) F_{22} F_{22}' - (n + 19) f_0'' F_{45} = F_{45}''' + \Phi_{45}$$

$$4[n\theta_0 F_{41}' + (n + 1)\Phi_{11} F_{31}' + (n + 2)\Phi_{21} F_{21}' + (n + 3)\, \Phi_{31} F_{11}' + (n + 4)\, \Phi_{41} f_0'] - [(n + 3) f_0 \Phi_{41}' + (n + 7)\, F_{11} \Phi_{31}' + (n + 11) F_{21} \Phi_{21}' + (n + 15) F_{31} \Phi_{11}' + (n + 19) F_{41} \theta_0'] = \Phi_{41}''/\mathrm{Pr}$$

$$4[n\theta_0 F_{42}' + (n + 1)\, \Phi_{11} F_{32}' + (n + 2)(\Phi_{22} F_{21}' + \Phi_{21} F_{22}') + (n + 3)\, \Phi_{32} F_{11}' + (n + 4) f_0'\, \Phi_{42}] - [(n + 3) f_0 \Phi_{42}' + (n + 7)\, F_{11} \Phi_{32}' + (n + 11)\, (F_{21} \Phi_{22}' + F_{22} \Phi_{21}') + (n + 15) F_{32} \Phi_{11}' + (n + 19) F_{42} \theta_0'] = \Phi_{42}''/\mathrm{Pr}$$

$$4[n\theta_0F'_{43}+(n+1)\Phi_{11}F'_{33}+(n+3)\Phi_{33}F'_{11}+(n+4)f'_0\Phi'_{43}]-$$
$$-[(n+3)f_0\Phi'_{43}+(n+7)F_{11}\Phi'_{33}+(n+15)F_{33}\Phi'_{11}+$$
$$+(n+19)F_{43}\theta'_0]=\Phi''_{43}/\mathrm{Pr}$$

$$4[n\theta_0F'_{44}+(n+4)\Phi_{44}f'_0]-[(n+3)f_0\Phi'_{44}+(n+19)F_{44}\theta'_0]=$$
$$=\Phi''_{44}/\mathrm{Pr}$$

$$4[n\theta_0F'_{45}+(n+2)\Phi_{22}F'_{22}+(n+4)\Phi_{45}f'_0]-[(n+3)f_0\Phi'_{45}+$$
$$+(n+11)F_{22}\Phi'_{22}+(n+19)F_{45}\theta'_0]=\Phi''_{45}/\mathrm{Pr}$$

The boundary conditions on the problem are

$$\left.\begin{array}{lll} F_{rj}=F'_{rj}=0 & & \text{for all } j \\ \Phi_{rr}=1, & \Phi_{rj}=0 & \text{for } j\neq r \end{array}\right\} \text{ at } \eta=0 \qquad (9a)$$

$$F'_{rj}=0, \quad \Phi_{rj}=0 \quad \text{for all } j \quad \text{at } \eta\to\infty. \qquad (9b)$$

In Table 2 we present the wall derivatives of the universal functions from which the rate of heat transfer and the skin friction can be calculated. Such tabulations serve as a reference against which other approximate solutions can be compared. We note that our results for $f''_0(0)$ and $\theta'_0(0)$ which correspond to the case of an isothermal flat plate agree very closely with those of Ostrach [4]. We also mention that for those who wish to follow similar configuration problems, the present tabulated data provide enough test cases for checking the computer program.

The velocity and temperature profiles associated with some universal functions are illustrated in Figures 1 and 2 for Pr = 0.733.

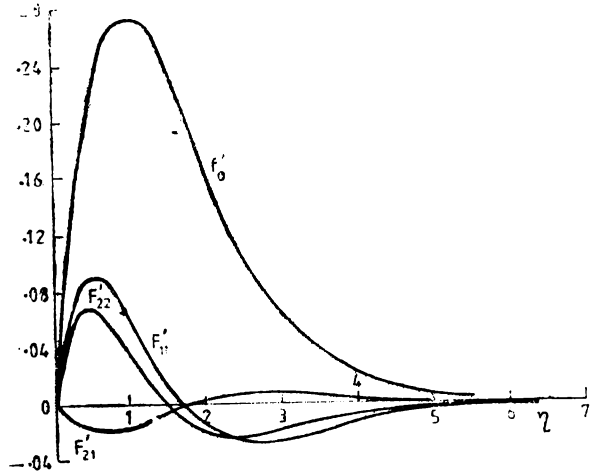

Fig. 1. Velocity function distributions for Pr = 0.733

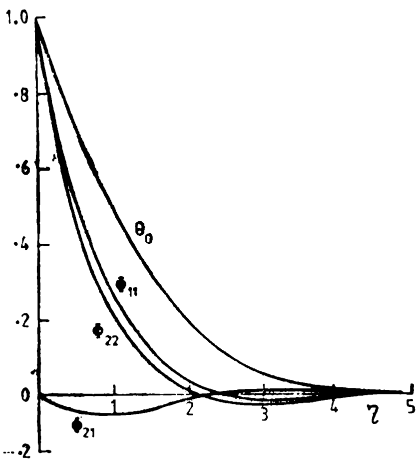

Fig. 2. Temperature function distribution for Pr = 0.733

Table 2

Values of the derivatives at the plate for $n = 0$

	Pr 0.733	Pr = 7
$f_0''(0)$	0.6741	0.4494
$\theta_0'(0)$	−0.5079	−1.0508
$F_{11}''(0)$	0.3873	0.2563
$\Phi_{11}'(0)$	−0.9286	−1.8539
$F_{21}''(0)$	−0.0382	−0.0251
$\Phi_{21}'(0)$	−0.1194	−0.2432
$F_{22}''(0)$	0.3361	0.2221
$\Phi_{22}'(0)$	−1.1175	−2.1941
$F_{31}''(0)$	−0.0021	−0.0024
$\Phi_{31}'(0)$	0.0492	0.0740
$F_{32}''(0)$	−0.0670	−0.0440
$\Phi_{32}'(0)$	−0.2563	−0.5193
$F_{33}''(0)$	0.3051	0.2016
$\Phi_{33}'(0)$	−1.2625	2.4557
$F_{41}''(0)$	0.0033	0.0018
$\Phi_{41}'(0)$	0.0016	−0.0130
$F_{42}''(0)$	0.0335	0.0218
$\Phi_{42}'(0)$	0.0864	0.1765
$F_{43}''(0)$	−0.0613	−0.0406
$\Phi_{43}'(0)$	−0.2724	−0.5498
$F_{44}''(0)$	0.2834	0.1874
$\Phi_{44}'(0)$	−1.3824	−2.6729
$F_{45}''(0)$	−0.0295	−0.0194
$\Phi_{45}'(0)$	−0.1297	−0.2636

REFERENCES

1. Jaluria, Y., *Natural convection heat and mass transfer*, Pergamon Press, New York, 1980
2. Kundu, K. M., *An analysis on free convection flow from a vertical plate with generalised wall temperature distribution*, Appl. Math. Modelling, 5 (1981), 122–126.
3. Soundalgekar, V. M., Takhar, H. S. and Singh, M., J. Non-Linear Analysis: Methods, Theory and Analysis, 7 (1983), 941–944.
4. Ostrach, S., *An analysis of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force*. NACA Report, 1111, 1953.

CRONICĂ

I. Publicaţii ale seminariilor de cercetare ale catedrelor de Matematică (serie de preprinturi)

Preprint 1—1985, Seminar of Functional Analysis and Numerical Methods.
Preprint 2—1985, Seminar of Stellar Structure and Stellar Evolution.
Preprint 3—1985, Seminar on Fixed Point Theory.
Preprint 4—1985, Seminar of Numerical and Statistical Calculus.
Preprint 5—1985, Seminar on Optimisation Theory.
Preprint 6—1985, Itinerant Seminar on Functional Equations, Approximation and Convexity.
Preprint 7—1985, Seminar of Mathematical Analysis.
Preprint 8—1985, Seminar on Best Approximation and Mathematical Programming.
Preprint 9—1985, Seminar of Numerical Solutions of Ordinary and Partial Differential Equations.
Preprint 10—1985, Seminar of Celestial Mechanics and Space Research.

II. Participări la manifestări ştiinţifice

1. Probleme actuale ale cosmologiei, Timişoara, 25 mai 1985: A Pál, Problema neutrinilor masivi şi a „materiei exotice cenuşii" în cosmologie şi astronomie galactică.
2. Coloquium on Applications of Mathematics on the Occasion of the 75-th Birthday of Lothar Collatz, Hamburg, 5—6 iunie 1985 (R.F.G.). Au participat: W. Breckner şi D. D. Stancu.
3. Seminarul de Ecuaţii diferenţiale şi control optimal, Bucureşti, septembrie, 1985: I. A. Rus, Realizări clujene în domeniul ecuaţiilor diferenţiale.
4. Conferinţa naţională de geometrie şi topologie, Călimăneşti, 26—29 septembrie 1985: P. Enghiş, Probleme asupra unor conexiuni semi-simetrice şi sfert-simetrice: P. Enghiş, M. Ţarină, Asupra unor conexiuni aproape produs: A. Vasiu, Asupra unei clase de prostructuri şi structuri de incidenţă cu elemente învecinate.
5. Sesiune Anuală a Institutului Central de Fizică, Iaşi 3—5 octombrie 1985: V. Ureche, T. Oproiu, Diagrame de imersiune la modele stelare relativiste de tip stepenar.
6. Colectivul Naţional de Informatică, Iaşi, 18—19 octombrie 1985: F. Boian, O metodă eficientă de implementare a deexecuţiei. D. Dumitrescu, Hierarchical Classification in Pattern Recognition. Gr. Moldovan, Unele probleme privind bazele de date distribuite.
7. Sesiunea ştiinţifică a Universităţii „Al. I. Cuza", din Iaşi, Iaşi, 24—26 octombrie 1985: M. Balázs, G. Goldner, O metodă de rezolvare aproximativă a ecuaţiilor funcţionale: Gh. Micula, Soluţii numerice pentru probleme eliptice liniare prin funcţii spline; P. Mocanu, On strangly-convex and strongly-starline functions; I. A. Rus, O măsură a proprietăţii de punct fix; Gr. Sălăgean, Asupra unor clase de funcţii univalente.
8. Progrese în Astronomie, Bucureşti, 25—26 octombrie 1985: V. Ureche, A. Imbroane, Momentul unghiular pentru stele relativiste liniare în rotaţie lentă; V. Ureche, T. Oproiu, Energia gravitaţională la stele relativiste de tip stepenar.
9. Simpozionul de Matematici aplicate, Timişoara, 1—2 noiembrie 1985: L. Ţâmbulea, Utilizarea calculatorului numeric la analiza formelor de undă biologice cu aplicaţii la curgerea sîngelui în artere; D. D. Stancu, Metode probabilistice în teoria aproximării funcţiilor.
10. Şedinţele de comunicări lunare ale Facultăţii de Matematică din Bucureşti: P. Mocanu, Demonstraţia lui L. de Branges a conjecturii lui Bieberbach.
11. Şedinţele de comunicări ale Secţiei de Matematică a Academiei R.S.R.: L. Lupşa, Mulţimi convexe faţă de o mulţime dată; P. Mocanu, Operatorii integrali şi funcţii stelate.

III. Manifestări ştiinţifice organizate de catedrele de matematică

1. Şedinţele de comunicări lunare ale catedrelor de matematică.
2. Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate, Cluj-Napoca, 23—25 mai 1985.
3. Conferinţa Naţională de Algebră, 6—8 iunie, 1985.
4. Conferinţa de ecuaţii diferenţiale, Cluj-Napoca, 21—23 noiembrie 1985.
5. Simpozionul de informatică şi aplicaţii, Cluj-Napoca, 6 decembrie, 1985.

RECENZII

R. Mneimmé, F. Testard, **Introduction à la théorie des groupes de lice classique**, Hermann Paris, Collection Méthodès, 1986, p. 345

C'est un ouvrage distingué par l'intention des auteurs à réaliser un exposé de la théorie de groupes de Lie plus accessible que celui de la plupart des livres consacrés. La table de la matière dont nous rappellons les principaux chapitres rend compte sur cet projet. À savoir on y présente.

1. Les premières propriétés des groupes GL(n, K), (K = R ou C) 2. Groupes topologiques opérant sur un ensemble. Application à l'étude de la topologie de GL(n, K). 3. La function exponentielle. Applications. 4. Étude des groupes orthogonaux. 5. Étude des groupes unitaires; géométries réelle et symplectique associées. 6. Étude des groupes symplectiques. 7. Intégration sur les variétés. Polynomes harmoniques.

Le livre continent aussi une liste de problèmes, un index terminologique, un index des notations et une bibliographie éssentielle.

La théorie générale vient d'etre ilustrée par des exemples concretes du domaine des groupes classiques, dont certaines propriétés sont traités d'une manière originale, inédite.

Le texte est adréssé aux étudiants qui préparent la licence en topologie et géométrie diffèrentielle, mais il offre une lecture instructive et attrayante à tous ceux qui s'interessent sur le sujet.

M. ŢARINĂ

L. Lovasz, M. D. Plummer, **Matching Theory**, Akadémiai Kiado, Budapest, 1986, 544 + XXXIII pp.

This book deals with the matchings (sets of edges without common points) in graphs. In the theory of matchings a lot of applied problems can be modelled, from which the entire theory was really born.

A complete treatment of this and related subjects is divided into twelve chapters. These chapters are the followings: 1. Matchings in bipartite graphs, 2. Flow theory, 3. Size and structure of maximum matchings, 4. Bipartite graphs with perfect matchings, 5. General graphs with perfect matchings, 6. Some graph-theoretical problems related to matchings, 7. Matchings and linear programmings, 8. Determinants and matchings, 9. Matching algorithms, 10. The f-factor problem, 11. Matroid matching, 12. Vertex packing and covering, and References with an impressive number of titles. Algorithmical aspects are also considered.

This well-written book is recommanded to all, who are interested in matching problems.

Z. KÁSA

F. Gécseg — M. Steinby, Tree Automata Akadémiai Kiado, Budapest 1984, 235 pages.

The book presents a rigorous mathematical discussion of the theory of tree automata, recognizable forests and tree transformations using, primarily, the language of universal algebra. It consists of four chapters. The first one contains topics of universal algebra, lattice, theory, finite automata and formal languages. Chapters II—IV present the basic results of tree automata theory: tree recognizers, tree grammars, recognizable forests and context-free languages, tree transducers and tree transformations.

The book is a good and systematic presentation of the results of the subject presented above and it is recomended to all who are interested in this field.

M. FRENŢIU

ÎNTREPRINDEREA POLIGRAFICĂ CLUJ,
Municipiul Cluj-Napoca, Cd. nr. 577/1986

Revista ştiinţifică a Universităţii din Cluj-Napoca, **STUDIA UNIVERSITATIS BABEŞ-BOLYAI,** apare începînd cu anul 1986 în următoarele condiţii:

matematică — trimestrial

fizică — semestrial

chimie — semestrial

geologie-geografie — semestrial pentru geologie şi anual pentru geografie

biologie — semestrial

filosofie — semestrial

ştiinţe economice — semestrial

ştiinţe juridice — semestrial

istorie — semestrial

filologie — semestrial

STUDIA UNIVERSITATIS BABEŞ-BOLYAI, the scientific journal of the University of Cluj-Napoca, starting with 1986 is issued as follows:

mathematics: quarterly

physics: biannually

chemistry: biannually

geology-geography: biannualy on geology and yearly on geography

biology: biannually

philosophy: biannually

economic sciences: biannually

juridical sciences: biannually

history: biannually

philology: biannually

43 875

Abonamentele se fac la oficiile poştale, prin factorii poştali şi prin difuzorii de presă, iar pentru străinătate prin „ROMPRESFILATELIA", sectorul export-import presă, P. O. Box 12—201, telex. 10376 prsfir, Bucureşti Calea Griviţei nr. 64—66.

Lei 35