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ŞCOALA MATEMATICĂ CLUJEANĂ — REALIZĂRI ŞI PERSPECTIVE

Şcoala matematică clujeană are vechi şi bogate tradiţii. Avînd ca şi fondatori pe Nicolae Abramescu, Aurel Angelescu, Theodor Angheluţă, Georghe Bratu, Gheorghe Iuga şi Petre Sergescu, după 23 August 1944, sub impulsul noilor transformări sociale conduse de Partidul Comunist Român, cercetările de matematică şi învăţămîntul matematic clujean au cunoscut o mare dezvoltare. Menţionăm în acest sens contribuţiile aduse de Gheorghe Călugăreanu (Teoria funcţiilor de variabilă complexă şi Teoria nodurilor), Tiberiu Popoviciu (Analiză numerică şi Teoria aproximării), Dumitru V. Ionescu (Analiză numerică şi Ecuaţii diferenţiale), Tiberiu Mihăilescu (Geometrie diferenţială), Gergely Eugen (Geometrie), Gheorghe Chiş (Astronomie) şi Gheorghe Pic (Algebră).

În ultimii ani cercetarea matematică clujeană s-a dezvoltat şi diversificat. La domeniile de cercetare cu bogată tradiţie s-au adăugat domenii noi ca : Logica matematică, Teoria categoriilor, Analiză funcţională, Cercetări operaţionale şi optimizare, Mecanică cerească şi Informatică. În cadrul celor 20 de seminarii care funcţionează pe lîngă colectivele de catedră (Matematica de bază, Didactica matematică, Informatica de bază, Analiza matematică, Teoria celei mai bune aproximări şi programarea matematică, Teoria optimizării, Teoria geometrică a funcţiilor analitice, Algebră abstractă, Geometrii pe inele, Geometrie diferenţială, Rezolvarea numerică şi aproximativă a ecuaţiilor diferenţiale şi cu derivate parţiale, Metode variaţionale, Teoria punctului fix, Calcul numeric şi statistic, Metode de aproximare numerică în hidrodinamică, Structura şi evoluţia stelelor, Mecanica cerească şi cercetări spaţiale, Institutul de calcul (Metode ale analizei funcţionale în analiza numerică), Centrul de calcul electronic (Modele, structuri şi prelucrări de informaţii) şi Laboratorul de cercetare interdisciplinară (Probleme actuale ale cercetării interdisciplinare) sînt abordate teme actuale de cercetare fundamentală şi teme legate de aplicaţiile matematicii. În afară de aceste seminarii au loc lunar şedinţe de comunicări. Pe lîngă aceste şedinţe de comunicări, Facultatea de Matematică organizează periodic, în colaborare cu facultăţile similare din ţară şi Societatea de Ştiinţe Matematice din R. S. România, următoarele manifestări ştiinţifice : Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate, Colocviul de cercetări operaţionale, Colocviul de geometrie şi topologie, Colocviul de mecanică şi Colocviul de astronomie, astro-fizică şi cercetări spaţiale.

Congresul al XIII-lea al Partidului Comunist Român a pus în faţa lucrătorilor din domeniul ştiinţei şi învăţămîntului sarcini de mare importanţă pentru dezvoltarea societăţii româneşti. Astfel, secretarul general al Partidului Comunist Român, tovarăşul Nicolae Ceauşescu, în raportul prezentat la Congres preciza următoarele :

„Cercetarea ştiinţifică românească are marea răspundere de a soluţiona mai rapid o serie de probleme de importanţă hotărîtoare pentru dezvoltarea economico-socială a patriei noastre".

„Programul de cercetare asigură îmbinarea organică a cercetării aplicative cu cercetarea fundamentală, în matematică, fizică, chimie, biologie, medicină şi alte domenii, sporind aportul ştiinţei la înfăptuirea cincinalului 1986—1990, cît şi la asigurarea de soluţii tehnice pentru înfăptuirea obiectivelor dezvoltării economico-sociale a ţării în perioada de după 1990."

Alături de întregul nostru popor, cadrele didactice şi studenţii facultăţii se angajează să facă totul pentru traducerea în viaţă a sarcinilor ce le revin din documentele şi hotărîrile Congresului al XIII-lea al Partidului Comunist Român.

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SUR L'APPROXIMATION DES FONCTIONS SEMI-CONTINUES PAR DES SUITES DE POLYNÔMES

SORIN GH. GAL*

1. **Introduction.** Soit $C_{[a,b]}$, l'ensemble des fonctions réelles, continues dans $[a, b]$. Dans [2], [3], en utilisant un résultat simple de „separation" des fonctions continues (voir par exemple [2], théorème 2.1.), j'ai démontré que pour une fonction $f \in C_{[a,b]}$, on peut construir une suite de polynômes convergeant uniformément vers f monotone décroissante (et croissante) sur $[a, b]$.

Le but de cette note est d'utiliser ce résultat de „séparation", pour les fonctions semi-continues, en obtenant ainsi, des résultats similaires.

D'ailleurs, le résultat obtenu pour les fonctions semi-continues, donne, dans le cas particulier des fonctions continues, des extensions des résultats de [2].

On obtient des résultats similaires aussi bien les fonctions monotones que pour les fonctions à variation bornée par exemple.

2. Voici le résultat de „séparation" avec la démonstration de [2]:

LEMME 2.1. *Si $f, g \in C_{[a,b]}$ ont la propriété $f(x) - g(x) \geqslant d > 0$, $\forall x \in [a, b]$, alors il existe un polynôme P, tel que*

$$g(x) < P(x) < f(x), \quad \forall x \in [a, b]. \tag{1}$$

Démonstration. En appliquant le théorème de Weierstrass pour la fonction $h(x) = (f(x) + g(x))/2$ et $\varepsilon = d/4$, il existe un polynôme P en vérifiant

$$P(x) - d/4 < h(x) < P(x) + d/4, \ \forall x \in [a, b]. \tag{2}$$

Mais $h(x) - g(x) = f(x) - h(x) = (f(x) - g(x))/2 \geqslant d/2 > 0$, $\forall x \in [a, b]$, donc $h(x) \geqslant d/2 + g(x)$ et $f(x) \geqslant d/2 + h(x)$, $\forall x \in [a, b]$.

En tenant compte de (2), nous obtenons

$P(x) + d/4 > h(x) \geqslant d/2 + g(x)$ et $f(x) \geqslant d/2 + h(x) > d/2 + P(x) - d/4$, d'où $g(x) < P(x) < f(x)$ $\forall x \in [a, b]$, c.q.e.d..

Soit maintenant $(u_n)_n$ une suite de fonctions réelles, définies sur $[a, b]$. On dit que la suite est „convexe" („concave") sur $[a, b]$ si

$$u_{n+2}(x) + u_n(x) - 2u_{n+1}(x) > 0 \ (<0), \ \forall n \in N, \ \forall x \in [a, b].$$

Alors, a lieu le

THEOREME 2.2. *Soit $f: [a, b] \to R$, supérieur bornée. Alors il existe une suite de polynômes $(P_n)_n$, monotone décroissante et „convexe" sur $[a, b]$, $P_n(x)$ en convergeant vers $f(x)$, dans chaque point $x \in [a, b]$ où f est supérieur semi-continue.*

Démonstration. Considérons la suite $f_n(x) = \sup\{f(t) - n \cdot |x - t|;\ t \in [a, b]\}$, $n \in N$, $x \in [a, b]$.

D'après le résultat de Baire (voir [1]), chaque fonction f_n est continue sur $[a, b]$ et la suite $(f_n)_n$ est non-croissante (c'est-à-dire $f_{n+1}(x) \leqslant f_n(x)$, $\forall x \in [a, b]$,

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$\forall n \in N$), $f_n(x)$ en convergeant vers $f(x)$, dans chaque point $x \in [a, b]$ où f est supérieur semi-continue.

Maintenant, je soutiens que la suite $(f_n)_n$ vérifie aussi

$$f_{n+2}(x) + f_n(x) - 2f_{n+1}(x) \geq 0, \ \forall n \in N, \ \forall x \in [a, b]. \tag{3}$$

En effet, si pour $x \in [a, b]$ fixé, quelconque, notons $h_n(x, t) = f(t) - n \cdot |x - t|$, $n \in N$, $t \in [a, b]$, alors évidemment $2h_{n+1}(x, t) = h_{n+2}(x, t) + h_n(x, t)$ et nous obtenons $2 \sup \{h_{n+1}(x, t) ;\ t \in [a, b]\} = \sup \{h_{n+2}(x, t) + h_n(x, t) ;\ t \in [a, b]\} \leq \sup \{h_{n+2}(x, t) ;\ t \in [a, b]\} + \sup \{h_n(x, t) ;\ t \in [a, b]\}$, donc la relation (3). Notons maintenant $F_n(x) = f_n(x) + 1/n, n \in N$, $x \in [a, b]$. Évidemment $(F_n)_n$ est décroissante et $F_n(x) - F_{n+1}(x) = f_n(x) - f_{n+1}(x) + 1/n - 1/(n + 1) > 1/(n(n+1)) > 0$, $\forall n \in N$, $\forall x \in [a, b]$.

Puis $F_{n+2}(x) + F_n(x) - 2F_{n+1}(x) = f_{n+2}(x) + f_n(x) - 2f_{n+1}(x) + 1/(n + 2) + (1/n) - 2/(n + 1) = f_{n+2}(x) + f_n(x) - 2f_{n+1}(x) + 2/(n(n + 1)(n + 2))$, d'où compte teur compte de (3), il résulte

$$F_{n+2}(x) + F_n(x) - 2F_{n+1}(x) > 2/(n(n + 1)(n + 2)) > 0, \ \forall n \in N, \ \forall x \in [a,b], \tag{4}$$

donc la suite (F_n) est aussi „convexe".

Evidemment $F_n(x)$ converge vers $f(x)$, dans chaque point $x \in [a, b]$ où $f_n(x)$ converge vers $f(x)$.

Maintenant, en tenant compte du lemme 2.1. pour

$$0 < d = d_n = 1/(n(n + 1)(n + 2)), \tag{5}$$

il existe un polynôme P_n(n — fixé) tel que

$$F_{n+1}(x) < P_n(x) < F_{n+1}(x) + d_n, \ \forall x \in [a, b], \ \forall n \in N. \tag{6}$$

Mais comme $F_n(x) - F_{n+1}(x) > 1/(n(n + 1)) > 1/(n(n + 1)(n + 2)) = d_n$, $\forall n \in N$, $\forall x \in [a, b]$, il résulte

$$F_{n+1}(x) < P_n(x) < F_{n+1}(x) + d_n < F_n(x), \ \forall n \in N, \ \forall x \in [a, b]. \tag{6'}$$

En remplaçant dans (6′) n avec $n + 1$ et $n + 2$, nous obtenons

$$F_{n+2}(x) < P_{n+1}(x) < F_{n+2}(x) + d_{n+1} < F_{n+1}(x) \text{ et} \tag{7}$$

$$F_{n+3}(x) < P_{n+2}(x) < F_{n+3}(x) + d_{n+2} < F_{n+2}(x), \ \forall n \in N, \ \forall x \in [a, b]. \tag{8}$$

Alors, de (6′), (7), (8) nous avons $P_{n+2}(x) + P_n(x) - 2P_{n+1}(x) > F_{n+3}(x) + F_{n+1}(x) + 2(- F_{n+2}(x) - d_{n+1}) = F_{n+3}(x) + F_{n+1}(x) - 2F_{n+2}(x) - 2d_{n+1} > 0$, $\forall n \in N$, $\forall x \in [a, b]$, en tenant compte de (4) et (5).

Puis, de (6′) et (7) la suite $(P_n)_n$ est aussi décroissante et il est évident que $P_n(x)$ converge vers $f(x)$, dans point $x \in [a, b]$ où $F_n(x)$ converge vers $f(x)$, donc dans chaque point où f est supérieur semi-continue, c.q.f.d.

COROLLAIRE 2.3. *Soit f: $[a, b] \to R$, inférieur bornée. Alors, il existe une suite de polynômes $(Q_n)_n$, monotone croissante et „concave" sur $[a, b]$, $Q_n(x)$ en convergeant vers $f(x)$, dans chaque point $x \in [a, b]$ où f est inférieur semi-continue.*

Démonstration. Si f est inférieur bornée et inférieur semi-continue dans un point $x \in [a, b]$, alors $-f$ est supérieur bornée et supérieur semi-continue dans x, donc on peut appliquer le théorème 2.2. pour la fonction $-f$, d'où le corollaire résulte facilement.

COROLLAIRE 2.4. *Soit $f: [a, b] \to R$, monotone sur $[a, b]$ (croissante ou décroissante).*

Alors, il existe les suites de polynômes $(P_n)_n$, $(Q_n)_n$, $P_n(x)$, $Q_n(x)$ en convergeant vers $f(x)$, presque partout dans $[a, b]$, $(P_n)_n$ — décroissante et „convexe" sur $[a, b]$ et $(Q_n)_n$ — croissante et „concave" sur $[a, b]$.

Démonstration. Si f est monotone sur $[a, b]$, alors f est évidemment bornée (supérieur et inférieur) sur $[a, b]$ et presque partout continue dans $[a, b]$, donc presque partout supérieur et inférieur semi-continue.

Alors, le corollaire 2.4. résulte du théorème 2.2. et du corollaire 2.3.

Remarque. En raissonnant comme ci-dessus, il est évident que le corollaire 2.4. a lieu aussi pour f à variation bornée sur $[a, b]$.

Voici maintenant l'extension des résultats de [2]:

COROLLAIRE 2.5. *Si $f \in C_{[a,b]}$, alors il y a suites de polynômes $(P_n)_n$ — décroissante et „convexe" et $(Q_n)_n$ — croissante et „concave" sur $[a, b]$, uniformément convergentes vers f.*

Démonstration. Comme les suites $(P_n)_n$, $(Q_n)_n$ du théorème 2.2. et du corollaire 2.3. sont monotones, d'aprés un résultat connu de Dini, il résulte la convergence uniforme des suites, c.q.e.d.

Remarque. Au fond, le théorème 2.2. nous montre l'existence d'une suite de polynômes qui conservent quelques propriétés de la suite de fonctions $(f_n(x))_n$, considérée par R. Baire (la convergence vers $f(x)$, la monotonie et la „convexité" („concavité") de la suite). Alors, il serait intéressant d'étudier des autres propriétés de la suite $(f_n)_n$, qu'on peut conserver par des suites de polynômes.

D'ailleurs, dans cet ordre idées, dans [4] on montre que si f est non-concave des ordres un, deux, trois et quatre sur $[0, 1]$, la suite des polynômes de Bernstein $(B_n(f; x))_n$, vérifie

$$B_{n+2}(f; x) - 2B_{n+1}(f; x) + B_n(f; x) \geqslant 0, \quad \forall n \in N, \quad \forall x \in [0, 1].$$

Mais comme de f non-concave d'ordre un, nous avons aussi la relation

$$B_{n+1}(f; x) \leqslant B_n(f; x), \quad \forall n \in N, \quad \forall x \in [0, 1],$$

il résulte que la suite des polynômes de Bernstein (ou plutôt $B_n(f; x) + 1/n$) représente une solution constructive de la suite $(P_n)_n$ du corollaire 2.5. (dans les hypothèses annonçées).

(Manuscrit reçu le 25 mars 1981)

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ASUPRA APROXIMĂRII FUNCŢIILOR SEMI-CONTINUE PRIN ŞIRURI DE POLINOAME

(R e z u m a t)

Fiind dată o funcţie continuă $f \in C_{[a,b]}$, în [2], [3] am construit un şir de polinoame convergent uniform către f, monoton descrescător (respectiv crescător) pe $[a, b]$. În prezenta notă, folosind rezultatele precedente, se construiesc şiruri cu proprietăţi asemănătoare pentru o funcţie $f: [a, b] \to R$ semi-continuă.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXX, 1985

THE PROXIMAL POINTS ALGORITHM FOR REFLEXIVE BANACH SPACES

G. KASSAY

1. **Introduction.** In 1976 appeared two articles by R. T. Rockafellar, of which the first one in "SIAM J. Control and optimization" ([3]), entitled "Monotone operators and the proximal point algorithm", the second one appeared in "Mathematics of operation research ([4]) and was entitled "Augmented Lagrangians and application of the proximal point algorithm in convex programming".

In the first work an algorithm based upon the theory of monotone operators is presented, in the second one were to be giving certain applications of this theory, relative to optimization problems.

The environment in which the problem is treated is a real Hilbert space H, where a maximal monotone operator $T: H \to 2^H$ is defined.

The purpose of the algorithm is to solve the operatorial equation $0 \in T(x)$ (see [3]). The main idea that suggested the construction of the algorithm was G. Minty's theorem (see [1]), namely that if $T: H \to 2^H$ is maximal monotone, then the operator P — called proximal (see Moreau [2]) — defined by $P := (I + cT)^{-1}$ is single-valued, nonexpansive, its domain being the whole Hilbert space H, for anyconstant $c > 0$. (Here I denotes the identity operator on H).

2. **Preliminaries.** As monotone operators are defined on more general space than Hilbert space, following Rockafellar's results I proposed to establish methods for solving equation $O \in T(x)$, where T is an operator defined on a reflexive Banach space X, maximal monotone:

$$T: X \to 2^{X^*} \quad (X^* \text{ denotes the dual space of } X)$$

In the construction of a „proximal" operator (of type $(I + cT)^{-1}$ in Hilbert case, see [3]) I based on the existence of dual applications $J_\varphi: X \to X^*$ for any $\varphi: [0, +\infty) \to R$ with properties:

1. φ continuous and increasing
2. $\varphi(0) = 0$
3. $\varphi(r) \to +\infty$, $r \to +\infty$

J_φ is called duality application with regulation function φ, and is defined by: $\forall x \in X$, $J_\varphi(x) = x^*$, where $x^* \in X^*$ so that:

i) $\langle x, x^* \rangle = \varphi(||x||) \cdot ||x||$

ii) $||x^*|| = \varphi(||x||)$

The existence of $x^* \in X^*$ for any $x \in X$ satisfying i) and ii) is assured by Hahn-Banach's theorem (upon extension of liniar and continuous functions), and it is unique when X^* is strictly convex (see [5] p. 54, Theorem 1.2) As any reflexive Banach space admits an equivalent strictly convex norm ([5]), p. 91, Theorem 3.2), this assumption upon X^* will not constrain the generality of the problem.

Let J be the duality application for which $\varphi(r) = r$. (normalized duality map). There was proved the following result (more general then Minty's theorem, see [11]) : If $T : X \to 2^{X^*}$ is maximal monotone, where X is a reflexive Banach space, and $c > 0$ an arbitrary constant, then $J + cT$ is surjective. Moreover, the operator $(J + cT)^{-1} : X^* \to X$ is single-valued, maximal monotone and demi-continuous (see [6] p. 122, Prop. 2.11)

Having these results, the „proximal" operator will be defined as following: Let $(\varphi_k)_{k\in N}$ and $(c_k)_{k\in N}$ any sequences of positive numbers, with $\varphi_k \to +\infty$, (c_k) bounded away from O; Let J_k, $k \in N$ be the duality applications defined by :

$$J_k(x) = \frac{1}{\varphi_k} \cdot J(x), \ \forall \, x \in X$$

Than the regularity function of J_k is $\varphi_{(r)} = \frac{1}{\varphi_k} (r)$ and obviously satisfies properties 1—3.

For any $k \in N$, we define : $P_k : X^* \to X$, $P_k : = (J_k + c\,T)^{-1}$.

These operators furnish the sequence $(x^k)_{k\in N}$, $x^k \in X$, called the sequence of proximal points, shortly the proximal sequence, by the following algorithm :

$$x^\circ \in X \text{ arbitrary}$$

$$\forall \, k \in N : x^{k+1} \approx P_k \circ J_k(x^k) \tag{1}$$

Composing P_k with J_k was necessary because P_k is defined on X^*. In (1) we have different methods to approximate $P_k \circ J_k(x^k)$. These methods depend especially on the results we proposed to obtain relative to the convergence of $(x^k)_{k\in N}$.

We'll show that under certain assumptions, the proximal sequence approximates the solution of $0 \in T(x)$.

In the following, we'll treat three cases :

— approximation in the weak topology of X

— ,, ,, ,, strong ,, ,,

— establishing the solution after a finite number of iterations

3. The weak approximation. From now on X will denote a reflexive Banach space, with its dual X^* strictly convex, and $T : X \to 2^{X^*}$ a maximal monotone operator.

We will also consider sequences (φ_k), (c_k), the operators J_k and P_k for any $k \in N$, defined in the preceeding paragraph and having all the properties stated before. In addition we shall make use of the mappings :

$$Q_k : X \to X^*, \quad Q_k = J_k - J_k \cdot P_k \cdot J_k, \qquad \forall \, k \in N$$

Thus the following properties are valid:

PROPOSITION: i) *For any* $k \in N$, $\frac{1}{c_k} Q_k(x) \in T(H_k(x))$, $\forall x \in X$, *where* H_k *denotes the composed operator* $P_k \cdot J_k$.

ii) $0 \in T(x) \Leftrightarrow Q_k(x) = 0$, $\forall k \in N$

Proof. i) Let $k \in N$ be arbitrary. We have $H_k(x) = (J_k + c_k T)^{-1} \circ J_k(x)$ for any $x \in X$, or $J_k(x) \in (J_k + c_k T) \circ H_k(x) = J_k \circ H_k(x) + c_k T \circ H_k(x)$, which is equivalent to $J_k(x) - J_k H_k(x) \in c_k T \circ H_k(x)$. Dividing this relation with c_k, we obtain i).

ii) $0 \in T(x) \Leftrightarrow 0 \in c_k T(x)$, $\forall_k \in N \Leftrightarrow J_k(x) \in J_k(x) + c_k T(x) = (J_k + c_k T)(x) \Leftrightarrow x = (J_k + c_k T)^{-1} \circ J_k(x) \Leftrightarrow x = H_k(x) \Leftrightarrow J_k(x) = J_k \circ H_k(x)$, $\forall_k \in N$, or $Q_k(x) = 0$. (The duality operators J_k are injectives, see [6]). Having this properties we can state the following theorem:

THEOREM 1. *Let* (x^k) *be a proximal sequence for* T, *obtained by the following selection criterion in* (1), § 2:

$$(A): \|x^{k+1} - H_k(x^k)\| \leqslant \varepsilon_k, \ \varepsilon_k \xrightarrow{k\to\infty} 0$$

Assuming (x^k) *is bounded, the equation* $0 \in T(x)$ *admits at least one solution; Moreover, the sequence* (x^k) *admits a weak cluster point, solution of the equation* $0 \in T(x)$.

Proof. The sequence (x^k) being bounded, there is a number $M > 0$ so that $\|x^k\| \leqslant M$ for any $k \in N$, and $\varepsilon_k < M(\varepsilon_k \to 0)$.

As in a reflexive Banach space any closed and bounded set is weakly compact, there exists an $x' \in X$ which is a weak cluster point of (x^k); so that: $\|x'\| \leqslant M$.

We'll show that the sequence $(Q_k(x^k))_k$ converges to 0 in the strong topology of X^*. We have:

$$\|Q_k(x^k)\| = \|J_k(x^k) - J_k \circ H_k(x^k)\| \leqslant \|J_k(x^k)\| + \|J_k \circ H_k(x^k)\| =$$
$$= \frac{1}{\varphi_k} \|x^k\| + \frac{1}{\varphi_k} \|H_k(x^k)\| = \frac{1}{\varphi_k} [\|x^k\| + \|x^{k+1} - H_k(x^k) + x^{k+1}\|] \leqslant$$
$$\leqslant \frac{1}{\varphi_k} [\|x^k\| + \varepsilon_k + \|x^{k+1}\|] \leqslant \frac{1}{\varphi_k} \cdot 3M \to 0, \ k \to \infty.$$

From monotonicity of T comes:

$\langle x - H_k(x^k), y - \frac{1}{c_k} \cdot Q_k(x^k)\rangle \geqslant 0$, $\forall k \in N$, $x \in X$ and $y \in T(x)$ (see prop. 1, i)). (1)

As $\|x^{k+1} - H_k(x^k)\| \to 0$ $(\varepsilon_k \to 0)$, x' is a weak cluster point for the sequence $(H_k(x^k))$ too.

Also, as $Q_k(x^k) \xrightarrow{k} 0$ strongly, $\frac{1}{c_k} Q_k(x^k) \xrightarrow{k} 0$ strongly (see the assumptions made upon (c_k)). Having these stated, from (1) we obtain:

$$\langle x - x', y\rangle \geqslant 0, \ \forall x \in X, \ y \in T(x) \quad \text{(see [5])} \tag{2}$$

Taking the maximality of T, this implies that $0 \in T(x')$

Remarks. 1. In this case, the operators $P_k = (J_k + c_k T)^{-1}$ are not nonexpansive as in Hilbertian case, because the duality operators J_k are not linear. It will be shown that of the duality operator is linear, then the Banach space is a Hilbert space and in this case the duality operator J reduces to the identity operator I (see [5], p. 40, theorem 3.1.).

Rockafellar uses the nonexpansivity of the proximal operators in theorem 1 in order to show that $Q_k(x^k) \to 0$ (Q_k was defined as $I - P_k$, see [3]) and also that the sequence (x^k) has a unique weak cluster point.

As it could be seen this property was not essential in the proof of the convergence of $Q_k(x^k)$. On the other hand, the uniqueness of the cluster point in the case of reflexive Banach spaces was not shown, this remaining an open question.

However, we mention that in our case x^{k+1} has to approximative $H_k(x^k)$ "less good", because in criterion (A) it is claimed that $\varepsilon \to 0$, without any assumption upon the convergence of the sery $\sum_{k=1}^{k} \varepsilon_k$ (see [3], criterion (A)). This fact is favorable in the effective construction of the proximal sequence.

2. The operators J_k were introduced in the expression of P_k in order to assure the strong convergence of $Q_k(x^k)$ to 0.

4. **Strong approximation.** In this paragraph will be treated the case in which the proximal sequence converges strongly to the solution of the equation $0 \in T(x)$. This case was studied for reflexive Banach spaces with property (H).

DEFINITION 1. (see [5]) *Let X be a Banach space. We say that X has the property (H) if it is strictly convex and if for any sequence $(x^n)_{n\in N}$ from X satisfying $x^n \rightharpoonup x$ and $\|x^n\| \to \|x\|$, we have $x^n \to x$.* (The symbols "$\rightharpoonup$" and "$\to$" denotes the convergence in the weak, respectively strong topology of X).

We remark that in any reflexive Banach space, exists an equivalent norm through which X and X^* has the property (H).

DEFINITION 2. *An operator $U: X \to 2^{X^*}$ ($X^* \to 2^X$) is Lipschitz-continuous at the origin, with modulus $a > 0$, if it satisfies the following two properties:*

i) $U(0) = \{\bar{x}\}$ (*U is single-valued in* 0)

ii) $\exists \tau > 0$, $\forall y \in X$ (resp. X^*) *with* $\|y\| \leqslant \tau$, $\forall x \in U(y)$,

we have:

$$\|x - \bar{x}\| \leqslant a \cdot \|y\| \tag{1}$$

In the rest, the notations will be those used in the preceeding paragraphs.

THEOREM 2. *Let (x^k) be a proximal sequence for T (maximal monotone). Suppose that $T^{-1}: X^* \to 2^X$ is Lipschitz — continuous at the origin with modulus $a > 0$, and also (x^k) is bounded. The following affirmations are valid:*

1° *If (x^k) is constructed due to criterion* (A) (see § 3), *then it converges strongly to $\bar{x} \in X$, the unique solution of* $0 \in T(x)$.

2° *If X^* has the property* (H), *and (x^k) is obtained using criterion* (B), *namely:*

$$\forall k \geqslant 1: \|x^{k+1} - H_k(x^k)\| \leqslant \delta_k \|x^1 - x^0\|, \ \delta_k \xrightarrow{k} 0$$

with $x' \neq x^\circ$ choosen arbitrarily, the sequence (x^k) converges strongly to $\bar{x}$, the unique solution of $0 \in T(x)$, moreover, we have the following "apriori" estimation:

$$\forall k \in N \text{ sufficiently large } \|x^{k+1} - \bar{x}\| \leqslant \mu_k \cdot \|x^1 - x^0\|, \ \mu_k \xrightarrow{k} 0$$

3° *If X^* has property* (H) *and (x^k) is obtained by criterion* (C):

$$\|x^{k+1} - H_k(x^k)\| \leqslant \delta'_k \cdot \|J(x^{k+1}) - J(x^k)\|, \ \delta'_k \xrightarrow{k} 0$$

and

$$\|J(x^{k+1}) - J \circ H_k(x^k)\| \leqslant \delta''_k \ \|J(x^{k+1}) - J(x^k)\|, \ \to 0$$

is bounded, then its converges strongly to $\bar{x}$ (the unique solution), moreover, we have the following "aposteriori" estimation:

$$\|x^{k+1} - \bar{x}\| \leqslant 0_k \|J(x^{k+1}) - J(x^k)\|, \ \forall_{k \in N} \text{ sufficiently large, with } 0_k \xrightarrow{k} 0.$$

Proof. 1° As T^{-1} is Lipschitz — continuous at 0, the equation $0 \in T(x)$ admits a unique solution. Let this be $\bar{x}$. We remark that the assumptions of theorem 1 are satisfied, which means that $Q_k(x^k) \xrightarrow{k} 0$. Let $\tau > 0$ be a number for which the relation $\|x - \bar{x}\| \leqslant a \cdot \|y\|$, $\forall y \in X^*$, $\|y\| \leqslant \tau$ and $x \in T^{-1}(y)$ is satisfied (see (1)). We select an order $k_0 \in N$, so that for any $k \geqslant k_0$: $\left\|\frac{1}{c_k} Q_k(x^k)\right\| \leqslant \tau$. Proposition 1/i implies:

$$H_k(x^k) \in T^{-1}\left(\frac{1}{c_k} Q_k(x^k)\right), \ k \in N. \text{ Thus:}$$

$\forall k \geqslant k_0$: $\|H_k(x^k) - \bar{x}\| \leqslant a \cdot \frac{1}{c_k} \|Q_k(x^k)\|$, relation that brings to:

$$\|H_k(x^k) - \bar{x}\| \xrightarrow{k} 0 \tag{2}$$

Let's estimate $\|x^{k+1} - \bar{x}\|$ in the following way:

$$\|x^{k+1} - \bar{x}\| \leqslant \|x^{k+1} - H_k(x^k)\| + \|H_k(x^k) - \bar{x}\|$$

This relation implies $\|x^{k+1} - \bar{x}\| \xrightarrow{k} 0$ due to criterion (A) and (2); thus the affirmation is proved.

2° We use a known result, namely that if X is a reflexive Banach space for which X^* has the property (H), any duality application on X is continuous (from X to X^* in the strong topologies) (see [5], p. 121, th. 5.1.).

As criterion (B) implies (A), due to 1°, (x^k) converges strongly to $\bar{x}$. The application $J: X \to X^*$ being continuous, we have $J(x^k) \xrightarrow{k} J(\bar{x})$, and thus $\|J(x^{k+1}) - J(x^k)\| \xrightarrow{k} 0$. Let's consider an order $k_1 \in N$ such that for any $k \geqslant k_1$:

$$\|J(x^{k+1}) - J(x^k)\| \leqslant \|x^1 - x^0\| \tag{3}$$

On the other hand, the strong convergence of (x^k) implies the strong convergence of $(H_k(x^k))$ to the same $\bar{x}$. Really, as

$$||H_k(x^k) - x^{k+1}|| \geq \Big| ||x^{k+1} - \bar{x}|| - ||H_k(x^k) - \bar{x}|| \Big| \text{ and}$$

$$||H_k(x^k) - x^{k+1}|| \xrightarrow{k} 0 \text{ (crit. } (B)), \text{ we have } ||H_k(x^k) - \bar{x}|| \xrightarrow{k} 0$$

The operator J being continuous, we also have $||J \circ H_k(x^k) - J(\bar{x})|| \xrightarrow{k} 0$, and thus $||J(x^{k+1}) - J \circ H_k(x^k)|| \xrightarrow{k} 0$.

Let's consider an order $k_2 \in N$, so that for any $k \geq k_2$ we have:

$$||J(x^{k+1}) - J \circ H_k(x^k)|| \leq ||x^1 - x^0|| \tag{4}$$

Let $\bar{k} = \max\{k_0, k_1, k_2\}$ (k_0 is the order from the proof of 1°). Thus for any $k \geq \bar{k}$:

$$||x^{k+1} - \bar{x}|| \leq ||x^{k+1} - H_k(x^k)|| + ||H_k(x^k) - \bar{x}|| \leq \delta_k ||x^1 - x^0|| + \frac{a}{c_k} ||Q_k(x^k)|| =$$
$$= \delta_k ||x^1 - x^0|| + \frac{a}{c_k} ||J_k(x^k) - J_k \circ H_k(x^k)|| = \delta_k ||x^1 - x_0|| + \frac{a}{c_k \cdot \varphi_k} ||J(x^k) -$$
$$- J \circ H_k(x^k)|| \leq \delta_k ||x^1 - x_0|| + \frac{a}{c_k \varphi_k} \cdot ||J(x^{k+1}) - J(x^k)|| + \frac{a}{c_k \varphi_k} ||J(x^{k+1}) -$$
$$- J \circ H_k(x^k)|| \leq \delta_k ||x^1 - x^0|| + \frac{2a}{c_k \cdot \varphi_k} ||x^1 - x^0|| = \left(\delta_k + \frac{2a}{c_k \cdot \varphi_k}\right) \cdot ||x^1 - x^0||.$$

Denoting $\mu_k = \delta_k + \frac{2a}{c_k \varphi_k}$, we have $\mu_k \to 0$ and also

$$||x^{k+1} - \bar{x}|| \leq \mu_k ||x^1 - x^0||, \text{ for any } k \geq \bar{k}.$$

3° Before proving this statement, we must show that criterion (C) is well-defined; In other words, at each iteration, $x^{k+1} \in X$ can be selected so that both inequalities should be satisfied simultaneously. In order to prove that the first inequality has sense, we must show that for any $\varepsilon > 0$ and $a, b \in X$, $a \neq b$

$$\exists x \in X \text{ so that } ||x - a|| \leq \varepsilon ||J(x) - J(b)|| \tag{5}$$

As operator J is injective (is strictly monotone, see [6] p. 116, Th 2.6), $J(a) \neq J(b)$. Let $||J(a) - J(b)|| = d < 0$. As J is continuous in a, we have:

$$\forall \delta > 0: \exists \tau > 0: \forall x \in X, ||x - a|| \leq \tau \Rightarrow ||J(x) - J(a)|| \leq \delta.$$

The right side of inequality (5) can be written:

$$||J(x) - J(b)|| \geq ||J(a) - J(b)|| - ||J(a) - J(x)|| = d - ||J(a) - J(x)|| \tag{6}$$

Let $\delta := \frac{d}{2}$. In accordance with (6) we have:

$$\exists \tau > 0: ||J(x) - J(b)|| \geq \frac{d}{2}, \forall x \in X \text{ with } ||x - a|| \leq \tau \tag{7}$$

Taking $\varepsilon > 0$ arbitrarily, let $x \in X$ be so that:

$$||x - a|| \leqslant \min\left\{\tau, \varepsilon \cdot \frac{d}{2}\right\} \text{ and thus we obtain inequality} \qquad (5)$$

The second inequality from criterion (C) is compatible with the first one, J being continuous. As in the preceeding case (2°) the strong convergence of (x^k) to $\bar{x}$, unique solution of the equation $0 \in T(x)$ is immediate, as the boundedness of (x^k) is equivalent with the boundedness of $(J(x^k))$ and thus the first inequality from (C) implies criterion (A).

In order to obtain the wished estimation, let's write the followings:

$$||x^{k+1} - \bar{x}|| \leqslant ||x^{k+1} - H_k(x^k)|| + ||H_k(x^k) - \bar{x}|| \leqslant \delta'_k||J(x^{k+1}) - J(x^k)|| +$$
$$+ \frac{a}{c_k\varphi_k}||J(x^{k+1}) - J(x^k)|| + \frac{a}{c_k\varphi_k}||J(x^{k+1}) - J \circ H_k(x^k)|| \leqslant \delta'_k||J(x^{k+1}) -$$
$$- J(x^k)|| + \frac{a}{c_k\varphi_k}||J(x^{k+1}) - J(x^k)|| + \frac{a}{c_k\varphi_k}\delta''_k||J(x^{k+1}) - J(x^k)|| =$$
$$= \left[\delta'_k + \frac{a}{c_k\varphi_k}(1 + \delta''_k)\right] \cdot ||J(x^{k+1}) - J(x^k)||, \ \forall k \geqslant k_0 \text{ (see also 2°)}.$$

Denoting $\theta_k = \delta'_k + \frac{a}{c_k\varphi_k}(1 + \delta''_k)$ it becomes:

$$\forall k \geqslant k_0: ||x^{k+1} - \bar{x}|| \leqslant \theta_k||J(x^{k+1}) - J(x^k)||, \quad \theta_k \xrightarrow{k} 0$$

Remarks. 1) As the first two terms of the proximal sequence (x^k) are arbitrary, it is convenient that in the apriori estimation from 2°, x^0 and x^1 should be choosen sufficiently close to each-other, namely so that $||x^1 - x^0||$ should be little. This implies the diminishing of the error $||x^{k+1} - \bar{x}||$ at iteration $k + 1$.

2) Both in the cases 2° and 3°, the convergence of the proximal sequence can be "accelerated" by choosing the sequences (φ_k) and (δ_k) respectively (φ_k), (δ'_k) and (δ''_k) so that (μ_k) and (θ_k) should converge to 0 is fastly as possible. (For example (φ_k) is taken so that $|\varphi_{k+1} - \varphi_k| \to +\infty$ and (δ_k) so that $\frac{\delta k}{k} \to 0$, in 2°).

3) In criterion (C) the convergence of (δ''_k) is not essential because the boundedness of (δ''_k) assures the convergence of θ_k to 0. But in order to obtain a better approximation it is recommended that this sequence should also be convergent to 0.

5. Establishing the solution of the $0 \in T(x)$ equation after a number of finite steps. We will study a case in which the equation $0 \in T(x)$ has a unique solution, which can be obtained by the proximal sequence $(x^k)_{k \in N}$ in its exact form (i.e. $x^{k+1} = H_k(x^k)$) after a finite number of iterations.

THEOREM 3. *Let (x^k) be a proximal sequence obtained by $x^{k+1} = H_k(x^k)$, $\forall k \in N$ with x^0 arbitrary* (1)

Suppose there exists $\bar{x} \in X$ *so that* $0 \in \operatorname{int} T(x)$ *and* (x^k) *is bounded. Then* $\exists k_0 \in N$ *with:*

$$\forall k \geqslant k_0 : x^k = \bar{x} \tag{2}$$

Proof. Notice that the definition of the proximal sequence (1) implies any criterion studied. We'll show that the operator $T^{-1}: X^* \to 2^X$ is constant and single-valued on a neighbourhood of 0. Since $0 \in T(\bar{x})$, $\exists \varepsilon > 0$ so that $\forall y' \in X^*$ with $\|y'\| \leqslant \varepsilon : y' \in T(\bar{x})$; in other words, $(\bar{x}, y') \in G(T)$ (Here $G(T)$ denotes the graph of T).

The monotonicity of T implies:

$\langle x - \bar{x}, y - y' \rangle \geqslant 0$, $\forall (x, y) \in G(T)$, $\forall y' \in X^*$ with $\|y'\| \leqslant \varepsilon$, or in other form $\langle x - \bar{x}, y' \rangle \leqslant \langle x - \bar{x}, y \rangle$.

Taking supremum from the left member of this inequation after y', we have:

$$\sup_{\|y'\| \leqslant \varepsilon} \langle x - \bar{x}, y' \rangle \leqslant \langle x - \bar{x}, y \rangle, \quad \forall (x, y) \in G(T) \tag{3}$$

Let $x \in X$, $x \neq \bar{x}$, and $y' := \frac{\varepsilon}{\|x - \bar{x}\|} \cdot J(x - \bar{x})$. Obviously, $\|y'\| = \varepsilon$ Thus from (3) comes:

$\varepsilon \|x - \bar{x}\| \leqslant \langle x - \bar{x}, y \rangle \leqslant \|x - \bar{x}\| \cdot \|y\|$, $\forall (x, y) \in G(T)$, $x \neq \bar{x}$, or $\|y\| \geqslant \varepsilon$. This implies that for any $(x, y) \in G(T)$, with $\|y\| < \varepsilon$ we have $x = \bar{x}$, which means that T^{-1} is constant in the neighbourhood $V := \{y / \|y\| < \varepsilon\}$ of the origin;

As a consequence of Proposition 1/i (§ 2) we have:

$$\forall k \in N : H_k(x^k) \in T^{-1}\left(\frac{1}{c_k} Q_k(x^k)\right) \tag{4}$$

The assumptions of Theorem 1 being satisfied, $\frac{1}{c_k} Q_k(x^k) \to 0$, which means that there is an order $k_0 \in N$ so that

$\forall k \geqslant k_0 : \left\| \frac{1}{c_k} Q_k(x^k) \right\| < \varepsilon$. In accordance with relation (4): $\forall k \geqslant k_0$: $H_k(x^k) = x^{k+1} = \bar{x}$.

Remarks. 1. When the proximal sequence (x^k) is obtained by one of the criteria (A), (B) or (C), the sequence $(H_k(x^k))$ will have the properties from Theorem 3, namely:

$\forall k \geqslant k_0 : H_k(x)^k = \bar{x}$, assuming that the other conditions of the theorem are satisfied.

In case (x^k) is obtained by criterion (A), we'll have:

$$\forall k \geqslant k_0 : \quad \|x^k - \bar{x}\| \leqslant \varepsilon_k.$$

2) The disavantage of this theorem is that assumption $0 \in \operatorname{int} T(x)$ is too restrictive and thus excludes a large game of problems.

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ALGORITMUL DE PUNCTE PROXIMALE PENTRU SPAȚII BANACH REFLEXIVE

(Rezumat)

Fiind dat un spațiu Banach reflexiv X și un operator maximal monoton $T : X \to 2^X$, se descrie un algoritm pentru rezolvarea ecuației operatoriale $0 \in T(x)$. Se tratează aproximarea soluției acestei ecuații în topologia slabă și tare a spațiului X; de asemenea se descrie un procedeu pentru aflarea soluției după un număr finit de iterații, într-un caz particular.

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STUDIUL CONVERGENŢEI UNOR ŞIRURI DE OPERATORI LINIARI ŞI POZITIVI

OCTAVIAN AGRATINI

1. Introducere. În [4] D. D. Stancu prezintă o metodă probabilistică pentru construirea de operatori liniari şi pozitivi. Vor deriva din această metodă cunoscuţii operatori Bernstein, Mirakyan, Baskakov, Feller, o variantă a operatorului Meyer—König şi Zeller cit şi doi operatori introduşi de acelaşi autor în [3] cu schema lui Markov—Polya. G. C. Jain şi S. Pethe obţin in [1] noi generalizări probabilistice ale operatorilor de tip Bernstein şi Szász. De asemenea in [2] J. P. King construieşte tot pe considerente probabilistice două generalizări ale operatorilor lui Bernstein şi Baskakov.

În această lucrare ne propunem să studiem cele două şiruri de operatori liniari şi pozitivi din [2] stabilind condiţii necesare şi suficiente care asigură convergenţa acestora in spaţiul de funcţii $C[0, a]$, a fiind un număr pozitiv. Instrumentul de lucru il constituie celebra teoremă a lui Bohman—Korovkin.

În generalizarea operatorului Bernstein, în [2] J. P. King pleacă de la variabilele aleatoare:

$$X_i\begin{pmatrix} 1 & 0 \\ p_i(x) & 1-p_i(x)\end{pmatrix}, \quad (i=\overline{1,n})$$

şi defineşte numerle reale $a_{ni}(x)$, $i=\overline{0,n}$ prin egalitatea:

$$\prod_{i=1}^{n}(p_i(x)z+1-p_i(x))=\sum_{k=0}^{n}a_{nk}(x)z^k, \quad z>0. \tag{1}$$

Primul membru al egalităţii reprezintă valoarea medie a variabilei aleatoare z^{Y_n} unde $Y_n=\sum_{i=1}^{n}X_i$ şi coeficienţii $a_{nk}(x)$, $k=\overline{0,n}$, indică probabilitatea ca Y_n să ia valoarea k. Astfel, pentru orice funcţie f continuă pe intervalul $[0,1]$ autorul defineşte operatorul L_n prin relaţia:

$$L_n(f;x):=\sum_{k=0}^{n}a_{nk}(x)f\left(\frac{k}{n}\right). \tag{2}$$

Notăm că pentru X_i se poate imagina următoarea interpretare probabilistică. Fie n discuri muzicale şi x fixat. Încredinţate spre ascultare unui meloman $p_i(x)$ reprezintă probabilitatea ca persoana să asculte faţa I a discului numărul i. Astfel X_i descrie modul de audiere a feţelor discului i.

Într-un mod similar a fost generalizat operatorul Baskakov in aceeaşi lucrare [2]. Fie x fixat şi considerăm n monede unde la prima monedă iese cap cu probabilitatea $p_i(x)$. Fie U_i variabila aleatoare care exprimă numărul de aruncări efectuate cu primele i monede pînă cînd la moneda a-i-a obțin cap. Notăm faptul că trecerea de la moneda j la moneda $j + 1$ se realizează în momentul obținerii feței cap la moneda j.

$$U_i\begin{pmatrix} k \\ q_i^k(x)p_i(x)\end{pmatrix}_{k\geq 0}, \quad (i = \overline{1, n}).$$

Sint definite numerele $b_{nk}(x)$, $k \geq 0$, prin relația:

$$\prod_{i=1}^{n} \frac{p_i(x)}{1 - q_i(x)\theta} = \sum_{k=0}^{\infty} b_{nk}(x)\theta^k, \quad |\theta| < 1. \tag{3}$$

Primul membru reprezintă valoarea medie a variabilei θ^{W_n} unde $W_n = \sum_{i=1}^{n} U_i$ iar coeficienții $b_{nk}(x)$ indică probabilitatea ca W_n să ia valoarea k. Pentru orice funcție f continuă pe $[0, \infty)$ autorul definește operatorul T_n prin relația:

$$T_n(f; x) := \sum_{k=0}^{\infty} b_{nk}(x) f\left(\frac{k}{n}\right). \tag{4}$$

2. **Rezultate.** Ne propunem să stabilim condiții necesare și suficiente de convergență a șirului de operatori definiți în (2). Vom demonstra în prealabil următoarea lemă:

LEMA 1. *Dacă șirul $(L_n)_{n>0}$ este definit prin relația* (2) *atunci au loc:*

(i) $L_n(e_0; x) = 1$

(ii) $L_n(e_1; x) = \frac{1}{n}\sum_{i=1}^{n} p_i(x)$

(iii) $L_n(e_2; x) = \frac{1}{n^2}\left(\sum_{i=1}^{n} p_i(x) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i(x)p_j(x)\right)$

Demonstrație. Identificarea realizată în (1) o consider ca o funcție în z; deci fie:

$$G(z) = \prod_{i=1}^{n} (p_i(x)z + 1 - p_i(x)) = \sum_{k=0}^{n} a_{nk}(x)z^k. \tag{5}$$

Deoarece $G(1) = 1$ în mod evident are loc (i). Derivăm (5) în raport cu z:

$$\frac{dG(z)}{dz} = \sum_{i=1}^{n} \frac{p_i(x)}{p_i(x)z + 1 - p_i(x)} \cdot G(z) = \sum_{k=1}^{n} ka_{nk}(x)z^{k-1}.$$

Luînd $z = 1$ vom obține:

$$G'(1) = \sum_{i=1}^{n} p_i(x) = \sum_{k=1}^{n} ka_{nk}(x),$$

de unde prin împărțire cu n rezultă (ii). Calculăm acum a doua derivată a lui G; vom avea:

$$\frac{d^2G(z)}{dz^2} = \sum_{i=1}^{n} \left[\frac{d}{dz} \left(\frac{p_i(x)}{p_i(x)z + 1 - p_i(x)} \right) G(z) + \frac{p_i(x)}{p_i(x)z + 1 - p_i(x)} \cdot \frac{dG(z)}{dz} \right] =$$

$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{p_i(x)}{p_i(x)z + 1 - p_i(x)} \cdot \frac{p_j(x)}{p_j(x)z + 1 - p_j(x)} \cdot G(z).$$

Pe de altă parte:

$$\frac{d^2G(z)}{dz^2} = \sum_{k=0}^{n} k(k-1)a_{nk}(x)z^{k-2}.$$

Luînd din nou $z = 1$ se obține:

$$G''(1) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} p_i(x)p_j(x) = \sum_{k=0}^{n} (k^2 - k)a_{nk}(x).$$

Pentru a ajunge la (iii) se împarte egalitatea de mai sus cu n^2, se separă suma ce conține pe k^2 și se folosește (ii). Astfel lema este complet demonstrată.

Vom demonstra următoarea teoremă:

TEOREMA 1. *Fie șirul $(L_n)_{n>0}$ definit în (2) și $f \in C[0, 1]$. Condiția necesară și suficientă ca $(L_n(f; \cdot))_{n>0}$ să convergă uniform spre f pe $[0, 1]$ este:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} p_i(x) = x, \qquad \forall\, x \in [0, 1]. \tag{6}$$

Demonstrație. Conform teoremei Bohman—Korovkin este necesar și suficient să verific că:

$$\lim_{n\to\infty} L_n(e_i;\ x) = e_i(x), \quad x \in [0, 1], \quad i = 0, 1, 2,$$

unde $e_i(x) = x^i$. Examinînd egalitățile (i), (ii), (iii) practic rămîne să arătăm că (6) implică:

$$\lim_{n\to\infty} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{p_i(x)p_j(x)}{n} = x^2.$$

Dar

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{p_i(x)p_j(x)}{n^2} = \left(\sum_{i=1}^{n} \frac{p_i(x)}{n} \right)^2 - \sum_{i=1}^{n} \frac{p_i^2(x)}{n^2} \quad \text{și} \quad 0 \leqslant \sum_{i=1}^{n} \frac{p_i^2(x)}{n^2} \leqslant \frac{1}{n} \to 0, \quad n \to \infty.$$

Astfel teorema este complet demonstrată.

Vom stabili rezultate similare pentru operatorul definit prin (4).

LEMA 2. *Dacă șirul* $(T_n)_{n>0}$ *este dat de relația* (4) *atunci au loc egalitățile:*

$$\text{(a)}\quad T_n(e_0\,;\,x) = 1$$

$$\text{(b)}\quad T_n(e_1\,;\,x) = \frac{1}{n}\sum_{i=1}^{n}\frac{q_i(x)}{p_i(x)}$$

$$\text{(c)}\quad T_n(e_2\,;\,x) = \frac{1}{n^2}\sum_{i=1}^{n}\left(\frac{q_i(x)}{p_i(x)} + \frac{q_i^2(x)}{p_i(x)}\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\frac{q_i(x)}{p_i(x)}\right)^2.$$

Demonstrație. Vom folosi aceeași metodă ca și în demonstrarea lemei precedente. Relația (3) o considerăm ca o funcție în θ; definim deci:

$$G(\theta) : = \prod_{i=1}^{n}\frac{p_i(x)}{1 - q_i(x)\theta} = \sum_{k=0}^{\infty} b_{nk}(x)\,\theta^k.$$

Evident $G(1) = 1$ și (a) este imediat verificată. Derivăm funcția G în raport cu θ și obținem:

$$\frac{dG(\theta)}{d\theta} = \sum_{i=1}^{n}\frac{q_i(x)}{1 - q_i(x)\theta}\cdot G(\theta) = \sum_{k=0}^{\infty} k b_{nk}(x)\,\theta^{k-1}.$$

În egalitățile de mai sus luăm $\theta = 1$, împărțim cu n și înlocuim $1 - q_i(x)$ cu egalul său $p_i(x)$, $(i = \overline{1, n})$; astfel se obține relația (b). Cu aceeași tehnică se obțin următoarele egalități:

$$\left.\frac{d^2G(\theta)}{d\theta^2}\right|_{\theta=1} = \sum_{i=1}^{n}\left(\frac{q_i^2(x)}{1 - q_i(x)} + \frac{q_i(x)}{1 - q_i(x)}\sum_{j=1}^{n}\frac{q_j(x)}{1 - q_j(x)}\right) = \sum_{k=0}^{\infty} k^2 b_{nk}(x) - \sum_{k=0}^{\infty} k b_{nk}(x)$$

Prin împărțire cu n^2 și folosind (b) se obține:

$$T_n(e_2\,;\,x) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{q_i(x)}{1 - q_i(x)}(q_i(x) + 1) + \frac{1}{n^2}\left(\sum_{i=1}^{n}\frac{q_i(x)}{1 - q_i(x)}\right)\left(\sum_{j=1}^{n}\frac{q_j(x)}{1 - q_j(x)}\right).$$

Avînd în vedere că $1 - q_i(x) = p_i(x)$, $i = \overline{1, n}$, deducem că relația la care s-a ajuns coincide cu (c). Astfel lema este complet demonstrată.

Acum putem demonstra următorul rezultat:

TEOREMA 2. *Fie șirul de operatori* $(T_n)_{n>0}$ *definit în* (4). *Fie* $a > 0$ *și* f *o funcție continuă pe intervalul* $[0, a]$. *Condiția necesară și suficientă ca* $(T_n(f\,;\cdot))_{n>0}$ *să conveargă uniform spre* f *pe* $[0, a]$ *este:*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i(x)} = x + 1, \quad \forall\, x \in [0, a]. \tag{7}$$

Demonstrație. Observăm că (7) se mai poate pune sub forma:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{q_i(x)}{p_i(x)} = x, \quad \forall\, x \in [0, a]. \tag{8}$$

Vom folosi încă o dată teorema lui Bohman—Korovkin, operatorii introduşi fiind liniari şi pozitivi. Relaţiile (a), (b) din lema precedentă asigură:

$$\lim_{n\to\infty} T_n(e_i\,;x) = e_i(x), \quad \forall\, x \in [0, a],\ i = 0,1.$$

Avînd în vedere următoarele delimitări evidente:

$$0 \leqslant \frac{1}{n^2}\sum_{i=1}^{n}\frac{q_i^2(x)}{p_i(x)} \leqslant \frac{1}{n^2}\sum_{i=1}^{n}\frac{q_i(x)}{p_i(x)}$$

şi din relaţia (8):

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\frac{q_i(x)}{p_i(x)} = 0, \quad \forall\, x \in [0, a] \tag{9}$$

vom obţine:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\frac{q_i^2(x)}{p_i(x)} = 0, \quad \forall\, x \in [0, a]. \tag{10}$$

Vom trece la limită în egalitatea (c) din lema 2 şi folosind (8), (9), (10) deducem:

$$\lim_{n\to\infty} T_n(e_3\,;x) = e_3(x), \quad \forall\, x \in [0, a]$$

şi teorema este complet demonstrată.

Observaţie. Este bine cunoscută următoarea afirmaţie: dacă un şir numeric $(a_n)_{n>0}$ converge spre a atunci $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n} a_i = a$. Aplicînd acest rezultat putem afirma:

1. dacă $\lim\limits_{k\to\infty} p_k(x) = x$ atunci are loc (6).
2. dacă $\lim\limits_{k\to\infty}\frac{q_k(x)}{p_k(x)} = x$ atunci are loc (7).

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THE STUDY OF THE CONVERGENCE OF SOME SEQUENCES OF LINEAR POSITIVE OPERATORS

(Summary)

In [4] D. D. Stancu has presented probabilistic methods for construction and study of some general classes of approximating linear positive operators.

In this paper we investigate two sequences of positive linear operators introduced by a probabilistic method by J. P. King [2]. We establish necessary and sufficient conditions for the convergence of the sequences of these operators in the space $C[0, a]$ $(a > 0)$. By using the well-known theorem of Bohman—Korovkin we prove the following two theorems.

(i) Let (L_n) be the sequence of linear positive operators defined at (2) for any $f \in C[0, 1]$. A necessary and sufficient condition that $(L_n f)$ converges uniformly to f on $[0, 1]$ is given at (6).

(ii) A necessary and sufficient condition that the operator T_n, defined at (4), applied to any $f \in C[0, a]$ $(a > 0)$, to converge uniformly to f is given at (7).

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ASUPRA T-RECURENŢEI UNOR CONEXIUNI SEMI-SIMETRICE ŞI SFERT-SIMETRICE

P. ENGHIŞ, I. LUP

Fie A_n un spaţiu cu conexiune afină. Într-un sistem de coordonate notăm cu Γ^i_{jk} componentele conexiunii afine cu $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ componentele tensorului de torsiune şi cu $T_k = T^i_{ik}$ componentele vectorului de torsiune.

Conexiunea Γ se numeşte semi-simetrică [5], dacă există un cîmp vectorial covariant S_k astfel ca

$$T^i_{jk} = S_{[j}\delta^i_{k]} \tag{1}$$

Dacă în (1) se aplică o contracţie în i şi j se obţine

$$T_k = (1 - n)S_k \tag{2}$$

iar dacă în (1) se ţine seama de (2) se obţine

$$(1 - n)T^i_{jk} = T_j\delta^i_k - T_k\delta^i_j \tag{3}$$

Un spaţiu cu conexiune afină A_n se numeşte T-recurent, dacă există un cîmp vectorial covariant ψ_r astfel ca

$$T^i_{jk,r} = \psi_r T^i_{jk} \tag{4}$$

unde prin virgulă este notată derivarea covariantă în raport cu conexiunea Γ.

Dacă în (4) se aplică o contracţie în i şi j se obţine

$$T_{k,r} = \psi_r T_k \tag{5}$$

şi deci vectorul de torsiune este şi el recurent de vector ψ_r.

Să observăm acum că în spaţiile semi-simetrice are loc reciproca acestei afirmaţii, adică un spaţiu semi-simetric cu vectorul de torsiune recurent de vector ψ_r este T-recurent de vector ψ_r.

Într-adevăr, dacă în (3) se aplică derivarea covariantă în raport cu Γ se obţine

$$(1 - n)T^i_{jk,r} = T_{j,r}\delta^i_k - T_{k,r}\delta^i_j \tag{6}$$

şi ţinînd seama de (5) rezultă

$$(1 - n)T^i_{jk,r} = \psi_r T_k\delta^i_k - \psi_r T_k\delta^i_j = \psi_r(T_j\delta^i_k - T_k\delta^i_j) = (1 - n)\psi_r T^i_{jk}$$

de unde

$$T^i_{jk,r} = \psi_r T^i_{jk}$$

Avem deci:

PROPOZIȚIA 1. *Într-un spațiu A_n cu conexiune semi-simetrică tensorul de torsiune și vectorul de torsiune sînt recurenți în același timp și cu același vector de recurență.*

Un spațiu A_n semi-simetric s-a numit [5] semi-simetric special, dacă cîmpul vectorial S_k este gradient. În acest caz rezultă

$$S_{[j,k]} = 0 \tag{7}$$

și din (2) s-a arătat [4] că spațiul este cu conexiune E n g h i ș [5], deci conexiunea Γ este o E-conexiune [4] și deci avem [2]

$$T_{[j,k]} = 0 \tag{8}$$

Dacă spațiul semi-simetric special este T-recurent rezultă din (8)

$$\psi_j T_k - \psi_k T_j = 0 \tag{9}$$

și avem:

PROPOZIȚIA 2. *Într-un spațiu semi-simetric T-recurent dotat cu o E-conexiune, vectorul de T-recurență este proporțional cu vectorul de torsiune.*

Se știe [5] că intr-un spațiu semi-simetric dotat cu o E-conexiune are loc relația

$$T^h_{[ij,k]} = 0 \tag{10}$$

Din (4) și (10) rezultă

$$T^h_{[ij}\Psi_{k]} = 0 \tag{11}$$

care este o relație de tip W a l k e r [7], avem deci:

PROPOZIȚIA 3. *Într-un spațiu semi-simetric T-recurent, dotat cu o E-conexiune are loc relația* (11).

O conexiune Γ se numește sfert-simetrică [5] dacă tensorul de torsiune are forma

$$T^i_{jk} = t^i_{[j}S_{k]} \tag{12}$$

unde t^i_j și S_k sînt cîmpuri tensoriale arbitrare.

Să presupunem acum cîmpul tensorial t^i_j covariant constant. Derivînd covariant (12) avem:

$$T^i_{jk,r} = t^i_{[j}S_{k],r} \tag{13}$$

și dacă presupunem cîmpul S_k recurent de vector ψ_r avem:

$$T^i_{jk,r} = \psi_r t^i_{[j}S_{k]} = \psi_r T^i_{jk}$$

și spațiul este T-recurent. Avem deci:

PROPOZIȚIA 4. *Un spațiu A_n sfert-simetric cu cîmpul tensorial t^i_j convariant constant și cîmpul vectorial S_k recurent de vector ψ_r, este T-recurent de vector ψ_r.*

Considerînd acum în spaţiul A_n sfert-simetric, cu cîmpul t^i_j covariant constant, tensorul

$$T^h_{jkp} = \frac{1}{3}\,[t^h_p S_k t^i_j S_i - t^h_p S_j t^i_k S_i + S_p t^h_j t^i_k S_i - S_p t^h_k t^i_j S_i + t^i_p S_i t^h_k S_j - t^i_p S_i t^h_j S_k] \quad (14)$$

[5] şi derivînd covariant (14) rezultă

$$T^h_{jkp,r} = \frac{1}{3}\,[t^h_p S_{k,r} t^i_j S_i + t^h_p S_k t^i_j S_{i,r} - t^h_p S_{j,r} t^i_k S_i - t^h_p S_j t^i_k S_{i,r} + S_{p,r} t^h_j t^i_k S_i +$$
$$+ S_p t^h_j t^i_k S_{i,r} - S_{p,r} t^h_k t^i_j S_i - S_p t^h_k t^i_j S_{i,r} + t^i_p S_{i,r} t^h_k S_j + t^i_p S_i t^h_k S_{j,r} - \quad (15)$$
$$- t^i_p S_{i,r} t^h_j S_k - t^i_p S_i t^h_j S_{k,r}]$$

Presupunînd spaţiu semi-simetric A_n cu cîmpul S_k recurent de vector ψ_r, din (15) rezultă că tensorul T^h_{jkp} este recurent cu vectorul $2\psi_r$. Avem deci:

PROPOZIŢIA 5. *Într-un spaţiu A_n sfert-simetric cu cîmpul tensorial t^i_j covariant constant şi cîmpul vectorial S_k recurent de vector ψ_r, tensorul T^h_{jkp} este recurent de vector $2\psi_r$.*

Într-o lucrare anterioară [3] am arătat că dacă un spaţiu A_n este T-recurent cu vector ψ_r, tensorii obţinuţi din tensorul de torsiune prin produs tensorial şi produs tensorial contractat, sînt recurenţi de vector $2\psi_r$. Din acest rezultat şi din propoziţia 4 rezultă

PROPOZIŢIA 6. *Într-un spaţiu A_n sfert-simetric cu cîmpul tensorial t^i_j covariant constant şi cîmpul S_k recurent, tensorii obţinuţi din tensorul de torsiune prin produs tensorial şi produs tensorial contractat, sînt recurenţi de vector $2\psi_r$.*

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SUR LA T-RÉCURRENCE DE CERTAINES CONNECTIONS SEMI-SYMÉTRIQUES ET QUART-SYMÉTRIQUES

(Résumé)

On montre que dans les espaces à connection semi-symétrique le tenseur de torsion et le vecteur de torsion sont récurrents en même temps, et que, si l'espace est doué d'une E-connection, le vecteur de torsion est proportionnel avec le vecteur de T-récurrence et on a la relation (11). Pour les espaces quart-symétriques avec le champ t^i_j covariant-constant, on montre que, de la récurrence de champ S_k, il réslute la T-récurrence ainsi que la récurrence des tenseurs obtenus à l'aide du tenseur de torsion.

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SUR LA CONVERGENCE QUASIUNIFORME

VASILE CÂMPIAN

En considérant, à la place des suites usuelles de fonctions définies sur un espace métrique et prenant des valeurs dans un autre espace métrique, des g-suites (suites généralisées) de fonctions définies sur un espace topologique X et prenant des valeurs dans un espace uniforme Y, I. Muntean [4] a étendu la notion de convergence quasiuniforme dans le sens de B. Gagaeff [3] et P. S. Alexandrov [1] et a démontré un critère de continuité pour la limite d'une g-suite de fonctions continues.

Dans cette note on démontre la continuité de la limite d'une g-suite convergente quasiuniformément de fonctions, dans le sens introduit par I. Muntean [4], en renonçant à l'hypothèse de la continuité des termes de la g-suite.

D'après [4], une g-suite de fonctions $f_i : X \to Y$, pour i appartenant à l'ensemble dirigé I, définies sur l'espace topologique X et prenant des valeurs dans l'espace uniforme Y est convergente quasiuniformément vers la fonction $f: X \to Y$ si :

1) la g-suite $(f_i)_{i \in I}$ converge en chaque point $x \in X$ vers $f(x)$, c'est—à—dire

$$f(x) = \lim_{i \in I} f_i(x) ;$$

2) pour tout entourage symétrique ouvert U dans la topologie de l'espace produit $X \times Y$, il existe une g-suite $(g_j)_{j \in J}$ de fonctions continues définies sur X et prenant des valeurs dans Y, ainsi qu'une g-suite $(G_j)_{j \in J}$ d'ensembles ouverts dans X, avec $\bigcup_{j \in J} G_j = X$, de sorte que l'on ait pour chaque $j \in J$ et pour n'importe quel $x \in G_j$

$$(f(x),\ g_j(x)) \in U.$$

On donne dans [2] une définition de la convergence quasiuniforme équivalente à celle-ci.

THÉORÈME: *La limite ponctuelle d'une g-suite de fonctions (f_i) définies sur l'espace topologique X et prenant des valeurs dans l'espace uniforme Y est une fonction continue $f: X \to Y$ si et seulement si la g-suite $(f_{i \in I})$ est convergente quasiuniformément vers f.*

Démonstration.

Nécessité. Soient f une fonction continue, U un entourage symétrique ouvert dans la topologie de l'espace produit $Y \times Y$ et J un ensemble dirigé quelconque (par exemple, $J = \{1\}$). On prend $g_i = f$ et $G_j = X$ pour n'importe quel $j \in J$; alors $(f(x), g_j(x)) = (f(x), f(x)) \in U$, parce que $(f(x), f(x))$est contenue dans la diagonale Δ du produit $Y \times Y$ et $\Delta \subset U$ pour tout entourage U.

Suffisance. Soient $(f_i)_{i \in I}$ une g-suite de fonctions convergeant quasiuniformément vers $f: X \to Y$, x_0 un point dans X, V un voisinage de $f(x_0)$ et U_1 un entourage de l'espace uniforme Y pour lequel

$$V_1 = \{y \mid y \in Y,\ (f(x_0),\ y) \in U_1\}. \tag{1}$$

Notons par K un entourage de l'espace Y de sorte que $K \circ K \subset U_1$ et par U un entourage symétrique ouvert dans la topologie de l'espace produit $Y \times Y$ pour lequel $U \circ U \cup K$. On a $U \subset U \circ U \subset K$ parce que $(x, y) = (x, y) \circ (y, y)$ pour toute $(x, y) \in U$, donc $U \circ U \circ U \subset K \circ K \subset U_1$, c'est-à-dire

$$U^3 \subset U_1. \tag{2}$$

Soient $(g_j)_{i \in I}$ et $(G_j)_{j \in J}$ les g-suites correspondant à l'entourage U et $j_0 \in J$ de sorte que $x_0 \in G_{j_0}$. Pour tout $x \in G_{j_0}$ on a

$$(f(x), g_{j_0}(x)) \in U, \tag{3}$$

en résultant particulièrement pour $x = x_0$

$$(f(x_0), g_{j_0}(x_0)) \in U. \tag{4}$$

Il résulte de la continuité de la fonction g_{j_0} dans le point x_0 qu'il existe un voisinage W de x_0 de sorte que

$$(g_{j_0}(x_0), g_{j_0}(x)) \in U \tag{5}$$

pour tout $x \in W$.

L'ensemble $W_1 = W \cap G_{j_0}$ est un voisinage pour x_0. En tenant compte de (4) et de (5), il résulte

$$(f(x_0), g_{j_0}(x)) \in U^2 \tag{6}$$

pour tout $x \in W_1$, tandis que (3), (6) et (2) nous conduisent à

$$(f(x_0), f(x)) \in U \tag{7}$$

pour tout $x \in W_1$. Par conséquent $f(x) \in V_1$ pour tout $x \in W_1$, c'est à dire f est continue dans le point x_0.

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ASUPRA CONVERGENŢEI CVASIUNIFORME

(Rezumat)

În această notă se demonstrează continuitatea limitei unui g-şir de funcţii convergent cvasiuniform, în sensul dat de I. Muntean, renunţîndu-se la ipoteza continuităţii termenilor g-şirului.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXX, 1985

ON THE GENERAL SOLUTION OF THE LINEAR ALGEBRAIC SYSTEMS

O. C. DOGARU

1. **Introduction.** Let $\mathbf{R}_{m\times n}$ be the set of $m \times n$ matrices over the field $\mathbf{R}$ and $M \in \mathbf{R}_{m\times n}$.

DEFINITION. A matrix $M^+ \in \mathbf{R}_{n\times m}$ is called a g_1-inverse of the matrix M if it satisfies the relation

$$MM^+M = M. \tag{1}$$

The relation (1) is the first from the system of four matrix equations that define in an unique way the generalized inverse M^g of the matrix M from the definition of R. Penrose [9]. The matrix M^+ is not unique.

Now let the linear algebraic system

$$AXB = C \tag{2}$$

be, where $A \in \mathbf{R}_{m\times n}$ and of rank $r \leqslant \min(m, n)$, $B \in \mathbf{R}_{p\times q}$ and of rank $s \leqslant \min(p, q)$ and $C \in \mathbf{R}_{m\times q}$.

It is used the following result [10]

THEOREM 1. *A necessary and sufficient condition for the system of equations (2) to be compatible is*

$$AA^+CB^+B = C. \tag{3}$$

In this case the general solution of the system (2) *is*

$$X = A^+CB^+ + Z - A^+AZBB^+ \tag{4}$$

where $Z \in \mathbf{R}_{n\times p}$ is an arbitrary matrix.

Here $A^+ \in \mathbf{R}_{n\times m}$, $B \in \mathbf{R}_{q\times p}$ denote the g_1-inverse of A and B respectively.

The calculus of the matrices A^+ and B^+ is not easy. In this case present interest to express A^+ and B^+ in terms more simple. Afterwards these terms are used to express the compatibility condition and the general solution of the system (2). This is the purpose of this paper. In the following it is used the factorization of the matrices A and B based on the orthogonal matrices.

2. **Orthogonal factorization.** Let $M \in \mathbf{R}_{m\times n}$ be of rank $r \leqslant \min(m, n)$. The matrix M may be factorized as

$$M = EF \tag{5}$$

where $E \in \mathbf{R}_{m\times r}$ and $F \in \mathbf{R}_{r\times n}$, with $E^TE = I_r$, where E^T denote the transpose of the matrix E, I_r is the unity matrix of order r, but the principal minor of order r in the matrix F, different from zero is in the first r rows and columns.

Hence

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad F = (F_1 F_2), \ \det (F_1) \neq 0. \tag{6}$$

where the matrices E_1 and F_1 are of order r and $E_2 \in \mathbf{R}_{m-r,r}$, $F_2 \in \mathbf{R}_{r,n-r}$.

3. **A form for the matrix M⁺.** Let $M \in \mathbf{R}_{m\times n}$ be and M^+ a g_1-inverse of M. In [3] a form for M^+ was find in the following manner. The matrix M^+ may be found in the form

$$M^+ = \begin{pmatrix} X & Y \\ U & V \end{pmatrix} \tag{7}$$

and the matrix M, using the decompositions (6) for E, may be written as

$$M = EF = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} (F_1 F_2) = \begin{pmatrix} E_1F_1 & E_1F_2 \\ E_2F_1 & E_2F_2 \end{pmatrix}. \tag{8}$$

Using (7) and (8) in the matrix equation $MM^+M = M$ four matrix equations are obtained

$$\begin{cases} E_1F_1XE_1F_1 + E_1F_2UE_1F_1 + E_1F_1YE_2F_1 + E_1F_2VE_2F_1 = E_1F_1 \\ E_1F_1XE_1F_2 + E_1F_2UE_1E_2 + E_1F_1YE_2F_2 + E_1F_2VE_2F_2 = E_1F_2 \\ E_2F_1XE_1F_1 + E_2F_2UE_1F_1 + E_2F_1YE_2F_1 + E_2F_2VE_1F_1 = E_2F_1 \\ E_2F_1XE_1F_2 + E_2F_2UE_1F_2 + E_2F_1YE_2F_2 + E_2F_2VE_2F_2 = E_2F_2. \end{cases}$$

If in this system we put $U = V = 0$, $X = F_1^{-1}E_1^T$, $Y = F_1^{-1}E_2^T$ and taking $E_1^T E_2 + E_2^T E_2 = I$, into consideration, a g_1-inverse for M is

$$M^+ = \begin{pmatrix} E^{-1} E_1^T & F_1^{-1} E_2^T \\ 0 & 0 \end{pmatrix} \tag{9}$$

where E_1^T, E_2^T denote the transpose of the matrix E_1 and E_2 respectively, while F_1^{-1} is the inverse of the matrix F_1.

4. **Factorization of the matrix A and B.** Now let $A \in \mathbf{R}_{m\times n}$ be of rank $r \leqslant \min(m, n)$, $B \in \mathbf{R}_{p\times q}$ be of rank $s \leqslant \min(p, q)$ and $C \in \mathbf{R}_{m\times q}$. For A and B we use a factorization of the form (5):

1) $A = GH$, $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, $H = (H_1 H_2)$, $G^T G = I_r$, $\det(H_1) \neq 0$ (10)

where the matrices G_1 and H_1 are of order r, $G_2 \in \mathbf{R}_{m-r,r}$, $H_2 \in \mathbf{R}_{n-r,r}$. The matrix

$$A^+ = \begin{pmatrix} H_1^{-1} G_1^T & H_1^{-1} G_2^T \\ 0 & 0 \end{pmatrix}, \tag{11}$$

$A^+ \in \mathbf{R}_{n\times m}$ is a g_1-inverse of A.

2) $B = RS,\ R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix},\ S = (S_1 S_2),\ R^T R = I_s,\ \det(S_1) \neq 0$ (12)

where the matrices R_1, S_1 are of order s, $R_2 \in \mathbf{R}_{p-s,s}$, $S_2 \in \mathbf{R}_{q-s,s}$. The matrix

$$B^+ = \begin{pmatrix} S_1^{-1} R_1^T & S_1^{-1} R_2^T \\ 0 & 0 \end{pmatrix} \tag{13}$$

is a g_1-inverse of B, $B^+ \in \mathbf{R}_{q\times p}$.

Using (10) and (12) the matrices A and B may be written as at (8):

$$A = \begin{pmatrix} G_1 H_1 & G_1 H_2 \\ G_2 H_1 & G_2 H_2 \end{pmatrix} \text{ respectively } B = \begin{pmatrix} R_1 S_1 & R_1 S_2 \\ R_2 S_1 & R_2 S_2 \end{pmatrix}. \tag{14}$$

For the matrix $C \in \mathbf{R}_{m\times q}$ from the system (2) it is used a suitable block decomposition

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{15}$$

thus the products where it appears to be possible.

5. **Results.** Using the factorization of 4° one may establish

THEOREM 2. *A necessary and sufficient condition for the system* (2) *to be compatible is*

$$\begin{cases} C_{11} S_1^{-1} S_2 = C_{12} \\ C_{21} S_1^{-1} S_2 = C_{22}. \end{cases} \tag{16}$$

The general solution is

$$X = \left[\begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} HZR\right] R^T + Z \tag{17}$$

where $W = H_1^{-1}(G_1^T C_{11} + G_2^T C_{21}) S_1^{-1}$ *and* Z *is arbitrary matrix.*

Proof. Using in (2) the notation (11) — (15), by calculus one obtains the system (16). For the general solution are used the fololwing intermediate results

$$A^+ C B^+ = \begin{pmatrix} W \\ 0 \end{pmatrix} R^T, \text{ where } W = H_1^{-1}(G_1^T C_{11} + G_2^T C_{21}) S_1^{-1},$$

$$A^+ A = \begin{pmatrix} H_1^{-1} H_1 & H_1^{-1} H_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} (H_1 H_2) = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} H,$$

$$B^+ B = RR^T.$$

Hence

$$X = \begin{pmatrix} W \\ 0 \end{pmatrix} R^T + Z - \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} HZRR^T = \left[\begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} HZR\right] R^T + Z.$$

COROLLARY 1. *For the system* $AX = C$ *the compatibility condition is* $GG^T C = C$ *and the general solution is*

$$X = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} (G^T C - HZ) + Z \tag{18}$$

where Z is an arbitrary matrix.

Proof. From the Theorem 2 if $B = I$ then the compatibility condition is $AA^+C = C$ that is $AA^+C = GG^TC = C$.

The general solution (4), in our terms, is

$$X = A^+C + Z - A^+AZ = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} [(G_1^T C_{11} + G_2^T C_{22}, G_1^T C_{12} + G_2^T C_{22})] -$$

$$- (H_1 H_2) Z + Z = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} (G^T C - HZ) + Z.$$

COROLLARY 2. *For equations* $Ax = c$ *the compatibility condition is* $GG^Tc = c$ *and the general solution is*

$$x = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} (G^T c - Hz) + z, \tag{19}$$

where z is an arbitrary vector.

Proof. From (3) one obtains $AA^+c = c$, that is $GG^T\ c = c$ and the general solution is

$$x - A^+c + z - A^+Az = \begin{pmatrix} H_1^{-1} \\ 0 \end{pmatrix} (G^T c - Hz) + z$$

where z is an arbitrary vector.

COROLLARY 3. *For equations* $XB = C$ *the compatibility conditions are* (16) *and the general solution is*

$$X = \left[\begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} S_1^{-1} - Z \begin{pmatrix} S_1^{-1} \\ 0 \end{pmatrix} \right] R^T + Z \tag{20}$$

where Z is an arbitrary matrix.

Proof. From (3) the compatibility condition is $CB^+B = C$, which in our terms is (16). The general solution, according to (4), is

$$X = CB^+ + Z - ZBB^+ = \left[\begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} S_1^{-1} - Z \begin{pmatrix} S_1^{-1} \\ 0 \end{pmatrix} \right] R^T + Z.$$

COROLLARY 4. *For equations* $xB = c$ *the compatibility condition is*

$$c_1 S_1^{-1} S_2 = c_2 \tag{21}$$

and the general solution is

$$x = (c_1 S_1^{-1} - zR) R^T + z, \tag{22}$$

where z is an arbitrary vector.

Proof. From (3) the compatibility condition is $cB^+B = c$, which may be written in the form (21). The general solution is

$$x = cB^+ + z - zBB^+ = c_1S_1^{-1}R^TR = (c_1S_1^{-1} - zR)R^T + z.$$

Remark. For the equations of the form $xB = c$, the compatibility condition is very simple to verify. It is expressed in terms of the matrix S and the vector c and one requires only the inversion of a $r \times r$ matrix.

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ASUPRA SOLUȚIEI GENERALE A SISTEMELOR ALGEBRICE LINIARE

(Rezumat)

Se dă soluția generală a sistemelor algebrice liniare compatibile de forma $AXB = C$ și a altor cazuri particulare, folosind g_1 – inversa matricelor A și B. g_1 – inversa este determinată în ideea din [8] prin factorizări ale matricilor A și B în formele $A = GH$, $B = RS$ unde $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, $G^TG = I_r$, $R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$, $R^TR = I_s$. $r(A) = r$, $r(B) = s$, $H = (H_1, H_2)$, $\det(H_1) \neq 0$, $S = (S_1, S_2)$, $\det(S_1) \neq 0$, G^T, R^T fiind matricele transpuse ale lui G respectiv R, iar I_r, I_s sînt matricele unitate de ordinul r respectiv s.

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GÉNÉRALISATIONS DE CERTAINES CLASSES DE FONCTIONS UNIVALENTES

TEODOR BULBOACĂ

1. Soit $\mathcal{F}_\alpha$ la classe des fonctions holomorphes dans le disque unité, normées par les conditions $f(0) = f'(0) - 1 = 0$, qui satisfont à la relation Re $(f'(z) + \alpha z f''(z)) > 0$, $\forall z \in U$ et soit $\mathcal{G}_\alpha$ la classe des fonctions holomorphes dans le disque unité, normées par les conditions $f(0) = f'(0) - 1 = 0$ qui satisfont à la relation

$$\mathrm{Re}\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > 0, \quad \forall z \in U.$$

Ces classes ont été introduites par P. N. Chichra [1]. Dans la deuxième partie de cette note on présente des généralisations de celles-ci.

En [2], les auteurs ont obtenu des résultats concernant la dérivée de Schwarz, la convexité et l'étoilement d'une fonction holomorphe et usuellement normée dans le disque unité.

Les résultats contenus dans la deuxième partie permettent leur généralisation, qui est donnée dans la troisième partie de cette note. Dans la dernière partie on présente une généralisation du théorème de Marx et Strohhäcker et un résultat concernant les fonctions convexes d'un certain ordre.

2. Soit $G_{\alpha,\beta}(g)$ la classe des fonctions holomorphes dans le disque unité U, normées par les conditions $f(0) = f'(0) - 1 = 0$ et qui vérifient la relation:

$$\mathrm{Re}\left[(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} + \frac{\alpha\beta}{2}\cdot\frac{g(z)}{zg'(z)}\cdot\left(1 - \frac{f^2(z)}{g^2(z)}\right)\right] > 0, \quad \forall z \in U.$$

où $\alpha \geqslant 0$, $\beta \leqslant 1$ et $g \in S^*$, où S^* est la classe des fonctions étoilées et usuellement normées dans U.

Soit $F_{\alpha,\beta}(g)$ la classe des fonctions holomorphes dans le disque unité U, normées par $f(0) = f'(0) - 1 = 0$ et qui vérifient la relation

$$\mathrm{Re}\left[f'(z) + \frac{\alpha g(z)}{g'(z)}\cdot f''(z) + \frac{\alpha\beta}{2}\cdot\frac{g(z)}{zg'(z)}\cdot(1 - f'^2(z))\right] > 0, \ \forall z \in U.$$

où $\alpha \geqslant 0$, $\beta \leqslant 1$ et $g \in S^*$.

On remarque que $G_{\alpha,0}(z) = \mathcal{G}_\alpha$, $F_{\alpha,0}(z) = \mathcal{F}_\alpha$.

THÉORÈME 1. *Si* $f \in G_{\alpha,\beta}(g)$ *alors* $\mathrm{Re}\,\frac{f(z)}{g(z)} > 0$, $\forall z \in U$.

Démonstrations. Soit $p(z) = \frac{f(z)}{g(z)}$, $z \in U$ Si $f \in G_{\alpha,\beta}(g)$, alors

$$\mathrm{Re}\left[p(z) + \alpha\, zp'(z)\cdot\frac{g(z)}{zg'(z)} + \frac{\alpha\beta}{2}\cdot\frac{g(z)}{zg'(z)}\,(1 - p^2(z))\right] > 0, \quad \forall z \in U.$$

Soit donc $\psi(r, s) = r + \alpha s \frac{g(z)}{zg'(z)} + \frac{\alpha\beta}{2} \cdot \frac{g(z)}{zg'(z)} (1 - r^2)$.

On peut vérifier que $\psi \in \Psi[1]$ (voir [2]) et, en employant le théorème A de [2], on obtient le résultat.

En employant le théorème précédent on obtient le résultat suivant.

Corollaire 1. *Si* $\alpha' > \alpha \geq 0$ *alors* $G_{\alpha'\beta}(g) \subset G_{\alpha,\beta}(g)$.

Remarques 1° Si $\alpha \geq 0$ et $\beta = 0$ alors le théorème I. devient

$$\mathrm{Re}\left[(1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)}\right] > 0, \quad \forall z \in U \Rightarrow \mathrm{Re}\frac{f(z)}{g(z)} > 0, \quad \forall z \in U, \ g \in S^*,$$

c'est-à-dire le lemme I de [1].

2°. Si $\alpha = 1$ on obtient

$$\mathrm{Re}\left[\frac{f'(z)}{g'(z)} + \frac{\beta}{2} \cdot \frac{1}{zg(z)g'(z)} (g^2(z) - f^2(z))\right] > 0, \ \forall z \in U \Rightarrow \mathrm{Re}\,\frac{f(z)}{g(z)} > 0, \ \forall z \in U,$$

$$g \in S^*, \ \beta \leq 1$$

qui est une forme plus générale du lème de Sakaguchi [5].

3°. Si $g(z) = z \in S^*$, on obtient du théorème 1

$$\mathrm{Re}\left[(1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) + \frac{\alpha\beta}{2}\left(1 - \frac{f^2(z)}{z^2}\right)\right] > 0, \ \forall z \in U \Rightarrow \mathrm{Re}\,\frac{f(z)}{z} > 0, \ \alpha \geq 0, \beta \leq 1,$$

qui pour $\beta = 0$ représente un résultat de Chichra [1].

Théorème 2. *Si* $f \in F_{\alpha,\beta}(g)$ *alors* $\mathrm{Re}\, f'(z) > 0, \ \forall z \in U$.

Démonstration. Soit $p(z) = f'(z), \ z \in U$. Si $f \in F_{\alpha,\beta}(g)$ alors

$$\mathrm{Re}\left[p(z) + \alpha z p'(z) \frac{g(z)}{zg'(z)} + \frac{\alpha\beta}{2} \cdot \frac{g(z)}{zg'(z)} (1 - p^2(z))\right] > 0, \quad \forall z \in U.$$

On considère $\psi(r, s) = r + \alpha \cdot s \frac{g(z)}{zg'(z)} + \frac{\alpha\beta}{2} \cdot \frac{g(z)}{zg'(z)} (1 - r^2)$ qui appartient à la classe $\Psi[1]$ et en employant le théorème A de [2] on obtient le résultat.

En employant le théorème precédent on obtient le résultat suivant.

Corollaire 2. *Si* $\alpha' > \alpha \geq 0$ *alors* $F_{\alpha',\beta}(g) \subset F_{\alpha,\beta}(g)$.

Remarque. Dans le théorème précédent, en prenant $g(z) = z \in S^*$ on obtient

$$\mathrm{Re}\left[f'(z) + \alpha z f''(z) + \frac{\alpha\beta}{2} (1 - f'^2(z))\right] > 0, \forall z \in U \Rightarrow \mathrm{Re}\, f'(z) > 0, \forall z \in U, \ \alpha \geq 0, \beta \leq 1$$

ou bien $\mathrm{Re}[f'(z) + \alpha z f''(z) - \frac{\alpha\beta}{2} f'^2(z)] > -\frac{\alpha\beta}{2}$, $\forall z \in U \Rightarrow \mathrm{Re}\, f'(z) > 0, \ \forall z \in U$, $\alpha \geq 0$, $\beta \leq 1$, qui pour $\beta = 1$ représente le théorème 7 de [4], et pour $\beta = 0$ représente le théorème 5 de [1].

Proposition. $f \in F_{\alpha,\beta}(g) \Leftrightarrow gf' \in G_{\alpha,\beta}(g)$.

On peut facilement vérifier cette propriété en employant la définition de ces classes; pour $\beta = 0$ et $g(z) = z$ on obtient le résultat de [1].

3. THÉORÈME 3. *Si f est une fonction holomorphe dans le disque unité normée par les conditions* $f(0) = f'(0) - 1 = 0$ et $\alpha \geqslant 0$, $\beta \leqslant 1$ alors

$$\mathrm{Re}\left[(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha z^2\left[\left\{\int_0^z f, z\right\} - \frac{\beta-1}{2}\cdot\frac{f'^2(z)}{f^2(z)}\right]\right] > -\frac{\alpha\beta}{2}, \forall z \in U \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z)} > 0, \forall z \in U$$

c'est-à-dire $f \in S^*$.

Démonstration. On applique le théorème 1 à la fonction définie par la relation $\frac{z^2f'(z)}{f(z)}$ dans le cas où $g(z) = z$.

COROLLAIRE 3. *Si f est une fonction holomorphe dans le disque unité, normée par les conditions* $f(0) = f'(0) - 1 = 0$ *et* $\alpha \geqslant 0$, *alors*

$$\mathrm{Re}\left[(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha z^2\left\{\int_0^z f, z\right\}\right] > -\frac{\alpha}{2}, \quad \forall z \in U \Rightarrow f \in S^*.$$

On obtient cela du théorème précédent pour $\beta = 1$. En employant ce résultat on peut montrer facilement que si $\alpha' > \alpha \geqslant 0$

$$\mathrm{Re}\left[(1+\alpha')\frac{zf'(z)}{f(z)} + \alpha' z^2\left\{\int_0^z f, z\right\}\right] > -\frac{\alpha'}{2}, \quad \forall z \in U \Rightarrow$$

$$\Rightarrow \mathrm{Re}\left[(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha z^2\left\{\int_0^z f, z\right\}\right] > -\frac{\alpha}{2}, \quad \forall z \in U.$$

THÉORÈME 4. *Si f est une fonctions holomorphe dans le disque unité normée par les conditions* $f(0) = f'(0) - 1 = 0$ et $\alpha \geqslant 0$, $\beta \leqslant 1$, *alors*

$$\mathrm{Re}\left[1 + (1+\alpha-\alpha\beta)\frac{zf''(z)}{f'(z)} + \alpha z^2\left[\{f, z\} - \frac{\beta-1}{2}\cdot\frac{f''^2(z)}{f'^2(z)}\right]\right] > 0, \quad \forall z \in U \Rightarrow$$

$$\Rightarrow \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad \forall z \in U$$

cèst-à-dire $f \in S^c$, *où* S^c *est la classe des fonctions convexes et usuellement normées dans* U.

Démonstration. On applique le théorème 1 à la fonctions f définie par la relation $z\left(\frac{zf''(z)}{f'(z)} + 1\right)$ dans le cas où $g(z) = z \in S^*$.

Si $\beta = 1$ on obtient le résultat de [2].

Pour $\beta = 1$ on peut montrer facilement que si $\alpha' > \alpha \geqslant 0$, alors

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)} + \alpha' z^2\{f, z\}\right] > 0, \ \forall z \in U \Rightarrow \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)} + \alpha z^2\{f, z\}\right] > 0, \forall z \in U.$$

4. Soit f, g des fonctions holomorphes dans le disque unité U. On dit que la fonction f est subordonnée à la fonction $g(f \prec g)$ si g est univalente dans le disque U, $f(0) = g(0)$ et $f(U) \subset g(U)$.

THÉORÈME 5. *Si h est une fonction convexe et univalente dans le disque unité U, $h(0) = 1$ et $\alpha \geqslant 0$, $\alpha + \beta \geqslant 0$ alors*

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + \beta)z^2\{f, z\} + \beta\frac{zf''(z)}{f'(z)}\left(1 + \frac{1}{2} \cdot \frac{zf''(z)}{f'(z)}\right) \prec h(z) \Rightarrow 1 + \frac{zf''(z)}{f'(z)} \prec h(z)$$

si pour $\alpha > 0$ on a $\inf\limits_{|z|=1} \operatorname{Re} \frac{1 - h^2(z)}{zh'(z)} \geqslant -2\left(1 + \frac{\beta}{\alpha}\right)$.

Démonstration. Soit la fonction $p(z) = 1 + \frac{zf''(z)}{f'(z)}$, $z \in U$.

La relation de l'énoncé est équivalente à $\psi(p(z), zp'(z)) \prec h(z)$ où

$$\psi(r, s) = r + \frac{\alpha}{2}(2s - r^2 + 1) + \beta s = r + (\alpha + \beta) \cdot s + \frac{\alpha}{2}(1 - r^2).$$

La fonction ψ est holomorphe dans le domaine $D = \mathbf{C}^2$,

$$\psi(h(0), 0) = 1 \in h(U).$$

Soit $\psi_0 = \psi(r_0, s_0)$ où $r_0 = h(z_0)$, $s_0 = m z_0 h'(z_0)$ où $m \geqslant 1$, $|z_0| = 1$.

$\psi_0 \notin h(U) \Leftrightarrow \operatorname{Re} \frac{\psi_0 - h(z_0)}{z_0 h'(z_0)} \geqslant 0$, quel que soit z_0 avec $|z_0| = 1$.

$$\operatorname{Re} \frac{\psi_0 - h(z_0)}{z_0 h'(z_0)} = (\alpha + \beta)m + \frac{\alpha}{2} \operatorname{Re} \frac{1 - h^2(z_0)}{z_0 h'(z_0)} \geqslant (\alpha + \beta)(m - 1) \geqslant 0$$

pour $m \geqslant 1$. En appliquant le théorème A de [3] on obtient que $p(z) \prec h(z)$.

COROLLAIRE 4. *Si $\alpha \geqslant 0$ et $\gamma < 1$ alors*

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + \beta)z^2\{f, z\} + \beta\frac{zf''(z)}{f'(z)}\left(1 + \frac{1}{2} \cdot \frac{zf''(z)}{f'(z)}\right) \prec \frac{1 + (2\gamma - 1)z}{1 + z} \Rightarrow$$

$$\Rightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + (2\gamma - 1)z}{1 + z}$$

où $\alpha|\gamma| \leqslant \beta$.

Démonstration. La fonction $h(z) = \frac{1 + (2\gamma - 1)z}{1 + z}$ est convexe et univalente dans le disque unité U, $h(0) = 1$ et

$\operatorname{Re} \frac{1 - h^2(z)}{zh'(z)} = -2 - 2\gamma \operatorname{Re} z \geqslant -2\left(1 + \frac{\beta}{\alpha}\right)$ si et seulement si $\alpha|\gamma| \leqslant \beta$.

Remarques 1°. Si $\beta = 0$ on obtient $\gamma = 0$ c'est-à-dire

$$1 + \frac{zf''(z)}{f'(z)} + \alpha z^2\{f, z\} \prec \frac{1 - z}{1 + z} \Rightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 - z}{1 + z}, \ \alpha \geqslant 0,$$

résultat déjà connu de [2].

2°. Si $\beta=\alpha|\gamma|$, $\gamma<1$ on obtient

$$1+\frac{zf''(z)}{f'(z)}+\alpha(1+|\gamma|)z^2\{f,z\}+\alpha|\gamma|\frac{zf''(z)}{f'(z)}\left(1+\frac{1}{2}\cdot\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+(2\gamma-1)z}{1+z}\Rightarrow$$

$$\Rightarrow 1+\frac{zf''(z)}{f'(z)}\prec\frac{1+(2\gamma-1)z}{1+z}\quad \alpha\geqslant 0,\quad \gamma<1.$$

3°. Si $0\leqslant\alpha<\beta$ *et* $|\gamma|\leqslant 1$ a lieu aussi l'implication du Corollaire 4.

COROLLAIRE 5. *Si* $0\leqslant\alpha<\beta$ *et* $1<M\leqslant\frac{\beta+\sqrt{\beta^2-\alpha^2}}{\alpha}$ alors

$$1+\frac{zf''(z)}{f'(z)}+(\alpha+\beta)z^2\{f,z\}+\beta\frac{zf''(z)}{f'(z)}\left(1+\frac{1}{2}\cdot\frac{zf''(z)}{f'(z)}\right)\prec M\frac{Mz+1}{M+z}\Rightarrow$$

$$\Rightarrow 1+\frac{zf''(z)}{f'(z)}\prec M\frac{Mz+1}{M+z}.$$

Démonstration. La fonction $h(z)=M\cdot\frac{Mz+1}{M+z}$ est convexe et univalente dans le disque unité U, $h(0)=1$ et

$$\operatorname{Re}\frac{1-h^2(z)}{zh'(z)}=-2-\frac{M^2+1}{M}\operatorname{Re}z\geqslant-2-2\frac{\beta}{\alpha}$$

si et seulement si $1<M\leqslant\frac{\beta+\sqrt{\beta^2-\alpha^2}}{\alpha}$

THÉORÈME 6. *Si* Re $\alpha\geqslant 0$ *et* f *est une fonction holomorphe dans le disque unité* U, *normée par les conditions* $f(0)=f'(0)-1=0$, *alors*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}+2\alpha\left(\frac{f(z)}{z}-f'(z)+zf''(z)\right)\right]>\frac{1}{2},\forall z\in U\Rightarrow\operatorname{Re}\frac{f(z)}{z}>\frac{1}{2},\quad\forall z\in U.$$

Si $\alpha=0$ on obtient le théorème de Marx et Strohhäcker [6], [7].

Démonstration. Soit la fonction $p(z)=\frac{2f(z)}{z}-1$, $z\in U$.
Le relation de l'énoncé est équivalente à

$$\operatorname{Re}\left[\frac{1}{2}+\frac{zp'(z)}{p(z)+1}+\alpha(zp'(z)+z^2p''(z))\right]>0,\ \forall z\in U.$$

Soit donc $\psi(r,s)=\frac{1}{2}+\frac{s}{r+1}+\alpha(s+t)$ avec le domaine correspondant $D=(\mathbb{C}\setminus\{-1\})\times\mathbb{C}\times\mathbb{C}$. On vérifie que $\psi\in\Psi[1]$ et en employant le théorème A de [2] on obtient le résultat cherché.

Remarque. Si l'on prend dans le théorème précédent $\frac{f(z)}{z}=p(z)$, on obtient que, si Re $\alpha\geqslant 0$ et $p(0)=1$ alors

$$1+\frac{zp'(z)}{p(z)}+2\alpha(zp'(z)+z^2p''(z))\prec\frac{1}{1+z}\Rightarrow p(z)\prec\frac{1}{1+z}.$$

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GENERALIZĂRI ALE UNOR CLASE DE FUNCŢII UNIVALENTE

(Rezumat)

În prezenta lucrare se generalizează clasele $\mathcal{F}_\alpha$ şi $\mathcal{G}_\alpha$ din [1] obţinindu-se cu ajutorul metodei date în [2] generalizări ale unor rezultate ale lui P. N. Chichra [1] şi K. Sakaguchi [5]. Sînt prezentate cîteva rezultate referitoare la derivata lui Schwarz şi convexitatea unei funcţii olomorfe, normate uzual în discul unitate. În ultima parte este dată o generalizare a teoremei lui Marx şi Strohhächer [6], [7] şi rezultate referitoare la funcţiile convexe de un anumit ordin.

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ON A PROBLEM OF SOBOLEV TYPE

VIORICA MUREŞAN

1. In the present paper we suggest to study the following problem of Sobolev type ([1]):

$$u_x(t, y, x) = F(t, y, x, u(t, y, x), u(t, x, x), u(x, y, x)) \quad (1)$$

$$u(t, y, 0) = f(t, y) \quad (2)$$

where F and f are given functions.

Varied problems of Sobolev type make the subject of many investigations. Thus V. Laksmikantham and M. Lord [1] and M. E. Lord [3] established the existence and uniqueness Theorems, as well V. Lakshmikantham, A. S. Vatsala and R. L. Vaughn [2] and A. S. Vatsala and R. L. Vaughn [4] established existence theorems. In the papers mentioned above there are some other considerations on this type of problems.

The purpose of the present paper is to demonstrate the existence and uniqueness theorem for the problem (1) + (2) through application of Banach's fixed point theorem and with convenient choise of metric in the spaces of considered functions, as well as to study the dependence of solution of the problem (1) + (2) from the dates of it.

2. We now deal with the existence and uniqueness of the solution of the problem. Let I denote the interval $[0, a]$. We consider the set

$$S_f = \{r \in R / |r - f(t, y)| \leq b, \ \forall t, y \in I\},$$

where $b > 0$ and we suppose that b and f are such that S_f is nonvoid set.

We suppose that the following conditions take place

$$\text{(i)} \quad f \in C[I \times I, \mathbf{R}],$$

$$\text{(ii)} \quad F \in C[I \times I \times I \times S_f \times S_f \times S_f, \mathbf{R}],$$

and Lipschitz's condition

$$\text{(iii)} \quad |F(t, y, x, u_1, v_1, w_1) - F(t, y, x, u_2, v_2, w_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2| + L_3 |w_1 - w_2|,$$

for any $(t, y, x, u_i, v_i, w_i) \in I \times I \times I \times S_f \times S_f \times S_f$, $i = 1, 2$.

We have

THEOREM 1 (see [1]). *If the conditions* (i), (ii), (iii) *are carried out and* $M = \sup\{|F(t, y, x, u, v, w)| / (t, y, x, u, v, w) \in I \times I \times I \times S_f \times S_f \times S_f\}$, $\delta = \min\left\{a, \frac{b}{M}\right\}$ again $\Delta = [0, \delta]$, *then the problem* (1) + (2) *has in the set* $B_f = \{u : \Delta \times \Delta \times \Delta \to S_f\}$ *one solution and only one, solution which can be obtai-*

ned through the succesive approximation method starting from any element of B_f.

Proof. We endow the set B_f with the metric determined by Bielecki norm

$$||u||_B = \max_{(t,y,x)\in\Delta\times\Delta\times\Delta} (|u(t, y, x)|e^{-\tau x}), \text{ where } \tau \in \mathbf{R}_+.$$

Thus $(B_f, ||\cdot||_B)$ becomes complete metric space.

The problem (1) + (2) is equivalent with the following integral equation

$$u(t, y, x) = f(t, y) + \int_0^x F(t, y, s, u(t, y, s), u(t, s, s), u(s, y, s))ds.$$

We consider the operator $A : B_f \to B_f$ defined by

$$(Au)(t, y, x) = f(t, y) + \int_0^x F(t, y, x, u(t, y, s), u(t, s, s), u(s, y, s))ds \text{ for all}$$

$(t, y, x) \in \Delta \times \Delta \times \Delta$.

Because $\delta \leqslant \frac{b}{M}$, it results that the condition of invariance takes place for B_f, that is $A(B_f) \subseteq B_f$. The mapping $A : (B_f, ||\cdot||) \to (B_f, ||\cdot||_B)$ is Lipschitz with the constant of Lipschitz $\frac{1}{\tau}(L_1 + L_2 + L_3)$. Therefore for sufficiently large τ, the mapping A is a contraction. Thus the Theorem 1 follows from Banach's fixed point Theorem.

Remark. An analogous theorem was demonstrated by V. Lakshmikantham *and* M. Lord in [1] through the succesive approximation method.

3. In sequel we study the dependence of the solution of the problem (1) + + (2) from F and f. To this we also consider the following problem

$$u_x(t, y, x) = G(t, y, x, u(t, y, x), u(t, x, x), u(x, y, x)) \tag{3}$$

$$u(t, y, 0) = g(t, y), \tag{4}$$

where $g \in C[I \times I, \mathbf{R}]$ and $G \in C[I \times I \times I \times S_g \times S_g \times S_g, \mathbf{R}]$.

We have

THEOREM 2. *In the conditions* (i) — (iii) *let* $u^* \in B_f$ *be the unique solution of problem* (1) + (2). *We suppose that* $u \in B_g$ *is a solution of problem* (3) + (4). *If the following*

(iv) b, f and g *are so that the* $S_{fg} = S_f \cap S_g$ *is a nonvoid set,*

(v) *there is* $\eta_1 > 0$ *so that* $|f(t, y) - g(t, y)| \leqslant \eta_1$, *for all* $(t, y) \in I \times I$,

(vi) *there is* $\eta_2 > 0$ *so that*

$|F(t, y, x, u, v, w) - G(t, y, x, u, v, w)| \leqslant \eta_2$,

for all $(t, y, x, u, v, w) \in I \times I \times I \times S_{fg} \times S_{fg} \times S_{fg}$,

(vii) $u^*(t, y, x)$, $u(t, y, x) \in S_{fg}$, $\forall\, t, y, x \in \Delta$, *take place, then we have*

$$\|u^* - u\|_C \leqslant (\eta_1 + \eta_2\delta)\ e^{(L_1+L_2+L_3)\delta},$$

where $\|\cdot\|_C$ *is the Cebisev norm.*

Proof. Estimating the difference of the solutions and applying an inequality of Bellman-Gronwall type ([1]) it obtain

$$|u^*(t, y, x) - u(t, y, x)| \leqslant \eta_1 + \eta_2\delta + L_1\int_0^x |u^*(t, y, s) - u(t, y, s)|\, ds +$$

$$+ L_2\int_0^x |u^*(t, s, s) - u(t, s, s)|\, ds + L_3\int_0^x |u^*(s, y, s) - u(s, y, s)|\, ds \leqslant$$

$$\leqslant (\eta_1 + \eta_2\delta)\ e^{(L_1+L_2+L_3)\delta}, \text{ for all } (t, y, x) \in \Delta \times \Delta \times \Delta.$$

Therefore

$$\|u^* - u\|_C \leqslant (\eta_1 + \eta_2\delta)\ e^{(L_1+L_2+L_3)\delta} \tag{5}$$

and thus the theorem is proved.

Remark. If we used the proof of Theorem 1.1 from [6], then it follows

$$\|u^* - u\|_C \leqslant \frac{(\eta_1 + \eta_2\delta)e^{\tau\delta}}{1 - \dfrac{L_1 + L_2 + L_3}{\tau}}. \tag{6}$$

As

$$\frac{e^{\tau\delta}}{\dfrac{L_1 + L_2 + L_3}{\tau}} > e^{(L_1 \times L_2 + L_3)\delta},$$

we can assert that the evaluation given by (5) is better than that given by (6).

In conclusion, the solution of the problem (1) + (2) depends on the continuous way by f and F.

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ASUPRA UNEI PROBLEME DE TIP SOBOLEV

(Rezumat)

În prezenta lucrare se demonstrează o teoremă de existenţă şi unicitate pentru o problemă de tip Sobolev şi se studiază dependenţa soluţiei problemei de datele iniţiale.

THE GENERAL n-DIMENSIONAL BETA DISTRIBUTION

ION MIHOC

1. Let $[\Omega, \mathcal{A}, P]$ be a probability space in the sense of Kolmogorov i.e. Ω an arbitrary abstract space, $\mathcal{A}$ a σ-algebra of subsets of Ω and P a probability measure in Ω and on $\mathcal{A}$.

A random variable Z, defined on $[\Omega, \mathcal{A}, P]$ has a general beta distribution if its probability density function has the following form:

$$\bar{\beta}_1(z;\ a_1, a_2) = \frac{1}{B(a_1, a_2)} \cdot \frac{(z-b_1)^{a_1-1}(b_2-z)^{a_2-1}}{(b_2-b_1)^{a_1+a_2-1}}, \tag{1.1}$$

where

$$0 < b_1 < z < b_2,\ a_1,\ a_2 > 0 \tag{1.1a}$$

and

$$B(a_1, a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, dt. \tag{1.1b}$$

If for the parameters b_1 and b_2 we consider the particular values: $b_1 = 0$, $b_2 = 1$, then we obtain just the probability density function

$$\beta_1(y;\ a_1,\ a_2) = \frac{1}{B(a_1, a_2)} \cdot y^{a_1-1}(1-y)^{a_2-1}, \tag{1.2}$$

where

$$0 < y < 1,\ a_1,\ a_2 > 0, \tag{1.2a}$$

for a random variable Y with a beta distribution and with the pair of parameters (a_1, a_2).

Now, if $Y^{(n)}$ is a n-dimensional random vector (by a random vector of n-dimensions we understand an n-dimensional vector $Y^{(n)} = (Y_1, Y_2, \ldots, Y_n)$ whose components Y_j are random variables on the same probability space), which follows an n-dimensional beta distribution (known as the Dirichlet distribution), then its probability density function will be [2]:

$$\beta_n(y_1, y_2, \ldots, y_n;\ a_1, a_2, \ldots, a_n, a_{n+1}) = \frac{1}{B_n(a_1, a_2, \ldots, a_n;\ a_{n+1})} \prod_{i=1}^{n} y_i^{a_i-1}\left(1 - \sum_{i=1}^{n} y_i\right)^{a_{n+1}-1}, \tag{1.3}$$

where

$$Y^{(n)} \in D_n,\ a_j > 0,\ j = \overline{1, n+1}, \tag{1.3a}$$

$$D_n = \left\{(y_1, y_2, \ldots, y_n) \mid y_i > 0,\ i = \overline{1, n},\ \sum_{i=1}^{n} y_i < 1\right\}, \tag{1.3b}$$

and

$$B_n(a_1, a_2, \ldots, a_n; a_{n+1}) = \frac{\prod_{i=1}^{n} \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^{n+1} a_i\right)}, \tag{1.3c}$$

is n-dimensional beta function or the Dirichlet function [1].

2. In this section we will define a new distribution which may be called the general n-dimensional beta distribution.

DEFINITION. We say that an n-dimensional random vector $Z^{(n)} = (Z_1, Z_2, \ldots, Z_n)$, follows a general n-dimensional beta distribution if its probability density function has the following form:

$$\bar{\beta}_n(z_1, z_2, \ldots, z_n; a_1, a_2, \ldots, a_n, a_{n+1}) = \frac{1}{B_n(a_1, \ldots, a_n, a_{n+1})} \prod_{i=1}^{n} \frac{\left(z_i - b_1^{(i)}\right)^{a_i-1}}{\left(b_2^{(i)} - b_1^{(i)}\right)^{a_i}} \cdot \left[1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}}\right]^{a_{n+1}-1} \tag{2.1}$$

if $Z^{(n)} \in \bar{D}_n$, $a_j > 0$, $j = \overline{1, n+1}$,

where

$$\bar{D}_n = \left\{(z_1, \ldots, z_n), 0 < b_1^{(i)} < z_i < b_2^{(i)};\ i = \overline{1, n};\ 1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}} > 0\right\} \tag{2.1a}$$

and $B_n(a_1, a_2, \ldots, a_n, a_{n+1})$ is the Dirichlet function.

The above definition is based on the following theorem.

THEOREM 1. *The function* $\bar{\beta}_n(z_1, \ldots, z_n; a_1, a_2, \ldots, a_{n+1})$, *defined by the formula* (2.1), *with the condition* (2.1a), *is a probability density function.*

Proof. To prove Theorem 1 it is suffices to show that $\bar{\beta}_n$, given by (2.1), satisfies the following conditions:

1° $\bar{\beta}_n(z_1, \ldots, z_n; a_1, \ldots, a_n, a_{n+1}) \geqslant 0$, for all $Z^{(n)} \in \bar{D}_n$,

2° $I_n = \int_{\bar{D}_n} \bar{\beta}_n (z_1, z_2, \ldots, z_n; a_1, \ldots, a_{n+1}) dz_1\, dz_2 \ldots dz_n = 1.$

From the above definitions, for the probability density and for the domain $\bar{D}_n$, we can see that the first condition is verified.

For to verify the second condition we make the following transformation:

$$z_i = b_1^{(i)} + (b_2^{(i)} - b_1^{(i)}) \cdot v_i \prod_{j=1}^{i-1} (1 - v_j),\ i = \overline{1, n}. \tag{2.2}$$

Taking into account that

$$\frac{\partial z_i}{\partial v_i} = (b_2^{(i)} - b_1^{(i)}) \cdot \prod_{j=1}^{i-1} (1 - v_j), \ i = \overline{1, n}, \tag{2.2a}$$

$$\frac{\partial z_i}{\partial v_k} = - (b_2^{(i)} - b_1^{(i)}) \cdot \prod_{\substack{j=1 \\ j \neq k}}^{i-1} (1 - v_j) \cdot v_i ; \ i = \overline{2, n} ; k = \overline{1, n-1}, i > k, \tag{2.2b}$$

$$\frac{\partial z_i}{\partial v_k} = 0, \ i < k, \ i = \overline{1, n-1}, \ k = \overline{2, n}, \tag{2.2c}$$

we obtain, for the Jacobian J_n, of the transformation (2.2), the form

$$J_n = \prod_{i=1}^{n} (b_2^{(i)} - b_1^{(i)}) \cdot \prod_{j=1}^{n-1} (1 - v_j)^{n-j}, \tag{2.3}$$

or, the echivalent form

$$J_n = \prod_{i=1}^{n} (b_2^{(i)} - b_1^{(i)}) \cdot \prod_{i=1}^{n} \left(\prod_{j=1}^{i-1} (1 - v_j) \right) \cdot \tag{2.3a}$$

Also, from

$$0 < b_1^{(i)} < z_i < b_2^{(i)}, \ i = \overline{1, n}, \tag{2.4}$$

the following conditions follow:

$$v_i \cdot \sum_{j=1}^{i-1} (1 - v_j) > 0, \ i = \overline{1, n}, \ \text{because} \ b_2^{(i)} - b_1^{(i)} > 0, \ \ i = \overline{1, n} \tag{2.4a}$$

$$v_i \cdot \sum_{j=1}^{i-1} (1 - v_j) < 1, \ i = \overline{1, n}. \tag{2.4b}$$

From the condition

$$1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}} > 0, \tag{2.5}$$

we obtain the condition

$$\sum_{i=1}^{n} \left[v_i \cdot \prod_{j=1}^{i-1} (1 - v_j) \right] < 1. \tag{2.5a}$$

From (2.4a), (2.4b) and (2.5a) it follows that

$$0 < v_i < 1, \ i = \overline{1, n}. \tag{2.6}$$

Such, if we make the transformation (2.2), the domain $\bar{D}_n$ becomes the following open hypercube:

$$\Delta_n = \{(v_1, v_2, \ldots, v_n) \mid 0 < v_i < 1, \ i = \overline{1, n}\}. \tag{2.7}$$

Finally, we need the following relations:

$$\frac{\left(z_i - b_1^{(i)}\right)^{a_i - 1}}{\left(b_2^{(i)} - b_1^{(i)}\right)^{a_i}} = \frac{v_j \cdot \prod_{j=1}^{i-1} (1 - v_j)^{a_i - 1}}{b_2^{(i)} - b_1^{(i)}}, \quad i = \overline{1, n}, \tag{2.8}$$

$$\left[1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}}\right]^{a_{n+1} - 1} = \left[1 - \sum_{i=1}^{n} v_i \prod_{j=1}^{i-1} (1 - v_j)\right]^{a_{n+1} - 1}. \tag{2.8a}$$

From the definition formula (2.1), using the relations (2.8), (2.8a), the transformation (2.2) and the absolute value of the Jacobian J_n, the above integral becomes

$$I_n = \frac{1}{B_n(a_1, a_2, \ldots, a_{n+1})} \cdot I'_n, \tag{2.9}$$

where

$$I'_n = \int_{\Delta_n} \prod_{i=1}^{n} \left\{ v_i^{a_i - 1} (1 - v_i)^{a_{n+1} - 1} \cdot \prod_{j=1}^{i-1} (1 - v_j)^{a_i} \right\} dv_1 \, dv_2 \ldots dv_n \tag{2.10}$$

because

$$1 - \sum_{i=1}^{n} v_i \cdot \prod_{j=1}^{i-1} (1 - v_j) = \prod_{j=1}^{n} (1 - v_j). \tag{2.10a}$$

Denoting

$$A_n(v_1, v_2, \ldots, v_n) = \prod_{i=1}^{n} \left\{ v_i^{a_i - 1} (1 - v_i)^{a_{n+1} - 1} \cdot \prod_{j=1}^{i-1} (1 - v_j)^{a_i} \right\} \tag{2.11}$$

it follows that:

— for $n = 2$, we have

$$A_2(v_1, v_2) = \prod_{i=1}^{2} v_i^{a_i - 1} (1 - v_i)^{\sum_{j=i+1}^{3} a_j - 1}; \tag{2.11a}$$

— for $n = 3$, we have

$$A_3(v_1, v_2, v_3) = \prod_{i=1}^{3} v_i^{a_i - 1} (1 - v_i)^{\sum_{j=i+1}^{4} a_j - 1}; \tag{2.11b}$$

— for n, it follows by induction that

$$A_n(v_1, v_2, \ldots, v_n) = \prod_{i=1}^{n} v_i^{a_i - 1} (1 - v_i)^{\sum_{j=i+1}^{n+1} a_j - 1}. \tag{2.12}$$

Using this new form for $A_n(v_1, v_2, \ldots, v_n)$, the above integral, I_n, becomes

$$I_n = \frac{1}{B_n(a_1, \ldots, a_{n+1})} \cdot \int_{\Delta_n} \prod_{i=1}^{n} v_i^{a_i-1}(1-v_i)^{\sum_{j=i+1}^{n+1} a_j - 1} dv_1\, dv_2 \ldots dv_n, \tag{2.13}$$

or

$$I_n = \frac{1}{B_n(a_1, \ldots, a_{n+1})} \cdot \prod_{j=1}^{n} \left(\int_0^1 v_i^{a_i-1}(1-v_i)^{\sum_{j=i+1}^{n+1} a_j - 1} dv_i \right), \tag{2.14}$$

f we use the definition of Δ_n.

On the basis of the relations

$$\int_0^1 v_i^{a_i-1}(1-v_i)^{\sum_{j=i+1}^{n+1} a_j - 1} dv_i = B_i\left(a_i, \sum_{j=i+1}^{n+1} a_j\right), \quad i = \overline{1, n}, \tag{2.15}$$

it follows that

$$I_n = \frac{1}{B_n(a_1, a_2, \ldots, a_{n+1})} \cdot \prod_{i=1}^{n} B_i\left(a_i, \sum_{j=i+1}^{n+1} a_j\right). \tag{2.16}$$

Finally, taking into account that

$$B_i\left(a_i, \sum_{j=i+1}^{n+1} a_j\right) = \frac{\Gamma(a_i)\Gamma\left(\sum_{j=i+1}^{n+1} a_j\right)}{\Gamma\left(\sum_{j=1}^{n+1} a_j\right)}, \quad i = \overline{1, n}, \tag{2.17}$$

from (2.16) we obtain $I_n = 1$.

Thus it has been shown that the function defined by the formula (2.1), is a probability density. Theorem 1 is proved.

3. Ordinary moments of the general n-dimensional beta distribution.

We now state and prove the following theorem.

THEOREM 2. *If $Z^{(n)} = (Z_1, Z_2, \ldots, Z_n)$ is a random vector with the probability density* (2.1), *then the ordinary moment of order $(r_1, r_2, \ldots, r_n)$ will be*

$$M_{r_1, r_2, \ldots, r_n}(Z^{(n)}) = \frac{1}{B_n(a_1, a_2, \ldots, a_{n+1})} \cdot I_{n, n-1, \ldots, 2, 1} \tag{3.1}$$

where

$$I_{n,n-1,\ldots,n-k,\ldots,2,1} = \sum_{s_0=0}^{r_n}\Big\{A_{s_0}\Big[\sum_{s_1=0}^{r_1}\Big\{A_{s_0 s_1}\ldots\Big[\sum_{s_k=0}^{r_{n-k}}\Big\{A_{s_0 s_1\ldots s_k}\ldots$$

$$\ldots\Big[\sum_{s_{n-2}=0}^{r_2}\Big\{A_{s_0 s_1\ldots s_k\ldots s_{n-2}}\Big[\sum_{s_{n-1}=0}^{r_1}\Big\{\cdot$$

$$\cdot\Big\{A_{s_0 s_1\ldots s_k\ldots s_{n-2}s_{n-1}}\Big\}\Big]\ldots\Big\}\Big]\Big\} \tag{3.2}$$

$$A_{s_0 s_1\ldots s_k} = A_{\prod\limits_{t=0}^{k} s_t} = B\left(a_{n-k}+r_{n-k}-s_k,\ \sum_{i=n-(k-1)}^{n+1} a_i + \sum_{i=n-(k-1)}^{n} r_i\right)\cdot$$

$$\cdot\, C^{s_k}_{r_{n-k}}\left(b_1^{(n-k)}\right)^{s_k}\left(b_2^{(n-k)}-b_1^{(n-k)}\right)^{r_{n-k}-s_k},\ k=\overline{0,\ n-1} \tag{3.3}$$

and $B_n(a_1,\ a_2,\ \ldots,\ a_n,\ a_{n+1})$ *is the Dirichlet function.*

Proof. The proof is by direct computation.

Making use of the definition of ordinary moment we have

$$M_{r_1,r_2,\ldots,r_n}(Z^{(n)}) = \frac{1}{B_n(a_1,\ a_2,\ \ldots,\ a_n,\ a_{n+1})}\cdot \bar{I}_n, \tag{3.4}$$

$$\bar{I}_n = \int_{D_n}\prod_{i=1}^{n} z_i^{r_i}\cdot\prod_{i=1}^{n}\frac{\left(z_i-b_1^{(i)}\right)^{a_i-1}}{\left(b_2^{(i)}-b_1^{(i)}\right)^{a_i}}\left[1-\sum_{i=1}^{n}\frac{z_i-b_1^{(i)}}{b_2^{(i)}-b_1^{(i)}}\right]^{a_{n+1}-1} dz_1\ldots dz_n. \tag{3.5}$$

To evaluate the above integral, $\bar{I}_n$, we apply the same transformation (2.2). Using the fact that for Jacobian we have the form (2.3) and that the new domain Δ_n is defined by (2.7), we can prove this theorem.

Because (2.8), (2.8a) and

$$\prod_{i=1}^{n} z_i^{r_i} = \prod_{i=1}^{n}\left[b_1^{(i)}+\left(b_2^{(i)}-b_1^{(i)}\right)\cdot v_i\cdot\prod_{j=1}^{i-1}(1-v_j)\right]^{r_i} =$$

$$= \prod_{i=1}^{n}\left[\sum_{t=0}^{r_i} C^t_{r_i}\left(b_1^{(i)}\right)^t\left(b_2^{(i)}-b_1^{(i)}\right)^{r_i-t}\cdot v_i^{r_i-t}\cdot\prod_{j=1}^{i-1}(1-v_j)^{r_i-t}\right], \tag{3.6}$$

the above integral, $\bar{I}_n$, becomes

$$I_n = \left(\int_0^1\sum_{r_1}\left(\int_0^1\sum\left(\cdots\left(\int_0^1\sum_{r_{n-1}}\left(\int_0^1\sum_{r_n} dv_n\right)_n dv_{n-1}\right)_{n-1}\ldots\right)_2 dv_1\right)_1 \tag{3.7}$$

where

$$\sum_{r_i} = \sum_{t=0}^{r_i} C_{r_i}^t (b_1^{(i)})^t (b_2^{(i)} - b_1^{(i)})^{r_i - t} \cdot v_i^{a_i + r_i - t - 1} \cdot (1 - v_i)^{\sum_{s=i+1}^{n+1} a_s - 1} \cdot \prod_{j=1}^{i-1} (1 - v_j)^{r_i - t}, \quad i = \overline{1, n}. \tag{3.8}$$

From (3.7) we see that we must compute the following integrals:

$$I_n = \int_0^1 \sum_{r_n} dv_n$$

$$_{n,n-1} = \int_0^1 \sum_{r_{n-1}} I_n \ dv_{n-1}$$

. . . .

$$I_{n,n-1,\ldots,n-k} = \int_0^1 \sum_{r_{n-k}} I_{n,n-1,\ldots,n-k+1} \ \mathrm{dv}_{n-k} \tag{3.9}$$

. . . .

$$I_{n,n-1,\ldots,n-k,\ldots,2,1} = \bar{I}_n$$

First we consider the integral I_n. We have

$$I_n = \int_0^1 \left(\sum_{s_0=0}^{r_n} C_{r_n}^{s_0} (b_1^{(n)})^{s_0} (b_2^{(n)} - b_1^{(n)})^{r_n - s_0} \cdot v_n^{a_n + r_n - s_0 - 1} \cdot (1 - v_n)^{a_{n+1} - 1} \cdot \prod_{j=1}^{n-1} (1 - v_j)^{r_n - s_0} \right) dv_n = \sum_{s_0=0}^{r_n} C_{r_n}^{s_0} (b_1^{(n)})^{s_0} (b_2^{(n)} - b_1^{(n)})^{r_n - s_0} \cdot \prod_{j=1}^{n-1} (1 - v_j)^{r_n - s_0} \cdot \int_0^1 v_n^{a_n + r_n - s_0 - 1} (1 - v_n)^{a_{n+1} - 1} dv_n$$

or

$$I_n = \sum_{s_0=0}^{r_n} B(a_n + r_n - s_0, \ a_{n+1}) \cdot C_{r_n}^{s_0} (b_1^{(n)})^{s_0} (b_2^{(n)} - b_1^{(n)})^{r_n - s_0} \cdot \prod_{j=1}^{n-1} (1 - v_j)^{r_n - s_0} \tag{3.10}$$

where

$$B(a_n + r_n - s_0, \ a_{n+1}) = \int_0^1 v_n^{a_n + r_n - s_0 - 1} (1 - v_n)^{a_{n+1} - 1} \ dv_n. \tag{3.10a}$$

For the next integral $I_{n,n-1}$ we have

$$I_{n,n-1} = \sum_{s_0=0}^{r_n} \Big\{ B(a_n + r_n - s_0,\ a_{n+1}) \cdot C_{r_n}^{s_0} (b_1^{(n)})^{s_0} (b_2^{(n)} - b_1^{(n)})^{r_n - s_0} \cdot \Big._0$$

$$\cdot \Big[\sum_{s_1=0}^{r_{n-1}} B(a_{n-1} + r_{n-1} - s_1,\ a_n + a_{n+1} + r_n - s_0) \cdot (b_1^{(n-1)})^{s_1} \cdot (b_2^{(n-1)} - b_1^{(n-1)})^{r_{n-1}-s_1} \cdot \Big._0$$

$$\cdot \prod_{i=1}^{n-2} (1 - v_j)^{r_{n-1}+r_n-s_0-s_1} \Big]_1 . \tag{3.11}$$

Generally, for the integral $I_{n,n-1,\ldots,n-k}$, $k = \overline{0,\ n-1}$, we have

$$I_{n,n-1,\ldots,n-k} = \sum_{s_0=0}^{r_n} \Big\{ B(a_n + r_n - s_0,\ a_{n+1}) \cdot C_{r_n}^{s_0} (b_1^{(n)})^{s_0} (b_2^{(n)} - b_1^{(n)})^{r_n - s_0} \cdot \Big._0$$

$$\cdot \Big[\sum_{s_1=0}^{r_{n-1}} \Big\{ B(a_{n-1} + r_{n-1} - s_1,\ a_n + a_{n+1} + r_n - s_0) \cdot C_{r_{n-1}}^{s_1} (b_1^{(n)})^{s_1} \cdot \Big._1 \Big._0$$

$$\cdot (b_2^{(n-1)} - b_1^{(n-1)})^{r_{n-1}-s_1} \cdot \Big]_1 \ldots$$

$$\ldots \Big[\sum_{s_k=0}^{r_{n-k}} \Big\{ B\Big(a_{n-k} + r_{n-k} - s_k,\ \sum_{i=n-(k-1)}^{n+1} a_i + \sum_{i=n-(k-1)}^{n} r_i - s_{k-1}\Big) \cdot \Big._k \Big._{k-1}$$

$$\cdot C_{r_{n-k}}^{s_k} (b_1^{(n-k)})^{s_k} (b_2^{(n-k)} - b_1^{(n-k)})^{r_{n-k} - s_k} \cdot \prod_{j=1}^{n-(k+1)} (1 - v_j)^{\sum_{i=n-k}^{n} r_i - \sum_{i=0}^{k} s_i} \Big]_k . \tag{3.12}$$

Thus, for $\bar{I}_n$ we obtain

$$\bar{I}_n = I_{n,n-1,\ldots,n-k,\ldots,2,1} = \sum_{s_0=0}^{r_n} \Big\{ B(a_n + r_n - s_0,\ a_{n+1}) \cdot C_{r_n}^{s_0} (b_1^{(n)})^{s_0} (b_2^{(n)} - b_1^{(n)})^{r_n - s_0} \cdot \Big._0$$

$$\cdot \Big[\sum_{s_1=0}^{r_{n-1}} \Big\{ B(a_{n-1} + r_{n-1} - s_1,\ a_n + a_{n+1} + r_n - s_0) \cdot C_{r_{n-1}}^{s_1} (b_1^{(n-1)})^{s_1} \cdot \Big._1 \Big._0$$

$$\cdot (b_2^{(n-1)} - b_1^{(n-1)})^{r_{n-1} - s_1} \Big[_1 \ldots$$

$$\ldots \Big[\sum_{s_k=0}^{r_{n-k}} \Big\{ B\Big(a_{n-k} + r_{n-k} - s_k,\ \sum_{i=n-(k-1)}^{n+1} a_i + \sum_{i=n-(k-1)}^{n} r_i - s_{k-1}\Big) \cdot C_{r_{n-k}}^{s_k} \cdot \Big._k \Big._{k-1}$$

$$\cdot \left(b_1^{(n-k)}\right)^{s_k} \left(b_2^{(n-k)} - b_1^{(n-k)}\right)^{r_{n-k}-s_k} \Bigg[_k \cdots \Bigg[_{n-3}$$

$$\Bigg[_{n-3} \sum_{s_{n-3}=0}^{r_3} \Bigg\{_{n-2} B\left(a_2 + r_2 - s_{n-2}, \sum_{i=3}^{n+1} a_i + \sum_{i=3}^{n} r_i - s_{n-3}\right) \cdot C_{r_2}^{s_{n-2}} \cdot$$

$$\cdot \left(b_1^{(2)}\right)^{s_{n-2}} \left(b_2^{(2)} - b_1^{(2)}\right)^{r_2-s_{n-2}} \cdot \Bigg[_{n-2} \sum_{s_{n-2}=0}^{r_1} \Bigg\{_{n-1} B\left(a_1 + r_1 - s_{n-1}, \sum_{i=2}^{n+1} a_i + \sum_{i=2}^{n} r_i - s_{n-2}\right) \cdot$$

$$\cdot C_{r_1}^{s_{n-1}} \left(b_1^{(1)}\right)^{s_{n-1}} \cdot \left(b_2^{(1)} - b_1^{(1)}\right)^{r_1-s_{n-1}} \Bigg\}_{n-1} \Bigg]_{n-2} \cdots \Bigg]_1 \Bigg\}_1 \Bigg]_0 \Bigg\}_0. \tag{3.13}$$

Introducing the notation (3.3) and substituting the above integral, $\bar{I}_n$, in (3.4) we get just the form (3.1) of the moment $Mr_1, r_2, \ldots, r_n(Z^{(n)})$, when the random vector $Z^{(n)}$ follows the general n-dimensional beta distribution.

Remark 1. If in the definition (1.1) of the probability density, for the random vector $Z^{(n)}$, we consider that

$$b_2^{(i)} = 1, \; b_1^{(i)} = 0, \; i = \overline{1, n}, \tag{3.14}$$

then we obtain, as a particular case, just the probability density function (1.3) of the random vector $Y^{(n)}$ which follows the n-dimensional beta distribution.

Remark 2. If we examine the terms $A_{s_0 s_1 \ldots s_k}$, we can see that they haven't the factor $\left(b_1^{(n-k)}\right)^{s_k}$ only if

$$s_k = 0, \;\; k = \overline{0, n-1}. \tag{3.15}$$

Taking into account the above remarks we can consider the following particular cases:

Case 1. If $b_1^{(i)} = 0, \; b_2^{(i)} = 1, \; i = \overline{1, n}$ then

$\left(b_1^{(n-k)}\right)^{s_k} = 0$, for all s_k, $s_k = 1, 2, \ldots, r_{n-k}$, $k = \overline{0, n-1}$

$\left(b_2^{(n-k)}\right) - b_1^{(n-k)}\Big)^{r_{n-k}-s_k} = 1$, for all s_k, $s_k = 0, 1, 2, \ldots, r_{n-k}$, $k = \overline{0, n-1}$

and therefore

$$I_{n, n-1, \ldots, 2, 1} = \prod_{k=0}^{n-1} B\left(a_{n-k} + r_{n-k}, \sum_{i=n-(k-1)}^{n+1} a_i + \sum_{i=n-(k-1)}^{n} r_i\right). \tag{3.16}$$

The ordinary moment of order $(r_1, r_2, \ldots, r_n)$ which coresponds to this particular case, will be

$$M_{r_1,\ldots,r_n}(Z^{(n)})\Bigg|_{\substack{b_1^{(i)}=0 \\ b_2^{(i)}=1}} {}_{i=\overline{1,n}} = \frac{\prod_{k=0}^{n-1} B\left(a_{n-k}+r_{n-k}, \sum_{i=n-(k-1)}^{n+1} a_i + \sum_{i=n-(k-1)}^{n} r_i\right)}{B_n(a_1, a_2, \ldots, a_n, a_{n+1})} =$$

$$= \frac{\Gamma(a_1+r_1)\cdot\Gamma(a_2+r_2)\cdot\ldots\cdot\Gamma(a_n+r_n)\cdot\Gamma(a_1+a_2+\ldots+a_{n+1})}{\Gamma(a_1)\cdot\Gamma(a_2)\ldots\cdot\Gamma(a_n)\cdot\Gamma(a_1+a_2+\ldots+a_{n+1}+r_1+\ldots+r_n)} =$$

$$= \prod_{i=1}^{n}\frac{\Gamma(a_i+r_i)}{\Gamma(a_i)}\cdot\frac{\Gamma\left(\sum_{i=1}^{n+1} a_i\right)}{\Gamma\left(\sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n} r_i\right)} = M_{r_1 r_2\ldots r_n}(Y^{(n)}). \tag{3.17}$$

If we examine the above moment, we can see that, as a matter of fact, it is just the moment of order $(r_1, r_2, \ldots, r_n)$ for the random vector $Y^{(n)}$ which n-dimensional beta distribution.

Case 2. If $r_1 = 1$, $r_i = 0$, $i = \overline{2, n}$, then

$$I_{n,n-1,\ldots,2,1} = \left[B\left(a_1+1, \sum_{i=2}^{n+1} a_i\right)(b_2^{(1)} - b_1^{(1)}) + B\left(a_1, \sum_{i=2}^{n+1} a_i\right)\cdot b_1^{(1)}\right]\cdot$$

$$\cdot\prod_{k=2}^{n} B\left(a_k, \sum_{i=k+1}^{n+1} a_i\right), \tag{3.18}$$

and the moment (3.1) becomes

$$M_{1,0,\ldots,0}(Z^{(n)}) = \frac{a_1}{a_1+a_2+\ldots+a_{n+1}}(b_2^{(1)} - b_1^{(1)}) + b_1^{(1)} = M(Z_1). \tag{3.19}$$

This is the ordinary moment or order one of the component Z_1 from the random vector $Z^{(n)} = (Z_1, Z_2, \ldots, Z_n)$.

Case 3. If we have in view the conditions from cases 1 and 2 we get

$$M_{1,0,\ldots,0}(Z^{(n)})\Bigg|_{\substack{b_1^{(i)}=0 \\ b_2^{(i)}=1}} {}_{i=\overline{1,n}} = \frac{a_1}{a_1+a_2+\ldots+a_n+a_{n+1}} = M(Y_1), \tag{3.20}$$

that is, just the ordinary moment of the order one of the component Y_1 from the random vector $Y^{(n)} = (Y_1, Y_2, \ldots, Y_n)$ [2].

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DISTRIBUŢIA BETA GENERALĂ *n*-DIMENSIONALĂ

(Rezumat)

În această lucrare se introduce şi se studiază o nouă distribuţie de probabilitate, numită distribuţia beta generală *n*-dimensională. Acest prim studiu cuprinde definiţia acestei distribuţii precum şi calculul momentelor de diferite ordine.

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NODAL PERIOD PERTURBATIONS DUE TO THE FIFTH ZONAL HARMONIC OF THE GEOPOTENTIAL

VASILE MIOC and ÁRPÁD PÁL

1. **Introduction.** The perturbations of the artificial satellites motion can also be studied by estimating the difference caused by a disturbing factor between the orbital period and the keplerian one. Such a method was used by different authors which have considered as a disturbing factor the non-centrality of the terrestrial gravitational field. Analytical formulae for the difference:

$$\Delta T_{\Omega} = T_{\Omega} - T_0 \tag{1}$$

(T_{Ω} = the nodal period, T_0 = the corresponding keplerian period) caused by the zonal harmonics of the geopotential were given, for instance, by Blitzer [1], Zhongolovich [12], Oproiu [10], Mioc [2] in the case of the second zonal harmonic, Mioc [2], [3], [5] in the case of the third zonal harmonic, Oproiu [10], Oproiu and Mioc [11], Mioc [2] in the case of the fourth zonal harmonic.

In this paper we shall consider as a disturbing factor the fifth zonal harmonic of the geopotential. Only the case of the satellites moving on quasi-circular orbits will be considered.

2. **Motion Equations.** We shall use the system Newton-Euler written in the form:

$$\begin{aligned}
dp/du &= 2Zr^3T/\mu,\\
d\Omega/du &= Zr^3BW/\mu pD,\\
di/du &= Zr^3AW/\mu p,\\
dq/du &= Zr^3kB(C/D)W/\mu p + Zr^2[r(A+q)/p + A]T/\mu + Zr^2BS/\mu,\\
dk/du &= -Zr^3qB(C/D)W/\mu p + Zr^2[r(B+k)/p + B]T/\mu - Zr^2AS/\mu,\\
dt/du &= Zr^2/\sqrt{\mu p},
\end{aligned} \tag{2}$$

where:

$$Z = 1/[1 - (r^2Cd\Omega/dt)/\sqrt{\mu p}], \tag{3}$$

$$q = e\cos\omega;\ k = e\sin\omega, \tag{4}$$

the following notations being used:

$$A = \cos u;\quad B = \sin u, \tag{5}$$

$$C = \cos i;\quad D = \sin i, \tag{6}$$

all the other notations appearing in equations (2) — (6) having their usual significances.

Taking into account the orbit equation in polar coordinates:

$$r = p/(1 + e\cos v), \tag{7}$$

the fact that $u = \omega + v$, the notations (4) and (5), and the fact that only the quasi-circular orbits are considered, we can write:

$$r = p(1 - Aq - Bk), \tag{8}$$

and:

$$r^n = p^n(1 - nAq - nBk). \tag{9}$$

3. **Disturbing Acceleration.** The components of the disturbing acceleration undergone by a satellite under the influence of the fifth zonal harmonic of the geopotential have the expressions [13]:

$$\begin{aligned} S &= -(3/4)c_{50}R^5\mu r^{-7}(63\ B^5D^5 - 70\ B^3D^3 + 15\ BD), \\ T &= (15/8)c_{50}R^5\mu r^{-7}(21\ AB^4D^5 - 14\ AB^2D^3 + AD), \\ W &= (15/8)c_{50}R^5\mu r^{-7}(21\ B^4CD^4 - 14\ B^2CD^2 + C), \end{aligned} \tag{10}$$

where c_{50} is a parameter characterizing the fifth zonal harmonic of the geopotential, and R is the mean terrestrial equatorial radius.

4. **Motion Equations in the Considered Case.** Substituting (9) and (10) in (2) and neglecting all the terms containing in factor q^n, k^n and $(qk)^m$ for $n \geqslant 2$, $m \geqslant 1$, the equations characterizing the disturbed motion of the satellite become:

$$\begin{aligned} dp/du &= 2ZKp^{-4}\{21\ D^5AB^4 - 14\ D^3AB^2 + DA + [-84\ D^5B^6 + (84\ D^5 + 56\ D^3) \\ &\quad B^4 + (-56\ D^3 - 4D)B^2 + 4D]q + (84\ D^5AB^5 - 56\ D^3AB^3 + 4\ DAB)k\}, \\ d\Omega/du &= ZKp^{-5}\{21\ CD^3B^5 - 14\ CDB^3 + (C/D)B + [84\ CD^3AB^5 - 56\ CDAB^3 + \\ &\quad + 4(C/D)AB]q + [84\ CD^3B^6 - 56\ CDB^4 + 4(C/D)B^2]k\}, \\ di/du &= ZKp^{-5}\{21\ CD^4AB^4 - 14\ CD^2AB^2 + CA + [-84\ CD^4B^6 + (84\ CD^4 + \\ &\quad + 56\ CD^2)B^4 + (-56\ CD^2 - 4C)B^2 + 4C]q + (84\ CD^4AB^5 - \\ &\quad - 56\ CD^2AB^3 + 4\ CAB)k\}, \\ dq/du &= ZKp^{-5}\{-(336/5)D^5B^6 + (42\ D^5 + 56\ D^3)B^4 + (-28\ D^3 - 8D)B^2 + \\ &\quad + 2D + [-315\ D^5AB^6 + (210\ D^5 + 266\ D^3)AB^4 + (-140\ D^3 - 39\ D)AB^2 + \\ &\quad + 10\ DA]q + [-315\ D^5B^7 + (168\ D^5 + 287\ D^3)B^5 + (-112\ D^3 - \\ &\quad - 53\ D)B^3 + (8D - 1/D)B]k\}, \\ dk/du &= ZKp^{-5}\{(336/5)D^5AB^5 - 56\ D^3AB^3 + 8\ DAB + [-315\ D^5B^7 + \\ &\quad + (336\ D^5 + 245\ D^3)B^5 + (-280\ D^3 - 25\ D)B^3 + (40\ D - 1/D)B]q + \\ &\quad + [315\ D^5AB^6 + (21\ D^5 - 266\ D^3)AB^4 + (-14D^3 + 39\ D)AB^2 + DA]k\}, \\ dt/du &= Zp^{3/2}\mu^{-1/2}(1 - 2Aq - 2Bk), \end{aligned} \tag{11}$$

where we have used the notation:

$$K = (15/8)c_{50}R^5. \tag{12}$$

5. **Variation of the Orbital Elements.** The variation of an orbital element y, $y \in \{p, \Omega, i, q, k\}$, between the initial ($u_0$) and current ($u$) positions can be calculated with the equation:

$$\Delta y = \int_{u_0}^{u} (dy/du)\ du, \tag{13}$$

the integrand being given by Newton-Euler equations. We have performed the integrals (13), with the integrands given by (11), by the successive approximations method, with $Z \cong 1$, limiting us to the first order approximation. We have found:

$$\Delta p = 2Kp_0^{-4}[K_1^0 I_{14} + K_2^0 I_{12} + K_3^0 I_{10} + (-4K_1^0 I_{06} + K_4^0 I_{04} + K_5^0 I_{02} + 4K_3^0 I_{00})q_0 + (4K_1^0 I_{15} + 4K_2^0 I_{13} + 4K_3^0 I_{11})k_0],$$

$$\Delta\Omega = Kp_0^{-5}[K_6^0 I_{05} + K_7^0 I_{03} + K_8^0 I_{01} + (4K_6^0 I_{15} + 4K_7^0 I_{13} + 4K_8^0 I_{11})q_0 + (4K_6^0 I_{06} + 4K_7^0 I_{04} + 4K_8^0 I_{02})k_0],$$

$$\Delta i = Kp_0^{-5}[K_9^0 I_{14} + K_{10}^0 I_{12} + K_{11}^0 I_{10} + (-4K_9^0 I_{06} + K_{12}^0 I_{04} + K_{13}^0 I_{02} + 4K_{11}^0 I_{00})q_0 + (4K_9^0 I_{15} + 4K_{10}^0 I_{13} + 4K_{11}^0 I_{11})k_0],$$

$$\Delta q = Kp_0^{-5}[K_{14}^0 I_{06} + K_{15}^0 I_{04} + K_{16}^0 I_{02} + 2K_3^0 I_{00} + (-15K_1^0 I_{16} + K_{17}^0 I_{14} + K_{18}^0 I_{12} + 10K_3^0 I_{10})q_0 + (-15K_1^0 I_{07} + K_{19}^0 I_{05} + K_{20}^0 I_{03} + K_{21}^0 I_{01})k_0],$$

$$\Delta k = Kp_0^{-5}[-K_{14}^0 I_{15} + 4K_2^0 I_{13} + 8K_3^0 I_{11} + (-15K_1^0 I_{07} + K_{22}^0 I_{05} + K_{23}^0 I_{03} + K_{24}^0 I_{01})q_0 + (15K_1^0 I_{16} + K_{25}^0 I_{14} + K_{26}^0 I_{12} + K_3^0 I_{10})k_0], \qquad (14)$$

where we have used the notations:

$$K_1 = 21D^5,\ K_2 = -14D^3,\ K_3 = D,\ K_4 = 84D^5 + 56D^3,\ K_5 = -56D^3 - 4D,$$
$$K_6 = 21CD^3,\quad K_7 = -14CD,\quad K_8 = C/D,\quad K_9 = 21CD^4,\quad K_{10} = -14CD^2,$$
$$K_{11} = C,\ K_{12} = 84CD^4 + 56CD^2,\ K_{13} = -56CD^2 - 4C,\ K_{14} = -(336/5)D^5,$$
$$K_{15} = 42D^5 + 56D^3,\quad K_{16} = -28D^3 - 8D,\quad K_{17} = 210D^5 + 266D^3,$$
$$K_{18} = -140D^3 - 39D,\quad K_{19} = 168D^5 + 287D^3,\quad K_{20} = -112D^3 - 53D,$$
$$K_{21} = 8D - 1/D,\ K_{22} = 336D^5 + 245D^3,\ K_{23} = -280D^3 - 25D,\ K_{24} = 40D - 1/D,$$
$$K_{25} = 21D^5 - 266D^3,\ K_{26} = -14D^3 + 39D,\ K_{27} = (189/4)D^3, \qquad (15)$$

the superior index "o" added to K_j ($j = \overline{1,27}$) in equations (14) signifying the respective value for $u = u_0$. It is the same for the inferior index "o" added to p, q and k.

The quantities I_{mn} appearing in equations (14) are the values of the integrals:

$$I_{mn} = \int_{u_0}^{u} A^m B^n\, du, \qquad (16)$$

which were calculated by using the recurrence formula:

$$I_{mn} = -A^{m+1}B^{n-1}/(m+n) + (n-1)\ I_{m,n-2}/(m+n). \qquad (17)$$

These values are:

$$I_{00} = u - u_0,$$
$$I_{01} = -(A - A_0),$$
$$I_{02} = -(AB - A_0B_0)/2 + (u - u_0)/2,$$
$$I_{03} = (A^3 - A_0^3)/3 - (A - A_0),$$
$$I_{04} = -(AB^3 - A_0B_0^3)/4 - 3(AB - A_0B_0)/8 + 3(u - u_0)/8,$$
$$I_{05} = -(A^5 - A_0^5)/5 + 2(A^3 - A_0^3)/3 - (A - A_0),$$

$$I_{06} = -(AB^5 - A_0B_0^5)/6 - 5(AB^3 - A_0B_0^3)/24 - 5(AB - A_0B_0)/16 + + 5(u - u_0)/16,$$
$$I_{07} = (A^7 - A_0^7)/7 - 3(A^5 - A_0^5)/5 + (A^3 - A_0^3) - (A - A_0),$$
$$I_{10} = B - B_0,$$
$$I_{11} = (B^2 - B_0^2)/2,$$
$$I_{12} = (B^3 - B_0^3)/3,$$
$$I_{13} = (B^4 - B_0^4)/4,$$
$$I_{14} = (B^5 - B_0^5)/5,$$
$$I_{15} = (B^6 - B_0^6)/6,$$
$$I_{16} = (B^7 - B_0^7)/7, \quad (18)$$

the index "o" added to A, B also signifying the position u_0.

6. **Difference Between the Two Periods.** The difference caused by a disturbing factor between the nodal and keplerian periods of an artificial satellite can be written in the form (see, for instance, [2], [3], [4], [6], [7], [8], [9], [11]):

$$\Delta T_\Omega = \sum_{j=1}^{4} I_j, \quad (19)$$

with:

$$I_j = \alpha p_0^\beta \mu^{-1/2} \int_0^{2\pi} (1 + Aq_0 + Bk_0)^\gamma \Delta y \; du \quad (j = \overline{1,3}), \quad (20)$$

$$I_4 = \int_0^{2\pi} \{\partial[(r^4 C \; d\Omega/dt)/\mu p]/\partial\sigma\}\sigma \; du. \quad (21)$$

Here σ is a small parameter characterizing the disturbing factor (we can take in our case $\sigma = c_{50}$), while α, β, γ have the following values for $y \in \{p, q, k\}$:

$$\begin{aligned} j = 1 &\Rightarrow y = p, & \alpha &= 3/2, & \beta &= 1/2, \; \gamma = -2; \\ j = 2 &\Rightarrow y = q, & \alpha &= -2A, & \beta &= 3/2, \; \gamma = -3; \\ j = 3 &\Rightarrow y = k, & \alpha &= -2B, & \beta &= 3/2, \; \gamma = -3. \end{aligned} \quad (22)$$

With these values and with the assumption that only the quasi-circular orbits are considered, equations (20) become:

$$I_1 = (3/2)p_0^{1/2}\mu^{-1/2} \int_0^{2\pi} (1 - 2Aq_0 - 2Bk_0)\Delta p \; du,$$

$$I_2 = -2p_0^{3/2}\mu^{-1/2} \int_0^{2\pi} (1 - 3Aq_0 - 3Bk_0)A \; \Delta q \; du, \quad (23)$$

$$I_3 = -2p_0^{3/2}\mu^{-1/2} \int_0^{2\pi} (1 - 3Aq_0 - 3Bk_0)B \; \Delta k \; du.$$

7. **Results.** By introducing the expressions (18) in equations (14) and by replacing the quantities Δy ($y \in \{p, q, k\}$) obtained in this manner in equations (23), we have calculated the integrals (23). The integral (21) can also be calculated, by taking into account equations (9) and (11). The results are:

$$
\begin{aligned}
I_1 &= \pi K p_0^{-7/2} \mu^{-1/2} \{L_1^0 B_0^5 + L_2^0 B_0^3 + L_3^0 B_0^3 + [L_4^0 A_0 B_0^5 + L_5^0 A_0 B_0^3 + L_6^0 A_0 B_0 + \\
&\quad + L_7^0(\pi - u_0)] q_0 + (L_8^0 B_0^6 + L_9^0 B_0^4 + L_{10}^0 B_0^2 + L_{11}^0) k_0\},\\
I_2 &= \pi K p_0^{-7/2} \mu^{-1/2} \{[L_{12}^0 A_0 B_0^5 + L_{13}^0 A_0 B_0^3 + L_{14}^0 A_0 B_0 + L_{15}^0(\pi - u_0)] q_0 + L_{16}^0 k_0\},\\
I_3 &= \pi K p_0^{-7/2} \mu^{-1/2} (L_{17}^0 B_0^6 + L_{18}^0 B_0^4 + L_{19}^0 B_0^2 + L_{20}) k_0,\\
I_4 &= \pi K p_0^{-7/2} \mu^{-1/2} L_{21}^0 k_0,
\end{aligned} \tag{24}
$$

where we have used the notations:

$$
\begin{aligned}
&L_1 = -6K_1/5, \quad L_2 = -2K_2, \quad L_3 = -6K_3, \quad L_4 = L_8 = -4K_1,\\
&L_5 = -5K_1 + 3K_4/2, \quad L_6 = -15K_1/2 + 9K_4/4 + 3K_5,\\
&L_7 = -15K_1/2 + 24K_3 + 9K_4/4 + 3K_5, \quad L_9 = L_{18} = -6K_2,\\
&L_{10} = -12K_3, \quad L_{11} = 5K_1/4 + 9K_2/4 + 6K_3, \quad L_{12} = L_{17} = K_{14},\\
&L_{13} = 5K_{14}/4 + 3K_{15}/2, \; L_{14} = 15K_{14}/8 + 9K_{15}/4 + 3K_{16},\\
&L_{15} = 12K_3 + 15K_{14}/8 + 9K_{15}/4 + 3K_{16}, \quad L_{16} = -525K_1/32 - 6K_3 -\\
&105K_{14}/64 - 15K_{15}/8 - 9K_{16}/4 + 5K_{10}/4 + 3K_{20}/2 + 2K_{21}, \; L_{19} = -24K_3,\\
&L_{20} = -75K_1/32 + 15K_2/4 + 16K_3 - 35K_{14}/64 - K_{25}/4 - K_{26}/2,\\
&L_{21} = -5K_1/4 - 7K_3 - 2K_{21} + K_{27}.
\end{aligned} \tag{25}
$$

The superior index „o" added to L_j ($j = \overline{1,21}$) in equations (24) has the same significance as previously.

Now, performing the sum (19), with I_j ($j = \overline{1,4}$) given by (24), replacing L_j^0 ($j = \overline{1,21}$) by their expressions given by (25), and then replacing K_j^0 ($j = \overline{1,27}$) by their expressions given by (15), we obtain:

$$
\begin{aligned}
\Delta T_\Omega^{(c_{50})} &= \pi K p_0^{-7/2} \mu^{-1/2} [-126\ D_0^5 B_0^5/5 + 28\ D_0^3 B_0^3 - 6\ D_0 B_0 +\\
&+ (-756\ D_0^5 A_0 B_0^5/5 + 168\ D_0^3 A_0 B_0^3 - 36\ D_0 A_0 B_0) q_0 +\\
&+ (-756\ D_0^5 B_0^6/5 + 168\ D_0^3 B_0^4 - 36\ D_0 B_0^2 - 483\ D_0^5/4 +\\
&+ 371\ D_0^3/2 - 72\ D_0) k_0],
\end{aligned} \tag{26}
$$

where $\Delta T_\Omega^{(c_{50})}$ denotes the difference ΔT_Ω caused only by the fifth zonal harmonic of the geopotential.

Equation (26) can also be written in a more concentrated form:

$$\Delta T_\Omega^{(c_{50})} = X^0 [Y_1^0 + 6\ Y_1^0 A_0 q_0 + (6\ Y_1^0 B_0 + Y_2^0) k_0], \tag{27}$$

where:

$$X^0 = (15\pi/8) c_{50} (R/p_0)^5 p_0^{3/2} \mu^{-1/2}, \tag{28}$$

and:

$$Y_1^0 = -126\ D_0^5 B_0^5/5 + 28\ D_0^3 B_0^3 - 6\ D_0 B_0,$$
$$Y_2^0 = -483\ D_0^5/4 + 371\ D_0^3/2 - 72\ D_0. \tag{29}$$

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PERTURBAŢII ÎN PERIOADA NODALĂ DATORATE CELEI DE-A CINCEA ARMONICE ZONALE A GEOPOTENŢIALULUI

(Rezumat)

Variaţiile cauzate de cea de-a cincea armonică zonală a geopotenţialului în perioada nodală a unui satelit artificial cu excentricitate orbitală mică sînt estimate prin stabilirea unei expresii a diferenţei dintre perioada nodală şi perioada kepleriană corespunzătoare.

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ASUPRA COEFICIENŢILOR POLINOAMELOR DE APROXIMARE A FUNCŢIILOR CONTINUE DE DOUĂ VARIABILE

AURELIA ŢOCA

În această notă se extinde, în cazul funcţiilor de două variabile, problema M-aproximării, introdusă în [1]. Se folosesc, în acest scop, polinoamele de tip Bernstein pe patrat. Se obţine, totodată, o valoare mai bună pentru constanta a din teorema 1 a lucrării citate.

Fie $f \in C_0\,[0,1] \times [0,1]$, adică $f \in C[0,1] \times [0,1]$ şi $f(0,0) = 0$, iar $\{M_{ij}\}_{i,j=0}^{\infty}$ un şir dublu, pozitiv, nedescrescător. Atunci f admite o M_{ij}-aproximare, dacă pentru orice $\varepsilon > 0$, există un polinom $P_{m(\varepsilon)\,n(\varepsilon)}(f;\ x,\ y) = \sum_{i=0}^{m(\varepsilon)} \sum_{j=0}^{n(\varepsilon)} a_{ij}\, x^i y^j$, $a_{00} = 0$, astfel încît condiţiile:

$$\max_{(x,y)\in[0,1]\times[0,1]} |f(x,\ y) - P_{m(\varepsilon)\,n(\varepsilon)}(f;\ x,\ y)| < \varepsilon,$$

$$|a_{ij}| \leqslant M_{ij},\ \ (i,\ j) \in \{0,\ 1,\ \ldots,\ m(\varepsilon)\} \times \{0,\ 1,\ \ldots,\ n(\varepsilon)\}$$

să fie indeplinite.

TEOREMA 1. *Fie* $f \in C_0\,[0,1] \times [0,1]$, *astfel ca* $f(x,y) = 0$ pentru $(x, y) \in [0, \alpha] \times [0, \beta]$, $0 < \alpha,\ \beta < 1$. *Atunci există un* $a = a(\alpha) > 0$ *şi un* $b = b(\beta) > 0$ *astfel ca, dacă un şir dublu, pozitiv, nedescrescător oarecare* $\{M_{ij}\} \nearrow \infty$ *satisface una din condiţiile:*

a) $$\lim_{i\to\infty}\lim_{j\to\infty} \frac{a^i b^j}{M_{ij}} = 0,$$

b) $$\lim_{j\to\infty}\lim_{i\to\infty} \frac{a^i b^j}{M_{ij}} = 0,$$

atunci f *admite o* M_{ij}*-aproximare.*

Demonstraţie. Fie $N_{ij} = \frac{a^i b^j}{M_{ij}}$, $i,\ j = 0,\ 1,\ \ldots$ Dacă şirul $\{M_{ij}\}$ satisface una din condiţiile teoremei, atunci există un şir simplu regulat $\{N_{i_\nu j_\nu}\}_{\nu=1}^{\infty}$, astfel ca

$$\lim_{\nu\to\infty} N_{i_\nu j_\nu} = 0. \tag{1}$$

Să considerăm şirurile simple $\{m_\nu\}_{\nu=1}^{\infty}$, $\{n_\nu\}_{\nu=1}^{\infty}$, determinate prin condiţiile:

$$\begin{cases} i_\nu = [\alpha,\ m_\nu] + 1 \\ j_\nu = [\beta, n_\nu], + 1 \end{cases},\ \ \nu = 1, 2, \ldots$$

Fie $B_{m_\nu n_\nu}$ polinomul Bernstein pe patrat, de grad m_ν în x şi n_ν în y, corespunzător funcţiei f:

$$B_{m_\nu n_\nu}(f;\, x, y) = \sum_{k=0}^{m_\nu}\sum_{l=0}^{n_\nu}\binom{m_\nu}{k}\binom{n_\nu}{l} x^k y^l (1-x)^{m_\nu-k}(1-y)^{n_\nu-l} f\left(\frac{k}{m_\nu}, \frac{l}{n_\nu}\right).$$

Ordonînd după puterile lui x şi y, polinomul $B_{m_\nu n_\nu}$ se rescrie sub forma:

$$B_{m_\nu n_\nu}(f;\, x, y) = \sum_{i=0}^{m_\nu}\sum_{j=0}^{n_\nu} a_{ij}\, x^i y^j,$$

unde

$$a_{ij} = \binom{m_\nu}{i}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{n_\nu}{j}\sum_{l=0}^{j}(-1)^l\binom{j}{l} f\left(\frac{i-k}{m_\nu}, \frac{j-l}{n_\nu}\right).$$

Fie $A = \max\limits_{(x,y)\in[0,1]\times[0,1]} |f(x, y)|$. Atunci

$$|a_{ij}| \leqslant A\binom{m_\nu}{i}\sum_{k=0}^{i}\binom{i}{k}\binom{n_\nu}{j}\sum_{l=0}^{j}\binom{j}{l} = A\binom{m_\nu}{i}2^i\binom{n_\nu}{j}2^j < A\, 3^{m_\nu} 3^{n_\nu} = A\, 3^{m_\nu+n_\nu},$$

deci

$$|a_{ij}| < A\, 3^{m_\nu+n_\nu}, \text{ pentru } i = i_\nu,\ i_\nu+1, \ldots, m_\nu;\ j = j_\nu,\ j_\nu+1, \ldots, n_\nu,$$

adică

$$|a_{ij}| < A\, 3^{\frac{i_\nu}{\alpha}+\frac{j_\nu}{\beta}}.$$

Luînd $a = 3^{\frac{1}{\alpha}}$, $b = 3^{\frac{1}{\beta}}$, se obţin inegalităţile:

$$|a_{ij}| < A\, a^{i_\nu} b^{j_\nu}, \tag{2}$$

pentru $i = i_\nu, \ldots, m_\nu$ şi $j = j_\nu, \ldots, n_\nu$.
Din (1) rezultă că există un ν_0 astfel ca pentru orice $\nu > \nu_0$ să aibă loc inegalitatea

$$\frac{a^{i_\nu} b^{j_\nu}}{M_{i_\nu j_\nu}} < \frac{1}{A},$$

de unde rezultă că

$$A\, a^{i_\nu} b^{j_\nu} < M_{i_\nu j_\nu}, \tag{3}$$

pentru orice $\nu > \nu_0$. Din relaţiile (2) şi (3) rezultă că există un $\nu > \nu_0$ astfel ca $|a_{ij}| < M_{i_\nu j_\nu}$ pentru $i = i_\nu, \ldots, m_\nu$; $j = j_\nu, \ldots, n_\nu$. Deoarece şirul $\{M_{ij}\}$ este nedescrescător, rezultă că

$$|a_{ij}| < M_{i_\nu+k,\, j_\nu+l}, \text{ oricare ar fi } k \geqslant 0,\ l \geqslant 0,$$

prin urmare $|a_{ij}| < M_{ij}$, $i = i_\nu, \ldots, m_\nu$; $j = j_\nu, \ldots, n_\nu$, oricare ar fi $\nu > \nu_0$. Acest lucru înseamnă că pentru orice $\nu > \nu_0$, există șirurile $m_\nu, m_{\nu+1}, \ldots$ și $n_\nu, n_{\nu+1}, \ldots$ pentru care $\{B_{m_\nu n_\nu}(f; x, y)\}$converge uniform către $f(x, y)$, pentru $\nu \to \infty$ și că $|a_{ij}| < M_{ij}$, $i = i_\nu, \ldots, m_\nu$; $j = j_\nu, \ldots, n_\nu$.

Observația 1. În cazul în care f este o funcție de o singură variabilă, teorema este demonstrată în lucrarea [1], pentru $a = 6^{1/\alpha}$.

Observația 2. Teorema de mai sus are loc și dacă se folosesc polinoamele $P_{mn}^{\langle\lambda,\theta\rangle}$, considerate în [3]:

$$P_{mn}^{\langle\lambda,\theta\rangle}(f; x, y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}^{\langle\lambda\rangle}(x)\, p_{n,j}^{\langle\theta\rangle}(y)\, f\left(\frac{k}{m}, \frac{j}{n}\right),$$

unde

$$p_{l,i}^{\langle\tau\rangle}(z) = \binom{l}{i}\frac{z(z+\tau)\ldots(z+(i-1)\tau)(1-z)(1-z+\tau)\ldots(1-z+(l-i-1)\tau)}{(1+\tau)(1+2\tau)\ldots(1+(l-1)\tau)}$$

iar λ, θ și τ sînt parametri nenegativi. Pentru a și b se iau în acest caz valorile $[(m-2)\lambda + 3]^{\frac{1}{\alpha}}$, respectiv $[(n-2)\theta + 3]^{\frac{1}{\beta}}$.

TEOREMA 2. *Dacă funcția* $f \in C_0[0,1] \times [0,1]$ *admite o* M_{ij}*-aproximare, unde* $M_{ij} < A\, a^i b^j$ $(A > 0,\ a \geqslant 1,\ b \geqslant 1;\ i, j = 0, 1, \ldots)$, *atunci* f *se prelungește analitic în polidiscul* $P(0, \rho)$, *unde* $\rho = \left(\frac{1}{a}, \frac{1}{b}\right)$.

Demonstrație. Fie, pentru un $\varepsilon > 0$ oarecare, $P_{m(\varepsilon)\,n(\varepsilon)}$ polinomul de M_{ij}-aproximare al lui f:

$$P_{m(\varepsilon)\,n(\varepsilon)}(f; x, y) = \sum_{i=0}^{m(\varepsilon)}\sum_{j=0}^{n(\varepsilon)} b_{ij}\, x^i y^j,$$

Au loc inegalitățile:

$$|P_{m(\varepsilon)\,n(\varepsilon)}(z)| = \left|\sum_{i=0}^{m(\varepsilon)}\sum_{j=0}^{n(\varepsilon)} b_{ij} z_1^i z_2^j\right| \leqslant A \sum_{i=0}^{m(\varepsilon)}\sum_{j=0}^{n(\varepsilon)} a^i b^j \left(\frac{1}{a} - \delta_1\right)^i \left(\frac{1}{b} - \delta_2\right)^j =$$

$$= A \sum_{i=0}^{m(\varepsilon)} a^i \left(\frac{1}{a} - \delta_1\right)^i \sum_{j=0}^{n(\varepsilon)} b^j \left(\frac{1}{b} - \delta_2\right)^j \leqslant$$

$$\leqslant A \frac{a\left(\frac{1}{a} - \delta_1\right)}{1 - a\left(\frac{1}{a} - \delta_1\right)} \frac{b\left(\frac{1}{b} - \delta_2\right)}{1 - b\left(\frac{1}{b} - \delta_2\right)} = A\frac{(1 - a\delta_1)(1 - b\delta_2)}{ab\delta_1\delta_2},$$

ceea ce înseamnă că $P_{m(\varepsilon)\,n(\varepsilon)}$ este uniform mărginită în polidiscul $P\left(0, \left(\frac{1}{a} - \delta_1\ \frac{1}{b} - \delta_2\right)\right)$, oricare ar fi $\delta = (\delta_1, \delta_2) \in \mathbf{R}^2$ dat, $\delta_1 > 0$, $\delta_2 > 0$ $(z = (z_1, z_2) \in \mathbf{C}^2)$. Aplicînd acum teorema Montel, extinsă în cazul funcțiilor de mai multe variabile (vezi, de exemplu, [4]) și ținînd cont de faptul că δ este arbitrar, rezultă imediat că afirmația din enunțul teoremei este adevărată.

TEOREMA 3. *Fie* $\{M_{ij}\}_{i,j=0}^{\infty}$ *un şir dublu pozitiv, nedescrescător. Condiţia necesară şi suficientă pentru ca o funcţie oarecare* $f \in C_0[0,1] \times [0,1]$ *să admită o* M_{ij}*-aproximare, este să se îndeplinescă una din condiţiile teoremei 1, pentru* $a > 1$, $b > 1$ *arbitrari.*

Demonstraţie. Ţinînd cont de teorema 2, necesitatea rezultă urmînd acelaşi raţionament ca în lucrarea [1]. La fel se procedează în cazul suficienţei, în demonstrarea căreia se foloseşte funcţia auxiliară $\bar{f}$, dată prin:

$$\bar{f}(x,y) = \begin{cases} \bar{f}(x,y)\ e^{-\left(\frac{1}{\gamma(x-\alpha)^2}+\frac{1}{\gamma(y-\beta)^2}\right)}, & \text{pentru } (x,y) \in (\alpha,\ 1] \times (\beta,\ 1] \\ 0 & , \text{pentru } (x,y) \in [0,\ \alpha) \times [0,\ \beta), \end{cases}$$

unde α, β, γ sînt numere reale strict pozitive. Se vede imediat că $\bar{f}$ satisface condiţiile teoremei 1.

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ON THE COEFFICIENTS OF APPROXIMATION POLYNOMIALS OF CONTINUOUS FUNCTIONS OF TWO VARIABLES

(Summary)

In this note the problem of M-approximation, as introduced in [1], is extended in the case of the functions of two variables. A better value for the constant a of theorem 1 [1] is also given.

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TWO COINCIDENCE THEOREMS FOR CONTRACTIVE TYPE MULTIVALUED MAPPINGS

TOMASZ KUBIAK*

Let (X, d) be a metric space. We denote by $CB(X)$ the set of all nonempty closed and bounded subsets of X. The well-known Hausdorff metric on $CB(X)$ is defined by letting

$$H(A, B) = \max \{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}$$

for each $A, B \in CB(X)$ where $D(a, B) = \inf_{b\in B} d(a, b)$.

In [2] Olga Hadžic proved the following coincidence theorem.

THEOREM 1. *Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A a closed mapping from X into $CB(SX \cap TX)$ such that $ATx = TAx$, $ASx = SAx$, for every $x \in X$ and:*

$$H(Ax, Ay) \leqslant qd(Sx, Ty),$$

for every $x, y \in X$ where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}_{n\in N}$ such that:

1. *For every $n \in N$, $Sx_{2n+1} \in Ax_{2n}$, $Tx_{2n} \in Ax_{2n-1}$.*
2. *There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$.*
3. *$Tz \in Az$, $Sz \in Az$.*

Now, we shall prove a similar coincidence theorem for a pair of multivalued mappings satisfying much weaker contractive definition.

THEOREM 2. *Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX \cap \cap TX)$ such that $ASx = SAx$, $BTx = TBx$, for every $x \in X$ and:*

$$H(Ax, By) \leqslant q \max\left\{d(Sx, Ty), D(Sx, Ax), D(Ty, By), \frac{1}{2}[D(Sx, By) + D(Ty, Ax)]\right\},$$

for every x, $y \in$ where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}_{n\in N}$ in X such that:

1. *For every $n \in N$, $Sx_{2n+1} \in Bx_{2n}$, $Tx_{2n} \in Ax_{2n-1}$.*
2. *There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$.*
3. *$Sz \in Az$, $Tz \in Bz$.*

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Proof. Let $x_0 \in X$. Since $Bx_0 \subset SX$ there exists $x_1 \in X$ such that $Sx_1 \in Bx_0$. Since $\frac{1}{\sqrt{q}} > 1$, then it is a simple consequence of the definition of H that there exists an element $u_1 \in Ax_1$ so that

$$d(u_1, Sx_1) \leqslant \frac{1}{\sqrt{q}} H(Ax_1, Bx_0).$$

Now, since $Ax_1 \subset TX$ there exists $x_2 \in X$ such that $u_1 = Tx_2$. Similarly, there exists $u_2 \in Bx_2$ such that

$$d(u_2, Tx_2) \leqslant \frac{1}{\sqrt{q}} H(Bx_2, Ax_1)$$

and $u_2 = Sx_3$ for some $x_3 \in X$ because $Bx_2 \subset SX$. Continuing in this fashion we define a sequence $\{x_n\}_{n \in N}$ in X where
$Tx_{2n} \in Ax_{2n-1}$ and $Sx_{2n+1} \in Bx_{2n}$ are such that

$$d(Tx_{2n}, Sx_{2n-1}) \leqslant \frac{1}{\sqrt{q}} H(Ax_{2n-1}, Bx_{2n-2}) \tag{1}$$

and

$$d(Sx_{2n+1}, Tx_{2n}) \leqslant \frac{1}{\sqrt{q}} H(Bx_{2n}, Ax_{2n-1}) \tag{2}$$

for every $n \in N$.

Now, we shall show that $\{Sx_{2n-1}\}_{n \in N}$ is Cauchy. For this purpose, observe first that (1) yields

$$d(Tx_{2n}, Sx_{2n-1}) \leqslant \frac{1}{\sqrt{q}} H(Ax_{2n-1}, Bx_{2n-2}) \leqslant \sqrt{q} \max\{d(Sx_{2n-1}, Tx_{2n-2}),$$

$$D(Sx_{2n-1}, Ax_{2n-1}), D(Tx_{2n-2}, Bx_{2n-2}), \frac{1}{2}[D(Sx_{2n-1}, Bx_{2n-2}) + D(Tx_{2n-2},$$

$$Ax_{2n-1})]\} \leqslant \sqrt{q} \max\{d(Sx_{2n-1}, Tx_{2n-2}), d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1}),$$

$$\frac{1}{2}[d(Sx_{2n-1}, Sx_{2n-1}) + d(Tx_{2n-2}, Tx_{2n})]\} = \sqrt{q} \max\{d(Sx_{2n-1}, Tx_{2n-2}),$$

$$d(Sx_{2n-1}, Tx_{2n}), \frac{1}{2} d(Tx_{2n-2}, Tx_{2n})\}.$$

Now, we shall examine the right-hand side of the above inequality. If the maximum is $\frac{1}{2} d(Tx_{2n-2}, Tx_{2n})$, then $d(Tx_{2n}, Sx_{2n-1}) \leqslant \frac{\sqrt{q}}{2}[d(Tx_{2n-2}, Sx_{2n-1}) + d(Sx_{2n-1}, Tx_{2n})]$ which implies $d(Tx_{2n}, Sx_{2n-1}) \leqslant \frac{\sqrt{q}}{2-\sqrt{q}} d(Tx_{2n-2}, Sx_{2n-1}) \leqslant \sqrt{q}\, d(Tx_{2n-2}, Sx_{2n-1})$.

Now, when $d(Sx_{2n-1}, Tx_{2n})$ is the maximum, we get $d(Tx_{2n}, Sx_{2n-1}) \leq \sqrt{q}\, d(Sx_{2n-1}, Tx_{2n})$, so that $d(Sx_{2n-1}, Tx_{2n}) = 0$ which implies $d(Sx_{2n-1}, Tx_{2n-2}) = 0$. Therefore, we can write

$$d(Tx_{2n}, Sx_{2n-1}) \leq \sqrt{q}\, d(Tx_{2n-2}, Sx_{2n-1}). \tag{3}$$

In a similar manner, from (2) it follows that

$$d(Sx_{2n+1}, Tx_{2n}) \leq \sqrt{q}\, d(Sx_{2n-1}, Tx_{2n}). \tag{4}$$

By induction we obtain from (3) and (4)

$$d(Tx_{2n}, Sx_{2n-1}) \leq q^n\, d(Sx_1, Tx_2) \tag{5}$$

and

$$d(Sx_{2n+1}, Tx_{2n}) \leq q^n\, d(Tx_0, Sx_1), \tag{6}$$

for every $n \in N$. By (5) and (6) we have $d(Sx_{2n-1}, Sx_{2n+1}) \leq d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1}) \leq q^n[d(Sx_1, Tx_2) + d(Tx_0, Sx_1)]$.
Finally, by a routine calculation we find that $\{Sx_{2n-1}\}_{n \in N}$ is Cauchy, hence convergent. Let $z = \lim_{n\to\infty} Sx_{2n-1}$. From (5) it follows that $\lim_{n\to\infty} d(Tx_{2n}, Sx_{2n-1}) = 0$, hence $z = \lim Tx_{2n}$.

It remains to show that $Sz \in Az$ and $Tz \in Bz$. Indeed, the continuity of S implies that $Sz = \lim_{n\to\infty} ST_{2n}$. Further, since $Tx_{2n} \in Ax_{2n-1}$, it follows that $STx_{2n} \in SAx_{2n-1} = ASx_{2n-1}$, which, allong with the fact that A is closed, implies that $Sz \in Az$. It can be similarly shown that $Tz \in Bz$. The proof is complete.

Remark. With $S = T = 1_X$, the identity mapping of X, the assumption of the closedness of A and B is superfluous. Indeed, we have: $x_{2n} \in Ax_{2n-1}$, $x_{2n+1} \in Bx_{2n}$ and $\lim_{n\to\infty} x_n = z$. Further, $d(x_{2n+1}, Az) \leq H(Az, Bx_{2n}) \leq q \max\{d(z, x_{2n}), D(z, Az), d(x_{2n}, x_{2n+1}), \frac{1}{2}[d(z, x_{2n+1}) + D(x_{2n}, Az)]\}$. Now, taking the limit as $n \to \infty$ it follows that $D(z, Az) \leq q\, D(z, Az)$ which implies $D(z, Az) = 0$, i.e., $z \in Az$. A similar argument shows that $z \in Bz$.

The following is the multivalued version of a coincidence theorem due to K. Goebel in [1]. It is an improvement over Theorem 1 under $S = T$. Notice that another extension of Goebel's result has been made recently by S. Park [4].

THEOREM 3. *Let M be a set, (X, d) a metric space, T a mapping from M into X such that TM is a complete subspace of X. Let A be a mapping from M into $CB(TM)$ such that*

$H(Ax, Ay) \leq q\ d(Tx, Ty)$ for every $x, y \in M$ where $0 \leq q < 1$. Then there exists $z \in M$ such that $Tz \in Az$.

Proof. Observe first that for each $a \in TM$ the set $AT^{-1}a$, where $T^{-1}a$ denotes the inverse image of a under T, is an one-element subset of $CB(TM)$. Indeed, let $U_1, U_2 \in AT^{-1}a$. Then there exist $x_1, x_2 \in T^{-1}a$ such that $U_1 =$

$= Ax_1$ and $U_2 = Ax_2$. So we have, $H(U_1, U_2) = H(Ax_1, Ax_2) \leqslant q\ d(Tx_1, Tx_2) = 0$ which implies $U_1 = U_2$. Therefore, $F : TM \to CB(TM)$ defined via $Fa = AT^{-1}a$, for all $a \in TM$, is a well-defined mapping. Now, for any $a, b \in TM$ and $x \in T^{-1}a$, $y \in T^{-1}b$ we obtain

$$H(Fa, Fb) = H(Ax, Ay) \leqslant q\ d(Tx, Ty) = q\ d(a, b).$$

Therefore, F satisfies the hypothesis of the well-known fixed point theorem of S. B. Nadler [3], and, thus, there exists $c \in TM$ such that $c \in Fc$. Finally, for every $z \in T^{-1} c$, we have $Az \in AT^{-1} c$, so that $Az = AT^{-1} c = Fc \in c = Tz$ which completes the proof.

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DOUĂ TEOREME DE COINCIDENŢĂ PENTRU APLICAŢII MULTIVOCE CONTRACTIVE

(Rezumat)

În lucrare se stabilesc teoreme de coincidenţă pentru aplicaţii multivoce. Aceste teoreme generalizează unele rezultate date de O. Hadžić şi K. Goebel.

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NICULAE ABRAMESCU (1884–1947)
Une époque de la didactique mathématique en Roumanie

Une centaine d'années est passée depuis la naissance de Niculae Abramescu, ancien professeur de mathématiques à l'Université de Cluj. L'activité qu'il a déployée au service de l'enseignement des mathématiques, à tous les niveaux, vient de marquer une époque dans la didactique mathématique de notre pays.

N. Abramescu est originaire de la ville de Tîrgovişte et a poursuivi ses études à l'Université de Bucarest. Au début de sa carrière, il a fonctionné quelques années comme professeur de lycée à Ploeşti, Botoşani. Vaslui et Galatzi. À la fin de la première guerre mondiale N. Abramescu fut nommé maître de conférences à l'Université de Cluj, pour la discipline de géometrie descriptive et infinitésimale. C'est ainsi qu'il a fait partie de la pleïade de professeurs, qui, avec ardeur et abnégation, ont mis les bases de l'enseignement universitaire en langue roumaine, en 1919 dans la capitale de la Transylvanie. Dès lors il a servi l'Université de Cluj sans interruption, même pendant la seconde guerre mondiale, dans les moments difficiles du refuge à Timişoara. Il est mort à Cluj en 1947 où il est enterré.

L'activité scientifique de N. Abramescu a subi, au début, l'influence des deux grands maitres qui giraient à ces commencements la section de mathématiques de la Faculté de Sciences de Cluj : D. Pompeiu et G. Tzitzeica. Ainsi, dans le domaine de l'analyse mathématique il a étudié les séries de polynômes dans le plan complexe, ainsi que l'univalence des fonctions analytiques et la répartition des zéros des fonctions complexes. Sa thèse, soustenue à la Faculté de Bucarest, en 1921, présente une systematisation de la théorie des polynômes orthogonaux. Dans le domaine de la géométrie, il a abordé divers problèmes de géométrie analytique et différentielle, liés à des courbes rémarquables et surtout des questions regardant la géométrie cinématique. Il a considéré aussi des questions de géométrie différentielle projective et affine des courbes et des surfaces.

A l'Université de Cluj. N, Abramescu a donné des cours de géométrie analytique, descriptive et infinitésimale, en rédigeant de sa propre main, avec une calligraphie rémarquable, tous ses manuscrits destinés à la lithographie. Son cours de géometrie issu à Cluj en quelques éditions a été préfacé par G. Tzitzeica. Il poursuit la lignée des traités de géométrie qui ont fait époque : G. Darboux, A. Tresse — A. Thibaut etc., ou bien de celui du matematicien T. Lalescu, écrit en roumain. Les questions spéciales présentées dans les annexes ou bien dans des tirés à part concernent la géométrie vectorielle, considérée aussi dans les travaux de R. Bricard, A. Chatelet, G. Bouligand, ainsi que des questions de géométrie non euclidienne. Parmis les compléments de géométrie de N. Abramescu, il faut citer son oeuvre *Leçons de géometric pure infinitésimale* avec applications dans la géometrie descriptive, parue à Cluj, en roumain, en 1930, et préfacée en français par M. d'Ocagne. Dans cet ouvrage, N. Abramescu présente les vertues du raisonnement syntétique qui rémonte à Lagrange, en mettant en valeur les avantages des méthodes directes dans l'étude infinitésimal des courbes, des surfaces et de la géometrie réglée. Le caractère monographique de l'exposé le rendait très utile en ce temps là pour le renseignement des mathématiciens dans un domaine moderne de la recherche.

En plein accord avec sa vocation pédagogique, N. Abramescu a déployé une prodigieuse activité sur le plan de l'enseignement secondaire. Il a continué cette activité à Cluj dans le cadre du Séminaire Pédagogique Universitaire, avec les professeurs Gh. Bratu, A. Angelescu et P. Sergescu. Il a rédigé environs 15 précis scolaires d'arithmetique, d'algèbre, de géométrie analytique, d'analyse mathématique, de mécanique et d'astronomie. Il a collaboré aussi avec d'autres professeurs, auteurs de précis, comme A. Manicatide, Gr. Orășanu etc. Plusieurs de ses précis ont eté très appreciés à l'époque, étant valables pour les programmes scolaires quelques dixaines d'années. La brochure *Formules pour la géométrie du triangle* paru en roumain a Bucarest en 1907, vise l'élargissement des connaissances mathématiques des élèves. Tous ces écrits élémentaires respectent les principes generaux de la didactique, en cultivant la passion pour l'étude, le rôle heuristique de la "propriété remarquable", la force d'expression des exemples bien choisis.

Ce bref exposé se propose de rendre hommage à la mémoire d'un professeur distingué de l'Université de Cluj, dont la vie a été étroitement liée au développement de l'enseignement mathématique dans notre pays.

M. ȚARINĂ

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IN MEMORIAM

Prof. dr. doc. DUMITRU V. IONESCU

În ziua de 20 ianuarie 1985 a încetat din viaţă profesorul emerit dr. doc. Dumitru V. Ionescu, reprezentant de frunte al şcolii matematice româneşti.

Născut la Bucureşti, la 14 mai 1901, Dumitru V. Ionescu şi-a făcut studiile la Liceul „Sfintul Sava". După examenul de bacalaureat s-a înscris la Facultatea de Ştiinţe a Universităţii din Bucureşti, secţia de Matematică, unde a avut ca profesori pe Gheorghe Ţiţeica, Anton Davidoglu, David Emmanuel, Nicolae Coculescu, Traian Lalescu şi Theodor Angheluţă. În anul 1923, Dumitru V. Ionescu pleacă la Paris, unde studiază la „École Normale Supérieure" şi la „Collège de France". A urmat, printre altele, cursurile lui Emil Picard, Henri Lebesque, Paul Montel şi Eduard Goursat. La 7 iunie 1927, şi-a susţinut teza de doctorat în matematică cu titlul „Sur une classe d'equations fonctionelles", în care studiază, folosind metoda aproximaţiilor succesive, problema lui Cauchy, Darboux, Picard şi Goursat pentru ecuaţii cu derivate parţiale cu argument modificat.

În toamna anului 1928 Dumitru V. Ionescu este numit conferenţiar la Universitatea din Cluj. La 15 mai 1931 devine profesor agregat şi la 1 iulie 1934 profesor titular. În perioada grea a celui de-al doilea război mondial, cind Facultatea de Ştiinţe din Cluj a funcţionat la Timişoara, Dumitru V. Ionescu a fost decanul acestei facultăţi, reuşind să învingă vicisitudinile vremii. În anii 1949—1955, Dumitru V. Ionescu a fost profesor şef de catedră şi la Politehnica din Cluj, iar din 1955 devine şeful catedrei de Ecuaţii diferenţiale, din Universitate, pînă la pensionarea sa, în anul 1971, dată de la care îşi continuă activitatea ca profesor consultant.

Dumitru V. Ionescu a fost un cercetător activ în domeniul matematicii. Astfel, a publicat peste 200 memorii, note şi monografii. Prin problemele tratate, acestea aparţin celor mai diverse domenii ale matematicii: aritmetică, algebră, geometrie, mecanică, ecuaţii funcţionale, ecuaţii integrale, ecuaţii diferenţiale, ecuaţii cu derivate parţiale şi analiză numerică. Menţionăm că aproape 100 de lucrări, din domeniul analizei numerice, sînt guvernate de o originală metodă de lucru, cunoscută sub denumirea de metoda funcţiei „fi".

Cu aceeaşi pasiune şi devotament s-a dăruit şcolii. Prin cursurile universitare de înaltă ţinută ştiinţifică şi măiestrie pedagogică, prin manualele şcolare şi universitare, în care a pus atita suflet, profesorul Dumitru V. Ionescu se numără printre dascălii de frunte ai învăţămîntului din ţara noastră.

Dumitru V. Ionescu a fost profesor consultant mai bine de 13 ani. În acest timp s-a ocupat în continuare de îndrumarea doctoranzilor şi de organizarea seminarului de cercetare ştiinţifică, de joia de la orele 18,00. A studiat şi a creat ca şi un cercetător activ.

Dumitru V. Ionescu nu mai este printre noi. Colectivul de Ecuaţii diferenţiale a pierdut un îndrumător al său, Şcoala matematică clujeană un cercetător şi reprezentant de frunte, Şcoala matematică românească un stilp al său, iar matematica mondială un specialist de prim rang al Analizei numerice.

Prof. dr. doc. GHEORGHE PIC

La 23 septembrie 1984 a încetat din viață profesorul dr. docent Gheorghe Pic, cel care timp de o jumătate de secol a slujit învățămîntul matematic clujean.

Profesorul Gheorghe Pic s-a născut la 18 martie 1907 în localitatea Szczacova din Silezia Superioară. După un an, tatăl său — care era inginer — se mută cu familia la Mediaș, unde tînărul Gheorghe Pic urmează școala primară și liceul, iar în 1925 își dă bacalaureatul la Blaj. După trei ani de studii la Facultatea de științe din Cluj își ia licența în matematici (Octombrie 1928). În continuare funcționează ca preparator la Institutul de fizică, iar în 1930 dă examenul de capacitate la Iași, pentru învățămîntul secundar. Timp de 15 ani, Gheorghe Pic lucrează ca profesor de matematică și fizică la liceul din Gherla. Între timp, după doi ani de studii la Roma, își susține teza de doctorat „Despre invarianții adiabatici ai sistemelor neolonome", din comisie făcînd parte Levi Civita (ca președinte) și Vito Volterra.

După efectuarea stagiului militar la Școala de ofițeri de artilerie din Craiova, între anii 1933–1936 profesorul Gheorghe Pic a cumulat și funcția de asistent onorific al profesorului Th. Angheluță de la Universitatea din Cluj, unde în 1945 este numit profesor. Din 1952, Gheorghe Pic ocupă funcția de director al Institutului de Studii Romîno-Sovietice și profesor la Institutul de construcții, iar din 1953, cea de profesor la Catedra de algebră a Universității din București. Reintors la Cluj, în anul 1957, profesorul Gheorghe Pic a continuat cu toată dăruirea munca de formare a tinerilor matematicieni la Universitatea clujeană, ocupînd și funcția de decan în perioada 1958–1962.

Domeniul de predilecție al Profesorului Gheorghe Pic l-au constituit grupurile finite. A fost inițiatorul școlii clujene de algebră modernă iar o pleiadă de tineri matematicieni, absolvenți sau doctoranzi ai profesorului Gheorghe Pic, răspîndiți în toată țara, desfășoară o prodigioasă activitate în acest domeniu. Școala clujeană de matematică îi datorează imens profesorului Gheorghe Pic și grație strădaniilor sale în dotarea Bibliotecii facultății, căreia i-a lăsat ca moștenire și valoroasa bibliotecă personală.

Plecarea dintre noi a profesorului dr. docent Gheorghe Pic constituie o grea pierdere pentru toți cei pe care i-a îndrumat și format, pentru Universitatea clujeană și pentru matematica românească.

Jens Wittenburg, **Dynamics of Systems of Rigid Bodies**, B. G. Teubner, Stuttgart, 1977, 224 p.

Monografia „Dinamica sistemelor de corpuri rigide", a profesorului dr. inginer Jens Wittenburg reprezintă o carte excelentă, unică în felul ei, adresată cercetătorilor şi studenţilor absolvenţi ce doresc să se specializeze în acest domeniu.

Primele patru capitole ale cărţii sînt destinate cinematicii, dinamicii şi problemelor clasice ale corpului rigid, intr-o scriere unitară tensorială.

Peste jumătate din carte este ocupată de capitolul cinci, partea esenţială a cărţii, care tratează formalismul general al dinamicii sistemelor de corpuri rigide. Relaţiile cinematice-ecuaţiile neliniare ale mişcării, cît şi celelalte mărimi legate de tratarea sistemelor de corpuri rigide, sint prezentate intr-o formă convenabilă atit studiului numeric cit şi analitic. Descrierea uniformă se sprijină în primul rînd pe conceptele teoriei grafurilor aplicată pentru prima dată în mecanică de către autor.

Expunerea este insoţită de 42 de probleme şi de multe exemple ilustrative netriviale care pun in evidenţă avantajul formalismului matematic cu notaţia sa simbolică vectorială şi tensorială.

AUREL TURCU

W. Müller, **Darstellungstheorie von endlichen Gruppen**, Teubner Stuttgart, 1980, IX + 211 pag.

Prin conţinutul ei, cartea se adresează în primul rind studenţilor de la facultăţile de matematică, dar este un mijloc folositor şi tuturor specialiştilor care in cercetarea lor au nevoie de teoria grupurilor, a modulelor şi a spaţiilor vectoriale. Folosirea cărţii presupune cunoaşterea unor noţiuni de bază din teoria grupurilor, a corpurilor şi a modulelor.

Partea întii a cărţii se ocupă cu conceptele fundamentale din teoria reprezentării grupurilor ca grupuri de transformări liniare ale unui spaţiu vectorial. Principalele probleme tratate in această parte a cărţii sînt: module şi inele semisimple, reprezentări liniare de algebre şi de grupuri finite, algebre grupale semisimple, teoria caracterelor reprezentărilor.

Partea a doua a cărţii este consacrată teoriei reprezentărilor modulare. La inceput sint prezentate clase speciale de inele (noetheriene, artiniene, proiective, injective etc.), de module şi de algebre. În continuare sînt dezvoltate rezultatele lui D. G. Higman, J. A. Green şi G. O. Michler referitoare la algebre grupale care nu sint semisimple. Ultimul punct al cărţii are ca scop studiul metodelor clasice ale teoriei modulare.

I. VIRAG

T. M. Flett, **Differential analysis**, Cambridge University Press, Cambridge 1980, 359 p.

Professor T. M. Flett was a distinguished mathematician, with fine and deep results in various branches of analysis, who, unfortunately, died early at the age of 52. He had almost completed the manuscript of this book and the task of finishing the work was taken by J. S. Pym.

As says Professor Pym in the Preface of the book, the text left by Professor Flett rested almost unaltered. The result is an excelent book on differential calculus.

The first two chapters deal with the differentiation of Banach spaces valued functions of one real variable with applications to ordinary differential equations. Here is worthy to mention the fine discussion on mean value and finite increment theorems for vector functions, containing some personal contributions of the author.

The third chapter is on Fréchet derivative and the fourth on Hadamard derivative. The Hadamard derivative is a notion between Gâteaux and Fréchet derivatives, defined via the cone of feasible directions. Although many pure analysts (e.g. the treatises of Dieudonné, Cartan) neglected this notion, it is very useful in the calculus of variations and optimization theory.

The book contains a wealth of exercises completing the main text. Each chapter ends with historical notes, reflecting the erudition and the encyclopaedic knowledge of the literature of Professor Flett.

In conclusion, this is a valuable book and we recommend it warmly to all who are interested in analysis.

S. COBZAŞ

K. Floret, **Mass- und Integrationstheorie. Eine Einführung.** B. G. Teubner, Stuttgart, 1981, 360 S.

Apărută în seria "Teubner-Studienbücher", cartea oferă o introducere în teoria integralei Daniell-Stone. Expunerea elegantă, clară și economicoasă este principala virtute a ei. Noțiunile și metodele expuse sînt simple, sugestive și suficient de flexibile pentru a putea fi aplicabile în multe situații. Ele se încadrează în mod organic în teoria măsurii și în teoria funcționalelor liniare și pozitive.

Deși conținutul cărții este deosebit de bogat, autorul are tot timpul în vedere caracterul introductiv al textului și nu urmărește să fie enciclopedic. Pentru a ajuta pe cititor la o parcurgere activă a materialului, în fiecare paragraf sînt incluse exerciții și probleme (în total 302).

Scopul principal al cărții, de a prezenta o teorie solidă a integralei care poate servi drept un pilon de bază al analizei funcționale, s-a realizat în mod strălucit.

KOLUMBÁN IOSIF

Wolfgang Polak, **Compiler Specification and Verification**, Springer-Verlag, Berlin, Heidelberg, New York, 1981, 269 pg.

The book is a revised version of the author's PhD thesis. It is organised into five chapters: 1. Introduction; 2. Theoretical framework; 3. Source and target languages; 4. The compiler proof; 5. Conclusions; and finishes with six appendices.

The author's purpose is to develop and prove a compiler for a PASCAL subset. The author thinks that "a program and its proof should be developed simultaneously from their specifications". He uses the stepwise refinement technique for designing and implementing programs to end with the verification of them. But the author insists too much on the mechanical verification aspect although he stresses that "verification should be an integral part of the development process". Also, only the partial correctness of the compiler is proved.

This book contains many ideas, tools and techniques used in programming. It is recommended to all those who are interested in the partial correctness of long real programs.

M. FRENȚIU

Erich Bohl, **Finite Modelle gewöhnlicher Randwertaufgaben.** Teubner Studienbücher, Mathematik, Stuttgart 1981, 318 S.

Das Buch ist der numerischen Behandlung einiger Randwertaufgaben der Form

$$-(px')' + \nu(kx)' = f(t, x, \lambda) \text{ in } (a, b),$$
$$\alpha_a x(a) - \beta_a x'(a) = \gamma_a, \quad \alpha_b x(b) + \beta_b x'(b) = \gamma_b$$

gewidmet. Der Verfasser wendet vor allem das Differenzenverfahren und das Verfahren finiter Elemente an, um die Lösung dieser Probleme auf lineare oder nichtlineare Gleichungssysteme zurückzuführen, in denen eine endliche Anzahl numerischer Unbekannte auftreten. Im dem Buch werden die grundlegenden Methoden zur Lösung der erhaltenen Systeme dargestellt und die Konvergenz der angewandten numerischen Verfahren untersucht. Gleichzeitig werden zahlreiche Probleme aus der Physik, Chemie und Biologie angeführt, die mit Hilfe der im Buch beschriebenen Methoden gelöst werden. Jedes Kapitel enthält Übungsaufgaben sowie einen Abschnitt mit Hinweisen betreffend die historische Entwiklung, Weiterentwicklungen und genaue Literaturangaben.

Das Buch wendet sich sowohl an Lehrnende, als auch an Lehrende, und ist mit geringen mathematischen Vorkentnissen veständlich.

SZILÁGYI PAUL

V. Schmidt, **Digitalschaltungen mit Mikroprozessoren**, B. G. Teubner, Stuttgart 1981.

Das Buch ist eine praxiserientierte Einführung für Ingenieure und Naturwissenschaftler in das Gebiet der Mikroprozessoren. Es werden Funktionskomponenten und Grundkonzepte von Mikroprozessoren, Bit-Slice 'Prozessoren', One-Chip-Mikroprozessoren, die Programmerstellung für Mikroprozessoren, Teste von Mikroprozessorgesteuerten Schaltungen sowie ein praktisches Beispiel beschrieben.

Die Vertrautheit des Lesers mit der 'normalen' Digitalelektronik vorausgesetzt, hat das Buch das Ziel, den Leser in die Lage zu versetzen, eigene Schaltungen mit wenigen aber komplexen Standard-Bauelementen zu realisieren.

FRIEDRICH LANDA

Josef Hainzl, **Mathematik für Naturwissenschaftler**, 3., durchgesehene und erweiterte Auflage, B. G. Teubner, Stuttgart, 1981, 376 Seiten.

Das vorliegende, in der Reihe „Leitfäden der angewandten Mathematik und Mechanik" erschienene, Buch ist die Ausarbeitung und Weiterentwicklung einer Vorlesung, die für Naturwissenschaftler an der Universität Freiburg gehalten wurde. Es umfasst die wichtigsten Kapitel der Analysis (Zahlbereiche und Funktionsbegriff, Differential- und Integralrechnung, elementare Funktionen, Fourierreihen, gewöhnliche Differentialgleichungen), ein Kapitel über analytische Geometrie und lineare Algebra (Vektorrechnung, lineare Abbildungen und Matrizen, lineare Gleichungssysteme und Determinanten, Symmetriegruppen), ein Kapitel über Wahrscheinlichkeitsrechnung (diskrete und stetige Wahrscheinlichkeitsverteilungen, Zufallsgrössen, Zufallsvektoren) sowie ein Kapitel über Statistik (Zufallsstichproben, Schätzen und Testen von Parametern).

Das Buch wendet sich vor allem an Studierende der Biologie, Chemie und verwandter Fächer. Sein Ziel ist dem Leser nicht nur grundlegende mathematische Kenntnisse zu vermitteln, sondern ihm auch deren Anwendbarkeit in den einzelnen Naturwissenschaften zu veranschaulichen. Dieses Ziel wird durch die leichtfassliche Darstellung erreicht, die bewusst auf mathematische Strenge und Allgemeinheit verzichtet, um die Rolle der Mathematik als Hilfswissenschaft für den Naturwissenschaftler besser hervorheben zu können.

WOLFGANG W. BRECKNER

R. Wagner, **Grundzüge der linearen Algebra**, Teubner, Stuttgart, 1981, 260 pag.

Cartea profesorului universitar R. Wagner de la Universitatea Würzburg este o introducere în algebra liniară destinată atît studenților de la facultățile de matematică, cît și profesorilor de gimnaziu. Autorul își propune nu numai o simplă transmitere de cunoștințe, ci mai ales relevarea unui mod de a înțelege construcția noțiunilor și rezultatelor de algebră liniară tratate. Metodele algebrei liniare sînt conduse pînă la aplicarea lor în alte domenii, cum ar fi geometria.

Obiectul cărții îl constituie spațiile vectoriale reale și aplicațiile lor liniare. Dezvoltînd o teorie a acestora, autorul se ocupă în cadrul celor șapte capitole ale cărții și de matrici, sisteme de ecuații liniare, teoria valorilor proprii, spații vectoriale euclidiene și determinanți. Materialul este prezentat cu o deosebită măiestrie didactică, clar, riguros, concis. Sînt date numeroase exemple, iar la sfîrșitul paragrafelor sînt propuse 129 exerciții și probleme. Cartea se încheie printr-un apendice și o tablă de materii.

Recomandăm cartea studenților și cadrelor didactice de la facultățile de matematică, precum și tuturor profesorilor de matematică, ca material bibliografic valoros în studierea și predarea algebrei liniare.

RODICA COVACI

Spisani Franco, **Teoria generale dei numeri relativi**. Vol. I. (Italian) [**General Theory of Directed Numbers**. Vol. I]. Con ingresso dei numeri moltiplicatori e divisori. [With introduction of multiplying and dividing numbers] Bilingual Italian/English text. Pubblicazioni a cura del Centro Superiore di Logica e Scienze Comparate. [Publications of the Center for Higher Studies in Comparate Logic and Science] International Logic Review, Bologna, 1983, 247 pp.

For a long time directed numbers were confounded with simple terms n preceded by „+" and „−" as operational signs. When we operate in an infinite numerical set, the sign taken into consideration is not only the one which precedes, but also that which follows the numbers in the progression of the series.

From the infinitistic point of view any general theory of directed numbers seemed bound to be dismissed out of hand as Cantor himself confirmed. The finitistic tendency begins to come to the fore. Founding a general theory of directed numbers at long last becomes a possibility.

The book gives a modern theory of directed numbers. A list of references and a list of symbols may be found.

The book is highly recommended for speciallists or nonspecialists in general logic as a fundamental text in this field.

DOREL I. DUCA

Harro Heuser, **Lehrbuch der Analysis**, Teil 2. Zweite Auflage. B. G. Teubner, Stuttgart 1983, 736 pag.

Prima parte a amplului tratat de analiză matematică al profesorului universitar Harro Heuser, pe care am recenzat-o în numărul din anul 1982 al revistei Studia Universitatis Babeș-Bolyai, cuprinde analiza funcțiilor reale de o variabilă reală. Partea a doua, a cărei a doua ediție o prezentăm aici, este dedicată studiului funcțiilor ale căror domenii și codomenii sînt submulțimi ale spațiilor $\mathbf{R}^p$. Dar, în afară de rezultatele clasice ale calculului diferențial și ale calculului integral în spațiul $\mathbf{R}^p$, sînt studiate și spațiile topologice, spațiile Banach, teoremele de punct fix ale lui Brouwer, Schauder și Kakutani, demonstrîndu-se cititorului că noțiunile și rezultatele abstracte ale topologiei și analizei funcționale s-au desprins în mod natural din bogatul material faptic acumulat de analiza matematică. Multe aplicații frumoase în diverse domenii științifice întregesc partea teoretică și pun în evidență influența stimulatoare pe care au avut-o problemele practice asupra dezvoltării analizei matematice.

Scrisă cu mult simţ pedagogic şi cu aceeaşi grijă ca şi partea intîi, cartea reprezintă o sursă de informare valoroasă pentru studenţii facultăţilor de matematică, iar pentru cadrele didactice ale acestor facultăţi un model de prezentare modernă a analizei matematice.

WOLFGANG W. BRECKNER

Banach Spaces Theory and its Applications, Bucharest 1981, Edited by A. Pietsch, N. Popa and I. Singer, Lectures Notes in Mathematics 991, Springer — Verlag, Berlin — Heidelberg — New York — Tokyo, 1983 (302 pp.).

These are the Proceedings of the First Romanian — G.D.R. Seminar on Banach Space Theory, held at Bucharest, Romania, from 31-st August to 5 th September 1981. The works of the Seminar were attended by eminent specialists from 15 countries giving and hearing talks on Banach space theory and related fields. The volume contains 26 written versions from the talks given at the Seminar. The papers contain original contributions of the authors and surveys of the main results obtained in this area of research. Many of the papers contain open problems, tracing ways for further investigations. The volume is a valuable contribution to the Banach space theory and its applications and we recommend it to all people working in analysis and functional analysis.

S. COBZAŞ

J. Diestel, **Sequences and Series in Banach Spaces.** Graduate Texts in Mathematics no. 92, Springer-Verlag, New York — Berlin — Heidelberg — Tokyo 1984 (260 pp.).

The author is well known to mathematical community by two previous books — Geometry of Banach Spaces, Selected Topics L.N.M. no. 484, Springer Verlag 1975 (Russian Translation Visca Skola, Kiew 1980), Vector Measures, Mathematical Surveys 15, A.M.S. 1977 (in cooperation with J.J. Uhl jr.) and by his deep contributions to the geometric theory of Banach Spaces, especially concerning the Radon — — Nikodym property. This new book covers plenty of topics related to sequences and series in Banach spaces, starting from the classical results of Banach, Schauder, Mazur and ending with some very recent discoveries of A. Pelczynski, C. Bessaga, H. P. Rosenthal, R. C. James, E. Odell, J. Lindenstrauss, L. Tzafriri et al. The aim of the book is to present in an accessible way to the working analysts some of the results of general analytic character from Banach space theory. The author accomplishes masterly this purpose. Each chapter ends with a list of references, hystorical comments, remarks on further developements of the subjects and numerous helpful exercises. The style is very clear, live and pleasant — the readers of two above quoted books will enjoy it again. The book is self-contained and makes accessible and collect together some of the very deep and difficult results obtained in the last years in Banach spaces theory. We recommend it warmly to all interested in these topics.

S. COBZAŞ

T. Kato, **Perturbation Theory for Linear Operators,** Springer — Verlag, 1984, 618 p.

This is the second corrected edition of the second edition of the now classical treatise of Professor Kato on perturbation theory. The first edition appeared in 1966. A Russian translation appeared in 1972. Springer — Verlag published also in 1982 a short version (161 p.) of the book containing the first two chapters and some additional material. With respect to the first edition this one is completed with suplementary notes on the recent development in perturbation theory. The bibliography is also substantially completed with papers published in the meantime in this very active field of research.

The book is a valuable contributions to the theory of perturbation of linear operators.

S. COBZAŞ

Adrian Bejan, **Convection Heat Transfer,** John Wiley and Sons, Inc., 1984, XV + 477 pp.

This textbook deals with topics being at the interface between heat transfer and fluid mechanics that may, at first sight, appear totally different, but which in fact are strongly interrelated and which are capable of cross-fertilizing each other. This area of research, known as convective heat transfer or, simply, convection is a fascinating one, and is one that is of fundamental importance in our technologycal society.

In 12 chapters, the book gradually covers classical as well as most recent topics of convective heat transfer. It provides a broad view within the subject area and includes, for the first time, important topics of heat transfer. Most of the currently used mathematical tehniques for analysing convection problems are included in this work: integral solutions, similarity solutions, scale analysis and modern numerical methods. However, a special emphasis is given on the utilization of the scale analysis fully developed by the author in his research papers. This method plays a central role in the formulation and analysis of various classes of problems to model natural phenomena and to use the results correctly and fruitfully.

Each chapter of the book is devoted to the description of a precise topic, together with the mathematical method employed for its analysis. But, the mathematical analysis is made in close relation with the physical essence of the phenomenon considered. The book includes a set of unsolved problems for each chapter which are fully worked out in the Solutions Manual. It offers an excellent account of the type of problems studied in the area of convective heat transfer and of the methods used. Consequently, new concepts and ideas are given a chance to filter through, and the reader has time to reflect about them. Each chapter is also suplemented by an important list of references on the topics considered. Therefore, the presentation of this textbook is organized to suit the students who need an introduction to the subject as well as the experienced researcher who needs a reference source for specific results. Giving a realistic and coherent overview of the research presently carried out in the area of convection, the present book will prove to be a valuable acquisition for the students and active researchers, too. The material is attractively presented and exceptionally readable; the figures and the printing are excellent.

As a conclusion, Professor Bejan's book is a complete account of the convective heat transfer topics, written by an expert who himself has made important and valuable contributions to the subject. It will surely take its place among the best textbooks in convective heat transfer. As such, it should belong to the library of every researcher and graduate student in the field of convection.

IOAN POP

J. France, J. H. M. Thornley, **Mathematical Models in Agriculture, Butterworths Borough Green, Sevenoaks,** Kent TN 15 8PH, London, 1984 (335 pag).

This book is an excellent aproach to the applications of mathematical methods to problems in agriculture and related sciences.

The actual agriculture and its related scientific disciplines are in a state of rapid evolution and quantitative methods of experimentation are becoming increasingly important. The Mathematics is now accepted as the most appropiate tool for the description of experimental results and the matching of these to current ideas and theories.

The aim of this book is to present the ideas, methods and recent applications of mathematical modelling in agriculture in such a way that agricultural scientists may learn when and how to attempt to express their ideas mathematically, how to solve the resulting mathematical problem and how to compare the predictions with experimental data.

The topics cover a range from animal and plant physiology to farm planning and control, and include crop growth, plant diseases and pests, and weather. Within each topic, attention is focussed on the most promising modelling approaches, and on those that are mathematically sound and instructive. In addition, those mathematical topics relevent to agriculture modelling: growth functions, dynamic modelling and mathematical programing, are brought together for the first time and treated in some detail.

The book containts the following 13 chapiters: 1. Role of mathematical models in agriculture and agricultural research. 2. Techniques: dynamic deterministic modelling. 3. Techniques: mathematical programming. 4. Testing and evaluation of models. 5. Growth functions. 6. Weather. 7. Plant and crop processes. 8. Crop responses and models. 9. Plant diseases and pests. 10. Animal processes. 11. Animal products. 12—13. Farm planning and control I—II.

The text is designed to be suitable also for self-tuition with the inclusion of nice exercises and worked outline solutions.

MARIA MICULA

V. S. Varadarajan, **Lie Groups, Lie Algebras and their Representations.** Graduate Texts in Mathematics 102. Springer — Verlag, 1984, New York, Berlin, Heidelberg, Tokyo, 430 pages.

The theory of Lie groups is treated usually in the books on Differential Geometry from the basic algebraic or analytic point of view without touching the representation theory. In this book, the author presents not only the general problems on Lie groups, but also he develops in detail the representation theory of semisimple Lie groups and Lie algebras. The book's contents is the following:

Preface.
Chapter 1. Differentiable and Analytic Manifolds.
Chapter 2. Lie groups and Lie algebras.
Chapter 3. Structure theory.
Chapter 4. Complex semisimple Lie algebras and Lie Groups: Structure and Representations.
Bibliography.
Index.

The Chapter 1 has an introductory character and Chapter 2 deals with the basic results and concepts on Lie groups. The Chapter 3 is mainly devoted to the structure theory of Lie algebras (without cohomology). The chapter 4 furnishes a fairly complete exposition of the representations theory of semisimple Lie algebras and Lie groups.

The algebraic treatment is based upon the ideas of Harish Chandra's papers and is followed up with the transcendental theory of the maximal tori. This duality is considered to be essential and significant even for the entire representation theory. Each chapter is followed by a lot of exercises, many of them from a theoretical importance.

The work was originally published in the Prentice-Hall Series in Modern Analysis, 1974. The present version of the textbook is almost selfcontained, asking only some acquaintance with topological groups and differential manifolds. It is to be recommended as very useful for graduate students, mathematical researchers and theoretical physicists.

M. ŢARINĂ

H. Grauert, R. Remmert, **Coherent Analytic Sheaves**, Grundlehren der Mathematischen Wissenschaften 265, Springer — Verlag, 1984, Berlin — Heidelberg — New York — Tokyo, 249 pages.

This volume is an excellent exposition of the theory of coherent sheaves in Complex Analysis. In fact, the work was prefigurated already in the early sixties, the authors being notorious specialists in this field of research. This printed version contains a systematically and exhaustive approach on the subject. After a breaf introduction with historical and methodological motivations of the problems, the book's contents is distributed in the following 10 chapters:

1. Complex spaces. 2. Local Weierstrass theory. 3. Finite holomorphic maps. 4. Analytic sets. Coherence of ideal sheaves. 5. Dimension theory. 6. Analyticity of the singular locus. Normalisation of the structure sheaf. 7. Riemann extension theorem and analytic coverings. 8. Normalisation of complex spaces. 9. Irreducibility and connectivity. Extension of analytic sets. 10. Direct image theorem.

Finnally an Annexe is given on the theory of sheaves and on the notion of coherence. The book contains also the essential bibliography, and Index of names, and a general index.

The work presents details about the four fundamental coherence theorems in Complex Analysis, namely the coherence of the structure sheaf $\mathcal{O}_X$ as a complex space X (ch. 2), the coherence of the ideal sheaf i(A) of any analitic set A (Ch. 4), the coherence of the Normalisation sheaf of any reduced structure sheaf $\mathcal{O}_X$ (Ch. 8) and the coherence of the direct image sheaves of a coherent analytic sheaf under a proper holomorphic map. (Ch. 10). Other topics in Complex Analysis are discussed, as the dimension theory of the complex spaces, analytic coverings (Ch. 5), normalisation spaces of reduced complex spaces (Ch. 8) and extension of the analytic sets into lower dimensional ones (Ch. 9).

The authors dedicated their book to Henri Cartan who initiated the new trends in Complex analysis by introducing in 1950 the notion of coherence sheaf.

The textbook is very useful for the graduate students as well as for the mathematicians working in Complex Analysis, Analytic spaces and Sheaf theory.

M. ŢARINĂ

Jacques Dixmier, **General Topology**, Springer — Verlag, New York, Berlin, Heidelberg, Tokyo, 1984, x + 140 pages.

This book is a concise introduction to general topology. It is intended for students and teaches them the basic concepts and results of topology, which are indispensable for understanding modern mathematics.

The main topics presented in the ten chapters of the book are topological spaces, limits and continuity, constructions of topological spaces, compact spaces, metric spaces, limits of functions, numerical functions, normed spaces, infinite sums, connected spaces. Each chapter contains numerous examples which illustrate key definitions and theorems. Additional exercises at the end of the book are a valuable completion of the text.

Well-organized and carefully written, the present book is very useful as a textbook for an advanced undergraduate or beginning graduate course in general topology. We recommend it to anyone who is studying for the first time general topology.

WOLFGANG W. BRECKNER

Interpolation Spaces and Allied Topics in Analysis, Lectures Notes in Mathematics 1070, 1984 (239 pp.)

These are the Proceedings of a Conference held in Lund, Sweden, from August 29 to September 1, 1983, edited by M. Cwikel and J. Peetre. The volume begins with an introductory paper by J. Peetre entitled „The theory of interpolation spaces — its origin, prospects for future", were the theory of interpolation spaces is outlined from its origins (M. Riesz and J. Marcinkiewicz) to present days. Some directions of further investigations are presented. The second paper is a translation (appearing for the first time in English) of a paper by B. Mityagin „An interpolation theorem for modular spaces' published originally in Mat. Sbornik 66 (108) (1965), 473 – 482. The volume contains also 15 contributed papers treating various aspects of interpolation theory (real and complex methods) in an abstract setting or in concrete spaces. The volume is a valuable contribution to interpolation theory and we recommend it warmly to all interested in analysis.

S. COBZAŞ

Infinite Dimensional Systems, Lectures Notes in Mathematics 1076, Springer — Verlag, Berlin — Heidelberg — New York — Tokyo, 1984

These are the proceedings of the "Conference on Operator Semigroups and Application" held in Retzhof (Styria) June 5—11 1983. The works of the Conference were attended by 42 specialists coming from 12 countries. The volume contains the written versions of 22 conferences presented at the Congress. It covers a variety of topics related mainly to differential and integral equations in abstract spaces, to semigroups of operators and applications. The papers contain many interesting new results and the book will be indispensable to all working in this area.

S. COBZAŞ

The following books have been received by the editorial staff that are to be reviewed:

Academic Press Inc.: 1. Approximation theory IV (Edited by C. K. Chui, L. L. Schumaker and J. D. Ward, 1983); 2. Elliptic problem solvers (Edited by G. Birkhoff and A. Schoenstadt, 1984); 3. O. Axelsson and V. A. Barker, Finite element solution of boundary value problems. Theory and computation.

Birkhäuser Verlag: 1. H. Aigner, Graphentheorie; 2. A. Fröhlich, Classgroups and Hermitian modules; 3. H. Schlichtkrull, Hyperfunctions and harmonic analysis on symmetric space; 4. Perspectives in Mathematics (Edited by W. Jaeger, J. Moser, R. Remmert).

Springer Verlag: 1. J. — P. Aubin and A. Cellina, Differential inclusions; 2. J. Conway, A course in functional analysis; 3. L. Hörmander, The analysis of partial differential operators I, II; 4. D. H. Luecking and L. A. Rubel, Complex analysis — A functional analysis approach; 5. H. Majima, Asymptotic analysis for integrable connections with irregular singular points; 6. J. Marsden and A. Weinstein, Calculus I; 7. Measure theory and its applications, Sherbrooke 1982; 8. P. Schapira, Microdifferential systems in the complex domain; 9. N.Z. Shor, Minimization methods for non-differentiable functions and applications; 10. E. W. Stredulinsky, Weighted inequalities and degenerate elliptic partial differential equations; 11. E. Zeidler, Nonlinear functional analysis and its applications III. Variational methods and optimization.

Cambridge Univ. Press: 1. J. D. Dollard and Ch. N. Friedman, Product integration; 2. S. Wagon, The Banach — Tarski paradox.

Akademie Verlag Berlin: 1. H. Baumgärtel, Analytic perturbation theory for matrices and operators; 2. G. M. Henkin and J. Leiterer, Theory of functions on complex manifolds; 3. H. Reichel, Structural induction in partial algebras.

Teubner Texte zur Mathematik Leipzig: 1. Algebraic and differential topology — Global differential geometry (ed. G. M. Rassias); 2. Global analysis — Analysis on manifolds (ed. T. Rassias); 3. H. Junek, Locally convex spaces and operator ideals; 4. S. L. Kruschkal and R. Kühnau, Quasikonforme Abbildungen; 5. J. Necas, Introduction to the theory of nonlinear elliptic equations; 6. Proceedings of the Second International Conference on Operator Algebras, Ideals and their applications in Theoretical Physics (Leipzig 1983); 7. Recent trends in mathematics; 8. H. — U. Schwarz, Banach lattices and operators; 9. Th. Zink, Cartiertheorie kommutativer formaler Gruppen.

CRONICĂ

I. Publicaţii ale seminariilor de cercetare ale Facultăţii de Matematică (serie de preprinturi)

Preprint 1—1984, Seminar of Functional Analysis and Numerical Methods.

Preprint 2—1984, Seminar of Celestral Mechanics and Space Research.

Preprint 3—1984, Seminar on Fixed Point Theory.

Preprint 4—1984, Seminar on Computer Sciences.

Preprint 5—1984, Models structures and dates processing.

Preprint 6—1984, Intinerant Seminar on Functional Equations, Approximation and Convexity.

Preprint 7—1984, Variational Methods.

Preprint 8—1984, Seminar on Best Approximation and Mathematical Programming.

II. Participări la manifestări ştiinţifice organizate în afara facultăţii

1. *Al III-lea seminar de spaţii Finsler, Braşov, 9—15 februarie 1984:*

M. Ţarină, P. Enghiş, *Formalism exterior în geometria unui fibrat vectorial.*

2. *Consfătuirea naţională de cercetare-proiectare asistată de calculator, I.C.I. Bucureşti, iunie—iulie 1984:*

Gr. Moldovan, Gr. Mureşan, Gh. Părău, T. Toadere, T. Tökés, *Model matematic şi programe privind extragerea sării prin dizolvare.*

S. Damian, I. Parpucea, M. Topliceanu, *Produs informatic pentru testarea plăcilor cu circuite integrate.*

3. *Al IV-lea Simpozion Naţional de Aplicaţii ale informaticii în proiectarea şi cercetarea în construcţii:*

A. Chisăliţă, B. Pârv, *Programul THACS*

4. *A VII-a Consfătuire a personalului de la institutele de informatică din reţeaua M.E.I., Gura Humorului, 30 iulie—5 august 1984:*

Gr. Moldovan, Gh. Mureşan, T. Toadere, *Model matematic privind extracţia sării prin dizolvare.*

A. Chisăliţă, B. Pârv, *Analiza răspunsului dinamic neliniar al sistemelor pe cabluri.*

S. Damian, B. Pârv, P. Pop, M. Vincze, *SIMPLOMCO*

D. Chiorean, I. Chiorean, E. Muntean, *Gestiunea memoriei la sistem de baze de date de tip Socrate pentru minicalculatoare.*

5. *A XV-a Conferinţă Naţională de Geometrie şi Topologie, Timişoara, 2—7 iulie 1984:*

F. Radó, *Caracterizarea semiizometriilor unui spaţiu Galois.*

A. Vasiu, *Caracterizarea grupală a unei structuri de translaţie cu elemente vecine.*

V. Groze, A. Vasiu, *Asupra unor aplicaţii ale unui plan Galois.*

B. Orban, *Transformări pătratice ale unui plan proiectiv pappusian.*

M. Ţarină, *Cîmpuri Jacobi pe un spaţiu omogen şi elemente speciale ale algebrei Lie.*

P. Enghiş, *E-conexiuni recurente.*

6. *Colocviul Romîno-Japonez de Geometrie Finsler, Iaşi, Braşov, Bucureşti, 15—25 august 1984*

M. Ţarină, *Invariant Finsler connections on vector bundles.*

7. *Reuniunea Subcomisiei nr. 5 „Stele duble" din Comisia de colaborare multilaterală a academiilor de ştiinţe din ţări socialiste pe tema „Fizica şi evoluţia stelelor", de la Tbilisi, U.R.S.S., 20—24 august 1984*, care a avut două părţi:

a) partea ştiinţifico-organizatorică, la care V. Ureche, preşedintele Subcomisiei a prezentat raportul de activitate pe perioada 1982—1984.

b) partea ştiinţifică, desfăşurată sub titlul „Stele duble şi evoluţia lor" la care s-au prezentat comunicările:

V. Ureche, A. Imbroane, *Slowly rotating relativistic linear stellar model.*

I. Todoran, *Apsidal motion in close binary systems with wery evolved components.*

Reuniunea de lucru a Comisiei de colaborare multilaterală a academiilor de ştiinţe din ţări socialiste pe tema „Fizica şi evoluţia stelelor", Suceava, 25—28 septembrie 1984. Au participat V. Ureche şi I. Todoran. Lect. dr. V. Ureche, preşedintele Subcomisiei nr. 5 „Stele duble", a prezentat raportul de activitate al Subcomisiei pe perioada 1982—1984 şi proiectul de reorganizare a acesteia.

8. *Colocviul de Mecanica Fluidelor şi aplicaţiile ei tehnice, Iaşi, 12—13 octombrie 1984:*

T. Petrilă, *Aplicaţii ale metodei elementului finit la limită în mecanica fluidelor.*

I. Stan, *Profilul concentraţiei surfactantului pe o picătură liberă.*

9. *Simpozionul Matematica în ştiinţele universului, Cluj-Napoca, 7—8 decembrie 1984:*

Á. Pál, *Teorii de mişcare a sateliţilor artificiali ai Pămîntului.*

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