STUDIA

UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

4

1980

CLUJ-NAPOCA

ANUL XXV 1980

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

4

Redacţia : 3400 CLUJ-NAPOCA, str. M. Kogălniceanu, 1 ● Telefon 1 34 50

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

ASUPRA MIŞCĂRII FLUIDE TRIDIMENSIONALE PRODUSE DE ROTOTRANSLAŢIA ÎN MASA DE FLUID A UNUI CORP RIGID ÎN PREZENŢA UNUI PERETE NEMĂRGINIT

T. PETRILA

Studiul influenţei pereţilor nelimitaţi în mişcarea plană, ideală, incompresibilă şi potenţială a constituit obiectul cercetărilor noastre anterioare [1], [2], [3], [4], [5].

Prezenta lucrare îşi propune extinderea metodelor, elaborate pentru mişcări plane, la cazul tridimensional, generalizare firească cerută de realitatea înconjurătoare. Desigur ea nu epuizează toate aspectele problemei, o aprofundare teoretică, cu trecerea în revistă a tuturor detaliilor, impunîndu-se pe viitor.

În prima parte a lucrării, utilizînd pentru soluţia Neumann corespunzătoare o reprezentare prin potenţiale newtoniene, se ajunge la un sistem corespunzător de ecuaţii Fredholm căruia i se pot aplica, cel puţin teoretic, metode de lucru valabile şi în cazul plan.

În partea doua a lucrării este utilizată metoda cuplului de profile pentru cazul peretelui plan nemărginit. În cazul particular cînd profilul rigid este o sferă, soluţia se poate da prin teorema sferei a lui P. Weiss [6], generalizare a importantei teoreme a cercului din cazul plan.

1. Formularea matematică. Ecuaţiile integrale ataşate problemei. Să considerăm un corp solid rigid tridimensional, limitat de suprafaţa S, mobil într-un fluid ideal în prezenţa peretelui nemărginit π; suprafeţele S şi π vor satisface condiţiile lui Liapunov.

Vom raporta mişcarea solidului şi mişcarea corespunzătoare a fluidului, atît faţă de un sistem de axe fixe $O_1x_1y_1z_1$, cît şi în raport cu reperul mobil $Oxyz$, legat rigid de corp. Mişcarea solidului va fi definită, la fiecare moment t, prin viteza $\bar{v}_0(t)$ a punctului O şi prin viteza instantanee de rotaţie $\bar{\omega}(t)$.

Condiţiile la limită ale problemei vor fi:

$$\vec{V}_r \cdot \vec{n}|_s = 0 \quad \text{sau} \quad \vec{V} \cdot \vec{n}|_s = \vec{V}_t \cdot \vec{n}|_s$$

$$\vec{V} \cdot \vec{n}|_\pi = 0$$

ceea ce exprimă alunecarea fluidului ideal de-a lungul suprafeţelor S şi π, presupuse impermeabile.

Vom avea apoi condiţiile asimptotice, cum ar fi, de exemplu, condiţia exprimînd faptul că fluidul este animat de o viteză dată $\vec{V}_\infty(t)$ la mari distanţe adică

$$\lim_{r=OM\to\infty} \vec{V}(M, t) = \vec{V}_\infty(t);$$

În ceea ce ne privește, vom presupune că $\vec{V}_\infty = 0$, adică fluidul este în repaus la infinit; în plus se admite continuitatea funcției $\vec{V}(x, y, z, t)$, precum și a derivatelor sale de ordinul întîi, în întregul domeniu al mișcării fluide D.

În privința condițiilor inițiale, care trebuiesc și ele atașate acestei probleme nestaționare, ele sînt date, de obicei, astfel încît mișcarea fluidă să fie irotațională. Această circumstanță se realizează dacă, de exemplu, solidul S și fluidul încep mișcarea, la momentul inițial t_0, din repaus.

Să presupunem, în cele ce urmează, că mișcarea fluidă are un caracter irotațional, fluidul ideal fiind considerat și incompresibil. Existența atunci a unui potențial al vitezelor $\Phi(x_1, y_1, z_1, t) \equiv \Phi(x, y, z, t)$, funcție armonică atît în raport cu coordonatele reperului fix, cît și față de cele ale reperului mobil, ne va permite următoarea reformulare a problemei la limită de mai sus (în raport cu reperul mobil $Oxyz$):

$$\begin{cases} \Delta\Phi = O \text{ în } D; \\ \left.\dfrac{d\Phi}{dn}\right|_S = (\vec{v}_0 + \vec{\omega} \times \vec{r})\vec{n}\,|_S; \\ \left.\dfrac{d\Phi}{dn}\right|_\pi = 0; \\ \operatorname*{grad}_{r\to\infty} \Phi \to 0, \end{cases} \tag{1}$$

ultima condiție fiind în acord cu repausul fluidului la mari distanțe.

Sîntem astfel conduși la rezolvarea unei probleme Neumann, atașată operatorului lui Laplace, pentru domeniul tridimensional nemărginit D.

Se verifică imediat îndeplinirea condiției necesare de existență a soluției acestei probleme, adică

$$\iint_S (\vec{V}_0 + \vec{\omega} \times \vec{r})\vec{n}\, d\sigma = 0,$$

oricare ar fi vectorii $\vec{V}_0(t)$ și $\vec{\omega}(t)$. Introducînd de asemenea notațiile următoare:

— V_1, V_2, V_3 pentru componentele lui $\vec{V}_0$ pe axele Ox, Oy, Oz și V_4, V_5, V_6 pentru cele ale lui $\vec{\omega}$ pe aceleași axe;

— n_1, n_2, n_3 pentru componentele versorului normalei $\vec{n}$ și n_4, n_5, n_6 pentru cele ale lui $\vec{r} \times \vec{n}$ față de aceleași axe,

condiția la limită pe S se va putea scrie:

$$\frac{d\Phi}{dn} = \sum_{p=1}^{6} n_p V_p,$$

unde n_p sînt mărimi geometrice depinzînd de punctul P al lui S, dar și de t, în timp ce V_p sînt funcții de timp cunoscute, independente de punctul P al lui S

Admiţînd acum că există funcţii $\Phi^{(p)}(x, y, z)$ armonice în D astfel ca $\frac{d\Phi^{(p)}}{dn} = n_p$ pe S şi $\frac{d\Phi^{(p)}}{dn} = 0$ pe π, în timp ce grad $\Phi^{(p)}$ se anulează la infinit, dacă se pune

$$\Phi(x, y, z, t) = \sum_{p=1}^{6} V_p(t) \cdot \Phi^{(p)}(x, y, z),$$

atunci această funcţie Φ va satisface tuturor condiţiilor problemei, definind deci mişcarea fluidă în exteriorul obstacolului S, în prezenţa peretelui nemărginit π.

Prin analogie cu cazul bidimensional, soluţia problemei Neumann tridimensionale de mai sus se poate reprezenta sub forma unei sume de potenţiale Newtoniene de simplu strat

$$\Phi(x, y, z, t) = \iint_S \frac{\nu(Q\,;\,t)}{r_{MQ}}\, d\sigma_Q + \iint_\pi \frac{\mu(Q\,;\,t)}{r_{MQ}}\, d\tau_Q,$$

unde r_{MQ} este distanţa de la punctul $M(x, y, z)$ la punctul de integrare Q iar $\nu(Q\,;\,t)$ şi $\mu(Q; t)$ desemnează densitatea stratului respectiv. Derivatele normale ale acestui potenţial, calculate într-un punct $P \in S$ şi respectiv $P \in \pi$ (prin continuitate exterioară) vor fi:

$$\left(\frac{d\Phi}{dn}\right)_{P\in S} = 2\pi\nu(P\,;\,t) + \iint_S \nu(Q\,;\,t)\,\frac{\cos(\overrightarrow{n_P}, \overrightarrow{PQ})}{r_{PQ}^2}\, d\sigma_Q +$$

$$+ \iint_S \mu(Q\,;\,t)\,\frac{\cos(\overrightarrow{n_P} \cdot \overrightarrow{PQ})}{r_{PQ}^2}\, d\tau_Q,$$

pentru $P \in S$, respectiv

$$\left(\frac{d\Phi}{dn}\right)_{P\in \pi} = 2\pi\mu(P\,;\,t) + \iint_S \nu(Q\,;\,t)\,\frac{\cos(\overrightarrow{n_P}, \overrightarrow{PQ})}{r_{PQ}^2}\, d\sigma_Q +$$

$$+ \iint_\pi \mu(Q\,;\,t)\,\frac{\cos(\overrightarrow{n_P}, \overrightarrow{PQ})}{r_{PQ}^2}\, d\tau_Q, \text{ pentru } P \in \pi^{1)}.$$

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[1] S-a presupus că atît suprafaţa nemărginită π cît şi densitatea μ satisfac condiţii de „tip D. Przeworska-Rolewicz" suficiente atît pentru existenţa integralelor singulare cît şi pentru aplicabilitatea formulelor de salt [7].

Rezolvarea problemei la limită de mai sus revine atunci la satisfacerea de către densitățile ν și μ, a următorului sistem de ecuații integrale singulare Fredholm de speța a II-a:

$$\begin{cases} 2\pi\nu(P;t) + \iint\limits_S \nu(Q;t) \dfrac{\cos(\vec{n}_P, \overrightarrow{PQ})}{r_{PQ}^2} d\sigma_Q + \iint\limits_\pi \mu(Q;t) \dfrac{\cos(\vec{n}_P, \overrightarrow{PQ})}{r_{PQ}^2} d\tau_Q = \\ \qquad = (\vec{V}_0(t) + \vec{\omega}(t) \times \overrightarrow{OP})\vec{n}_P, \quad P \in S \\ 2\pi\mu(P;t) + \iint\limits_S \nu(Q;t) \dfrac{\cos(\vec{n}_P, \overrightarrow{PQ})}{r_{PQ}^2} d\sigma_Q + \iint\limits_\pi \mu(O;t) \dfrac{\cos(\vec{n}_P, \overrightarrow{PQ})}{r_{PQ}^2} d\tau_Q = \\ \qquad = 0, \quad P \in \pi. \end{cases} \tag{2}$$

Acestui sistem i se pot aplica metode de studiu analoage cu cele dezvoltate de noi în [1], [5]. Soluția sa va genera soluția problemei propuse inițial care, avînd în vedere reprezentarea sa prin potențiale newtoniene, va satisface și condiția la infinit.

2. **Cazul peretului plan nemărginit.** Să presupunem acum că suprafața π este o suprafață plană nemărginită; fie ea chiar planul $z_1 = 0$ în reperul fix $Ox_1y_1z_1$.

Problema determinării influenței prezenței peretelui nelimitat π asupra mișcării fluide generate de rototranslația rigidului S, ar putea fi atunci abordată și prin metoda cuplului de profile dezvoltată de noi, pentru cazul bidimensional, în lucrările [2], [3], [5].

Să considerăm, pentru aceasta, alături de rigidul $S \equiv S_1$, și simetricul său S_2, simetric care execută, în masa de fluid ideal, o „deplasare simetrică" comparativ cu cea a lui S_1; peste tot „simetria" trebuie înțeleasă în raport cu planul fix π. Odată rezolvată problema determinării mișcării fluide generate de rototranslația simetrică a cuplului de solide simetrice S_1 și S_2, va fi soluționată și problema, propusă inițial de noi, a mișcării fluide generate de deplasarea obstacolului tridimensional $S \equiv S_1$ în prezența peretelui plan nelimitat π.

Acceptînd atunci ipotezele făcute deja în paragraful precedent, potențialul vitezelor Φ, în domeniul Ω, exteriorul obstacolelor S_1 și S_2, poate fi căutat sub forma

$$\Phi = \Phi_1 + \Phi_2.$$

unde funcțiile Φ_1 și Φ_2 satisfac respectiv, următoarele probleme la limită:

$$\begin{cases} \Delta\Phi_1 = 0 \text{ în } \Omega \\ \left.\dfrac{d\Phi_1}{dn_1}\right|_{S_1} = (\vec{V}'_{0'} + \vec{\omega}' \times \vec{r}')\vec{n}_1 = \sum\limits_{p=1}^{6} V_p^{(1)}(t)\, n_p^{(1)} \\ \left.\dfrac{d\Phi_1}{dn_2}\right|_{S_2} = 0 \\ \operatorname{grad}_{(r'\to\infty)} \Phi_1 \to 0 \end{cases} \tag{3'}$$

şi

$$\begin{cases} \Delta \Phi_2 = 0 \text{ în } \Omega \\ \left.\dfrac{d\Phi_2}{dn_1}\right|_{S_1} = 0 \\ \left.\dfrac{d\Phi_2}{dn_2}\right|_{S_2} = (\vec{V}_{0''}'' + \vec{\omega}'' \times \vec{r}'')\vec{n}_2 = \sum_{p=1}^{6} V_p^{(2)}(t)\, n_p^{(2)} \\ \underset{(r'' \to \infty)}{\text{grad}\ \Phi_2} \to 0 \end{cases} \tag{3''}$$

În formularea de mai sus, cele două probleme la limită au fost raportate la reperele mobile corespunzătoare, adică la $O'x'y'z'$, reperul mobil legat rigid de S_1, şi respectiv la $O''x''y''z''$, reperul mobil legat rigid de S_2. De asemenea, remarcăm că semnificaţia tuturor funcţiilor din membrii drepţi este identică cu cea introdusă deja în paragraful precedent; este evident că, din considerentele existente de simetrie, oricare din aceste funcţii, prezente de pildă în prima problemă, se pot exprima cu ajutorul omoloagelor sale din a doua problemă şi reciproc.

Interpretarea mecanică a acestor ultime două probleme la limită ar fi următoarea: prima ar corespunde mişcării fluide generate de deplasarea profilului *mobil* S_1 în prezenţa profilului simetric *fix* S_2, în timp ce, în a doua problemă, S_1 şi S_2 îşi vor schimba rolurile între ele.

Este evident că odată soluţionată una din aceste probleme soluţia celeilalte se va obţine imediat prin considerente de simetrie. În sfîrşit dacă $\Phi_p^{(i)}$, $i = 1, 2$, $p = \overline{1,6}$ sînt soluţiile armonice ale problemelor Neumann „ataşate", corespunzător datelor $n_p^{(i)}$, $i = 1,2$, $p = \overline{1,6}$, pe frontiere, atunci, raţionînd ca şi în paragraful precedent, funcţia Φ căutată va admite reprezentarea

$$\Phi = \sum_{i=1}^{2} \sum_{p=1}^{6} V_p^{(i)}(t)\, \Phi_p^{(i)}$$

Să considerăm acum cazul simplificat, cînd obstacolele S_1 şi S_2 sînt sfere de rază R. Fixîndu-ne atunci asupra primei probleme la limită (3'), vom căuta să o rezolvăm aplicînd o tehnică care generalizează pe cea inspirată din aplicarea teoremei cercului, din cazul plan.

Fie deci φ_0 un potenţial al vitezelor, a unei mişcări fluide ideale, incompresibile, irotaţionale, raportat la reperul *fix* $O''x''y''z''$ şi care este o funcţie armonică şi regulată în S_2. Se ştie atunci [6], că, *oricare* ar fi un astfel de potenţial „original" φ_0, potenţialul

$$\varphi = \varphi_0 + \varphi_1 = \varphi_0 + \frac{R}{r}\varphi_0\left(\frac{R^2 x}{r^2}, \frac{R^2 y}{r^2}, \frac{R^2 z}{r^2}\right) - $$

$$- \frac{2}{Rr}\int_0^R \lambda \varphi_0\left(\frac{\lambda^2 x}{r^2}, \frac{\lambda^2 y}{r^2}, \frac{\lambda^2 z}{r^2}\right) d\lambda$$

(unde am omis pentru simplificarea scrierii dublul accent "pentru coordonatele în reperul $O''x''y''z''$) este o funcţie armonică ce satisface condiţiei de alunecare pe sfera fixă impermeabilă, S_2 în timp ce $\varphi - \varphi_0$ se comportă la mari distanțe ca și $\frac{1}{r^2}$. Dar atunci, dacă am reuși să-l alegem pe φ_0 astfel încît potențialul φ să satisfacă și problemei (raportată față de reperul mobil $O'x'y'z'$)

$$\begin{cases} \Delta\varphi = 0 \text{ în } \Omega^{2)} \\ \left.\dfrac{d\varphi}{dn_1}\right|_{S_1} = \vec{V}'_{0'} \cdot \vec{n}_1|_{S_1} \\ \operatorname{grad}_{r'\to\infty} \varphi \to 0 \end{cases} \tag{4}$$

am avea soluția căutată a problemei (3').

Cum însă soluția problemei (4) este

$\varphi(M, t) = -\frac{R^3}{2r'^3} \vec{V}'_{0'} \cdot \vec{r}'$, ea fiind unic determinată, abstracție făcînd de o constantă aditivă, determinarea lui φ_0 revine la rezolvarea unei ecuații integrale liniare. Abordarea acestei ecuații s-ar putea face considerînd pentru funcția armonică φ_0 o dezvoltare după polinoamele armonice omogene $H_n(x, y, z)$[3]; într-adevăr se verifică prin calcul direct că dacă $\varphi_0 = H_n(x, y, z)$ avem

$$\varphi_1 = \frac{n}{n+1} \frac{R^{2n+1}}{r^{2n+1}} H_n(x, y, z),$$

iar dacă

$$\varphi_0 = \frac{H_n(x, y, z)}{r^n}$$

va rezulta

$$\varphi_1 = \frac{n+1}{n} \frac{1}{R^{2n+1}} H_n(x, y, z).$$

Construind atunci și pentru soluția φ, a problemei (4), dezvoltarea corespunzătoare în seria de polinoame armonice omogene, problema revine la rezolvarea unui sistem algebric în coeficienții numerici ai dezvoltării funcției φ_0.

Problema ar putea fi rezolvată mai direct observînd că condițiile impuse asupra funcției φ_0, în teorema sferei a lui P. W e i s s [6], ar putea fi „slăbite" dacă l-am defini pe φ_0 prin următoarele condiții:

— φ_0 este o funcție armonică regulată în interiorul sferei S_2;
— $\varphi_0|_{S_2} = f(\theta, \varphi)$, unde f este o funcție continuă dată pe S_2;
— φ_0 este o funcție armonică regulată în exteriorul sferei S_2 (domeniu conținînd și punctul de la infinit).

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[2]) În cazul rotației unei sfere, fluidul fiind ideal, $\vec{\omega}$ nu joacă un rol efectiv.

[3]) Principala dificultate ar consta în imposibilitatea realizării simultane pentru φ_0 atît a regularității în vecinătatea originii, cerută de enunțul teoremei sferei [6], cît și a constanței la mari distanțe, cerută de sensul fizic al problemei. Am putea depăși acest obstacol considerînd pe φ_0 ca o funcție armonică în stratul sferic de raze ε și A, unde ε este suficient de mic iar A suficient de mare, ceea ce ar implica o dezvoltare de forma $\varphi = \sum_{n=0}^{\infty} \left[\frac{1}{A^n} + \frac{\varepsilon^{n+1}}{r^{2n+1}} H_n(x, y, z) \right]$.

O funcţie φ_0 astfel definită ar reprezenta deci soluţia unei probleme Dirichlet interioare şi a unei probleme Dirichlet exterioare, racordate continuu prin datele comune $f(\theta, \varphi)$ pe frontiera S_2; cu alte cuvinte φ_0 ar fi armonică şi regulată în întregul spaţiu, mai puţin suprafaţa S_2 unde ea este doar continuă. Dar atunci, cu un φ_0 astfel definit, urmînd procedeul lui P. Weiss, vom putea determina o funcţie φ_1, armonică şi regulată atît în interiorul cît şi în exteriorul sferei S_2 şi care satisface pe suprafaţa S_2 (unde este doar continuă) condiţia:

$$\frac{d\varphi_1}{dn} + \frac{d\varphi_0}{dn}\bigg|_{S_2} = 0.$$

Fie acum $f(\theta, \varphi)$, restricţia lui φ, soluţia problemei (4), la sfera S_2. Rezolvînd atunci problema Dirichlet interioară pentru sfera S_2, cu datele $f(\theta, \varphi) = \varphi|_{S_2}$ sîntem conduşi, în mod univoc, la o funcţie $\tilde{\varphi}_0$ armonică şi regulată în S_2. Considerînd atunci prelungirea prin continuitate a funcţiei $\tilde{\varphi}_0$ în exteriorul sferei S_2 (de fapt soluţia problemei Dirichlet exterioare cu aceleaşi date pe contur), funcţia

$$\Phi_1 = \tilde{\varphi}_0 + \frac{R}{r}\,\tilde{\varphi}_0\left(\frac{R^2x}{r^2}, \frac{R^2y}{r^2}, \frac{R^2z}{r^2}\right) - \frac{2}{Rr}\int_0^R \lambda\tilde{\varphi}_0\left(\frac{\lambda^2x}{r^2}, \frac{\lambda^2y}{r^2}, \frac{\lambda^2z}{r^2}\right)d\lambda \qquad (5)$$

va satisface condiţiile la limită cerute atît pe sfera S_1 cît şi pe sfera S_2, fiind soluţia căutată a problemei (3′).

Rezolvînd analog şi problema la limită (3″) vom avea imediat potenţialul vitezelor Φ al mişcării fluide generate de rototranslaţia simetrică a cuplului de sfere simetrice şi deci al mişcării fluide induse de deplasarea unei sfere $S \equiv S_1 \equiv S_2$ în prezenţa peretelui nelimitat π.

În cazul particular al mişcărilor fluide ideale, incompresibile, irotaţionale, axial simetrice, mişcarea fluidă poate fi determinată prin funcţia de curent a lui Stokes, ψ_0. Considerînd atunci, de pildă, mişcarea fluidă, axial simetrică în jurul unei sfere rigide în prezenţa peretelui plan, nemărginit, fix π, perpendicular pe axa de simetrie a configuraţiei (un diametru al sferei), funcţia de curent a acestei mişcări va fi căutată în clasa funcţiilor [8]

$$\psi = \psi_0(r, \theta) - \frac{r}{R}\,\psi_0\left(\frac{R^2}{r}, \theta\right)$$

unde ψ_0 poate fi construită analog, ca soluţia unei probleme Dirichlet relativ la sferă, datele pe suprafaţa sferei fiind restricţii ale soluţiei unei alte probleme Dirichlet pentru semispaţiul superior $z_1 > 0$ (admiţînd că domeniul fizic ocupă chiar acest semispaţiu).

(Intrat în redacţie la 5 ianuarie 1979

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SUR L'ÉCOULEMENT FLUIDE TRIDIMENSIONNEL PRODUIT PAR LA ROTOTRANSLATION DANS LA MASSE DE FLUIDE D'UN CORPS RIGIDE EN PRÉSENCE D'UNE PAROI ILLIMITÉE

(Résumé)

On donne un bref aperçu sur des possibilitées d'étude pour l'influence des parois illimitées dans le cas des écoulements fluides idéaux tridimensionnels, en utilisant des techniques introduites déjà par l'auteur pour les mêmes problèmes mais dans le cas plan.

Dans la première partie du travail, utilisant pour la solution du problème Neumann, auquel on est conduit, une représentation par des potentiels newtoniens on obtient un système correspondant d'équations Fredholm de deuxième espèce, auquel on peut appliquer les méthodes de travail valables dans le cas plan.

Dans la deuxième partie on développe la méthode du couple des profils pour le cas de la paroi illimitée plane. Dans le cas particulier où le profil rigide est une sphère, la solution s'obtient directement par le théorème de la sphère dû à P. Weiss.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

FLOW OF A DUSTY GAS IN A ROTATING CHANNEL

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1. **Introduction.** Soo [1] initiated the study of the flow of Dusty Gases taking some assumptions regarding dust particles which considerably simplified the equations for velocity fields of the fluid and dust particles.

In the present investigation, we have studied the flow of a dusty gas through a channel with arbitrary time varying pressure gradient in a rotating frame of reference. The flow of a dusty gas in a rotating medium has some bearing on the pollution problems as well as on the motion of aerosol over the rotating earth. We have discussed the results for two particular cases, e.g. when the pressure gradient changes impulsively and when it changes in an accelerated manner. It is found that the velocities of the gas and the dust particles decrease with increase in either rotation or mass concentration of dust. However, the secondary velocity components increase with increase in mass concentration of dust when the pressure gradient changes in an accelerated manner.

2. **Equations of the Problem.** Consider the motion of a dusty gas within a parallel plate channel $Z = \pm L$ in a rotating frame of reference such that the channel and the gas are rotating in unision with a uniform angular velocity Ω about the z-axis. The channel is considered to be infinite, so that the physical variables are functions of Z and t only. The equation of continuty for the incompressible gas is $\vec{\nabla} \cdot \vec{q} = 0$ which gives $w = 0$ everywhere in the gas; $\vec{q} \equiv (u, v, w)$ being the velocity of the gas. The equation of conservation of mass for the dust particles is given by (Saffman [2])

$$\frac{\partial N}{\partial t} + \vec{\nabla} \cdot (N\vec{q}_p) = 0,$$

where N is the number-density of the dust particles, and $\vec{q}_p \equiv (u_p, v_p, w_p)$ is the velocity of a dust particle. This is satisfied, if we take $N = N_0$, a constant (cf. Saffman [2]) and $w_p = 0$. The equations of conservation of momentum for the gas and the dust particles in a rotating frame of reference are given by (cf. Gupta and Pop [3])

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} + 2\Omega v + \frac{KN_0}{\rho}(u_p - u), \tag{1}$$

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial z^2} - 2\Omega u + \frac{KN_0}{\rho}(v_p - v), \tag{2}$$

$$\frac{\partial u_p}{\partial t} = \frac{K}{m}(u - u_p) + 2\Omega v_p, \tag{3}$$

$$\frac{\partial v_p}{\partial t} = \frac{K}{m}(v - v_p) - 2\Omega u_p. \tag{4}$$

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In the above equations, m is the mass of a dust particle, K is the Stoke's resistance coefficient, p is the pressure of the gas, ν and ρ are respectively kinematic viscosity and density of the gas.

Introducing non-dimensional variables

$$\left.\begin{aligned} u_1 &= \frac{uL}{\nu}, \quad v_1 = \frac{vL}{\nu}, \quad u_{p_1} = \frac{U_pL}{\nu}, \quad v_{p_1} = \frac{v_pL}{\nu}, \\ T &= \frac{t\nu}{L^2}, \quad \eta = \frac{z}{L}, \quad \xi = \frac{x}{L}, \quad p^* = \frac{pL^2}{\rho\nu^2}, \end{aligned}\right\} \tag{5}$$

the above equations (1) — (4) become

$$\frac{\partial u_1}{\partial T} = F(T) + \frac{\partial^2 u_1}{\partial \eta^2} + 2Ev_1 + f\nu(u_{p_1} - u_1), \tag{6}$$

$$\frac{\partial v_1}{\partial T} = \frac{\partial^2 v_1}{\partial \eta^2} - 2Eu_1 + f\nu(v_{p_1} - v_1), \tag{7}$$

$$\frac{\partial u_{p_1}}{\partial T} = \nu(u_1 - u_{p_1}) + 2Ev_{p_1}, \tag{8}$$

$$\frac{\partial v_{p_1}}{\partial T} = \nu(v_1 - v_{p_1}) - 2Eu_{p_1}, \tag{9}$$

$$\left.\begin{aligned} \text{where } E &= \frac{\Omega L^2}{\nu} \text{ (rotation parameter)}, \\ f &= \frac{mN_0}{\rho} \text{ (mass concentration of the dust)}, \\ \nu &= \frac{KL^2}{m\nu} \text{ (relaxation time parameter of the dust)}, \end{aligned}\right\} \tag{10}$$

and
$$-\frac{\partial p^*}{\partial \xi} = F(T), \tag{11}$$

$F(T)$ being an arbitrary function of the time representing the sudden change in the pressure gradient which causes the motion.

Initial and boundary conditions can be written as

$$\left.\begin{aligned} u_1 = v_1 = u_{p_1} = v_{p_1} = 0 \quad &\text{for } T = 0, \\ u_1 = v_1 = 0 \quad \text{at } \eta = \pm 1 \quad &\text{for } T > 0, \\ \frac{\partial u_1}{\partial \eta} = \frac{\partial v_1}{\partial \eta} = 0 \text{ at } \eta = 0. & \end{aligned}\right\} \tag{12}$$

3. Solution of the Problem. Multiplying the equations (6) — (9) by $\cos a_n\eta$ where

$$a_n = (2n+1)\frac{\pi}{2}, \tag{13}$$

and the integrating within the limit 0 to 1, using the conditions (12), we get

$$\frac{d\bar{u}_1}{dT} = \frac{(-1)^n}{a_n}F(T) + 2E\bar{v}_1 - a_n^2\bar{u}_1 + f\nu(\bar{u}_{p_1} - \bar{u}_1), \tag{14}$$

$$\frac{d\bar{v}_1}{dT} = -2E\bar{u}_1 - a_n^2\bar{v}_1 + f\nu(\bar{v}_{p_1} - \bar{v}_1), \tag{15}$$

$$\frac{d\bar{u}_{p_1}}{dT} = \nu(\bar{u}_1 - u_{p_1}) + 2Ev_{p_1}, \tag{16}$$

$$\frac{d\bar{v}_{p_1}}{dT} = \nu(\bar{v}_1 - \bar{v}_{p_1}) - 2E\bar{u}_{p_1}, \tag{17}$$

where the finite cosine transforms are defined as

$$\left.\begin{aligned} \bar{u}_1(n, T) &= \int_0^1 u_1 \cos a_n\eta\, d\eta, \quad \bar{v}_1(n, T) = \int_0^1 v_1 \cos a_n\eta\, d\eta, \\ \bar{u}_{p_1}(n, T) &= \int_0^1 u_{p_1} \cos a_n\eta\, d\eta, \quad \bar{v}_{p_1}(n, T) = \int_0^1 v_{p_1} \cos a_n\eta\, d\eta. \end{aligned}\right\} \tag{18}$$

The inversion formula for the transform $\bar{u}_1 = \int_0^1 u_1 \cos a_n\eta\, d\eta$ can be shown to be

$$u_1 = 2\sum_{n=0}^{\infty} \bar{u}_1(n, T)\cos a_n\eta, \tag{19}$$

and similary for v_1, u_{p_1}, v_{p_1}.

From (14), (15),

$$\frac{dH}{dT} = \frac{(-1)^n}{a_n}F(T) - (a_n^2 + 2iE)H - f\nu(H - h), \tag{20}$$

and similarly from (16), (17),

$$\frac{dh}{dT} = \nu(H - h) - 2iEh, \tag{21}$$

where $H = \bar{u}_1 + i\bar{v}_1, \quad h = \bar{u}_{p_1} + iv_{p_1}.$ (22)

Taking Laplace Transforms of (20) and (21), we get

$$(s + a_n^2 + 2iE)\overline{H} = \frac{(-1)^n}{a_n}\overline{F} - fr(\overline{H} - \overline{h}), \tag{23}$$

$$(s + \nu + 2iE)\overline{h} = \nu\overline{H}, \tag{24}$$

where

$$\left.\begin{aligned} \overline{H} &= \int_0^\infty He^{-sT}\,dT, \quad \overline{h} = \int_0^\infty hc^{-sT}\,dT, \\ \overline{F} &= \int_0^\infty Fe^{-sT}\,dT. \end{aligned}\right\} \tag{25}$$

From (23), (24),

$$\overline{H} = \frac{(-1)^n(s + \nu + 2iE)}{a_n(s - a)(s - b)}, \tag{26}$$

$$\overline{h} = \frac{(-1)^n\,\nu\overline{F}}{a_n(s - a)(s - b)}, \tag{27}$$

where

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = -\frac{1}{2}\left[(a_n^2 + \nu + f\nu + 4iE) \mp \{(a_n^2 + \nu + f\nu)^2 - 4a_n^2\nu\}^{\frac{1}{2}}\right]. \tag{28}$$

Using convolution theorem for obtaining inverse Laplace Transforms of (26), (27), we get

$$H = \frac{(-1)^n}{a_n(b - a)}\int_0^T F(T - \lambda))[(b + \nu + 2iE)e^{b\lambda} - (a + \nu + 2iE)e^{a\lambda}]d\lambda, \tag{29}$$

$$h = \frac{(-1)^n\nu}{a_n(b - a)}\int_0^T (e^{b\lambda} - e^{a\lambda})\,F(T - \lambda)d\lambda. \tag{30}$$

Applying inversion formula (19) for the cosine transforms, equations (29), (30) become

$$u_1 + iv_1 = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{a_n(b-a)} \left\{ \int_0^T F(T-\lambda)[(b+\nu+2iE)e^{b\lambda} - (a+\nu+2iE)e^{a\lambda}]d\lambda \right\} \cos a_n\eta, \tag{31}$$

$$u_{p_1} + iv_{p_1} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{a_n(b-a)} \left\{ \int_0^T F(T-\lambda)(e^{b\lambda} - e^{a\lambda})d\lambda \right\} \cos a_n\eta. \tag{32}$$

The velocities given by (31) and (32) has been examined in detail for two particular cases of initial pressure gradient.

4. **Particular Cases.** *Case* I. Pressure gradient changes impulsively. Here we let

$$F(T) = \Delta H(T), \tag{33}$$

where Δ is a constant and $H(T)$ is Heaviside unit function defined by

$$\left.\begin{aligned} H(T) &= 0 \quad \text{for } T < 0, \\ &= 1 \quad \text{for } T > 0. \end{aligned}\right\} \tag{34}$$

Substituing the function F in (31) and (32) and evaluating the integrals, we get

$$u_1 + iv_1 = 2\Delta H(T)\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{a_n(b-a)ab} \{a(b+\nu+2iE)e^{bT} - b(a+\nu+2iE)a^{aT} + (b-a)(\nu+2iE)\} \cos a_n\eta \right], \tag{35}$$

$$u_{p_1} + iv_{p_1} = 2\Delta H(T)\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{a_n(b-a)ab} \{ae^{bT} - be^{aT} + (b-a)\} \cos a_n\eta \right]. \tag{36}$$

Separating real and imaginary parts from (35) and (36), we obtain

$$u_1 = 2\Delta H(T)\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{a_nR} \{(e^{-BT}a_3 - e^{-AT}a_4)\cos 2ET - (e^{-BT}b_3 - e^{-AT}b_4)\sin 2ET + Ra_2\} \cos a_n\eta \right], \tag{37}$$

$$v_1 = -2\Delta H(T)\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{a_nR} \{(e^{-BT}b_3 - e^{-AT}b_4)\cos 2ET + (e^{-BT}a_3 - e^{-AT}a_4)\sin 2ET - Rb_2\} \cos a_n\eta \right], \tag{38}$$

$$u_{p_1} = -2\Delta H(T) \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{a_n R} \left\{ \left(\frac{Be^{-BT}}{B^2+4E^2} - \frac{Ae^{-AT}}{A^2+4E^2} \right) \cos 2ET - \right.\right.$$

$$\left.\left. -2E\left(\frac{e^{-BT}}{B^2+4E^2} - \frac{e^{-AT}}{A^2+4E^2} \right) \sin 2ET - \frac{Ra_1}{a_1^2+b_1^2} \right\} \cos a_n\eta \right], \tag{39}$$

$$v_{p_1} = 2\Delta H(T) \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{a_n R} \left\{ \left(\frac{Be^{-BT}}{B^2+4E^2} - \frac{Ae^{-AT}}{A^2+4E^2} \right) \sin 2ET + \right.\right.$$

$$\left.\left. +2E\left(\frac{e^{-BT}}{B^2+4E^2} - \frac{e^{-AT}}{A^2+4E^2} \right) \cos 2ET - \frac{Rb_1}{a_1^2+b_1^2} \right\} \cos a_n\eta \right], \tag{40}$$

where

$$\left\{ \begin{array}{ll} R = b - a = -\{(a_n^2 + \nu + f\nu)^2 - 4a_n^2\nu\}^{\frac{1}{2}}, & \\ A = \frac{1}{2}\left[(a_n^2 + \nu + f\nu) - \{(a_n^2 + \nu + f\nu)^2 - 4a_n^2\nu\}^{\frac{1}{2}} \right], & \\ B = \frac{1}{2}\left[(a_n^2 + \nu + f\nu) + \{(a_n^2 + \nu + f\nu)^2 - 4a_n^2\nu\}^{\frac{1}{2}} \right], & \\ a_1 = a_n^2\nu - 4E^2, & b_1 = 2E(a_n^2 + \nu + f\nu), \\ a_2 = \frac{a_1\nu + 2Eb_1}{a_1^2 + b_1^2}, & b_2 = \frac{2Ea_1 - \nu b_1}{a_1^2 + b_1^2}, \\ a_3 = 1 - Aa_2 + 2Eb_2, & b_3 = 2Ea_2 + Ab_2, \\ a_4 = 1 - Ba_2 + 2Eb_2, & b_4 = 2Ea_2 + Bb_2. \end{array} \right. \tag{41}$$

The non-dimensional components of shear stress on the wall $\eta = 1$ in the x- and y- directions are respectively by

$$\tau_x = -2\Delta H(T) \sum_{n=0}^{\infty} \left[\frac{1}{R} \{ (e^{-BT}a_3 - e^{-AT}a_4) \cos 2ET - \right.$$

$$\left. -(e^{-BT}b_3 - e^{-AT}b_4) \sin 2ET + Ra_2 \} \right], \tag{42}$$

$$\tau_y = 2\Delta H(T) \sum_{n=0}^{\infty} \left[\frac{1}{R} \{ e^{-BT}b_3 - e^{-AT}b_4) \cos 2ET + \right.$$

$$\left. + (a_3e^{-BT} - a_4e^{-AT}) \sin 2ET - Rb_2 \} \right]. \tag{43}$$

The non-dimensional resultant shear stress on the wall is given by

$$\tau = \sqrt{\tau_x^2 + \tau_y^2}. \tag{44}$$

Case II. Pressure gradient changes in an accelerated manner. Here we let
$F(T) = \Delta T H(T)$. (45)

In (31) and (32), substituting for the function F and integrating, we get

$$u_1 + iv_1 = 2\Delta H(T) \sum_{n=0}^{\infty} \Big(\frac{(-1)^n}{a_n (b-a) a^2 b^2} \{(\nu + 2iE) T (b-a) ab + a^2 (b + \nu + 2iE) e^{bT} - b^2 (a + \nu + 2iE) e^{aT} + (b-a)[ab + (a+b)(\nu + 2iE)]\} \cos a_n \eta \Big), \tag{46}$$

$$u_{p_1} + iv_{p_1} = 2\Delta H(T) \sum_{n=0}^{\infty} \Big[\frac{(-1)^n}{a_n (b-a) a^2 b^2} \{T ab(b-a) + a^2 e^{bT} - b^2 e^{aT} + (b^2 - a^2)\} \cos a_n \eta \Big]. \tag{47}$$

Separating real and imaginary parts from (46) and (47), we obtain

$$u_1 = 2\Delta H(T) \sum_{n=0}^{\infty} \Big[\frac{(-1)^n}{a_n R} \{R T a_2 + a_5 - a_6 + R a_7\} \cos a_n \eta \Big], \tag{48}$$

$$v_1 = 2\Delta H(T) \sum_{n=0}^{\infty} \Big[\frac{(-1)^n}{a_n R} \{R T b_2 - b_5 + b_6 - R b_7\} \cos a_n \eta \Big], \tag{49}$$

$$u_{p_1} = 2\Delta H(T) \sum_{n=0}^{\infty} \Big[\frac{(-1)^n}{a_n R} \Big\{ \frac{R T a_1}{a_1^2 + b_1^2} + a_8 - a_9 + a_{10} \Big\} \cos a_n \eta \Big], \tag{50}$$

$$v_{p_1} = -2\Delta H(T) \sum_{n=0}^{\infty} \Big[\frac{(-1)^n}{a_n R} \Big\{ \frac{R T b_1}{a_1^2 + b_1^2} + b_8 - b_9 + b_{10} \Big\} \cos a_n \eta \Big], \tag{51}$$

where

$$a_5 = \frac{(\nu - B) e^{-BT}}{(B^2 + 4E^2)^2} \{(B^2 - 4E^2) \cos 2ET - 4BE \sin 2ET\},$$

$$b_5 = \frac{(\nu - B) e^{-BT}}{(B^2 + 4E^2)^2} \{(B^2 - 4E^2) \sin 2ET + 4BE \cos 2ET\},$$

$$a_6 = \frac{(\nu - A) e^{-AT}}{(A^2 + 4E^2)^2} \{(A^2 - 4E^2) \cos 2ET - 4AE \sin 2ET\},$$

$$b_6 = \frac{(\nu - A) e^{-AT}}{(A^2 + 4E^2)^2} \{A^2 - 4E^2) \sin 2ET + 4AE \cos 2ET\},$$

$$a_7 = \frac{a_1}{(a_1^2 + b_1^2)} \{1 - (A + B) a_2 + 4E b_2\} - \frac{b_1}{(a_1^2 + b_1^2)} \{(A + B) b_2 + 4E a_2\},$$

$$b_7 = \frac{b_1}{(a_1^2 + b_1^2)} \{1 - (A + B) a_2 + 4E b_2\} + \frac{a_1}{(a_1^2 + b_1^2)} \{(A + B) b_2 + 4E a_2\},$$

$$a_8 = \frac{e^{-BT}}{(B^2+4E^2)^2}\{(B^2-4E^2)\cos 2ET - 4BE\sin 2ET\},$$

$$b_8 = \frac{e^{-BT}}{(B^2+4E^2)^2}\{4BE\cos 2ET + (B^2-4E^2)\sin 2ET\},$$

$$a_9 = \frac{e^{-AT}}{(A^2+4E^2)^2}\{A^2-4E^2)\cos 2ET - 4AE\sin 2ET\},$$

$$b_9 = \frac{e^{-AT}}{(A^2+4E^2)^2}\{4AE\cos 2ET + (A^2-4E^2)\sin 2ET\},$$

$$a_{10} = \frac{A^2-4E^2}{(A^2+4E^2)^2} - \frac{B^2-4E^2}{(B^2+4E^2)^2},$$

$$b_{10} = \frac{4AE}{(A^2+4E^2)^2} - \frac{4BE}{(B^2+4E^2)^2}. \tag{52}$$

The non-dimensional components of shear stress on the wall $\eta = 1$ in the x- and y- direstions are given respectively by

$$\tau_x = -2\Delta H(T)\sum_{n=0}^{\infty}\left[\frac{1}{R}\{RTa_2 + a_5 - a_6 + Ra_7\}\right], \tag{53}$$

$$\tau_y = -2\Delta H(T)\sum_{n=0}^{\infty}\left[\frac{1}{R}\{RTb_2 - b_5 + b_6 - Rb_7\}\right]. \tag{54}$$

The non-dimensional resultant shear stress τ on the wall is given by

$$\tau = \sqrt{\tau_x^2 + \tau_y^2}. \tag{55}$$

5. **Discussion.** In order to discuss the effects of the concentration of the dust particles and of the rotation, we have presented the dimensionless velocities at the non-dimensional time $T = 0{\cdot}2$ in figs. 1 to 8. The results for the first case, when the pressure-gradient changes impulsively, are given in figs. 1 to 4. It is observed that for fixed value of the rotation parameter E, the velocities of the gas and the dust particles decrease with increase in mass concentration f on the dust particles. For a fixed f, the velocities of the gas as well as dust particles decrease with increase in rotation parameter E.

Figs. 5 to 8 present the velocity profiles for the second case, when the pressure-gradient changes in an accelerated manner. It is observed that the effect of the rotation parameter E is to decrease the velocity as in the first case. The secondary velocity components increase with increase in the mass concentration of the dust particles. On the other hand, the primary velocity components decrease with the increase in f as in the first case.

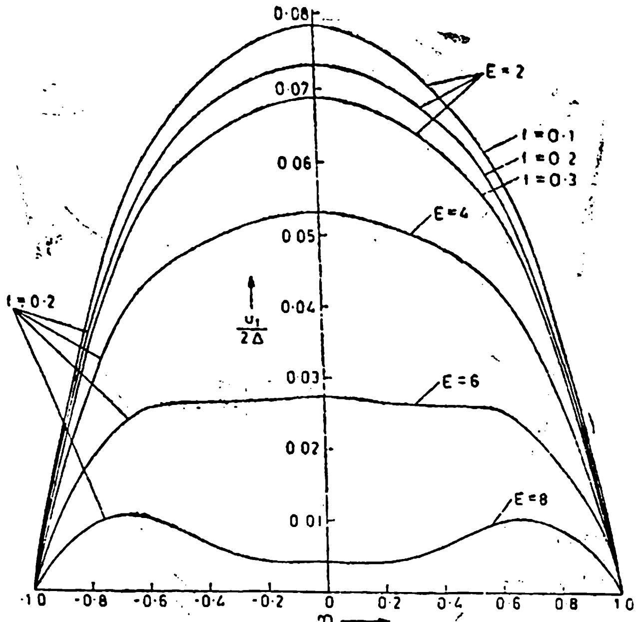


Fig. 1. Primary Velocity Profiles of gas when pressure gradient changes impulsively, for different values of f, E at $T = 0.2$.

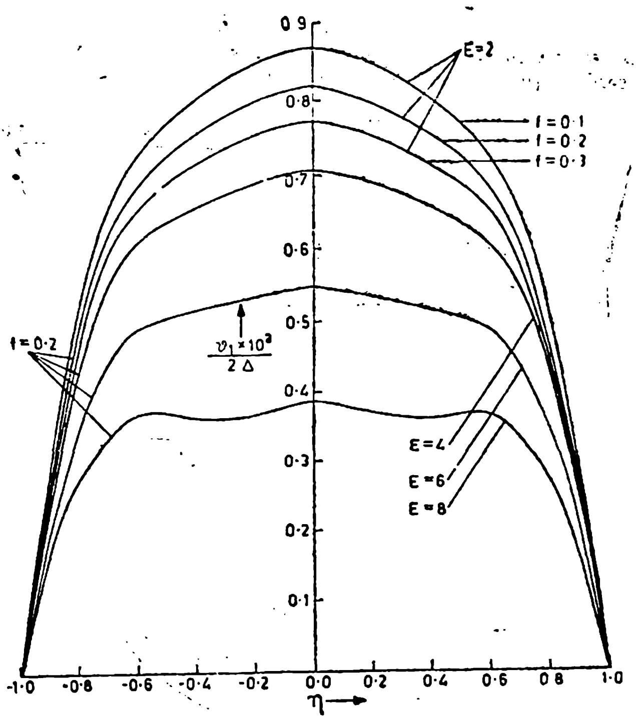


Fig. 2. Secondary Velocity Profiles of gas when pressure gradient. changes impulsively, for different values of f, E at $T = 0.2$.

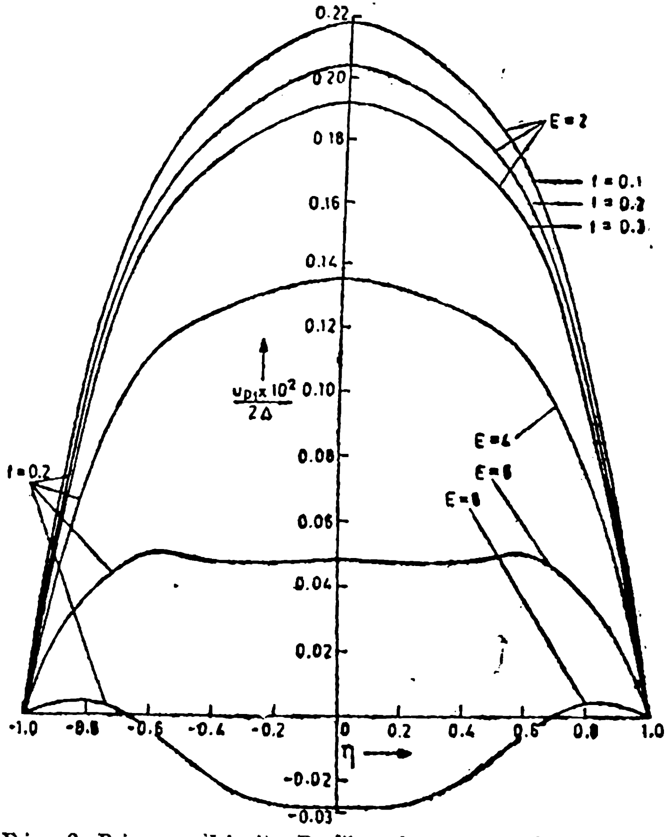
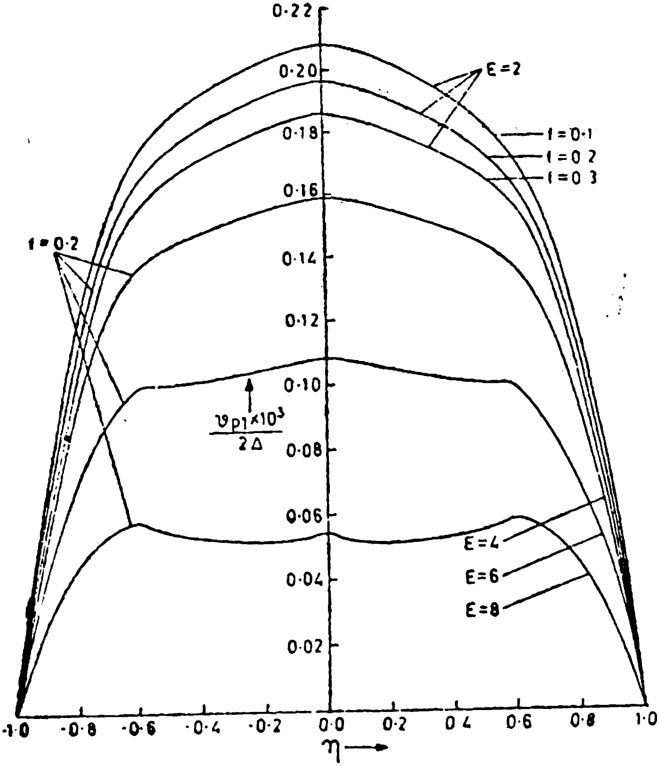


Fig. 3. Primary Velocity Profiles of particles when pressure gradient changes impulsively, for different values of f, E at $T = 0.2$.



F i g. 4. Secondary Velocity Profiles of particles when pressure gradient changes impulsively, for different values of f, E at $T = 0.2$.

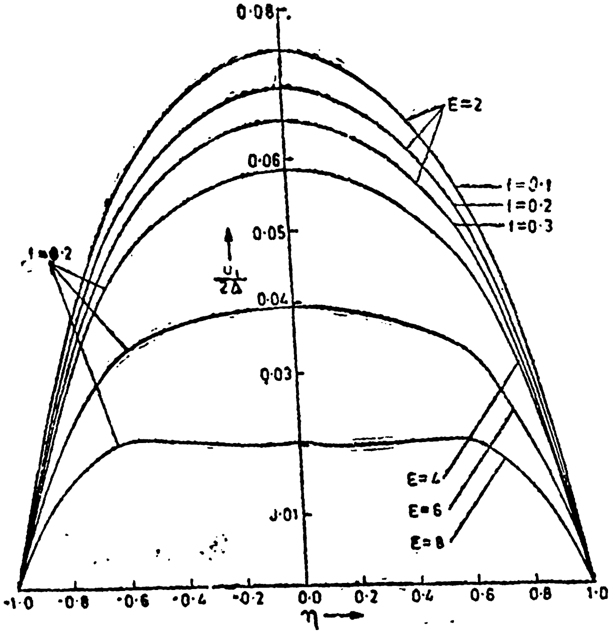


Fig 5. Primary Velocity Profiles of gas when pressure gradient changes in an accelerated manner, for different values of f, E at $T = 0.2$.

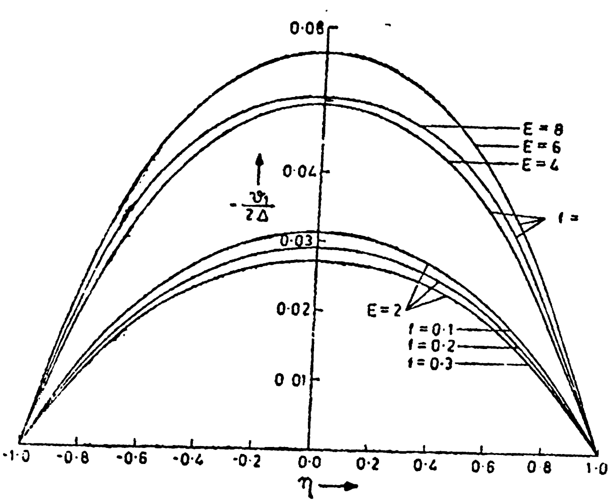
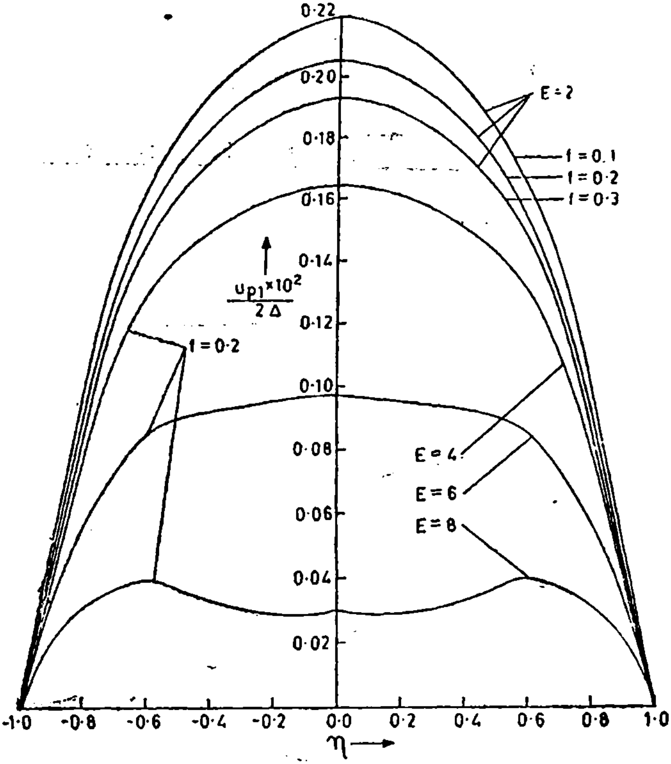


Fig. 6. Secondary Velocity Profiles of gas when pressure gradient changes in an accelerated manner, for different values of f, E at $T = 0.2$.



F i g. 7. Primary Velocity Profiles of particles when pressure gradient changes in an accelerated manner, for different values of f, E at $T = 0.2$.

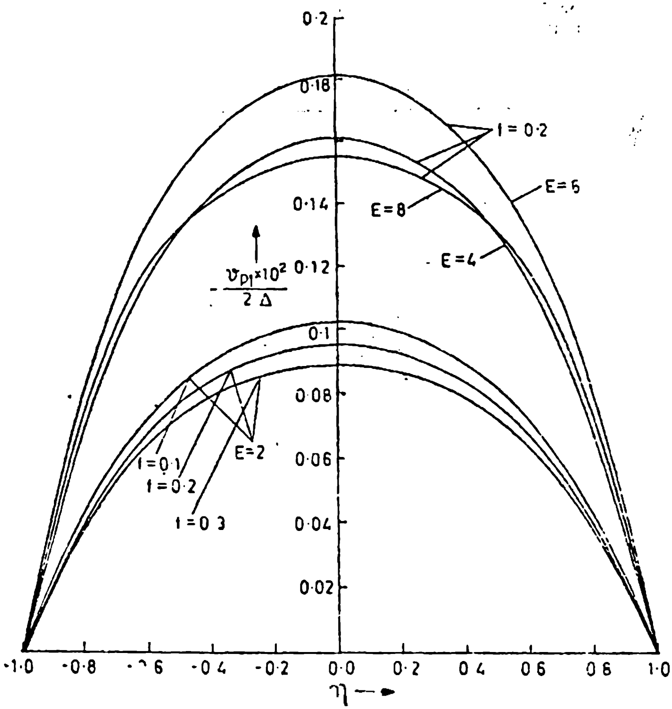


Fig. 8. Secondary Velocity Profiles of particles when pressure gradient changes in an accelerated manner, for different values of f, E at $T = 0.2$.

The non-dimensional resultant shear stress τ at the wall $\eta = 1$ have been presented in tables 1 and 2 corresponding to the two cases. In either case, it is found that τ decreases with increase in either f or E.

Table 1

The non-dimensional resultant shear stress τ for various values of f and E ($T = 0.2$)

$E = 2.0$				$f = 0.2$			
f	0.1	0.2	0.3	E	4.0	6.0	8.0
τ	0.20328	0.19396	0.18565	τ	0.18109	0.16110	0.13600

Table 2

The non-dimensional resultant shear stress τ for various values of f and E ($T = 0.2$)

$E = 0.2$				$f = 0.2$			
f	0.1	0.2	0.3	E	4.0	6.0	8.0
τ	0.02551	0.02442	0.02344	τ	0.02350	0.02206	0.02023

Acknowledgement. Authors are thankful to Prof. A.S. Gupta for this constant encouragement. One of the authors (S.K.N.) also wishes to express his thanks to U.G.C. for granting F.I.P. Fellowship and to Bolpur College for granting leave.

(Received August 30, 1979)

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MIŞCAREA UNUI PRAF DE GAZ ÎN ROTAŢIE

(Rezumat)

În lucrare se prezintă o soluţie exactă a mişcării unui fluid, de genul prafului de gaz, într-un canal în rotaţie datorită unui gradient de presiune care variază arbitrar în timp, folosind metoda transformatei Laplace.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

CONSIDERAŢII NUMERICE PENTRU MIŞCĂRI CU REACŢII CHIMICE PESTE UN DISC ÎN ROTAŢIE

IOAN STAN şi CĂLIN GHEORGHIU*

În două lucrări anterioare [1] şi [2] am încercat stabilirea prin metode numerice a profilului concentraţiei pentru scurgeri cu reacţii chimice peste un disc în rotaţie.

Lucrarea de faţă are drept scop studiul comparativ al metodelor descrise în [1] şi [2].

Astfel, scurgerea peste o suprafaţă este deseori însoţită de procese fizico-chimice în timpul cărora, în general, corpul îşi păstrează forma şi structura. În aceste condiţii difuzia componenţilor către şi de la suprafaţă are o mare importanţă. Reacţiile chimice pot avea .loc numai la limita de separare a celor două faze — reacţii eterogene, sau în masa de fluid — reacţii omogene, datorită cărora concentraţia variază rapid în această zonă, difuzia reactanţilor printre produşii de reacţie avînd aici un rol foarte important.

Unele lucrări recente investighează scurgerea peste o suprafaţă cu reacţii chimice, luînd în consideraţie numai difuzia moleculară în stratul limită. Aşa cum am arătat în [3, 4] barodifuzia este un agent mecanic care împreună cu difuzia moleculară cauzează modificări în profilul concentraţiei reactanţilor.

Vom stabili profilul concentraţiei pentru scurgeri peste un disc în rotaţie, cu ajutorul unei metode cu diferenţe finite şi comparativ şi cu metoda tirului cu pas variabil, luînd în considerare efectul barodifuziei, pentru un amestec fluid bicomponenţial atît în cazul reacţiilor eterogene cît şi omogene.

1. **Ecuaţiile ce guvernează fenomenul.** Fluidul fiind considerat incompresibil ecuaţia continuităţii are forma:

$$\nabla \cdot \vec{v} = 0 \tag{1.1}$$

unde $\vec{v}$ reprezintă viteza scurgerii. Presupunînd scurgerea permanentă a unui fluid vîscos ecuaţia mişcării a lui Navier-Stokes este:

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p + \nu \Delta \vec{v} \tag{1.2}$$

unde p presiunea, ν viscozitatea cinematică iar ρ densitatea.

Dacă notăm cu c concentraţia unui component, ecuaţia continuităţii acestui component este

$$\frac{dc}{dt} = - \nabla \cdot \vec{i} \tag{1.3}$$

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unde $\vec{i}$ este fluxul de masă al componentului şi este dat de relaţia

$$\vec{i} = -[D\nabla c - k\nabla p] \tag{1.4}$$

D fiind coeficientul de difuzie moleculară iar k de barodifuzie. Presupunînd coeficienţii fizici constanţi înlocuirea lui (1.4) în (1.3) ne dă:

$$(\vec{v} \cdot \nabla)c = D\Delta c - k\Delta p \tag{1.5}$$

ecuaţia difuziei unui component cu luarea în considerare a efectului barodifuziei.

Alegînd axa Oz în lungul axei de rotaţie a nunui disc plan infinit, ce se roteşte cu viteza unghiulară constantă ω, condiţia de aderenţă la suprafata discului impune următoarele condiţii la limită, considerînd coordonate cilindrice

$$z = 0, \quad v_r = 0, \quad v_\theta = r\omega, \quad v_z = 0$$
$$z = \infty, \quad v_r = 0, \quad v_\theta = 0, \quad v_z = -m$$

unde m este o constanţă ce se va determina ulterior.

Scurgerea hidrodinamică în jurul unui disc a fost studiată de W. G. C o c h r a n [5]. El a căutat soluţii de forma

$$v_r = r\omega F(\zeta), \qquad v_\theta = r\omega G(\zeta), \qquad v_z = \sqrt{\nu\omega}H(\zeta), \qquad p = -\rho\omega\nu P(\zeta) \tag{1.6}$$

unde

$$\zeta = \sqrt{\frac{\omega}{\nu}}\, z \tag{1.7}$$

În vecinătatea discului F, G, H sînt de forma

$$\begin{aligned} F &= a_0\zeta - \frac{1}{2}\zeta^2 - \frac{1}{3}b_0\zeta^3 - \frac{1}{12}b_9^2\zeta^4 - \frac{1}{60}a_0\zeta_5 + \left(\frac{1}{360} - \frac{a_0 b_0}{90}\right)\zeta_6 + \dots \\ G &= 1 + b_0\zeta + \frac{1}{3}a_0\zeta^3 + \frac{1}{12}(a^0 b^0 - 1)\zeta^4 - \frac{b_0}{15}\zeta_5 - \left(\frac{a^0}{90} + \frac{b_0}{45}\right)\zeta_6 + \dots \\ H &= -a_0\zeta^2 + \frac{1}{3}\zeta^3 + \frac{1}{6}b_0\zeta^4 + \frac{1}{30}b_0\zeta^5 + \frac{1}{180}a_0\zeta_6 + \dots \end{aligned} \tag{1.8}$$

unde $a_0 = 0{,}510$ iar $b_0 = -0{,}616$.

Grosimea stratului limită hidrodinamic este

$$\delta_H = 3{,}6\sqrt{\frac{\nu}{\omega}} \tag{1.9}$$

astfel că extremităţii superioare a stratului limită hidrodinamic creat peste un disc în rotaţie uniformă îi corespunde coordonata adimensională $\zeta = 3{,}6$.

2. Ecuațiile difuziei. Ecuația (1.5) în coordonate cilindrice se scrie [6]

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r} \cdot \frac{\partial c}{\partial \theta} + v_z \frac{\partial c}{\partial z} = D\left(\frac{\partial^2 c}{\partial z^2} + \frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial c}{\partial r} + \frac{1}{r} \cdot \frac{\partial^2 c}{\partial \theta^2}\right) - k \frac{\partial^2 p}{\partial z^2} \quad (2.1)$$

cu condițiile la limită

$$z = 0, \quad c = 0; \quad z = \infty, \quad c = c_0 \quad (2.2)$$

considerînd că ecuația chimică este rapidă.

Presupunînd că soluția este de forma

$$c = c(z)$$

și introducînd noua variabilă ζ, obținem ecuația diferențială

$$c'' - Sc \cdot H(\zeta) \quad c' + KSc \cdot f(\zeta) = 0 \quad (2.3)$$

unde $K = k\rho\omega$ iar $Sc = \frac{\nu}{D}$ și

$$f(\zeta) = P'' = H'^2(\zeta) + H(\zeta) \cdot H''(\zeta) - H'''(\zeta)$$

Încazul în care luăm în considerare și o reacție omogenă care are loc în masa fluidului, în membrul stîng al ecuației (2.3) apare un termen de forma $-M \cdot c(\zeta)$, unde $0,01 \leq M \leq 0,1$ este un termen ce caracterizează apariția reacțiilor chimice omogene. Avem astfel ecuația reacțiilor omogene

$$c'' - Sc \cdot H(\zeta) \cdot c' - M \cdot c + K \cdot Sc \cdot f(\zeta) = 0 \quad (2.4)$$

Grosimea stratului limită de difuzie este dat de relația [7]

$$\delta_0 = 1,61 \left(\frac{D}{\nu}\right)^{1/3} \sqrt{\frac{\nu}{\omega}},$$

care pentru numere Schmidt cuprinse între 0,1 și 10^3 poate fi aproximată de valoarea

$$\delta_0 = 0,05 \cdot \delta_H$$

care arată că grosimea stratului limită de difuzie creat la scurgerea peste un disc în rotație uniformă reprezintă 5% din grosimea stratului limită hidrodinamic. Ținînd cont de relația (1.9) avem

$$\delta_0 = 0,18 \sqrt{\frac{\nu}{\omega}}$$

deci extremității superioare a stratului limită de difuzie îi corespunde coordonata adimensională

$$\zeta_0 = 0,18$$

Limitîndu-ne la studiul profilului concentrației în stratul limită de difuzie, condițiile la limită (2.2) devin

$$\zeta = 0, \quad c = 0; \quad \zeta = 0,18, \quad c = c_0 \quad (2.5)$$

astfel că scurgerea unui fluid vîscos peste un disc în rotație uniformă conduce, în ceea ce privește reacțiile chimice, la o problemă bilocală liniară.

3. O metodă cu diferenţe pentru calculul profilului concentraţiei. Problemei bilocale liniare (2.4)–(2.5) i se ataşează schema cu diferenţe

$$\frac{c_{j+1} - 2c_j + c_{j-1}}{h^2} - Sc \cdot H(\zeta_j) \frac{c_{j+1} - c_{j-1}}{2h} - Mc_j + K \cdot Sc \cdot f(\zeta_j) = 0, \quad (3.1)$$

unde $j = \overline{1,N}$, $h = 0{,}18/(N + 1)$ pasul reţelei iar $H(\zeta)$ este dat de (1.8) trunchiat la termenul în ζ^6.

Condiţiile la limită (2.5) se transcriu în forma

$$c_0 = 0, \quad c_{N+1} = 1 \quad (3.2)$$

Schema cu diferenţe (3.1) cu condiţiile (3.2) se scrie în forma matricială astfel

$$A \cdot C = r, \quad (3.3)$$

unde

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{pmatrix}, \quad r = \begin{pmatrix} f(\zeta_1) \\ f(\zeta_2) \\ \vdots \\ f(\zeta_N) \end{pmatrix} (-K \cdot Sc)\left(-\frac{h^2}{2}\right) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -2 + H(\zeta_N) \end{pmatrix}$$

$$A = \begin{pmatrix} a_1 & -d_1 & 0 & \dots & 0 & 0 & 0 \\ -b_1 & a_2 & -d_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -b_{N-1} & a_{N-1} & -d_{N-1} \\ 0 & 0 & 0 & \dots & 0 & -b_N & a_N \end{pmatrix} \quad (3.4)$$

iar elementele a_j, b_j şi d_j ale matricii A au forma

$$a_j = 1 + \frac{h^2}{2} M, \quad b_j = \frac{1}{2}\left(1 + \frac{h}{2} H(\zeta_j)\right), \quad d_j = \frac{1}{2}\left(1 - \frac{h}{2} H(\zeta_j)\right) \quad (3.5)$$

Pentru $h = 0{,}02$, numărul Schmidt $Sc = 1$, constantele $K = 0{,}01$, $M = 0{,}1$ sint satisfăcute condiţiile [8]

$$\frac{h}{2} |H(\zeta_j)| > 1, \qquad j = \overline{1,N} \quad (3.6)$$

şi

$$\begin{gathered} |a_j| > |d_j| > 0, \\ |a_j| > |b_j| + |d_j| \qquad 2 \leqslant j \leqslant N - 1 \\ |a_N| > |b_N| > 0 \end{gathered} \quad (3.7)$$

deci schema cu diferenţe are o soluţie unică.

Pentru diferenţele dintre soluţia exactă şi aproximarea teoretică (eroarea de trunchiere) respectiv soluţia exactă şi aproximarea numerică (eroarea locală de rotunjire) au loc majorările

$$|c(\zeta_j) - c_j| \leqslant h^2 \left(\frac{M_4 + 2p^* M_3}{12 \cdot Q^*}\right) \qquad j = \overline{0, N+1}$$

$$|c(\zeta_j) - C_j| \leqslant h^2 \frac{M_4 + 2p^* M_3}{12 \cdot Q_*} + \frac{1}{h^2}\left(\frac{2}{Q_*}\right) \qquad j = \overline{1, N+1}$$

unde $Q = 0{,}1$; $p = 0{,}03$; $M_3 = \max\limits_{\zeta \in [0,1]} |c^{\mathrm{III}}(\zeta)|$, $M_4 = \max\limits_{\zeta \in [0,1]} |c_{\mathrm{IV}}(\zeta)|$, $c(\zeta)$ soluţia problemei (2.4)—(2.5), c_j soluţia ecuaţiei cu diferenţe (3.1)—(3.2) iar C_j aproximaţia numerică a soluţiei $c(\zeta)$ pe nodurile ζ_j $j = \overline{1, N}$.

Schema cu diferenţe folosită are deci ordinul doi de exactitate.

Sistemul (3.3) devine în condiţiile de mai sus un sistem algebric liniar de 8 ecuaţii cu 8 necunoscute. El a fost rezolvat fiind programat astfel încît să se apeleze în programul principal, scris în limbaj FORTRAN, la două subprograme aflate în biblioteca matematică a calculatorului FELIX C256 şi anume RESOL şi REBAN. Calculele au fost repetate apoi pentru $M = 0$, cazul reacţiilor omogene.

Tabel 1

Concentraţia c/c_0 pentru scurgeri cu reacţie

η	eterogenă	omogenă
0,00	0,000000	0,000000
0,02	0,110985	0,111044
0,04	0,222050	0,222164
0,06	0,333186	0,333346
0,08	0,444375	0,444569
0,10	0,555596	0,555803
0,12	0,666812	0,667011
0,14	0,777983	0,778149
0,16	0,889064	0,889165
0,18	1,000000	1,000000

Valorile concentraţiei pe cele 10 noduri folosite la integrare sînt date în tabelul 1, ele reprezentînd concentraţia în diferite puncte ale stratului limită de difuzie, cînd în acest strat nu au loc decît reacţii eterogene (prima coloană), spre deosebire de valorile din a doua coloană care semnifică concentraţia în aceleaşi puncte ale stratului limită de difuzie, dar în care au loc reacţii atît omogene cît şi eterogene.

Se observă că, deşi mare coeficientul $M(M = 0{,}1)$, reacţia omogenă influenţează foarte puţin profilul concentraţiei în stratul limită de difuzie, valorile corespunzătoare ale concentraţiei pentru cele două cazuri diferind doar de la zecimala a patra din cele şase exacte obţinute.

4. Metoda tirului cu pas variabil pentru calculul profilului concentraţiei. Această metodă constă în a rezolva problema bilocală liniară neomogenă (2.4)–(2.5) transformînd-o în două probleme cu condiţii iniţiale

$$\begin{cases} c'' - Sc \cdot H(\zeta) \cdot c' - M \cdot c = -K \cdot Sc \cdot f(\zeta) & (4.1) \\ c(0) = 0, \quad c'(0) = 0 & (4.1a) \end{cases}$$

şi

$$\begin{cases} c'' - Sc \cdot H((\zeta) \cdot c' - M \cdot c = 0 & (4.2) \\ c(0) = 0, \quad c'(0) = 1 & (4.2a) \end{cases}$$

care integrate numeric ne dau pe $C_{pj} + 0 \cdot C_{1j}$ respectiv C_{2j}, $j = \overline{0,N}$; soluţia problemei (2.4)–(2.5) pe nodurile

$$\zeta_0 = 0, \quad \zeta_j = \zeta_0 + j \cdot h; \quad j = 0, 1, \ldots, N \quad h = (0{,}18 - 0)/N \tag{4.3}$$

fiind dată de relaţia

$$C_j = C_{pj} + 0 \cdot C_{1j} + S_k \cdot C_{2j} \qquad j = \overline{0,N} \tag{4.4}$$

unde

$$S_k = (1 - C_{pN} - 0 \cdot C_{1N})/C_{2N} \tag{4.5}$$

Condiţiile la limită se reflectă asupra aproximaţiilor numerice în forma

$$C_0 = 0, \qquad C_N = 1,$$

Metoda tirului cu pas variabil [2] constă în continuare în a rezolva cele două probleme cu valori iniţiale (4.1)–(4.1a) şi (4.2)–(4.2a) transformîndule în cîte un sistem de două ecuaţii diferenţiale cu două funcţii necunoscute de ordinul întîi. Pasul reţelei va varia (se va înjumătăţi) în aşa fe lincît în nodurile în care la un moment dat procedura se opreşte soluţia să fie aflată cu precizia dorită, impusă apriori. În acest fel se poate întîmpla ca nodurile pe care s-au calculat soluţiile celor două sisteme de ecuaţii diferenţiale provenind din transformarea lui (4.1)–(4.1a) şi respectiv (4.2)–(4.2a) să nu coincidă.

Dificultatea se poate înlătura folosind formule de interpolare cu un grad de exactitate (dorit) potrivit pentru a găsi valorile aproximative ale soluţiilor pe nodurile ce nu coincid.

Cu relaţia (4.4) se poate afla atunci valoarea aproximativă a soluţiei problemei (2.4)–(2.5) cu gradul de exactitate dorit.

Pentru rezolvarea unui sistem de două ecuaţii diferenţiale cu două funcţii necunoscute căruia i se ataşează condiţii iniţiale se poate folosi, de exemplu, un program existent în biblioteca matematică a calculatorului Felix C256, program ce rezolvă o astfel de problemă printr-o metodă Runge-Kutta modificată de Gill, folosind pasul variabil pentru obţinerea preciziei dorite.

Cazul în care în masa de fluid nu au loc reacţii omogene se prezintă ca un caz particular al celui discutat mai sus ($M = 0$).

Pentru a rezolva ecuaţiile stratului limită de difuzie cu şi fără reacţii omogene avem de rezolvat următoarele patru sisteme de ecuaţii diferenţiale:

$$\begin{cases} C_1'(\zeta) = C_2(\zeta) \\ C_2'(\zeta) = Sc \cdot H(\zeta) \cdot C_2(\zeta) + M \cdot C_1(\zeta) - K \cdot Sc \cdot f(\zeta) \end{cases}$$

cu condiţii iniţiale

$$C_1(0) = 0, \quad C_2(0) = 0$$

$$\begin{cases} C_1'(\zeta) = C_2(\zeta) \\ C_2'(\zeta) = Sc \cdot H(\zeta) \cdot C_2(\zeta) + M \cdot C_1(\zeta) \end{cases}$$

cu

$$C_1(0) = 0, \quad C_2(0) = 1$$

$$\begin{cases} C_1'(\zeta) = C_2(\zeta) \\ C_2'(\zeta) = Sc \cdot H(\zeta) \cdot C_2(\zeta) - K \cdot Sc \cdot f(\zeta) \end{cases}$$

cu

$$C_1(0) = 0, \quad C_2(0) = 0$$

şi în fine

$$\begin{cases} C_2'(\zeta) = Sc \cdot H(\zeta) \cdot C_2(\zeta) \\ C_1'(\zeta) = C_1(\zeta) \end{cases}$$

cu

$$C_1(0) = 0, \quad C_2(0) = 1$$

Rezultatele obţinute în urma rulării programelor sînt date mai jos

Tabel 2

Concentraţia c/c_0 pentru scurgeri cu reacţie

ζ	eterogenă	omogenă
0,00	0,000000	0,000000
0,02	0,110004	0,110004
0,04	0,220016	0,220022
0,06	0,330036	0,330055
0,08	0,440064	0,440111
0,10	0,550100	0,550193
0,12	0,660144	0,660303
0,14	0,770201	0,770454
0,16	0,880267	0,880631
0,17	0,935300	0,935751
0,18	1,000000	1,000000

Comparînd tabelele 1 şi 2 se remarcă o bună coincidenţă a rezultatelor obţinute prin cele două metode.

De asemenea se observă încă o dată influenţa mică a reacţiei omogene asupra profilului concentraţiei în stratul limită de difuzie.

(Intrat în redacţie la 12 noiembrie 1979)

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NUMERICAL METHODS FOR THE FLOWS WITH CHEMICAL REACTIONS OVER A ROTATING DISK

(Summary)

Numerical methods are applied to the establishment of concentration profile in boundary layer. Barodiffusion equations are applied in the boundary layer of a rotating disk and the presence of a chemical reaction between fluid and disk are considered. Heterogene and homogene reactions are discussed.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

ASUPRA APROXIMĂRII FUNCŢIILOR DE DOUĂ ŞI MAI MULTE VARIABILE PRIN OPERATORI LINIARI ŞI POZITIVI

C. MANOLE

1. Fie $R_+ = (0, \infty)$ şi $R^+ = [0,\infty)$. Să notăm cu $\mathcal{E}_s$ spaţiul liniar al funcţiilor $f: R^s_+ \to R$, de tip exponenţial. Mai precis, $f \in \mathcal{E}_s$ dacă şi numai dacă există constantele $A \in R_+$ şi $B \in R^s_+$ astfel încît pentru orice $x \in R^s_+$ să avem

$$|f(x)| < A \cdot e^{Bx}$$

Prin $S_n: \mathcal{E}_1 \to \mathcal{E}_1$ notăm operatorul lui Favard-Szász

$$(S_nf)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1}$$

iar prin $S_{m,n}: \mathcal{E}_2 \to \mathcal{E}_2$ notăm operatorul

$$(S_{m,n}f)(x, y) = e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}f\left(\frac{i}{m}, \frac{j}{n}\right) \tag{2}$$

În [4] se dau expresii ale restului din formulele de aproximare ale funcţiilor de una şi două variabile prin operatorii S_n şi $S_{m,n}$. Unul dintre rezultatele găsite în lucrarea citată este reprezentarea restului din formula de aproximare

$$f(x) = (S_nf)(x) + (R_nf)(x)$$

cu diferenţe divizate. Mai precis,

$$(R_nf)(x) = -\frac{x}{n}\cdot e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\left[x, \frac{k}{n}, \frac{k+1}{n}; f\right]. \tag{3}$$

Extinderea la două variabile se poate face în modul următor:

TEOREMA 1. *Restul $(R_{m,n}f)(x, y)$ din formula de aproximare*

$$f(x,y) = (S_{m,n}f)(x, y) + (R_{m,n}f)(x, y)$$

este dat de

$$\begin{aligned}(R_{m,n}f)(x, y) = &-\frac{x}{m}\cdot e^{-mx}\sum_{i=0}^{\infty}\frac{(mx)^i}{i!}\left[x, \frac{i}{m}, \frac{i+1}{m}; f\right]_x \\ &-\frac{y}{n}\cdot e^{-ny}\sum_{j=0}^{\infty}\frac{(ny)^j}{j!}\left[y, \frac{j}{n}, \frac{j+1}{n}; f\right]_y \\ &-\frac{xy}{mn}\cdot e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\begin{bmatrix} x, & \frac{i}{m}, & \frac{i+1}{m} & \\ & & & ; f \\ y, & \frac{j}{n}, & \frac{j+1}{n} & \end{bmatrix}\end{aligned} \tag{4}$$

unde

$$\left[\begin{matrix} x, \frac{i}{m}, \frac{i+1}{m} \\ y, \frac{j}{n}, \frac{j+1}{n} \end{matrix}; f\right] = \left[x, \frac{i}{m}, \frac{i+1}{m}; \left[y, \frac{j}{n}, \frac{j+1}{n}; f(x, y)\right]_y\right]_x$$

este diferenţa divizată bidimensională a funcţiei $f(x, y)$ pe punctele puse în evidenţă.

Această reprezentare rezultă din (3) şi din formula (11.3) dată de către D. D. S t a n c u (a se vedea [3] p. 159).

TEOREMA 2. *Operatorul $S_{m,n}$ se poate reprezenta prin*

$$(S_{m,n}f)(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\begin{matrix} 0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{i}{m} \\ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{j}{n} \end{matrix}; f\right] x^i y^j \tag{5}$$

Demonstrarea acestui rezultat se poate face dezvoltînd în serie funcţia $e^{-(mx+ny)}$ şi punînd în evidenţă coeficientul lui x^iy^j din produsul seriilor.

Menţionăm că în cazul operatorului lui Favard-Szász, S, un rezultat analog cu cel de la (5) a fost stabilit de către A. L u p a ş în [1].

2. În lucrarea [2] a fost introdus operatorul

$$(L_n f)(x) = \sum_{k=0}^{\infty}\left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k}{n}; f\right]\cdot A_{k,n}(x)$$

ataşat unei funcţii $f \in \mathcal{E}_1$, unde pentru fiecare n fixat, polinoamele $A_{k,n}(x)$ verifică relaţia

$$A_{k,n}(x) = k \cdot A^{k-1,n}(x) \quad n = 1, 2, \ldots$$

Acest operator s-a folosit apoi pentru aproximarea funcţiei f presupusă continuă pe $[0,a]$, a fiind arbitrar, dar pozitiv. Ne propunem, în continuare, să dăm o extindere la două variabile a unor rezultate stabilite în lucrarea citată.

Pe spaţiul liniar $\mathcal{E}_2$ se definesc operatorii

$$L_{m,n}: f \to L_{m,n}f$$

prin

$$(L_{m,n}f)(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\begin{matrix} 0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{i}{m} \\ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{j}{n} \end{matrix}; f\right] A_{m,n}^{i,j}(x, y) \tag{6}$$

unde $A_{m,n}^{i,j}(x,y)$ sînt polinoame de două variabile.

Fie $K(t,s)$ o funcţie nenegativă, definită şi integrabilă pe $D = [0,1] \times [0,1]$ cu proprietatea că

$$\iint_D K(t,s)dtds = 1$$

În cele ce urmează

$$A_{m,n}^{i,j}(x, y) = \iint_D K(t, s)\left(x + \frac{t}{m}\right)^i \left(y + \frac{s}{n}\right)^j dtds \tag{7}$$

Şirul $(A_{m,n}^{i,j})_{i,j=1}^{\infty}$ este un şir de polinoame Appell de două variabile în sensul că

$$\frac{\partial A_{m,n}^{i,j}(x, y)}{\partial x} = i \cdot A_{m,n}^{i-1,j}(x, y) \quad \text{şi} \quad \frac{\partial A_{m,n}^{i,j}(x, y)}{\partial y} = j \cdot A_{m,n}^{i,j-1}(x, y)$$

pentru fiecare n şi m numere naturale fixate.

TEOREMA 3. *Operatorii $L_{m,n}$ definiţi la (6) sînt liniari şi pozitivi, cu proprietăţile:*

$$(L_{m,n}e_{0,0})(x, y) = 1 \tag{8}$$

$$(L_{m,n}e_{1,0})(x, y) = x + \frac{1}{m}\iint_D t \cdot K(t, s)dtds \tag{9}$$

$$(L_{m,n}e_{0,1})(x, y) = y + \frac{1}{n}\iint_D s \cdot K(t, s)dtds \tag{10}$$

$$\begin{aligned}(L_{m,n}e_{2,2})(x, y) &= x^3 + y^3 + \frac{x}{m}\iint_D (2t + 1)K(t, s)dtds + \\ &+ \frac{y}{n}\iint_D (2s + 1)K(t, s)dtds + \frac{1}{m^2}\iint_D (t + t^2)K(t, s)dtds + \\ &+ \frac{1}{n^2}\iint_D (s + s^2)K(t, s)dtds\end{aligned} \tag{11}$$

unde $e_{0,0}(u, v) = 1$, $e_{1,0}(u, v) = u$, $e_{0,1}(u, v) = v$ *şi* $e_{2,2}(u, v) = u^2 + v^2$.

Demonstraţie. Din (2), (6) şi (7) avem

$$(L_{m,n}f)(x, y) = \iint_D K(t, s)(S_{m,n}f)\left(x + \frac{t}{m}, y + \frac{s}{n}\right) dtds \tag{12}$$

ceea ce arată că pozitivitatea lui $S_{m,n}$ implică pozitivitatea operatorului $L_{m,n}$. Deoarece $(S_{m,n}e_{0,0})(x, y) = 1$, $(S_{m,n}e_{1,0})(x, y) = x$, $(S_{m,n}e_{0,1})(x, y) = y$ şi $(S_{m,n}e_{2,2})(x, y) = x^2 + y^2 + x/m + y/n$, formulele (8)—(11) se obţin din (12).

TEOREMA 4 *Dacă $f \in \mathcal{E}_2$ este continuă pe $H = [0, a] \times [0, b]$ unde a și b sînt numere pozitive arbitrare fixate, atunci*

$$\lim_{m,n\to\infty} \|f - L_{m,n}f\| = 0$$

unde $\|\cdot\| = \max_{H} |\cdot|$

Demonstrație. Avînd în vedere că în conformitate cu (8)—(11) avem uniform pe H

$$\lim_{m,n\to\infty} \|f - L_{m,n}f\| = 0$$

dacă $f(x, y)$ reprezintă pe rînd funcțiile $e_{0,0}(x, y)$, $e_{1,0}(x, y)$, $e_{0,1}(x, y)$ și $e_{2,2}(x, y)$, rezultă, în baza unei teoreme a lui V. I. Volkov, (a se vedea [5]) că atunci cînd m, n tind spre infinit șirul operatorilor (6) converge uniform pe H către funcția f.

În continuare ne vom ocupa de evaluarea ordinului de aproximare prin operatorul (6) a unei funcții f presupusă continuă pe H.

În acest scop vom face uz de modulul de continuitate al funcției f care se definește astfel

$$\omega(f;\ \delta_1, \delta_2) = \max |f(x_2, y_2) - f(x_1, y_1)|$$

unde $\delta_1 > 0$, $\delta_2 > 0$ iar (x_1, y_1) și (x_2, y_2) sînt puncte din H supuse condițiilor $|x_2 - x_1| < \delta_1$ și $|y_2 - y_1| < \delta_2$. Ne va fi utilă următoarea.

TEOREMA 5. *Fie M o mulțime compactă din plan. Dacă $T_n : C(M) \to C(M)$ este un operator liniar și pozitiv cu proprietatea că $T_n e_{0,0} = e_{0,0}$ atunci*

$$\|f - T_n f\| < \left(1 + \frac{1}{\delta_1} \|T_n Q_2\|^{1/2} + \frac{1}{\delta_2} \|T_n R_2\|^{1/2}\right) \omega(f;\ \delta_1, \delta_2) \tag{13}$$

unde $Q_2(t, x) = (t - x)^2$, $R_2(\tau\, y) = (\tau - y)^2$ și $\|\cdot\|$ este norma uniformă din M.

TEOREMA 6. *Dacă $f \in \mathcal{E}_2$ este continuă pe H are loc inegalitatea*

$$\|f - L_{m,n}f\| < C_{m,n}\, \omega\left(f;\ \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)$$

unde $C_{m,n} = (1 + \sqrt{a + 2/m} + \sqrt{b + 2/n})$.

Demonstrație. Fie $Q_2(t, x) = (t - x)^2$ și $R_2(\tau, y) = (\tau - y)^2$. Din (8) și (9) avem

$$(L_{m,n}Q_2(t, x))\,(x, y) = \frac{x}{m} + \frac{1}{m^2} \iint_D (t + t^2) K(t, s)\, dt\, ds$$

iar din (8) și (10)

$$(L_{m,n}R_2(\tau, y))(x, y) = \frac{y}{n} + \frac{1}{n^2} \iint_D (s + s^2) K(t, s)\, dt\, ds$$

Pentru $(x, y) \in H$, se obţine

$$\| L_{m,n} Q_2 \| \leqslant \frac{1}{m}\left(a + \frac{2}{m}\right)$$

şi

$$\| L_{m,n} R_2 \| \leqslant \frac{1}{n}\left(b + \frac{2}{n}\right)$$

În conformitate cu (13)

$$\| f - L_{m,n} f \| < \left(1 + \frac{1}{\delta_1}\sqrt{\frac{1}{m}\left(a + \frac{2}{m}\right)} + \frac{1}{\delta_2}\sqrt{\frac{1}{n}\left(b + \frac{2}{n}\right)}\right)\omega(f;\ \delta_1, \delta_2)$$

Alegerea $\delta_1 = \frac{1}{\sqrt{m}}$, $\delta_2 = \frac{1}{\sqrt{n}}$ conduce la inegalitatea din enunţ.

În încheierea acestui paragraf menţionăm că există operatori liniari şi pozitivi de forma celui de la (6) cu proprietăţile enunţate. Astfel, pentru

$$K(t, s) = \frac{e^{t+s}}{(e-1)^2}$$

operatorul corespunzător este

$$(L_{m,n} f)(x, y) = \frac{e^{-(mx+ny)}}{(e-1)^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(mx+1)^{i+1} - (mx)^{i+1}}{(i+1)!} \times \right.$$

$$\left. \frac{(ny+1)^{j+1} - (ny)^{j+1}}{(j+1)!}\right) f\left(\frac{i}{m}, \frac{j}{n}\right)$$

3. Rezultatele stabilite se pot extinde, fără dificultate, la un număr oarecare de variabile.

Astfel, dacă se consideră funcţia $f \in \mathcal{C}_s$, operatorul

$$L_{n_1, n_2, \ldots, n_s} : \mathcal{C}_s \to \mathcal{C}_s$$

va fi

$$(L_{i_1, \ldots, i_s} f)(x_1, \ldots, x_s) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_s=0}^{\infty} D^f_{i_1, \ldots, i_s} A^{i_1, \ldots, i_s}_{n_1, \ldots, n_s}(x_1, \ldots x_s) \tag{14}$$

unde

$$D^f_{i_1, \ldots, i_s} = \begin{bmatrix} 0, \frac{1}{n_1}, \frac{2}{n_1}, \ldots, \frac{i_1}{n_1} & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & ; f \\ 0, \frac{1}{n_s}, \frac{2}{n_s}, \ldots, \frac{i_s}{n_s} & \end{bmatrix}$$

este diferenţa divizată s-dimensională pe punctele puse în evidenţă, iar $A^{i_1,\dots,i_s}_{n_1,\dots,n_s}(x_1,\dots,x_s)$ sînt polinoamele

$$A^{i_1,\dots,i_s}_{n_1,\dots,n_s}(x_1,\dots,x_s)=\iint\limits_{D_s}\dots\int K(t_1,\dots,t_s)\left(x_1+\frac{t_1}{n_1}\right)^{i_1}\dots\left(x_s+\frac{t_s}{n_s}\right)^{i_s}dt_1\dots dt_s$$

$K(t_1,\dots,t_s)\geqslant 0$ fiind o funcţie definită şi integrabilă pe hipercubul $D_s: 0\leqslant t_k\leqslant 1$, $(k=\overline{1,s})$, cu proprietatea

$$\iint\limits_{D_s}\dots\int K(t_1,\dots,t_s)\,dt_1\dots dt_s=1.$$

Se pot enunţa teoreme analoge teoremelor 3, 4 şi 6. Astfel avem

TEOREMA 3'. *Operatorii $L_{n_1,\dots,n_s}$ definiţi la (14) sînt liniari şi pozitivi, cu proprietăţile*

$$(L_{n_1,n_2,\dots,n_s}1)(x_1,x_2,\dots,x_s)=1$$

$$(L_{n_1,n_2,\dots,n_s}t_j)(x_1,x_2,\dots,x_s)=x_j+\frac{1}{n_j}\iint\limits_{D_s}\dots\int t_j\cdot K(t_1,t_2,\dots,t_s)dt_1\dots dt_s,\ j=\overline{1,s}$$

$$\left(L_{n_1,n_2,\dots,n_s}\sum_{j=1}^{s}t_j^2\right)(x_1,x_2,\dots,x_s)=$$

$$=\sum_{j=1}^{s}x_j^2+\sum_{j=1}^{s}\frac{1}{n_j}\iint\limits_{D_s}\dots\int(2t_j+1)K(t_1,\dots,t_s)\,dt_1\dots dt_s+$$

$$+\sum_{j=1}^{s}\frac{1}{n_j^2}\iint\limits_{D_s}\dots\int(t_j+t_j^2)K(t_1,\dots t_s)dt_1\dots dt_s$$

TEOREMA 4'. *Dacă $f\in\mathfrak{S}_s$ este continuă pe hiperparalelipipedul $H_s=[0,a_1]\times[0,a_2]\times\dots\times[0,a_s]$ unde $a_1,\dots,a_s$ sînt numere pozitive arbitrare fixate, atunci şirul $L_{n_1,\dots,n_s}f$ converge uniform pe H_s către funcţia f cînd $n_1,\dots,n_s\to\infty$.*

TEOREMA 6' *Dacă $f\in\mathfrak{S}_s$ este continuă pe H_s şi $\omega(f;\delta_1,\dots,\delta_s)$ este modulul de continuitate al funcţiei f, atunci are loc inegalitatea*

$$\|f-L_{n_1,\dots,n_s}f\|<C_{n_1,\dots,n_s}\,\omega\left(f;\frac{1}{\sqrt{n_1}},\frac{1}{\sqrt{n_2}},\dots,\frac{1}{\sqrt{n_s}}\right)$$

unde

$$C_{n_1,\dots,n_s}=1+\sum_{j=1}^{s}\left(a_j+\frac{2}{n_j}\right)^{\frac{1}{2}}.$$

(Intrat în redacţie la 21 decembrie 1979)

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ON THE APPROXIMATION OF FUNCTIONS OF TWO AND SEVERAL VARIABLES BY POSITIVE LINEAR OPERATORS

(S u m m a r y)

The aim of this paper is to investigate the approximation properties of some linear positive operators. For instance, if $S_{m,n}$ is defined by (2), then the remainder is given by (4). Likewise, $S_{m,n}$ may be represented by means of divided differences as in (5).

Finaly, the operators $L_{m,n}$, $m, n = 1, 2, \ldots$ defined by (6) and (7) are considered and studied.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

SADDLEPOINT OPTIMALITY CRITERIA OF NONLINEAR PROGRAMMING IN COMPLEX SPACE WITHOUT DIFFERENTIABILITY

DOREL I. DUCA

Introduction. Consider the problem

$$\text{Minimize } \operatorname{Re} f(z, \bar{z}) \tag{1}$$

subject to

$$z \in X,\ g(z, \bar{z}) \in S,\ h(z, \bar{z}) = 0, \tag{2}$$

where X is a nonempty set in C^n, S is a polyhedral cone in C^m, $f: X \times \bar{X} \to C$, $g: X \times \bar{X} \to C^m$, and $h: X \times \bar{X} \to C^p$.

Abrams R. A. [1] has given Kuhn-Tucker type saddlepoint optimality criteria for a nonlinear programming problem in complex space. In the proof of the necessary optimality condition it is essential that the functions are analytical.

In this paper, some saddlepoint optimality criteria without differentiability are given for Problem (1)—(2).

2. Notation and Preliminary Results. Let $C^n(R^n)$ denote the n-dimensional complex(real) vector space with Hermitian (Euclidean) norm $\|\cdot\|$, $R^n_+ = \{x/x = (x_j) \in R^n,\ x_j \geq 0,\ j = 1, \ldots, n\}$ the non-negative orthant of R^n, and $C^{m\times n}$ the set of mxn complex matrices.

If A is a matrix or vector, A^T, $\bar{A}$, A^H denote its transpose, complex conjugate and conjugate transpose respectively. For $z = (z_j)$, $w = (w_j) \in C^n$, $\langle z, w\rangle = w^H z$ denotes the inner product of z and w, $\operatorname{Re} z = (\operatorname{Re} z_j) \in R^n$ denotes the real part of z, and $\operatorname{Im} z = (\operatorname{Im} z_j) \in R^n$ denotes the imaginary part of z.

If $X \subseteq C^n$ then $\bar{X} = \{z \in C^n / \bar{z} \in X\}$ and $-X = \{z \in C^n / -z \in X\}$.

The nonempty set S in C^m is a polyhedral cone if it is a finite intersection of closed half-spaces in C^m, each containing 0 in its boundary, i.e.

$$S = \bigcap_{k=1}^{q} H_{u_k}, \quad \text{where } H_{u_k} = \{v \in C^m / \operatorname{Re}\langle v, u_k\rangle \geq 0\},\ k = 1, \ldots, q. \tag{3}$$

The polar S^* of the nonempty set S in C^m is defined by

$$S^* = \{u \in C^m / v \in S \Rightarrow \operatorname{Re}\langle u, v\rangle \geq 0\}.$$

If S is a polyhedral cone (3), then

$$\operatorname{int} S = \{v \in C^m / \operatorname{Re}\langle v, u_k\rangle > 0,\ k = 1, \ldots, q\},$$

or equivalently,

$$\operatorname{int} S = \{v \in C^m / 0 \neq u \in S^* \Rightarrow \operatorname{Re}\langle v, u\rangle > 0\}$$

and $\operatorname{int} S \neq \emptyset$ iff $S^* \cap (-S^*) = \{0\}$.

Let X be a nonempty convex set in C^n and let S be a closed convex cone in C^m. The function $g : X \times \overline{X} \to C^m$ is said to be concave with respect to S if for any $z, w \in X$ and $0 \leqq \lambda \leqq 1$:

$$g[\lambda z + (1-\lambda)w,\ \lambda \bar{z} + (1-\lambda)\bar{w}] - \lambda g(z, \bar{z}) - (1-\lambda) g(w, \bar{w}) \in S.$$

The function $f : X \times \overline{X} \to C$ is said to have convex real part with respect to R_+ if it is concave with respect to $-CR_+ = \{x \in C / \operatorname{Re} x \leqq 0\}$.

If $x = (x_j)$, $y = (y_j) \in R^n$, we consider

$$x \leqq y (x < y) \text{ iff } x_j \leqq y_j (x_j < y_j) \text{ for any } j \in \{1, \ldots, n\},$$

$$x \leqslant y \text{ iff } x \leqq y \text{ and } x \neq y.$$

LEMMA 1. *Let* $S = \bigcap_{k=1}^{q} H_{u_k}$ *be a polyhedral cone in* C^m, *let* $k \in \{1, \ldots, q\}$ *be fixed, and let* X *be a nonempty convex set in* C^n. *If* $g : X \times \overline{X} \to C^m$ *is concave with respect to* S, *then the function* $h_k : X \times \overline{X} \to C$ *defined by the formula*

$$h_k(z, w) = -\langle g(z, w), u_k\rangle \text{ for all } (z, w) \in X \times \overline{X},$$

has convex real part with respect to R_+.

Proof. The function g being concave with respect to S, we have

$$g[\lambda z + (1-\lambda)w,\ \lambda \bar{z} + (1-\lambda)\bar{w}] - \lambda g(z, \bar{z}) - (1-\lambda) g(w, \bar{w}) \in S$$

for all $z, w \in X$ and $0 \leqq \lambda \leqq 1$.

Since $S = \bigcap_{k=1}^{q} \{v \in C^m / \operatorname{Re}\langle v, u_k\rangle \geqq 0\}$, it follows that

$$\operatorname{Re}\langle g[\lambda z + (1-\lambda)w,\ \lambda \bar{z} + (1-\lambda)\bar{w}] - \lambda g(z, \bar{z}) - (1-\lambda) g(w, \bar{w}), u_k\rangle \geqq 0,$$

or equivalently,

$$\operatorname{Re}\{\lambda h_k(z, \bar{z}) + (1-\lambda) h_k(w, \bar{w}) - h_k[\lambda z + (1-\lambda)w,\ \lambda \bar{z} + (1-\lambda)\bar{w}]\} \geqq 0$$

for all $z, w \in X$ and $0 \leqq \lambda \leqq 1$. Consequently, h_k has convex real part with respect to R_+.

LEMMA 2. *Let* $A \in C^{p \times n}$, $B \in C^{q \times n}$ *and* $D \in C^{r \times n}$ *be given matrices, with* A *being nonvacuous. Then exactly one of the following two systems has a solution:*

(I) $\quad \operatorname{Re}(Az) > 0, \quad \operatorname{Re}(Bz) \geqq 0, \quad Dz = 0, \quad z \in C^n$

(II) $\quad \begin{cases} A^H u + B^H v + D^H w = 0, & u \in R^p, \quad v \in R^q, \quad w \in C^r. \\ u \geqslant 0, \ v \geqq 0. \end{cases}$

Proof. If $z = x + iy$, $A = A' + iA''$, $B = B' + iB''$, $D = D' + iD''$, $(x, y \in R^n;\ A', A'' \in R^{p\times n};\ B', B'' \in R^{q\times n};\ D', D'' \in R^{r\times n})$, then the system (I) is equivalent to the real system:

$$\begin{cases} (A' - A'')\begin{pmatrix} x \\ y \end{pmatrix} > 0, & (B' - B'')\begin{pmatrix} x \\ y \end{pmatrix} \geqq 0, \\ (D' - D'')\begin{pmatrix} x \\ y \end{pmatrix} = 0, & (D''\ D')\begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad x, y \in R^n. \end{cases} \tag{4}$$

By Motzkin's theorem [5], either (4) has a solution $x, y \in R^n$, or

$$\begin{cases} (A')^T u + (B')^T v + (D')^T w^1 + (D'')^T w^2 = 0 \\ -(A'')^T u - (B'')^T v - (D'')^T w^1 + (D')^T w^2 = 0 \\ u \geqslant 0,\ v \geqq 0,\ w^1,\ w^2 \in R^r \end{cases} \tag{5}$$

has a solution (u, v, w^1, w^2), but never both.

Since the real system (5) is equivalent to the system (II), where $w = w^1 + iw^2$, the lemma is proved.

LEMMA 3. *Let X be a nonempty convex set in C^n, let $f_k : X \times \bar{X} \to C^{m_k}$ $k = 1, 2, 3$ be vector functions having convex real part with respect to $R_+^{m_k}$, $k = 1, 2, 3$, and let $g : C^n \times C^n \to C^p$ be an afine vector function.*

If the system:

$$\operatorname{Re} f_1(z, \bar{z}) < 0, \quad \operatorname{Re} f_2(z, \bar{z}) \leqslant 0, \quad \operatorname{Re} f_3(z, \bar{z}) \leqq 0, \quad g(z, \bar{z}) = 0, \tag{6}$$

has no solution $z \in X$, then there exist $\lambda^k \in R^{m_k}$, $k = 1, 2, 3$ and $\mu \in C^p$ such that

$$(\lambda^1, \lambda^2, \lambda^3, \mu) \neq 0, \tag{7}$$

and

$$\operatorname{Re}[(\lambda^1)^T f_1(z, \bar{z}) + (\lambda^2)^T f_2(z, \bar{z}) + (\lambda^3)^T f_3(z, \bar{z}) + \mu^H g(z, \bar{z})] \geqq 0, \tag{8}$$

for all $z \in X$.

Proof. Let $\varphi_k = \varphi_k(x, y)$, $k = 1, 2, 3$ and $\psi_j = \psi_j(x, y)$, $j = 1, 2$ denote the real vector functions of 2n variables $x, y \in R^n$, defined by the formulas:

$$\varphi_k(x, y) = \operatorname{Re} f_k(x + iy,\ x - iy),$$

for all $(x, y) \in Y = \{(x, y) \in R^{2n} / z = x + iy \in X\}$, $\quad k = 1, 2, 3;$

$$\psi_1(x, y) = \operatorname{Re} g(x + iy,\ x - iy), \text{ for all } (x, y) \in R^{2n},$$

and

$$\psi_2(x, y) = \operatorname{Im} g(x + iy,\ x - iy), \text{ for all } (x, y) \in R^{2n}.$$

System (6) has no solution $z \in X$ iff the system

$$\begin{cases}\varphi_1(x, y) < 0, & \varphi_2(x, y) \leqslant 0, \quad \varphi_3(x, y) \leqq 0, \\ \psi_1(x, y) = 0, & \psi_2(x, y) = 0,\end{cases} \tag{9}$$

has no solution $(x, y) \in Y$.

Since Y is nonempty and convex, the real vector functions φ_k, $k = 1,2,3$ are convex on the set Y, and the real vector functions ψ_1 and ψ_2 are afine on R^{2n}, the hypotheses of Corollary 4.2.2 in [5] are satisfied, thus there exist

$\lambda^k \in R_+^{m_k}$, $k = 1, 2, 3$ and $\mu^j \in R^p$, $j = 1, 2$ such that

$$(\lambda^1, \lambda^2, \lambda^3, \mu^1, \mu^2) \neq 0, \tag{10}$$

and

$$(\lambda^1)^T \varphi_1(x, y) + (\lambda^2)^T \varphi_2(x, y) + (\lambda^3)^T \varphi_3(x, y) + (\mu^1)^T \psi_1(x, y) + (\mu^2)^T \psi_2(x, y) \geqq 0$$

for all $(x, y) \in Y$,

or equivalently,

$$\mathrm{Re}[(\lambda^1)^T f_1(z, \bar{z}) + (\lambda^2)^T f_2(z, \bar{z}) + (\lambda^3)^T f_3(z, \bar{z}) + \mu^H g(z, \bar{z})] \geqq 0$$

for all $z \in X$, where $\mu = \mu^1 + i\mu^2 \in C^p$.

This proves the inequality (8). From (10) follows (7) and lemma is proved.

3. **Results.** THEOREM 1. *Let X be a nonempty convex set in C^n, let $S = \bigcap_{k=1}^{q} H_{u_k}$ be a polyhedral cone in C^m with nonempty interior. Let $f: X \times \bar{X} \to C$ be a function of convex real part with respect to R_+, let $g: X \times \bar{X} \to C^m$ be concave with respect to S, and let $h: C^n \times C^n \to C^p$ be afine.*

If z^0 is a solution of Problem (1)—(2), then there exists $(\tau, u^0, v^0) \in R_+ \times S^* \times C^p$, $(\tau, u^0, v^0) \neq 0$ such that

$$\mathrm{Re}\langle g(z^0, \bar{z}^0), u^0\rangle = 0, \tag{11}$$

$$\begin{aligned} &\mathrm{Re}[\tau f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u\rangle + \langle h(z^0, \bar{z}^0), v\rangle] \leqq \\ &\leqq \mathrm{Re}[\tau f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u^0\rangle + \langle h(z^0, \bar{z}^0), v^0\rangle] \leqq \\ &\leqq \mathrm{Re}\,[\tau f(z, \bar{z}) - \langle g(z, \bar{z}), u^0\rangle + \langle h(z, \bar{z}), v^0\rangle], \end{aligned} \tag{12}$$

for all $z \in X$, $u \in S^*$ and $v \in C^p$.

Proof. Let z^0 be a solution of Problem (1)—(2). Then the system

$$\begin{cases} \mathrm{Re}[f(z, \bar{z}) - f(z^0, \bar{z}^0)] < 0 \\ \mathrm{Re}[-\langle g(z, \bar{z}), u_k\rangle] \leqq 0, \ k = 1, \ldots, q \\ h(z, \bar{z}) = 0, \end{cases} \tag{13}$$

has no solution $z \in X$.

In view of Lemma 1, the hypotheses of Lemma 3 are satisfied. Then there exist $\tau \in R_+$, $\lambda = (\lambda_k) \in R_+^q$ and $v^0 \in C^p$ such that

$$(\tau, \lambda, v^0) \neq 0 \tag{14}$$

and

$$\mathrm{Re}\left\{\tau[f(z, \bar{z}) - f(z^0, \bar{z}^0)] - \sum_{k=1}^{q} \lambda_k \langle g(z, \bar{z}), u_k\rangle + (v^0)^H h(z, \bar{z})\right\} \geqq 0$$

for all $z \in X$.

Since $u^0 = \sum_{k=1}^{q} \lambda_k u_k \in S^*$, it follows that

$$\mathrm{Re}\{\tau[f(z, \bar{z}) - f(z^0, \bar{z}^0)] - \langle g(z^0, \bar{z}^0), u^0\rangle + \langle h(z, \bar{z}), v^0\rangle\} \geqq 0 \tag{15}$$

for all $z \in X$.

By letting $z = z^0 \in X$ in (15), we get that $\mathrm{Re}\langle g(z^0, \bar{z}^0), u^0\rangle \leqq 0$. But since $u^0 \in S^*$ and $g(z^0, \bar{z}^0) \in S$, we have $\mathrm{Re}\langle g(z^0, \bar{z}^0), u^0\rangle \geqq 0$. Hence $\mathrm{Re}\langle g(z^0, \bar{z}^0), u^0\rangle = 0$, which is the equality (11).

Now, from (15) and (11) and the fact that z^0 is a solution of Problem (1)—(2), it follows that

$$\mathrm{Re}[\tau f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u^0\rangle + \langle h(z^0, \bar{z}^0), v^0\rangle] \leqq$$
$$\mathrm{Re}[\tau f(z, \bar{z}) - \langle g(z, \bar{z}), u^0\rangle + \langle h(z, \bar{z}), v^0\rangle],$$

for all $z \in X$, which is the second ineguality of (12).

Since $g(z^0, \bar{z}^0) \in S$, we have $\mathrm{Re}\langle g(z^0, \bar{z}^0), u\rangle \geqq 0$ for all $u \in S^*$, and since $h(z^0, \bar{z}^0) = 0$, it follows that

$$\mathrm{Re}[\tau f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u\rangle + \langle h(z^0, \bar{z}^0), v\rangle] \leqq$$
$$\leqq \mathrm{Re}[\tau f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u^0\rangle + \langle h(z^0, \bar{z}^0), v^0\rangle]$$

for all $u \in S^*$ and $v \in C^p$, which is the first inequality of (12).

It remained to show that $(\tau, u^0, v^0) \neq 0$.

a) If $(\tau, v^0) \neq 0$, we have $(\tau, u^0, v^0) \neq 0$.

b) If $(\tau, v^0) = 0$, from (14) it follows that $\lambda = (\lambda_k) \neq 0$. If $u^0 = 0$, we get that the system

$$\sum_{k=1}^{q} \lambda_k u_k = 0, \quad \lambda = (\lambda_k) \geqslant 0,$$

has a solution. Then by Lemma 2, the system

$$\mathrm{Re}\,\langle t, u_k\rangle > 0,\ k = 1, \ldots, q$$

has no solution $t \in C^m$, which contradicts int $S \neq \emptyset$. Consequently $u^0 \neq 0$, and the theorem is proved.

THEOREM 2. *Consider Problem (1)—(2), and let the hypotheses of Theorem 1 be satisfied.*

If in addition

i) *there exists* $z^1 \in X$ *such that* $g(z^1, \bar{z}^1) \in int\ S$ *and* $h(z^1, \bar{z}^1) = 0$,

and

ii) $0 \in \text{int}\,\{h(z, \bar{z}) \,/\, z \in X\}$,

then a necessary condition for z^0 *to be a solution of Problem (1)—(2) is that there exist* $u^1 \in S^*$, $v^1 \in C^p$ *such that*

$$\text{Re}\langle g(z^0, \bar{z}^0), u^1\rangle = 0 \tag{16}$$

and

$$\begin{gathered}\text{Re}[f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u\rangle + \langle h(z^0, \bar{z}^0), v\rangle] \leqq \\ \leqq \text{Re}[f(z^0, \bar{z}^0) - \langle g(z^0, \bar{z}^0), u^1\rangle + \langle h(z^0, \bar{z}^0), v^1\rangle] \leqq \\ \leqq \text{Re}[f(z, \bar{z}) - \langle g(z, \bar{z}), u^1\rangle + \langle h(z, \bar{z}), v^1\rangle],\end{gathered} \tag{17}$$

for all $z \in X$, $u \in S^*$ *and* $v \in C^p$.

Proof. In view of Theorem 1, there exists $(\tau, u^0, v^0) \in R_+ \times S^* \times C^p$, $(\tau, u^0, v^0) \neq 0$ such that (11) and (12) are satisfied.

We shall show that if $\tau = 0$, then a contradiction arises. Let $\tau = 0$. Then

$$(u^0, v^0) \neq 0, \tag{18}$$

and from the second inequality of (12) we have

$$0 \leqq \text{Re}[-\langle g(z, \bar{z}), u^0\rangle + \langle h(z, \bar{z}), v^0\rangle] \text{ for all } z \in X. \tag{19}$$

In view of Condition i), there exists $z^1 \in X$ such that $h(z^1, \bar{z}^1) = 0$ and

$$0 \neq u \in S^* \Rightarrow \text{Re}\langle g(z^1, \bar{z}^1), u\rangle > 0. \tag{20}$$

By letting $z = z^1 \in X$ in (19), we get that

$$0 \geqq \text{Re}\langle g(z^1, \bar{z}^1), u^0\rangle. \tag{21}$$

If $u^0 \neq 0$, from (20) it follows that $\text{Re}\langle g(z^1, \bar{z}^1), u^0\rangle > 0$, which contradicts (21), therefore $u^0 = 0$.

Since $u^0 = 0$, from (18) we have $v^0 \neq 0$, and from (19) it follows that

$$\text{Re}\langle h(z, \bar{z}), v^0\rangle \geqq 0 \text{ for all } z \in X. \tag{22}$$

Since $0 \in \text{int}\,\{h(z,\bar{z}) \,/\, z \in X\}$, there exists $\delta > 0$ such that $B(0;\ \delta) = \{t \in C^p \,/\, ||t|| < \delta\} \subseteq \{h(z, \bar{z}) \,/\, z \in X\}$. Let $0 < a < \delta \,/\, ||v^0||$. Then the point $t^0 = -av^0$ belongs to $B(0\,;\ \delta)$, hence exists a $z^2 \in X$ such that $t^0 = -av^0 = h(z^2, \bar{z}^2)$. By letting $z = z^2 \in X$ in (22) we get that

$$\text{Re}\langle h(z^2, \bar{z}^2), v^0\rangle = \text{Re}\langle -av^0, v^0\rangle = -a\,||v^0||^2 \geqq 0,$$

which contradicts the fact that $v^0 \neq 0$ and $a > 0$. Consequently $\tau > 0$.

Dividing (11) and (12) by $\tau > 0$ and setting $u^1 = (1/\tau)u^0 \in S^*$, $v^1 = (1/\tau)v^0 \in C^p$ we get (16) and (17).

THEOREM 3. *Let X be a nonempty set in C^n, let $f: X \times \bar{X} \to C$, $g: X \times \bar{X} \to C^m$, $h: X \times \bar{X} \to C^p$, and let S be a closed convex cone in C^m.*

If there exist $z^0 \in X$, $\tau \in R_+$, $u^0 \in S^*$, $v^0 \in C^p$ such that $\tau > 0$ and the inequalities (12) hold for all $z \in X$, $u \in S^*$ and $v \in C^p$, then z^0 is a solution of Problem (1)—(2).

Proof. From the first inequality of (12) we have

$$\operatorname{Re}[\langle g(z^0, \bar{z}^0), u - u^0\rangle + \langle h(z^0, \bar{z}^0), v^0 - v\rangle] \geqq 0 \tag{23}$$

for all $u \in S^*$ and $v \in C^p$.

By letting $u = u^0 \in S^*$ in (23), we get that

$$\operatorname{Re}\langle h(z^0, \bar{z}^0), t\rangle \geqq 0 \text{ for all } t \in C^p,$$

therefore

$$h(z^0, \bar{z}^0) = 0. \tag{24}$$

From (23) and (24) we obtain

$$\operatorname{Re}\langle g(z^0, \bar{z}^0), w\rangle \geqq 0 \text{ for all } w = u - u^0 \in S^*, \text{ i.e.}$$

$$g(z^0, \bar{z}^0) \in (S^*)^* = S,$$

therefore $z^0 \in X$ is a feasible solution of Problem (1)—(2).

By letting $u = 0 \in S^*$ and $v = v^0 \in C^p$ in (23), we get that

$$\operatorname{Re}\langle g(z^0, \bar{z}^0), u^0\rangle = 0, \tag{25}$$

because $u^0 \in S^*$ and $g(z^0, \bar{z}^0) \in S$, and hence $\operatorname{Re}\langle g(z^0, \bar{z}^0), u^0\rangle \geqq 0$.

Let z be a feasible solution of Problem (1)—(2). Then $\operatorname{Re}\langle g(z, \bar{z}), u^0\rangle \geqq 0$ and $h(z, \bar{z}) = 0$. From the second inequality of (12) and from (24) and (25) we have

$$\operatorname{Re} \tau f(z, \bar{z}) \geqq \operatorname{Re}[\tau f(z, \bar{z}) - \langle g(z, \bar{z}), u^0\rangle + \langle h(z, \bar{z}), v^0\rangle] \geqq \operatorname{Re}[\tau f(z^0, \bar{z}^0) -$$
$$- \langle g(z^0, \bar{z}^0), u^0\rangle + \langle h(z^0, \bar{z}^0), v^0\rangle] = \operatorname{Re} \tau f(z^0, \bar{z}^0).$$

Since $\tau > 0$, we have $\operatorname{Re} f(z, \bar{z}) \geqq \operatorname{Re} f(z^0, \bar{z}^0)$, i.e. z^0 is a solution of Problem (1)—(2).

THEOREM 4. *Let X be a nonempty set in C^n, let $f: X \times \bar{X} \to C$, $g: X \times \bar{X} \to C^m$, $h: X \times \bar{X} \to C^p$, and let S be a closed convex cone in C^m.*

If there exist $z^0 \in X$, $u^1 \in S^*$, $v^1 \in C^p$ such that the inequalities (17) hold for all $z \in X$, $u \in S^*$, $v \in C^p$, then z^0 is a solution of Problem (1)—(2).

Proof. Apply Theorem 3 with $\tau = 1 > 0$, $u^0 = u^1 \in S^*$ and $v^0 = v^1 \in C^p$.

(Received March 18, 1980)

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CRITERII NEDIFERENŢIALE DE TIP PUNCT ŞA, DE OPTIM, ÎN PROGRAMAREA NELINIARĂ ÎN DOMENIUL COMPLEX

(Rezumat)

Se dau criterii nediferenţiale de tip punct şa, de optim, pentru problema de programare neliniară în domeniul complex

$$\min \operatorname{Re} f(z, \bar{z})$$

în condiţiile

$$z \in X, \quad g(z, \bar{z}) \in S, \quad h(z, \bar{z}) = 0,$$

unde X este o submulţime nevidă a lui C^n, $f: X \times \overline{X} \to C$, $g: X \times \overline{X} \to C^m$, $h: X \times \overline{X} \to C^p$, iar S este un con convex închis din C^m.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

APLICAŢII CU ITERATE φ — CONTRACŢII

IOAN A. RUS

0. **Introducere.** Fie $K \in C([a,b]\times[a,b]\times R)$. Considerăm operatorul integral neliniar de tip Volterra

$$f: C[a,b] \to C[a,b],\ x \mapsto f(x)$$

unde

$$f(x)(t) = \int_a^t K(t,s,x(s))ds, \qquad t \in [a,b]$$

În continuare presupunem că funcţia K satisface la următoarea condiţie Lipschitz în raport cu al treilea argument

$$|K(t,s,u) - K(t,s,v)| \leqslant L|u-v|,\ \forall t,s \in [a,b],\ u,v \in R;\ L > 0.$$

În aceste condiţii aplicaţia f se bucură de următoarele proprietăţi:

a) Operatorul

$$f:(C[a,b],\ \|\cdot\|_C) \to (C[a,b],\ \|\cdot\|_C)$$

este Lipschitz cu constanta, $\alpha = L(b-a)$. Prin $\|\cdot\|_C$ am notat norma lui Cebîşev.

b) Notăm prin

$$\|x\|_\tau = \max_{t\in[a,b]} |x(t)|e^{-\tau(t-t_0)}, \qquad \tau > 0,\ t_0 \in R$$

o normă Bielecki.

Operatorul

$$f: (C([a,b],\ \|\cdot\|_C) \to (C[a,b],\ \|\cdot\|_C)$$

este Lipschitz cu constanta $\dfrac{L}{\tau}$.

Deci există $\tau > 0$ astfel încît f este contracţie.

c) Operatorul f^n, $n \in N$,

$$f^n: (C[a,b],\ \|\cdot\|_C) \in (C[a,b],\ \|\cdot\|_C)$$

este Lipschitz cu constanta $\alpha_n = \dfrac{L^n(b-a)^n}{n!}$.

Deci există $n \in N$, astfel încît f^n este contracţie.

Acest exemplu a sugerat diverse probleme dintr-un punct de vedere abstract şi anume:

Problema 1. Dîndu-se o mulţime X şi $f: X \to X$ o aplicaţie. În ce condiţii asupra lui X şi f există o metrică d pe X astfel încît

$$f: (X, d) \to (X, d) \text{ să fie contracţie}$$

Pentru anumite soluţii date acestei probleme a se vedea C. Bessaga [1], R. D. Holmes [5], V. I. Opoižev [7], I. Rosenholdtz [9], I. A. Rus [11] (pag. 83—84).

Problema 2. Fie (X, d) un spaţiu metric. O aplicaţie, $f: (X, d) \to (X, d)$, se numeşte tare-contracţie dacă pentru orice $\varepsilon > 0$ există pe X o metrică d_ε, echivalentă cu d, pentru care avem

$$d_\varepsilon(f(x), f(y)) \leqslant \varepsilon d_\varepsilon(x,y), \ \forall\, x,y \in X$$

Se pune problema studierii proprietăţilor aplicaţiilor tare-contractive.

Aceste tipuri de aplicaţii au fost introduse şi studiate de K. Goebel [3] şi [4].

Problema 3. Fie $f: (X,d) \to (X,d)$ cu proprietatea că există $p \in N$, astfel încît f^p este contracţie. Să se studieze f folosind o metrică de forma

$$d_f(x,y) = d(x,y) + a_1 d(f(x), f(y)) + \ldots + a_{p-1} d(f^{p-1}(x), f^{p-1}(y))$$

unde $a_1, \ldots, a_{p-1}$ sînt numere reale nenegative.

Pentru rezultate de acest tip a se vedea W. Walter [12]. Menţionăm că Walter consideră aplicaţii de tip Lipschitz cu proprietatea că există o iterată ce este contracţie.

În prezenta lucrare ne propunem să studiem, după modelul propus în [10], următoarea teoremă.

TEOREMA 1. *Fie (X, d) un spaţiu metric complet şi $f: X \to X$ o aplicaţie. Presupunem că există $p \in N$, $\varphi, \psi: R_+ \to R_+$ astfel încît*

(a) *φ şi ψ sînt continue*

(b) *φ şi ψ sînt monoton crescătoare*

(c) *$\varphi(r) < r$, $\forall\, r > 0$ şi $\psi(r) \leqslant r$, $\forall\, r > 0$ şi $\psi(0) = 0$.*

(d) *$r - \varphi(r) \to +\infty$*

(e) *pentru orice $x, y \in X$*

$$d(f(x), f(y)) \leqslant \psi(d(x, y))$$

şi

$$d(f^p(x), f^p(y)) \leqslant \varphi(d(x, y))$$

În aceste condiţii

(i) *$F_f = \{x^*\}$*

(ii) *oricare ar fi $x_0 \in X$, şirul aproximaţiilor succesive, $x_n = f^n(x_0)$, $n \in N$, este convergent şi are ca limită pe x^**

(iii) *are loc estimarea*

$$d(x_n, x^*) \leqslant \psi^{n-1}\left(\varphi^{p\left[\frac{n}{p}\right]}(d(x_0, x^*))\right)$$

unde prin $\left[\frac{n}{p}\right]$ s-a notat partea întreagă a lui $\frac{n}{p}$.

1. Demonstraţia teoremei 1. (*i*) Pe baza teoremei 2.2.3. din [10], rezultă că f are un punct fix unic. Aplicaţia f comutînd cu f ([11], Corolarul 1.1.1., pag. 12) rezultă că f are un punct fix unic. Mai mult,

$$F_f = F_{f^p} = \{x^*\}$$

$(ii + (iii)$. Fie $x_0 \in X$. Notăm $x_n = f^n(x_0)$. Pe baza teoremei 2.2.3 din [10] rezultă că şirul $(f^{kp}(x_0))_{k \in N}$ este convergent, are ca limită pe x^* şi are loc estimarea

$$d(x_{np}, x^*) \leqslant \varphi^k(r_{x_0})$$

unde

$$r_{x_0} = \sup\{r \mid r - \varphi(r) \leqslant d(x_0, f^p(x_0))\}$$

Folosind acest rezultat şi condiţiile din teoremă avem

$$d(x_n, x^*) \leqslant \psi^{n-p\left[\frac{n}{p}\right]}\left(d\left(x_{p\left[\frac{n}{p}\right]}, x^*\right)\right) \leqslant \psi^{p-1}\left(\varphi^{p\left[\frac{n}{p}\right]}(d(x_0, x^*))\right).$$

2. Şiruri de aplicaţii şi puncte fixe. Fie $f: X \to X$ o aplicaţie ce are un punct fix unic x^* şi f_n un şir de aplicaţii ce converge uniform către f, $f_n \rightrightarrows f$. Dacă $x_n \in F_{f_n}$, în ce condiţii asupra lui f şi asupra şirului $(f_n)_{n \in N}$, avem $x_n \to x^*$?

Pentru diverse aspecte ale acestei probleme a se vedea [10].

În cazul în care existenţa şi unicitatea punctului fix al lui f este asigurată de teorema 1 avem

TEOREMA 2. *Fie (X, d) un spaţiu metric complet şi $f_n, f: X \in X$, astfel încît*

(a) $f_n \rightrightarrows f$

(b) *există $p \in N$, φ şi $\psi: R_+ \to R_+$, astfel încît sînt satisfăcute condiţiile din teorema* 1.

(c) $F_{f_n} \neq \emptyset$, *pentru orice $n \in N$*

În aceste condiţii oricare ar fi $x_n \in F_{f_n}$, şirul $(x_n)_{n \in N}$ este convergent şi are ca limită unicul punct fix, x^*, al lui f.

Demonstraţie. Din condiţia (a) rezultă că şirul $(f_n^p)_{n \in N}$ converge uniform către f^p. Se aplică teorema 2.2.3 din [10], aplicaţiei f^p şi şirului (f_n^p).

3. Dependenţa de parametru. Fie (X, d) un spaţiu metric, (Y, τ) un spaţiu topologic şi $f: X \times Y \to X$ o aplicaţie. Presupunem că pentru orice $y \in Y$ aplicaţia $f(.,y)$ are un punct fix unic, x_y^*. Se consideră aplicaţia

$$P: Y \to X, \quad y \mapsto x_y^*$$

Se pune problema de a vedea în ce condiţii aplicaţia P este continuă.

Pentru unele soluţii ale acestei probleme se poate vedea [10].

În cazul în care existenţa şi unicitatea punctului fix al lui $f(.,y)$ este asigurată de teorema 1, avem

TEOREMA 3. *Fie (X, d) un spaţiu metric complet, (Y, τ) un spaţiu topologic şi $f: X \times Y \to X$ o aplicaţie ce satisface la următoarele condiţii*
(a) *există $p \in N$, $\varphi, \psi: R^+ \to R^+$, astfel încît aplicaţia $f(.,y)$ satisface la condiţiile teoremei 1, oricare ar fi $y \in Y$ (p, φ şi ψ nu depind de y!).*
(b) *aplicaţia $f(x.,): Y \to X$ este continuă, pentru orice $x \in X$.*
În aceste condiţii aplicaţia P este continuă.

Demonstraţie. Fie y_1 şi $y_2 \in Y$, $x^*_{y_1}$ şi $x^*_{y_2}$ punctele fixe ale lui $f(.y_1)$ respectiv $f(.,y_2)$. Din condiţiile teoremei avem

$$d(x^*_{y_1}, x^*_{y_2}) = d(f^p(x^*_{y_1}, y_1), f^p(x^*_{y_2}, y_2) \leqslant$$
$$\leqslant d(f^p(x^*_{y_1}, y_1), f^p(x^*_{y_1}, y_2)) + d(f^p(x^*_{y_1}, y_2), f^p(x^*_{y_2}, y_2) \leqslant$$
$$\leqslant d(f^p(x^*_{y_1}, y_1), f^p(x^*_{y_1}, y_2)) + \varphi(d(x^*_{y_1}, y^*_{y_2}))$$

de unde obţinem că

$$d(x^*_{y_1}, x^*_{y_2}) - \varphi(d(x^*_{y_1}, x^*_{y_2})) \to 0, \text{ dacă } y_1 \to y_2$$

Prin urmare

$$d(x^*_{y_1}, x^*_{y_2}) \to 0, \text{ dacă } y_1 \to y_2 .$$

4. **Exemplu.** Dacă $\psi(r) = \beta r$, $\beta > 1$ şi $\varphi(r) = \alpha r$, $\alpha < 1$, atunci avem următorul rezultat.

TEOREMA 4. Fie (X, d) *un spaţiu metric complet şi $f: X \to X$ o aplicaţie. Presupunem că există $p \in N$, $\alpha \in [0,1[$, $\beta \in]1, +\infty[$, astfel încît*

(a) $d(f(x), f(y)) \leqslant \beta\, d(x, y), \qquad \forall x, y \in X$
(b) $d(f^p(x), f^p(y)) \leqslant \alpha\, d(x, y), \qquad \forall x, y \in X$

În aceste condiţii avem

(i) $F_f = \{x^*\}$
(ii) *oricare ar fi $x_0 \in X$, şirul aplicaţiilor succesive, $x_n = f^n(x_0)$, $n \in N$ este convergent şi are ca limită pe x^*.*
(iii) *are loc estimarea*

$$d(x_n, x^*) \leqslant \beta^{k-1} \frac{\alpha^{\left[\frac{n}{p}\right]}}{1 - \alpha} d(x_0, f(x_0))$$

(iv) *Fie $g: X \to X$, o aproximantă a lui f, adică*

$$d(f(x), g(x)) \leqslant \eta, \qquad \forall x \in X$$

Fie $x_n = f^n(x_0)$, $y_n = g^n(x_0)$
Avem estimarea

$$d(y_n, x^*) \leqslant \eta \frac{\beta^n - 1}{\beta - 1} + \beta \frac{\alpha^{\left[\frac{n}{p}\right]}}{1 - \alpha} d(x_0, f(x_0))$$

5. **Probleme deschise.** *Problema* 1. Fie $(X, ||\cdot||)$ un spaţiu Banach şi $f: X \to X$ o aplicaţie cu proprietatea că $\exists p \in N$, $\exists \alpha \in]0,1[$ astfel încît

$$||f^p(x) - f^p(y)|| \leqslant \alpha ||x - y||, \quad \forall x, y \in X.$$

În ce condiţii aplicaţia $1_X - f$ este surjecţie?

În cazul $p = 1$, a se vedea [10].

Problema 2. Se cunosc la ora actuală o gamă variată de aplicaţii de tip contracţii (a se vedea [10] şi [11]). Pentru care din acestea se poate aplica programul realizat în prezenta lucrare? Pentru $p = 1$, a se vedea [10].

(Intrat în redacţie la 18 martie 1980)

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MAPS WITH φ – CONTRACTIVE ITERATES

(Summary)

In this paper we present an analysis of the theorem 1 in connection with
(a) the method of successive approximations of the fixed point and with error estimations.
(b) various kinds of continuity of the fixed points.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

SUR LA TORSION DES COURBES QUI SONT TRACÉES SUR UNE SURFACE

L. BITAY

En partant des résultats de E. Franckx [1, 2] nous donnons une construction graphique pour le rayon de torsion d'une courbe tracée sur une surface quelconque et construisons l'indicatrice des torsions de cette courbe.

La théorie de courbure des courbes Γ qui sont tracées sur une surface par un point M_0 à une direction donnée a été élaborée depuis longtemps. Les résultats les plus importants en ce sens sont les suivants :

1. Les courbes Γ du point de vue de la courbure sont représentables par des sections planes de la surface en direction donnée.

2. Meusnier a démontré que le centre de courbure C d'une section oblique est la projection du centre de courbure C_n de la section normale en ce point sur le plan de la section oblique.

E. Franckx a démontré dans deux article successifs [1, 2] des résultats analogues pour la torsion des courbes qui sont tracées sur une surface par un point M_0 :

1. Il a considéré une famille de courbes $\mathcal{F}$, qui passent par le point M_0 et y ont un contact d'ordre deux. (Ces courbes ont donc le même plan osculateur, le même centre de courbure et la même axe polaire.)

2. Soit $\mathcal{S}$ l'ensemble des sections sphériques de la surface donnée qui se font avec des sphères osculatrices des courbes de la famille $\mathcal{F}$. E. Franckx a démontré que du point de vue de la torsion les courbes de la famille $\mathcal{F}$ sont représentables par des courbes $\mathcal{S}$: la torsion d'une courbe arbitraire Γ_0 de la famille $\mathcal{F}$ est égale à la torsion de la section sphérique avec la sphère osculatrice de Γ_0.

Il a donné même une construction pour la torsion des courbes $\mathcal{S}$. Nous proposons de simplifier cette construction en approchant à la construction de Meusnier mentionnée plus haut, puis nous construisons une indicatrice des torsions de la famille $\mathcal{S}$.

Soit Γ_0 une courbe tracée sur une surface par le point M_0 de celle-ci, tangente à une direction donnée. Soit C le centre de courbure, Δ l'axe polaire, S le centre de la sphère osculatrice, $\bar{m}$ et $\bar{n}$ les vecteurs unitaires de la normale à la surface ou bien de la normale principale de Γ_0. Nous désignons par θ l'angle de $\bar{n}$ et $\bar{m}$, $\varphi = CM_0S$. Franckx a démontré [1] que l'expression

$$k = \tau \frac{\cos \varphi}{\sin (\theta - \varphi)} \tag{1}$$

est une invariante scalaire pour toutes les courbes de la famille $\mathcal{F}$, qui ont un contact d'ordre deux avec Γ_0. (τ est le rayon de courbure de la courbe considérée). On observe que Franckx utilise l'invariante $\frac{1}{\tau} \frac{\sin (\theta - \varphi))}{\cos \varphi}$ à la place de l'invariante (1) et ainsi sa construction met en évidence la torsion au lieu de rayon de torsion. Notre modification simplifie d'une part la construction de Franckx et d'autre part nous permet à construire une indicatrice des rayons de torsion de la famille de courbe $\mathfrak{F}$ à l'aide de laquelle on peut poursuivre d'une façon intuitive la variation de rayon de torsion de ces courbes, exactement de la même manière que l'indicatrice de Dupin nous indique intuitivement la variation du rayon de courbure de la famille Γ.

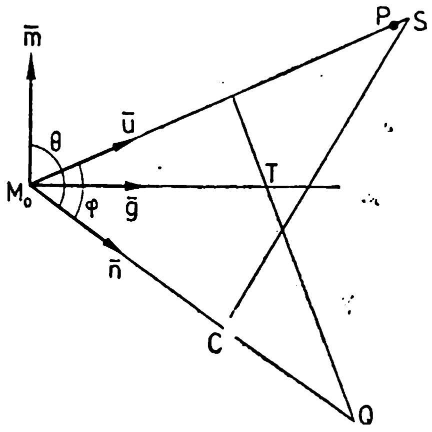

Fig. 1.

Sur la tangente de la surface qui appartient au plan normal de la famille $\mathfrak{F}$ nous prenons le point T défini par le vecteur

$$\overline{M_0 T} = k\bar{g}, \tag{2}$$

$\bar{g}$ étant le vecteur unitaire du trièdre de Darboux. (fig. 1) Du point T abaissons la perpendiculaire sur M_0S qui coupe la normale principale en un point Q defini par le vecteur

$$\overline{M_0 Q} = \tau\bar{n}. \tag{3}$$

Démonstration. Le vecteur unitaire $\bar{u}$ du vecteur $\overline{M_0 S}$ peut être décomposé comme il suit

$$\bar{u} = \sin(\theta - \varphi)\,\bar{g} + \cos(\theta - \varphi)\,\bar{m}$$

et

$$\overline{M_0 Q} = \lambda\bar{n} = \lambda \sin\theta\cdot\bar{g} + \lambda\cos\theta\,\bar{m},$$

où λ désigne la mesure algébrique du vecteur $\overline{M_0 Q}$ par rapport à $\bar{n}$. En exprimant que les vecteurs $\bar{u}$ et $\overline{TQ} = \overline{TM_0} + \overline{M_0 Q}$ sont perpendiculaires nous obtenons l'équation

$$0 = (\lambda\sin\theta - k)\sin(\theta - \varphi) + \lambda\cos\theta\cos(\theta - \varphi),$$

qui donne, tenant compte d'(1) $\lambda = \tau$, ce qu'il faut démontrer.

En prenant sur la sémidroite M_0S le vecteur $\overline{M_0 P} = \tau\bar{u}$ et faisant varier φ dans l'intervalle $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ nous obtenons une indicatrice des torsions comme le lieu géométrique des points P. Choisissant dans le plan normal un système des coordonnés polaire avec le pôle M_0 et l'axe polaire $\bar{n}$, nous obtenons l'équation d'indicatrice des torsions selon (1)

$$\rho = k\,\frac{\sin(\theta - \varphi)}{\cos\varphi}. \tag{4}$$

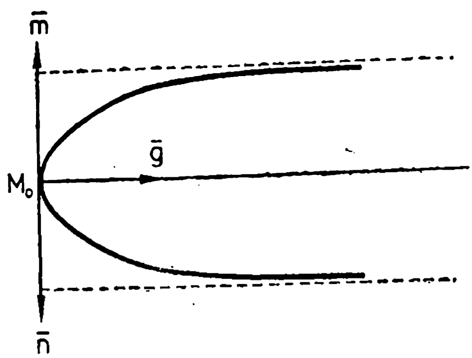

Fig. 2.

Cette courbe a deux asymptotes perpendiculaires sur la normale principale, tous les deux étant symétriques par raport à M_0 situés à distances égales à $\pm k \cos\theta$ de ceci. La courbe est tangente à la normale de la surface en M_0 et elle traverse l'un des asymptotes.

Dans le cas particulier $\theta = \pi$ (fig. 2) le plan osculateur de la famille contient la normale de la surface et l'équation (4) nous donne $\rho = k \text{ tg } \varphi$, qui est une courbe symétrique par rapport à $\bar{g}$. D'ici nous déduisons que les sections sphériques de la surface qui s'obtiennent par des sphères symétriques par rapport au plan osculateur de la famille ont des torsions égales en valeur absolue et de signes contraires.

(Manuscrit reçu le 20 mars 1980

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ASUPRA TORSIUNII CURBELOR TRASATE PE O SUPRAFAȚĂ

(Rezumat)

Pornind de la rezultatele obținute de E. Franckx [1, 2] autorul dă o construcție grafică pentru raza de torsiune a unei curbe trasată pe o suprafață și construiește o indicatoare a torsiunilor acestor curbe.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

NONLINEAR OPERATORS THAT TRANSFORM A WEDGE

A. B. NÉMETH

0. **Introduction.** Our paper aims to develop a method of transformation of a wedge into another one, so as to preserve some of its properties and to improve some others. To this end we have introduced special nonlinear operators, which are in fact resolvents of some sublinear operators, called here correct operators. Although the formalism and some of the results (see e.g. Lemmas 1 and 2) are related to the spectral theory, we have ignored throughout this aspect (which will be considered elsewhere). The developed principle is related to the method of Bishop and Phelps [1], largely extended by Ekeland [3], and also to the machinery used by Krasnosel'skii [5] in various problems on ordered B-spaces. As a consequence we have deduced that a wide class of wedges in a normed space have the property that they can be approximated indefinitely by completely regular cones. The complete regularity of a cone is important from both theoretical and computational points of view (see e.g. [4] and respectively [8]). Implicitely we have got a new method of introducing an ordering into a normed space for which the order convergence coincides with the norm convergence, and so we have completed the results in this direction obtained by DeMarr [2] and by Vulih and Danilenko [9].

1. **Auxiliary results.** We begin with some auxiliary results that are closely related to the spectral theory for nonlinear operators. Our special interest as well as the simplicity of the reasonings motivate their proof given here.

The operator S acting in the normed space Y is said to be *nonexpansive* if $||S(y_1) - S(y_2)|| \leqslant ||y_1 - y_2||$ for any y_1 and y_2 in Y.

LEMMA 1. *If S is a nonexpansive operator acting in the normed space Y and having its range in a complete subspace, then for each t in $(-1, 1)$ the operator*

$$S_t = I - tS$$

has a continuous inverse.

Proof. Since S is nonexpansive, we have for any y_1 and y_2 in Y

$$||S_t(y_1) - S_t(y_2)|| \geqslant ||y_1 - y_2|| - |t|\,||S(y_1) - S(y_2)|| \geqslant (1 - |t|)||y_1 - y_2||. \quad (1)$$

Hence S_t is injective for any t in $(-1, 1)$.

Suppose now that $S(Y) \subset Y_0$, where Y_0 is a complete subspace of Y. Let z be an arbitrary element of Y and put $Y_1 = \mathrm{sp}\{Y_0, z\}$. then Y_1 is complete. Consider the operator J defined by

$$J(y) = tS(y) + z.$$

Y_1 is an invariant space of J. For y_1 and y_2 elements in Y we have

$$||J(y_1) - J(y_2)|| = |t|\,||S(y_1) - S(y_2)|| \leqslant |t|\,||y_1 - y_2||.$$

If $|t| < 1$, then J is a contraction and hence it has a fixed point y in Y_1, that is, $y = tS(y) + z$. This means that $S_t(y) = z$ and the surjectivity of S_t is proved.

Thus S_t^{-1} exists. To prove its continuity, put $z_1 = S_t^{-1}(y_1)$ and $z_2 = S_t^{-1}(y_2)$. According to (1) we have

$$||S_t(z_1) - S_t(z_2)|| \geqslant (1 - |t|)\, ||z_1 - z_2||$$

and hence

$$||S_t^{-1}(y_1) - S_t^{-1}(y_2)|| \leqslant (1 - |t|)^{-1}\, ||y_1 - y_2||$$

and the continuity of S_t^{-1} follows. Q.E.D.

LEMMA 2. *Consider the set of operators acting in Y endowed with the topology of the uniform convergence on bounded sets. Then the family S_t^{-1}, $t \in (-1, 1)$, depends continuously on t.*

Proof. We fix t_0 in $(-1, 1)$ and shall prove that there exist the numbers C and D depending only on t_0 such that

$$||(S_t^{-1} - S_{t_0}^{-1})(y)|| < |t - t_0|\,(C||y|| + D) \tag{2}$$

for any t sufficiently close to t_0.

Put $z_0 = S_{t_0}^{-1}(y)$ and $z = S_t^{-1}(y)$. Then $y = z_0 - t_0S(z_0)$ and $y = z - tS(z)$ and hence

$$z - z_0 - tS(z) + t_0S(z_0) = 0,$$

or,

$$(t - t_0)S(z_0) = z - z_0 - t(S(z) - S(z_0)).$$

Using the fact that S is nonexpansive this relation yields

$$|t - t_0|\, ||S(z_0)|| \geqslant ||z - z_0|| - |t|\, ||S(z) - S(z_0)|| \geqslant (1 - |t|)\, ||z - z_0||. \tag{3}$$

We have also $||S(z_0)|| \leqslant ||z_0|| + ||S(0)||$ and using the definition of z_0 we get

$$||y|| \geqslant ||z_0|| - |t_0|\, ||S(z_0)|| \geqslant ||z_0|| - |t_0|\,(||z_0|| + ||S(0)||) =$$
$$= (1 - |t_0|)\, ||z_0|| - |t_0|\, ||S(0)||$$

wherefrom

$$||z_0|| \leqslant \frac{1}{1 - |t_0|}(||y|| + |t_0|\, ||S(0)||)$$

and hence

$$||S(z_0)|| \leqslant \frac{1}{1 - |t_0|}(||y|| + |t_0|\, ||S(0)||) + ||S(0)|| =$$
$$= \frac{1}{1 - |t_0|}\,(||y|| + S(0)||).$$

This together with (3) gives

$$||z - z_0|| \leqslant \frac{|t - t_0|}{1 - |t|} ||S(z_0)|| \leqslant \frac{|t - t_0|}{(1 - |t|)(1 - |t_0|)} (||y|| + ||S(0)||). \quad (4)$$

For t in a suitable neighbourhood of t_0 we have $(1 - |t|)(1 - |t_0|) > \frac{1}{2}(1 - |t_0|)^2$. Put $C = 2(1 - |t_0|)^{-2}$, $D = 2(1 - |t_0|)^{-2}||S(0)||$ and use the definitions of z and z_0. Then (4) implies (2). Q.E.D.

2. **Correct sublinear operators.** The subset W of the vector space Y is called a *wedge* if:

(i) ty is in W whenever $y \in W$ and t is a non-negative real number;

(ii) $u + v$ is in W whenever u and v are in W.

The wedge K is called a *cone* if

(iii) $y \in K$ and $-y \in K$ imply $y = 0$.

Let $\leqslant$ be a reflexive and transitive binary relation defined on Y, related to the linear structure of Y by the axioms:

(I) if $u \leqslant v$, then $tu \leqslant tv$ for any non-negative t;

(II) if $u \leqslant v$, then $u + y \leqslant v + y$ for any y in Y.

We shall refer in the sequel to a binary relation of this kind as to an *ordering* on the vector space Y.

There exists a correspondence between the famliy of orderings on Y and the family of wedges in Y in the following sense: Given a wedge W in Y we can define a binary relation $\leqslant$ on Y by putting $u \leqslant v$ if $v - u \in W$. The obtained relation is an ordering on Y, called the ordering *induced* by W, or *W-ordering*. Conversely, given the ordering $\leqslant$ on Y, the set $W = \{y \in Y : y \geqslant 0\}$ is a wedge in Y, that induces just the original ordering on Y.

If we suppose that the order relation $\leqslant$ on Y is in addition anti-symmetric, then $(Y, \leqslant)$ will be called an *ordered vector space.* In the same way as in the above paragraph, a correspondence can be estabilished between the family of reflexive, transitive and anti-symmetric binary relations which satisfy (I) and (II), and the family of cones in Y.

Let V be a wedge (a cone) in Y. The terms *V-order bounded, V-monotone,* etc. mean order boundendess, monotonicity etc. with respect to the ordering induced in Y by V.

The operator S acting in the space Y endowed with an ordering $\leqslant$ is called *sublinear* if it is positively homogeneous and

$$S(y_1 + y_2) \leqslant S(y_1) + S(y_2)$$

for any y_1 and y_2 in Y. In accordance with the above convention, S will be called *V-sublinear* if it is sublinear with respect to the ordering introduced in the space Y by the wedge (cone) V.

Let W be a wedge in the normed space Y. The W-sublinear operator S acting in Y will be called *W-correct* if it satisfies the following conditions

(a) S is continuous and its range is contained in W;

(b) the operator $S_t = I - tS$ is surjective and one-to-one for any t in $[0, 1)$;

(c) S_t^{-1} is a continuous operator for any t in $[0, 1)$;

(d) the family S_t^{-1}, $t \in [0, 1)$ depends continuously on t if we consider the set of operators acting in Y endowed with the topology of the uniform convergence on norm bounded sets.

As an immediate consequense of Lemmas 1 and 2 we have the

COROLLARY 1. *If S is a nonexpansive W-sublinear operator acting in Y with the range in W intersected with a complete subspace of Y, then S is a correct W-sublinear operator.*

We shall need in the sequel the notion of norm definitness. The operator Q acting in the normed space Y will be called *c-norm definite on the set M*, with c a positive real, if $\|Q(y)\| \geqslant c\|y\|$ for any y in M.

3. Examples of correct operators.

1° Let W be a wedge in the normed space Y and consider the element k in W of norm 1. Define the operator S by

$$S(y) = \|y\|k.$$

Then $S(Y)$ is contained in W and S is obviously sublinear and nonexpansive. S is also 1-norm definite. Hence S is a W-correct 1-norm definite operator by Corollary 1.

2° Let $k_1, \ldots, k_n$ be nonzero elements of W having the property that $\sum_{i=1}^{n} \|k_i\| = 1$. Suppose that $f_1, \ldots, f_n$ are functionals on Y and consider the following conditions on them:

a) f_i, $i = 1, \ldots, n$ are sublinear non-negative functionals on Y that are also Lipschitz with the constant 1.

b) Suppose that $k_1, \ldots, k_n$ generate a cone and that there exist some positive reals c_i, $i = 1, \ldots, n$ such that for any y in W there exists at least an $i (1 \leqslant i \leqslant n)$ such that $f_i(y) \geqslant c_i\|y\|$.

If the conditions in a) are satisfied, then the operator S defined by

$$S(y) = f_1(y)k_1 + \ldots + f_n(y)k_n$$

is a correct W-sublinear operator.

If there hold also the conditions in b), then S is also norm definite on W.

Let us verify the first assertion. S is obviously W-sublinear and has its range in W. Let y_1 and y_2 be arbitrary elements in Y. Then

$$\|S(y_1) - S(y_2)\| \leqslant \sum_{i=1}^{n} |f_i(y_1) - f_i(y_2)| \, \|k_i\| \leqslant \|y_1 - y_2\| \sum_{i=1}^{n} \|k_i\| =$$
$$= \|y_1 - y_2\|,$$

that is, S is nonexpansive. Apply now Corollary 1 to conclude that S is W-correct.

Consider now the second assertion. Because $k_1, \ldots, k_n$ generate a finite dimensional cone K, this will be a normal cone in Y. Denote by $\leqslant$ the ordering in Y induced by K. This ordering will be semi-monotone, i.e., there exists a positive real b such that $0 \leqslant y_1 \leqslant y_2$ implies $b||y_1|| \leqslant ||y_2||$. (The definition of the normal cone see in 5.) Let y be an arbitrary element of W. Then there exists an $i(1 \leqslant i \leqslant n)$ such that $f_i(y) \geqslant c_i||y||$. We have also

$$S(y) \geqslant f_i(y)k_i$$

and by the semi-monotonicity of the norm,

$$||S(y)|| \geqslant bf_i(y)||k_i|| \geqslant (bc_i||k_i||)||y||.$$

Putting $c = \min\{bc_i||k_i||\}$, the obtained relation implies that S is a c-norm definite operator on W.

3°. Suppose that Q is a compact Hausdorff space and denote by $C(Q)$ the vector space of the continuous real-valued functions on Q endowed with the uniform norm. Let f be a function from $Q \times \mathbf{R}$ to $\mathbf{R}$, and let consider the following conditions on it:

a) f is non-negative and continuous on $Q \times \mathbf{R}$;

b) f is sublinear with respect the second variable and is uniformly Lipschitz on Q with respect to this variable, the Lipschitz constant being 1;

c) there exists a positive c such that

$$f(q, x) \geqslant cx$$

for any q in Q and any positive x.

Let be W the cone in $C(Q)$ of the non-negative functions on Q. Then if a) and b) hold, the operator S defined by

$$S(y)(q) = f(q, y(q))$$

is a W-correct operator in $C(Q)$.

If it holds also c), then S is also c-norm definite on the cone W.

4° Let Q be a subset in $\mathbf{R}^n$ of finite Lebesgue measure and let f be a function from $Q \times \mathbf{R}$ to $\mathbf{R}$ having the property that $f(., x)$ is measurable for any real x. Consider the following conditions on f:

a) f is non-negative on its domain;

b) $f(q,.)$ is sublinear and Lipschitz with the constant 1 for almost all q in Q;

c) there exists a positive real c, such that

$$cx \leqslant f(q, x)$$

for any positive x and almost all q in Q.

Consider the space $L^p(Q)$ $(1 \leqslant p < \infty)$, and let W be the cone of elements of this space that are non-negative on Q (i.e., the equivalence classes that contain non-negative functions). Then the operator S defined by the relation

$$S(y)(q) = f(q, y(q))$$

is for f satisfying a) and b) a correct sublinear one.

Indeed, $f(., x)$ is measurable for any x by hypothesis and $f(q,.)$ is continuous for almost all q in Q by b). Thus the Charatheodory condition holds for f and hence using again b) it follows that S acts continuously in $L^p(Q)$ (see e.g. Theorem 10.1.5 and 10.1.6 in [10]). Condition a) ensures that S has its range in W, and the uniformly Lipschitz property of f with the constant 1 in b) implies that it is nonexpansive. S is also sublinear by b) and using Corollary 1 we conclude that it is W-correct.

If condition c) also holds, then it follows that S is in plus c-norm definite on the cone W.

5° Let Q be as in 4°, and let K be a real-valued function defined on $Q \times Q$. Suppose $1 < p < \infty$, and let p' be the conjugate of p, that is, the number having the property $1/p + 1/p' = 1$. Consider the following conditions on K:

a) K is non-negative on its domain;

b) $K(r, .)$ is in $L^{p'}(Q)$ for almost all r in Q;

c) $z(q) = \left(\int\limits_Q (K(r, q))^{p'}\, dq\right)^{1/p'}$ is in $L^p(Q)$ and has the norm ≤ 1;

d) there exists a positive c such that

$$K(r, q) \geq c$$

for almost all (r, q) in $Q \times Q$.

(A) Suppose for the sake of simplicity that the linear operator A defined by

$$(Ay)(r) = \int\limits_Q K(r, q) y(q) dq$$

acts in $L^p(Q)$.

If f is a function as in 4° which satisfies the conditions 4°a) and 4°b), if K has the property (A) and satisfies the conditions a), b) and c) above, then the Hammerstein operator S defined by

$$S(y)(r) = \int\limits_Q K(r, q) f(q, y(q))\, dq$$

is a W-correct operator acting in $L^p(Q)$, where W is the cone of non-negative elements of this space.

The continuity of S follows from Theorem 10.1.8 in [10]. The sublinearity of its is a consequence of 4° b). S has the range in W by 4° a) and by a). We shall check that S is nonexpansive. Let y_1 and y_2 be elements of $L^p(Q)$. We have, using Hölder's inequality and the Lipschitz property of f

$$\|S(y_1) - S(y_2)\|^p = \int\limits_Q \left|\int\limits_Q K(r, q)\,(f(q, y_1(q)) - f(q, y_2(q)))\, dq\right|^p dr \leq$$

$$\leq \int\limits_Q \left(\int\limits_Q K(r, q)\, |f(q, y_1(q)) - f(q, y_2(q))|\, dq\right)^p dr \leq \int\limits_Q \left(\int\limits_Q K(r, q)\, |y_1(q) - y_2(q)|\, dq\right)^p dr \leq$$

$$\leq \int\limits_Q \left(\left(\int\limits_Q (K(r, q))^{p'}\, dq\right)^{p/p'} \int\limits_Q |y_1(q) - y_2(q)|^p\, dq\right) dr = \|y_1 - y_2\|^p \int\limits_Q \left(\int\limits_Q K(r, q))^{p'}\, dq\right)^{p/p'} dr.$$

By the condition c) we have

$$\left(\int_Q \left(\int_Q (K(r, q))^{p'} dq\right)^{p/p'} dr\right)^{1/p} \leqslant 1$$

and hence the above relation yields

$$||S(y_1) - S(y_2)|| \leqslant ||y_1 - y_2||,$$

that is, S is nonexpansive.

It suffices now to apply Corollary 1 to conclude that the operator S is W-correct.

If f satisfies also condition 4°c) and K satisfies also c), then S is in plus c_0-norm definite with some positive c_0.

6° Let the set Q be as in 4° and let K be a function from $Q \times Q \times \mathbf{R}$ to $\mathbf{R}$. Suppose that $K(.,.,x)$ is for any x in $\mathbf{R}$ measurable on $Q \times Q$. Consider the following conditions on K:

a) K is non-negative on its domain;
b) K is sublinear and is uniformly Lipschitz on $Q \times Q$ with respect the last variable, having the Lipschitz constant 1;
c) there exists a positive c such that

$$cx \leqslant K(r, q, x)$$

for almost all (r, q) in $Q \times Q$ and any positive x.

Let W be the cone of the non-negative elements on $L^p(Q)$. Consider the Uryshon operator S defined on $L^p(Q)$ by

$$S(y)(r) = \int_Q K(r, q, y(q))dq.$$

The measurability hypothesis on K and the condition b) imply that S acts from $L(Q)$ in itself continuously (Theorem 10.1.13 in [10]). If it holds also a) then S is W-correct since it is nonexpansive by the uniform Lipschitzianity supposed in b).

If K satisfies also the condition in c), then S is also c-norm definite on W.

4. Nonlinear operators that transform a wedge.

PROPOSITION 1. *Let W be a wedge in the normed space Y and suppose that S is a W-correct operator acting in Y. Put $S_t = I - tS$. Then $S_t^{-1}(W)$ is a wedge for any t in $[0, 1)$. If $0 \leqslant t_1 \leqslant t_2 < 1$, then $S_{t_2}^{-1}(W) \subset S_{t_1}^{-1}(W) \subset W$.*

Proof. Let $W_t = S_t^{-1}(W)$. We have by definition

$$W_t = \{y \in Y : y - tS(y) \in W\}.$$

For any positive real number s and any y in W we have

$$sy - tS(sy) \in sW = W,$$

because S is positively homogeneous and W is a wedge. This means that sy is in W_t.

Suppose that y_1 and y_2 are in W_t, that is that $y_1 - tS(y_1)$ and $y_2 - tS(y_2)$ are in W. It follows that

$$y_1 + y_2 - t(S(y_1) + S(y_2)) \in W. \tag{5}$$

Since S is W-subadditive, we have

$$t(S(y_1) + S(y_2)) - tS(y_1 + y_2) \in W.$$

Adding this to (5), it follows that

$$y_1 + y_2 - tS(y_1 + y_2) \in W$$

and hence $y_1 + y_2$ is in W_t. This proves that W_t is a wedge for any t in $[0, 1)$.

We have for any $0 \leqslant t_1 \leqslant t_2 < 1$ and any y in Y

$$t_2S(y) - t_1S(y) \in W. \tag{6}$$

If $y \in W_{t_2}$, i.e., $y - t_2S(y) \in W$, then by adding this relation to (6) we get

$$y - t_1S(y) \in W,$$

which impies that y is in W_{t_1}. We have also $S_0^{-1} = I$, relation that completes the proof. Q.E.D.

PROPOSITION 2. *Let W be a wedge in the normed space Y and let S be a W-correct operator acting in this space. Then $S_t^{-1}(W) \cap (-S_t^{-1}(W))$, with $S_t = I - tS$, is for any t in $(0, 1)$ the maximal subspace W_0 of W with the property that $y \in W_0$ implies that $-S(y)$ and $-S(-y)$ are in W.*

Proof. Denote by W_0 the subset in W of the elements y having the property that

$$y, -y, -S(y), -S(-y) \text{ are in } W. \tag{7}$$

Then W_0 is a subspace of W. Indeed, if y is in W_0, then sy, $-sy$, $-S(sy)$, $-S(-sy)$ are for any real s in W and hence sy is in W_0. Suppose that y_1 and y_2 are in W_0. Then $y_1 + y_2$, $-y_1 - y_2$, $-S(y_1) - S(y_2)$ and $-S(-y_1) - S(y_2)$ are all in W. Because S is W-subadditive, $S(y_1) + S(y_2) - S(y_1 + y_2) \in W$ and $S(-y_1) + S(-y_2) - S(-y_1 - y_2) \in W$. Thus $y_1 + y_2$, $-(y_1 + y_2)$, $-S(y_1 + y_2)$ and $-S(-y_1 - y_2)$ are all in W and hence $y_1 + y_2$ is in W_0. This proves that W_0 is a subspace of W and by its construction it follows also that it is the maximal subspace in W with elements y for that $-S(y)$ and $-S(-y)$ are in the wedge W.

Suppose that y is in W_0. Then y and $-S(y)$ are in W and hence $y - tS(y)$ is in W for any positive t. Thus W_0 is in W_t for any t in $(0,1)$.

Suppose now that for some t in $(0,1)$ y and $-y$ are in W_1, i.e., $y - tS(y)$ and $-y - tS(-y)$ are in W. Since the range of S is contained in W, $tS(y)$ and $tS(-y)$ are in W. It follows then that y and $-y$ are in W and hence $-S(y)$ and $-S(-y)$ are in W too. This means that y is in the subspace W_0 and yields the inclusion $W_t \cap (-W) \subset W_0$. Q.E.D.

COROLLARY 2. *Let S be as in Proposition 2. If for any y in W at least one of the elements $-y$, $-S(y)$ and $-S(-y)$ is outside W, then $S_t^{-1}(W)$ is for any t in $(0,1)$ a cone.*

5. **Some permanence properties of the transformed wedge.** Let W be a wedge in the normed space Y. We have seen that if S is a W-correct operator acting in Y, then $S_t^{-1}(W)$, where $S_t = I - tS$, is a wedge for any t in $[0,1)$. We shall show that some properties of W are preserved by this transformation. As a direct consequence of the definition of the W-correct operators and of Proposition 1 we have

COROLLARY 3. *The interior of each wedge $S_t^{-1}(W)$ with t in $(0,1)$ is nonvoid if and only if W has this property. The element y is in* int W *if and only if $S_t^{-1}(y)$ is in* int $S_t^{-1}(W)$.

The cone K in Y is said to be *normal* if there exists a positive c such that $y_1, y_2 \in K$, $||y_1|| = ||y_2|| = 1$ imply that $||y_1 + y_2|| \geqslant c$. The cone K is normal if and only if the norm in Y is *K-semi monotone.* This means that denoting by $\leqslant$ the K-ordering in Y, there exists some positive b such that whenever $0 \leqslant y_1 \leqslant y_2$, then it holds $||y_1|| \leqslant b||y_2||$. (See e.g. Theorem 1.2 in [5].) Since the subcone of a normal cone is obviously normal, we have as a direct consequence of Proposition 1 the.

COROLLARY 4. *Let K be a normal cone in the normed space Y and let S be a K-correct operator. Then $S_t^{-1}(K)$ is for each t in $(0,1)$ a normal cone.*

The wedge W in Y is said to be *generating*, if for each y in Y there exist w_1 and w_2 in W such that $y = w_1 - w_2$.

PROPOSITION 3. *Let S be a W-correct operator. If W is a generating wedge in Y, then $S_t^{-1}(W)$ has for each t in $(0,1)$ the same property.*

We shall use in the proof the following

LEMMA 3. *For any w in W there exists some u in $S_t^{-1}(W)$ such to $u - w$ be in $S_t^{-1}(W)$.*

Proof. Let w be in W. Let us denote by $\leqslant$ the W-ordering in Y. We have to prove the existence of a u having the properties $u - tS(u) \geqslant 0$ and $u - w - tS(u - w) \geqslant 0$. Since S has its range in W, w and $tS(-w)$ are in $W = S_tS_t^{-1}(W)$. Hence there exist v_1 and v_2 in $S_t^{-1}(W)$ such that $w = S_t(v_1)$ and $tS(-w) = S_t(v_2)$, that is,

$$w = v_1 - tS(v_1)$$

and

$$tS(-w) = v_2 - tS(v_2).$$

Using the sublinearity of S we get

$$v_1 + v_2 - w - tS(-w) - tS(v_1 + v_2) \geqslant v_1 - tS(v_1) - w + v_2 - tS(v_2) - tS(-w) = 0.$$

Putting in this relation $u = v_1 + v_2$ then u is in $S_t^{-1}(W)$ and using again the sublinearity of S we conclude

$$u - w - tS(u - w) \geqslant u - w - tS(u) - tS(-w) \geqslant 0. \qquad \text{Q.E.D.}$$

Proof of Proposition 3. Let y be an element in Y. There exist by hypothesis w_1 and w_2 in W such that

$$y = w_1 - w_2.$$

In accordance with Lemma 3 there exist u_1 and u_2 in $S_t^{-1}(W)$ such that $u_1 - w_1$ and $u_2 - w_2$ are in $S_t^{-1}(W)$. Put $u = u_1 + u_2$. Then $z_1 = u - w_1 = u_2 + u_1 - w_1$ and $z_2 = u - w_2 = u_1 + u_2 - w_2$ are in $S_t^{-1}(W)$. On the other hand

$$z_2 - z_1 = (u - w_2) - (u - w_1) = w_1 - w_2 = y,$$

that is, $S_t^{-1}(W)$ is a generating wedge. Q.E.D.

The element e in W is said to be a *W-order unit* of Y, if for any y in Y there exists an $s > 0$ such that $se - y \in W$.

PPROPOSITION 4. *Let e be a W-order unit in Y. Then for each t in* $(0,1)$, $S_t^{-1}(e)$ *will be an $S_t^{-1}(W)$-order unit in Y.*

Proof. Put $e_1 = S_t^{-1}(e)$. Let y be an arbitrary element of Y. Since e is a W-order unit, there exists an $s > 0$ such that $se - (y + tS(-y)) \in W$. We have $e = S_t(S_t^{-1}(e)) = S_t(e_1) = e_1 - tS(e_1)$ and hence $tS(e_1) = e_1 - e$. On the other hand $stS(e_1) + tS(-y) - tS(se_1 - y) \in W$ by the W-sublinearity of S. Denoting by $\leqslant$ the W-ordering in Y and using the obtained relations we get

$$(se_1 - y) - tS(se_1 - y) \geqslant (se_1 - y) - stS(e_1) - tS(-y) =$$
$$= se_1 - y - s(e_1 - e) - tS(-y) = se - y - tS(-y) \geqslant 0.$$

This means that $se_1 - y$ is in $S_t^{-1}(W)$. Q.E.D.

6. Nonlinear operators that transform a wedge in a regular, respectively in a completely regular cone. Let W_0 be a wedge contained in W. We shall say that W_0 is *W-regular (completely W-regular)* (see 1.8.4 in [5]) if any W-increasing and W-bounded (norm bounded) sequence of elements in W_0 is fundamental. If W is itself W-regular (completely W-regular), then it is called simply *regular (completely regular)*. It is easy to see that if W_0 is W-regular (completely W-regular) for some wedge W containing it, then W_0 must be in fact a cone. In particular any regular (completely regular) wedge is in fact a cone. From the above definitions it follows also that if W_0 is a W-regular (completely W-regular) cone, then W_0 is itself regular (completely regular), because any W_0-increasing and W_0-bounded (norm bounded) sequence is also W-increasing and W-bounded (norm bounded) one. We shall need in the sequel the following characterization lemma, which generalizes Lemma in [7]:

LEMMA 4. *Let W_0 be a cone contained by the wedge W. Then W_0 is W-regular (completely W-regular) if and only if for any sequence (y_i) in W_0, the con-*

dition $||y_i|| \geqslant d$ *for any* i *and some positive* d *implies that the set* $\left\{\sum_{i=1}^{n} y_i : n = 1, 2, \ldots\right\}$ *cannot be W-bounded (norm bounded).*

Proof. Suppose that there exists a sequence in W_0 with the property that $||y_i|| \geqslant d$ for any i and some positive d, and the set $\left\{\sum_{i=1}^{n} y_i : n = 1,2, \ldots\right\}$ is W-bounded (norm bounded). Then $z_n = \sum_{i=1}^{n} y_i$, $n = 1, 2, \ldots$ form a W-increasing, W-bounded (norm bounded) sequence in W_0 that cannot be fundamental. Hence W_0 cannot be W-regular (completely W-regular).

Assume now that W_0 isn't W-regular (completely W-regular), that is, that W_0 contains a W-increasing, W-bounded (norm bounded) sequence (z_j) with the property that $||z_{j+1} - z_j|| \geqslant d$ for any j and for some positive d. Put $y_i = z_{i+1} - z_i$. Then $z_{n+1} - z_1 = \sum_{i=1}^{n} y_i$ and the set $\left\{\sum_{i=1}^{n} y_i : n = 1, 2, \ldots\right\}$ is W-bounded (norm bounded) because (z_n) has this property. Since $||y_i|| \geqslant d$ for any i, we get a contradiction with the condition in the lemma. Q.E.D.

PROPOSITION 5. *Let W be a wedge in the normed space Y and let S be a W-correct operator which is c-norm definite on W for some positive c. If $W_S =$ $=$ co $S(W)$ is a W-regular cone, then $S_t^{-1}(W)$ is a regular cone for each t in* $(0, 1)$.

Proof. Let be $W_t = S_t^{-1}(W)$ and assume that (y_n) is a W_t-increasing, W_t-bounded sequence in Y which isn't fundamental. Passing if necessary to a subsequence we can suppose that $||y_{n+1} - y_n|| \geqslant d$ for any n and for some positive d. Since (y_n) is W_t-increasing, we have

$$y_{n+1} - y_n \geqslant tS(y_{n+1} - y_n), \tag{8}$$

where we have denoted by $\leqslant$ the W-ordering in Y. Summing this relation from $n = 1$ to $n = m$, we get

$$y_{m+1} - y_1 \geqslant t(S(y_2 - y_1) + \ldots + S(y_{m+1} - y_m)). \tag{9}$$

On the other hand, the c-norm definiteness of S implies

$$||S(y_{n+1} - y_n)|| \geqslant c||y_{n+1} - y_n|| \geqslant cd > 0 \tag{10}$$

for any n. Because $S(y_{n+1} - y_n)$ is in W_S (we have $y_{n+1} - y_n \in S_t^{-1}(W) \subset S_0^{-1}(W) =$ $= W$ by Proposition 1), and since (y_n) is W-bounded (by the same reasoning), using Lemma 4 and the relation (9) we get a contradiction with the hypothesis that W_S is W-regular. Q.E.D.

PROPOSITION 6. *Let W be a normal cone in Y and suppose that S in as in Proposition 5. If $W_S = \mathrm{co}\, S(W)$ is a completely W-regular cone, then $S_t^{-1}(W)$ is a completely regular cone for each t in* $(0, 1)$.

Proof. Put as above $W_t = S_t^{-1}(W)$ and assume that (y_n) is a W_t-increasing norm bounded sequence in Y which isn't fundamental. We proceed as in proof of Proposition 5 to get the relations (8), (9) and (10). Using the semi-monotonity of the norm in Y (we have supposed that W is a normal cone), from the relation (9) it follows that

$$a||y_{m+1} - y_1|| \geqslant t||S(y_2 - y_1) + \ldots + S(y_{m+1} - y_m)||$$

for some positive a. Because (y_n) is norm bounded, $||y_{n+1} - y_n|| \geqslant d$ for any n, S is c-norm definite on W, the above relation and (10) furnish via Lemma 4 a contradiction with the hypothesis that W_S is a completely W-regular cone. Q.E.D.

Proposition 6 is rather weak in comparison with Proposition 5. The inconvenience is that we need in its proof the normality of W. To avoid this, we shall give sufficient conditions for complete regularity (as well as for regularity) of $S_t^{-1}(W)$ in other terms, using the methods in [5], 1.6. Let us introduce first some notions.

The functional f (defined either on W or on Y) is called *W-positive*, if $f(y) \geqslant 0$ for any y in W; it is called *strictly W-positive* if it is W-positive and $f(y) > 0$ for any y in $W \setminus \{0\}$. f is said to be *uniformly W-positive*, if $f(y) \geqslant \geqslant b||y||$ for some positive b and for any y in W. If from $y_n \in W$ and $||y_n|| \geqslant d$ for any n and for some positive d it follows that $\lim_m f\left(\sum_{i=1}^{m} y_i\right) = \infty$, then f is called *strictly W-increasing*. The functional f is called *W-monotone*, if $f(u) \leqslant f(v)$ whenever $v - u \in W$.

A functional f defined on W is called *superlinear* if it is positively homogeneous and $f(u + v) \geqslant f(u) + f(v)$ for any u and v in W.

PROPOSITION 7. *Let W be a wedge in Y and let S be a W-correct operator. If there exists a superlinear functional on Y which is bounded on bounded sets in Y, is W-positive and has the property that fS is uniformly W-positive, then $S_t^{-1}(W)$ is a completely regular cone for any t in* $(0,1)$.

Proof. Let be $W_t = S_t^{-1}(W)$ and assume that (y_n) is a W_t-increasing norm bounded sequence which isn't fundamental. Passing if necessary to a subsequence we can suppose that $||y_{n+1} - y_n|| \geqslant d$ for some positive d and for any n. Since (y_n) is W_t-increasing we have

$$y_{n+1} - y_n - tS(y_{n+1} - y_n) \geqslant 0, \quad n = 1, 2, \ldots, \tag{11}$$

where we have used $\leqslant$ for the W-ordering in Y. We apply f to the element in (11) and use the W-positivity of it and the superlinearity of it to conclude

$$f(y_{n+1} - y_n) - tfS(y_{n+1} - y_n) \geqslant f(y_{n+1} - y_n - tS(y_{n+1} - y_n)) \geqslant 0.$$

From this relation and the fact that fS is uniformly W-positive, we have

$$f(y_{n+1} - y_n) \geqslant tb||y_{n+1} - y_n|| \geqslant tbd > 0$$

with some positive b. Using now the idea in the proof of Theorem 1. 10 in [5], we have

$$f(y_m) = f\left(y_1 + \sum_{n=1}^{m-1} (y_{n+1} - y_n)\right) \geqslant f(y_1) + \sum_{n=1}^{m-1} f(y_{n+1} - y_n) \geqslant$$
$$\geqslant f(y_1) + (m-1)tbd.$$

The obtained inequality implies that $f(y_m) \to \infty$ for $m \to \infty$. But this contradicts the assumptions that (y_n) is norm bounded and that f is bounded on norm bounded sets. Q.E.D.

PROPOSITION 8. *Let S be as in Proposition 5. If there exists a functional f which is W-positive and superlinear on W, is bounded on W-order bounded sets of W and is strictly increasing on $W_S = \mathrm{co}\, S(W)$, then $S_t^{-1}(W)$ is a regular cone for each t in $(0,1)$.*

Proof. Let be $W_t = S_t^{-1}(W)$ and assume that (y_n) is a W_t-increassing, W_t-order bounded sequence in Y that isn't fundamental. Passing if necessary to a subsequence, we can suppose that $||y_{n+1} - y_n|| \geqslant d$ for any n and for some positive d. Since (y_n) is W_t-increasing, we have

$$y_{n+1} - y_n - tS(y_{n+1} - y_n) \geqslant 0, \; n = 1, 2, \ldots$$

where $\leqslant$ stands for the W-ordering in Y. Summing this relation from $n = 1$ to $n = m$, we get

$$y_{m+1} - y_1 - t(S(y_2 - y_1) + \ldots + S(y_{m+1} - y_n)) \geqslant 0.$$

Using the superlinearity and the W-positivity of f it follows

$$f(y_{m+1} - y_1) - tf(S(y_2 - y_1) + \ldots + S(y_{m+1} - y_m)) \geqslant 0.$$

Since $||S(y_{n+1} - y_n)|| \geqslant c||y_{n+1} - y_n|| \geqslant cd$ and since f is strictly increasing on W_S, it follows that

$$f(y_{m+1} - y_1) \geqslant tf(S(y_2 - y_1) + \ldots + S(y_{m+1} - y_m)) \to \infty \tag{12}$$

for $m \to \infty$. Now, $\{y_{m+1} - y_1 : m = 1, 2, \ldots\}$ is a W-order bounded set because it is W_t-order bounded by hypothesis. Hence (12) furnish a contradiction with the assumption that f is bunded on W-order bounded sets. Q.E.D.

7. **Examples and comments.** 1°. Suppose that W is a wedge in the normed space Y with the property that its closure $\overline{W}$ isn't a subspace of Y. Then there exists an element y in $\overline{W}$ such that $-y \notin \overline{W}$. If k is an element of W sufficiently close to y, then $-k \notin \overline{W}$. Let k be an element of W with this property which have norm 1. According the Theorem 1.4 in [6], there exists a continuous linear functional y' such that it is W-positive and $(y', k) > 0$. We

can suppose that $(y', k) = 1$. Define the correct 1-norm definite sublinear operator S as in the example 1° of 3, i.e., let be

$$S(y) = ||y||k \tag{13}$$

for any y in Y. Then $y'S$ is uniformly W-positive since $(y'S)(y) = ||y||$ for any y in W. Because y' is a W-positive linear functional, it follows according Proposition 7 that $S_t^{-1}(W)$, where $S_t = I - tS$, is a completely regular cone for any t in (0,1). Using also Lemma 1 and 2 and Proposition 1 we can state the

PROPOSITION 9. *Let W be a wedge in the normed space Y such that its closure $\overline{W}$ isn't a subspace. Then there exists a family S_t^{-1} ($t \in [0, 1)$) of continuous one-to-one operators such that:* (a) *S_t^{-1} depends continuously on t with respect to the topology of the uniform convergence of the operators on norm bounded sets in Y;* (b) *if $0 \leq t_1 \leq t_2 < 1$, then $S_{t_1}^{-1}(W) \supset S_{t_2}^{-1}(W)$;* (c) *$S_0^{-1} = I$;* (d) *$S_t^{-1}(W)$ is a completely regular cone for each t in* (0, 1). *S_t^{-1} also preserves some important properties of the wedge (cone) W: if W is* (i) *generating or (and)* (ii) *has an order unit or (and)* (iii) *is normal or (and)* (iv) *has a nonvoid interior, then so does the one $S_t^{-1}(W)$ for each t in* (0, 1).

For the last four assertions see the Propositions 3 and 4 and the Corollaries 4 and 3.

If W is a semispace, then $S_t^{-1}(W)$, where S_t is defined as above, will be for any t in (0, 1) a completely regular cone with a nonvoid interior. The described method is another way of introducing orderings in a normed space in order that the order convergence and the topological convergence coincide and thus it completes the results of DeMarr [2] and of Vulih and Danilenko [9].

The most important consequence of Proposition 9 is that a wide class of orderings (those induced by wedges whose closures aren't subspaces) have the property that after a „slight modification" they become completely regular ones. The regular and the completely regular orderings have both theoretical and computational use. For instance, any regular B-space has the chain completeness property and hence any convex mapping with values in a such space have subgradients (this is an immediate consequence of a result of Fel'dman in [4]). On the other hand, the regularity of an ordering is very imortant in the numerical analysis (see e.g. the paper [8] of Vandergraft).

The operator S_t^{-1} cannot be defined for $t > 1$. In some concrete cases the transformed wedge W_t can be defined directly. Let us consider the wedge W and the element k of its, having the above properties. Put

$$W_t = \{y \in Y : y - t||y||k \in W\}.$$

If W is a normal cone and the norm in Y is monotone (i.e., if $0 \leq y_1 \leq y_2$ implies $||y_1|| \leq ||y_2||$), then W_t reduces to $\{0\}$ for any $t > 1$. Indeed, if $y \in W_t$, then $y - t||y||k \geq 0$ and using the monotonity of the norm we conclude that either $y = 0$ or $t \leq 1$. In this case W_1 is the cone generated by the elements y of norm one in W, with the property that $y - k$ is in W.

2°. Consider the space $C(Q)$ of the real-valued continuous functions def ned on the compact Hausdorff space Q, and let W be the cone of the non-nega tive functions in $C(Q)$. Define the operator S by putting

$$S(y)(q) = |y(q)|.$$

Then S is a particular form of the oprator considered in the example 3° of 3 Hence it is W-correct. We shall check that S_t^{-1}, where $S_t = I - tS$, transform the cone W onto itself for any t in $[0, 1)$. Fix t and suppose that z is an arbitrary element of W. Put $y = (1 - t)z$. Then $S_t^{-1}(y) = z$. Indeed, we have $S_t(z) = z - tz = (1 - t)z = y$, and hence our assertion follows.

The cone W is normal but it isn't regular. Thus the considered example shows that the condition of the norm definitness of the correct operator S is essential in all the considerations in 5.

(Received May 7, 1980)

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OPERATORI NELINIARI CARE TRANSFORMĂ UN CON

(Rezumat)

Se dă o metodă de transformare a unui con într-un subcon al său astfel ca să se conserve unele proprietăţi (interiorul nevid, de a fi generator, de a poseda unitate de ordine etc.) ale conului original şi de a obţine îmbunătăţirea altora dintre proprietăţile sale. Operatorii cu ajutorul cărora se efectuează această transformare sînt de fapt rezolvenţi ai unor operatori subliniari cu proprietăţi speciale introduşi în lucrare. Se arată printre altele că orice con (în general nepropriu), a cărui închidere nu este un subspaţiu, poate fi aproximat oricît de bine cu subconuri complet regulare.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

A STUDY OF THE REMAINDER IN AN APPROXIMATION FORMULA USING A FAVARD-SZÁSZ TYPE OPERATOR*

D. D. STANCU

1. In our previous paper [11] we gave a method for constructing a class of linear positive operators, depending on a real parameter α, defined for any function $f: J \to R$, J being a certain interval of the real axis, by the following formula

$$(L_m^{\langle\alpha\rangle} f)(x) := \frac{1}{\varphi_m^{\langle\alpha\rangle}(0)} \sum_{k=0}^{\infty} (-1)^k \frac{x^{[k,-\alpha]}}{k!} \cdot D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \cdot f(x_{m,k}), \tag{1}$$

where $x \in I = [0, a](a > 0)$, $x^{[k,-\alpha]} = x(x+\alpha)\dots(x+(k-1)\alpha)$, $x_{m,k} \in J \supseteq I$, while $(\varphi_m^{\langle\alpha\rangle})$ is a sequence of functions depending on α, which are analytic in a region D containing the disk $|z - a| \leq a$ and which can be expanded in a Newton convergent series on D. By D_α^k we have denoted the Nörlund difference quotient, defined by

$$D_\alpha^k g(x) = D_\alpha(D_\alpha^{k-1} g(x)), \quad D_\alpha g(x) = \frac{g(x+\alpha) - g(x)}{\alpha}, \quad D_\alpha^0 g(x) = g(x).$$

If we assume that $\alpha \geqq 0$ and

$$\varphi_m^{\langle\alpha\rangle}(0) > 0, \ (-1)^k D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \geqq 0,$$

for any $x \in I$, $k = 0, 1, 2, \dots$ and $m = 1, 2, \dots$, one sees that $(L_m^{\langle\alpha\rangle})$ is a sequence of positive linear operators.

For $I = J = [0, 1]$, $x_{m,k} := \frac{k}{m}$ and $\varphi_m^{\langle\alpha\rangle}(x) := (1-x)^{[m,-\alpha]}$ the operator $L_m^{\langle\alpha\rangle}$ reduces to the operator of Bernstein-type $P_m^{\langle\alpha\rangle}$ introduced for the first time in our paper [8], namely:

$$(P_m^{\langle\alpha\rangle} f)(x) := \sum_{k=0}^{m} \binom{m}{k} \frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}} f\left(\frac{k}{m}\right).$$

These polynomials of Bernstein-type have been recently further investigated by G. Mastroianni and M. R. Occorsio [3], [4], who called them „the Stancu polynomials". We also mention that in our previous paper [9] we have studied the operator $P_m^{\langle\alpha\rangle}$ by probabilistic tools.

* The announcement of the results of this paper has been presented in our note [14].

In 1968, assuming that $\alpha > 0, 0 < x < 1$, we gave [8] an integral representation of this operator by means of the operator of Bernstein B_m, namely

$$(P_m^{\langle\alpha\rangle} f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(B_m f)(t)dt, \tag{2}$$

by using the beta function $B(a, b)$.

It is obvious that in the general case for $\alpha > 0$ we can write the operator (1) under the form

$$(L_m^{\langle\alpha\rangle} f)(x) = \frac{1}{\varphi_m^{\langle\alpha\rangle}(0)} \sum_{k=0}^{\infty} \binom{-\frac{x}{\alpha}}{k} (\Delta_\alpha^k \varphi_m^{\langle\alpha\rangle})(x) \cdot f(x_{m,k}).$$

Assuming that $x_{m,k} = \frac{k}{m}$, according to a result established in [11] we can give the following representation

$$(L_m^{\langle\alpha\rangle} f)(x) = \frac{1}{\varphi_m^{\langle\alpha\rangle}(0)} \sum_{j=0}^{\infty} (-1)^j \frac{x^{[j,-\alpha]}}{j!} (D_\alpha^j \varphi_m^{\langle\alpha\rangle}(0)) \cdot (\Delta_{\frac{1}{m}}^j f)(0). \tag{3}$$

When

$$\varphi_m^{\langle\alpha\rangle}(x) := \frac{1}{(1+x)^{[m,-\alpha]}}, \quad x_{m,k} := \frac{k}{m}, \ \alpha \geqq 0,$$

then one obtains a class of linear positive operators investigated in our earlier paper [10], which for $\alpha = 0$ reduces to the Baskakov operator.

2. Now we shall choose in (1):

$$\varphi_m^{\langle\alpha\rangle}(x) := (1+\alpha m)^{-\frac{x}{\alpha}}, \quad x_{m,k} = \frac{k}{m}. \tag{4}$$

Because we have

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^k \left(\frac{m}{1+\alpha m}\right)^k (1+\alpha m)^{-\frac{x}{\alpha}},$$

we obtain a class of operators of Favard-Szàsz type, defined by

$$(L_m^{\langle\alpha\rangle} f)(x) := \sum_{k=0}^{\infty} w_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}\right), \tag{5}$$

where

$$w_{m,k}^{\langle\alpha\rangle}(x) := (1+\alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1+\alpha m}\right)^k \cdot \frac{x^{[k,-\alpha]}}{k!}. \tag{6}$$

If we make in (5), (6) $\alpha \to 0$, then we get the operator of J. Favard [1] and O. Szász [16]:

$$(L_m f)(x) := e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right). \tag{7}$$

By applying the formula (3), we obtain a representation in terms of finite and divided differences:

$$(L_m^{\langle\alpha\rangle} f)(x) = \sum_{j=0}^{\infty} \frac{m^j x^{[j,-\alpha]}}{j!} \left(\Delta^j_{\frac{1}{m}} f\right)(0) = \sum_{j=0}^{\infty} x^{[j,-\alpha]}\left[0, \frac{1}{m}, \ldots, \frac{j}{m}; f\right],$$

since

$$\left(\Delta^j_{\frac{1}{m}} f\right)(0) = \frac{j!}{m^j}\left[0, \frac{1}{m}, \ldots, \frac{j}{m}; f\right].$$

In 1968 we gave for the operators $P_m^{\langle\alpha\rangle}$ the integral representation (2) and for an operator $W_m^{\langle\alpha\rangle}$, of Meyer-König and Zeller type, introduced in 1970 in [10], we gave an intergal representation by using the original Meyer-König and Zeller operator M_m. In the case of the opetaror $L_m^{\langle\alpha\rangle}$, from (5)—(6), such a representation, by using the operator (7), has been given in a paper published in 1972 by S. P. Pethe and G. C. Jain [7], namely

$$(L_m^{\langle\alpha\rangle} f(x) = \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)} \int_0^{\infty} e^{-t} t^{\frac{x}{\alpha}-1} (L_m f)(\alpha t) dt,$$

using the idea for such a representation from our paper [8] (mentioned in their paper), where we gave formula (2). It should be noticed that this operator has been only mentioned under this form in [7], but not investigated. In fact the authors have only been concerned there with the study of the operator $W_m^{\langle\alpha\rangle}$ introduced and investigated first in [10]. They were probably not aware of our paper [10].

3. We next turn to the task of evaluating the remainder term of the approximation formula

$$f(x) = (L_m^{\langle\alpha\rangle} f)(x) + (R_m^{\langle\alpha\rangle} f)(x). \tag{8}$$

First of all we wish to determine the degree of exactness of this formula. For this purpose we shall calculate the values of the operator $L_m^{\langle\alpha\rangle}$ for the monomials $e_j(x) = x^j$ $(j = 0, 1, 2)$ for any $x \geq 0$. The remainder of this formula in the case $\alpha = 0$ has been evaluated in different forms by A. Lupaş [2] and by F. Stancu [15]. So that we shall assume that $\alpha > 0$.

We can write succesively

$$(L_m^{\langle\alpha\rangle} e_0)(x) = (1+\alpha m)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \frac{x^{[k,-\alpha]}}{k!} \left(\frac{1+\alpha m}{m}\right)^k =$$

$$= (1+\alpha m)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \binom{\frac{x}{\alpha}+k-1}{k} \left(\frac{\alpha m}{1+\alpha m}\right)^k = (1+\alpha m)^{-\frac{x}{\alpha}} \left(1 - \frac{\alpha m}{1+\alpha m}\right)^{-\frac{x}{\alpha}} = 1,$$

since according to the binomial series we have the following expansion

$$(1-y)^{-\beta} = \sum_{k=0}^{\infty} \binom{\beta+k-1}{k} y^k \quad (|y|<1).$$

Consequently we have $(R_m^{\langle\alpha\rangle} e_0)(x) = 0$ for any $x \geqq 0$.

Next, we can write

$$(L_m^{\langle\alpha\rangle} e_1)(x) = (1+\alpha m)^{-\frac{x}{\alpha}} \sum_{k=1}^{\infty} m^k \frac{x(x+\alpha)\ldots(x+(k-1)\alpha)}{k!(1+\alpha m)^k} \cdot \frac{k}{m} =$$

$$= (1+\alpha m)^{-\frac{x}{\alpha}-1} \cdot x \sum_{j=0}^{\infty} \frac{(\alpha m)^j}{(1+\alpha m)^j} \binom{\frac{x}{\alpha}+j}{j} =$$

$$= (1+\alpha m)^{-\frac{x}{\alpha}-1} \cdot x \cdot \left(1 - \frac{\alpha m}{1+\alpha m}\right)^{-\left(\frac{x}{\alpha}+1\right)} = x.$$

Hence: $(R_m^{\langle\alpha\rangle} e_1)(x) = 0$ for any $x \geqq 0$.

Finally, by similar calculations we obtain

$$(L_m^{\langle\alpha\rangle} e_2)(x) = x^2 + \frac{1+\alpha m}{m} x,$$

and it follows that

$$(R_m^{\langle\alpha\rangle} e_2)(x) = -\frac{1+\alpha m}{m} x. \tag{9}$$

Hence, the approximation formula (8) has the degree of exactness $N = 1$.

In order to evaluate the remainder, we first notice that because $L_m^{\langle\alpha\rangle} e_0 = e_0$ we can write succesively

$$(R_m^{\langle\alpha\rangle} f)(x) = (1+\alpha m)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \left(\frac{m}{1+\alpha m}\right)^k \frac{x^{[k,-\alpha]}}{k!} \left[f(x) - f\left(\frac{k}{m}\right)\right] =$$

$$= \frac{1}{m} (1+\alpha m)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \left(\frac{m}{1+\alpha m}\right)^k \frac{x^{[k,-\alpha]}}{k!} (mx-k) \left[x, \frac{k}{m}; f\right],$$

in terms of divided differences of the first order.

By using the identity: $mx - k = m(x + k\alpha) - k(1 + \alpha m)$, we obtain

$$(R_m^{\langle\alpha\rangle} f)(x) = (1 + \alpha m)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \left(\frac{m}{1 + \alpha m}\right)^k \frac{(x + k\alpha) \cdot x^{[k, -\alpha]}}{k!} \left[x, \frac{k}{m}; f\right] -$$

$$- \frac{1}{m} (1 + \alpha m)^{-\frac{x}{\alpha}} \sum_{k=1}^{\infty} \left(\frac{m}{1 + \alpha m}\right)^k \frac{x^{[k, -\alpha]}}{k!} (1 + \alpha m) \left[x, \frac{k}{m}; f\right].$$

If we make in the last sum the change of index of summation: $k - 1 = j$, we find

$$\frac{1}{m} (1 + \alpha m)^{-\frac{\alpha}{x}} (1 + \alpha m) \sum_{j=0}^{\infty} \left(\frac{m}{1 + \alpha m}\right)^{j+1} \frac{x^{[j+1, -\alpha]}}{j!} \cdot \left[x, \frac{j+1}{m}; f\right] =$$

$$= (1 + \alpha m)^{-\frac{\alpha}{x}} \sum_{k=0}^{\infty} \left(\frac{m}{1 + \alpha m}\right)^k \frac{(x + k\alpha) x^{[k, -\alpha]}}{k!} \cdot \left[x, \frac{k+1}{m}; f\right].$$

It follows that we have

$$(R_m^{\langle\alpha\rangle} f)(x) = (1 + \alpha m)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \left(\frac{m}{1 + \alpha m}\right)^k \frac{(x + k\alpha) x^{[k, -\alpha]}}{k!} \cdot \left\{\left[x, \frac{k}{m}; f\right] - \left[x, \frac{k+1}{m}; f\right]\right\}.$$

Taking into account that

$$\left[x, \frac{k}{m}; f\right] - \left[x, \frac{k+1}{m}; f\right] = -\frac{1}{m} \left[x, \frac{k}{m}, \frac{k+1}{m}; f\right]$$

and using the notation (6), we can give finally the following expression for the remainder of the approximation formula (8):

$$(R_m^{\langle\alpha\rangle} f)(x) = -\sum_{k=0}^{\infty} \frac{x + k\alpha}{m} w_{m,k}^{\langle\alpha\rangle}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f\right] =$$

$$= -\frac{1 + \alpha m}{m} x \sum_{k=0}^{\infty} w_{m,k}^{\langle\alpha\rangle}(x + \alpha) \left[x, \frac{k}{m}, \frac{k+1}{m}; f\right], \tag{10}$$

which is the analogue of the formulas given by us for the remainders in the case of the operators B_m, $P_m^{\langle\alpha\rangle}$ and $W_m^{\langle\alpha\rangle}$ (see our papers [12] and [13]). For $\alpha = 0$ formula (10) has been given in [15].

From the representation (10) of the remainder there follows immediately that:

(i) Formula (8) has the degree of exactness $N = 1$;

(ii) If $\alpha \geq 0$, $m = 1, 2, \ldots$ and f is convex of first-order on $[0, \infty)$, without being linear, then we have $L_m^{\langle\alpha\rangle} f > f$ on $(0, \infty)$;

(iii) If $\alpha \geqq 0$ and all the divided differences of the second-order of f are bounded on $[0, \infty)$, then we have.

$$|(R_m^{\langle\alpha\rangle} f)(x)| \leqq \frac{1 + \alpha m}{m} x M_2(f),$$

where $M_2(f)$ is the least upper bound of the absolute values of the second-order divided differences of f on $[0, \infty)$.

(iv) If $\alpha \geqq 0$ and f is bounded and continuous on $[0, \infty)$, then for any fixed point $x \geqq 0$ we have:

$$(R_m^{\langle\alpha\rangle} f)(x) = -\frac{1 + \alpha m}{m} x [\xi_1, \xi_2, \xi_3; f],$$

where $\xi_i (i = 1, 2, 3)$ are distinct points of $(0, \infty)$, which might depend upon the function f. This formula can be obtained from (10) if we make use of a known theorem of T. Popoviciu [5].

Now if we assume that the second-order derivative of f exists on $[0,\infty)$, then we obtain the following representation for this remainder

$$(R_m^{\langle\alpha\rangle} f)(x) = -\frac{1 + \alpha m}{2m} x f''(\xi). \tag{11}$$

4. Finally, we shall give an integral representation of the remainder. Because formula (8) has the degree of exactness $N = 1$, according to a know theorem of Peano [6] we obtain the following formula

$$(R_m^{\langle\alpha\rangle} f)(x) = \int_0^\infty G_m^{\langle\alpha\rangle}(t; x) f''(t)\, dt, \tag{12}$$

where

$$G_m^{\langle\alpha\rangle}(t; x) = (R_m^{\langle\alpha\rangle} \psi_x)(t), \qquad \psi_x(t) = (x - t)_+ = \frac{x - t + |x - t|}{2},$$

the subscript x indicating that $R_m^{\langle\alpha\rangle}$ is to be performed with respect to x, while t is held fixed.

As in our paper [12], one can give immediately an explicit expression for the Peano kernel $G_m^{\langle\alpha\rangle}$ and it is easy to see that we have $G_m^{\langle\alpha\rangle}(t; x) \leqq 0$ on $[0, \infty) \times [0, \infty)$. Hence we can apply the mean value theorem of the integral calculus and we see that we can obtain in another way formula (11), since it is readily seen that we have

$$\int_0^\infty G_m^{\langle\alpha\rangle}(t; x) dt = \frac{1}{2} (R_m^{\langle\alpha\rangle} e_2)(x) = -\frac{1 + \alpha m}{2m} x.$$

It should be noticed that for a fixed $x \geqq 0$ the kernel $G_m^{\langle\alpha\rangle}(t; x)$ represents a spline function of first degree, having the knots $\frac{k}{m}$.

(Received May 8, 1980)

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UN STUDIU AL RESTULUI ÎNTR-O FORMULĂ DE APROXIMARE CARE FOLOSEŞTE UN OPERATOR DE TIP FAVARD-SZÁSZ

(Rezumat)

În anul 1971 autorul a dat în lucrarea [11] o metodă de construire a unui operator liniar pozitiv general, de forma (1), care depinde de un parametru mic α. În cazul special de la (4) sîntem conduşi la operatorul de tip Favard-Szász definit la (5)–(6), care generalizează operatorul de la (7). În lucrarea de faţă se urmăreşte evaluarea restului din formula de aproximare (8). Printr-o metodă bazată pe teoria diferenţelor divizate, folosită de autor şi în alte lucrări [12], [13], s-a obţinut mai întîi pentru acest rest reprezentarea de la (10) cu ajutorul diferenţelor divizate de ordinul al doilea, precum şi reprezentările de tip Cauchy (11) şi de tip Peano (12).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 4, 1980

SOME PROPERTIES OF THE FIXED POINTS SET FOR MULTIFUNCTIONS

MIRA-CRISTIANA ALICU and OTILIA MARK

Many fixed point theorems can be extended for multifunctions, elementary examples showing that in this case the unicity part is lost. We study the transfer of some properties of the sets that are values of a multifunction to the set of fixed points.

Let (X, d) be a metric space and $S(X) = \{A \in \mathfrak{P}(X) \mid A \neq \emptyset,\ A = \bar{A}\}$. We denote by H the Hausdorff metric on $S(X)$, and for $x \in X$, $D(x, A) = \inf\limits_{y \in A} d(x, y)$.

DEFINITION 1. The multifunction $f: X \to S(X)$ is *contractive* if

$$H(f(x), f(y)) < d(x, y),\ \forall\, x, y \in X,\ x \neq y.$$

DEFINITION 2. Let $\varphi: R_+^5 \to R_+$ be a function satisfying the following properties:

a) $\varphi(r) \leqslant \varphi(s)$, for $r \leqslant s$, $r, s \in R_+^5$

b) φ is continuous

c) $\varphi(r, r, r, r, r,) < r$, for $r > 0$

d) $r - \varphi(r, r, r, r, r) \to +\infty$, for $r \to +\infty$

The multifunction $f: X \to S(X)$ is a *φ-contraction* if

$$H(f(x), f(y)) \leqslant \varphi(d(x, y), D(x, f(x)), D(y, f(y)), D(x, f(y)), D(y, f(x))), \qquad \forall\, x, y \in X.$$

We consider the case of the multifunctions defined on the real axis R or on some subsets of R.

THEOREM 1. *Let $f: [a, b] \to \mathfrak{P}([a, b])$ be a contractive multifunction with $f(x)$ non-void, compact and convex, $\forall x \in [a, b]$. Then the fixed points set is non-void, compact and convex.*

Proof. The only non-void, compact and convex sets in R being the closed intervals, we have $f(x) = [m(x), M(x)]$, where m and M are functions which are defined on $[a, b]$ with values in the same interval. We obtain $H(f(x), f(y)) = \max\{|M(x) - M(y)|,\ |m(X) - m(y)|\}$. The multifunction f being contractive, we have for $x \neq y$

$$|M(x) - M(y)| < |x - y|$$
$$|m(x) - m(y)| < |x - y|.$$

We can apply the theorem of E d e l s t e i n [1], following which the functions m and M have a unique fixed point. Let x_m and x_M be the fixed points, which are also fixed points for f. It is easy to show that $x_m \leqslant x_M$ and that on the left of x_m and on the right of x_M there are not other fixed points. Any point x in] x_m, x_M [is also in $[m(x), M(x)]$, so it is a fixed point.

We have proved that the fixed points set is $[x_m, x_M]$, so it is non-void, compact and convex.

Remark. The statement of Theorem 1 is not true for $f: R \to S(R)$, as the following example shows.

Let $g: R \to R$ be a function defined by $g(x) = \begin{cases} \frac{5}{2}, \text{ for } x \in]-\infty, 2[\\ x + \frac{1}{n}, \text{for } x \in [2, +\infty[\end{cases}$

and $f: R \to S(R)$, $f(x) = [0, g(x)]$.

f satisfies the condition

$$H(f(x), f(y)) < |x - y|, \text{ for } x \neq y,$$

so it is contractive. It also has as values non-void, compact and convex sets, but the fixed points set is $[0, +\infty[$ and it is not compact.

THEOREM 2. *Let $\varphi: R_+^5 \to R_+$ be a function as in Definition 2, and $f: R \to \mathfrak{L}(R)$ a φ-contraction with $f(x)$ non-void, compact and convex, $\forall x \in R$. In this case the fixed points set is non-void, compact and convex.*

Proof. We have $f(x) = [m(x), M(x)]$ and $H(f(x), f(xy)) = \max\{|m(x) - m(y)|, |M(x) - M(y)|\}$. Because f is a φ-contraction, and φ satisfies the condition a), it follows that

$$|m(x) - m(y)| \leqslant \varphi(d(x, y), D(x, f(x)), D(y, f(y)), D(x, f(y)),$$
$$D(y, f(x)) \leqslant \varphi(d(x, y), d(x, m(x)), d(y, m(y)), d(x, m(y)), d(y, m(x))).$$

We obtain similarly

$$|M(x) - M(y)| \leqslant \varphi(d(x, y), d(x, M(x)), d(y, M(y)), d(x, M(y)), d(y, M(x))).$$

By theorem 1 in [2] it follows that $m: R \to R$ has a unique fixed point x_m and $M: R \to R$ a unique fixed point x_M. There are not other fixed points on the left of x_m on the right of x_M; any x in $[x_m, x_M]$ is a fixed point, so the fixed points set is compact and convex.

If we consider some other metric spaces, the properties mentioned above may not be trus. In [3] there are given examples of multifunctions in R^2 having as values convex and compact sets, whose fixed points set is however not convex; those multifunctions satisfy conditions of the type

$H(f(x), f(y)) \leqslant kd(x, y)$, with $0 < k < 1$, so they are a special case of the multifunctions considered here.

(Received June 10, 1980)

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UNELE PROPRIETĂŢI ALE MULŢIMII PUNCTELOR FIXE PENTRU APLICAŢII MULTIVOCE

(Rezumat)

În lucrare se studiază cazul în care proprietăţi ale imaginilor unei aplicaţii multivoce (convexitatea şi compactitatea) se transmit la mulţimea punctelor fixe. Rezultatele le generalizează pe cele obţinute de H. Schirmer pentru contracţii, aplicaţiile considerate aici fiind φ-contracţii sau contractive, definite pe R, respectiv pe intervale compacte.

RECENZII

H. Freund, Elemente der Zahlentheorie, B. G. Teubner, Stuttgart, 1979.

Cartea cuprinde noţiunile de bază din teoria numerelor, tratînd în cele trei capitole ale sale despre teoria divizibilităţii (divizori, multipli, numere prime), sisteme de numeraţie şi fracţii precum şi despre congruenţe (clase de resturi). De asemenea se evidenţiază unele paragrafe speciale despre ecuaţii diofantice, teorema lui Euler şi Fermat, congruenţe neliniare etc.

Profesor de matematică şi didactica matematicii, autorul a reuşit să selecteze cele mai semnificative subiecte ale temei şi mai ales să le dea o formă sugestivă de prezentare.

Conţinutul cărţii poate să intereseze atît pe studenţii facultăţilor de matematici cît şi (îndeosebi) pe profesorii de matematici, care pot găsi aici multe soluţii şi procedee ingenioase în predarea noţiunilor aferente.

Cele 18 figuri, 17 exemple şi 56 exerciţii (împreună cu soluţiile lor) întregesc în mod fericit expunerea. Un pronunţat caracter de originaltate emană în special din interpretările geometrice sugestive şi din exerciţiile-jocuri incluse.

Alături de scheme sinoptice şi demonstraţii riguroase, cititorul va găsi de asemenea preţioase indicaţii metodologice.

N. BOTH

B. Rauhut, N. Schmitz, E. W. Zachow, Spieltheorie. B. G. Teubner, Stuttgart, 1979, 400

Lucrarea este consacrată prezentării teoriei jocurilor. În prima parte sînt date diferite modele matematice ale situaţiilor conflictuale care conduc la jocuri strategice în general cu n parteneri. Se definesc strategiile jucătorilor şi se stabilesc diferite proprietăţi ale acestora, folosindu-se forma normală a jocurilor şi noţiunea de mulţime informaţională.

În continuare sînt tratate jocurile cu doi parteneri cu sumă nulă, demonstrîndu-se teorema fundamentală de minimax şi legătura acesteia cu existenţa unui punct şa pentru funcţia valoarea medie de cîştig. Pentru jocurile matriciale se stabileşte teorema corespunzătoare ce există între rezolvarea unui astfel de joc şi soluţiile a două probleme de programare liniară. Într-un capitol sînt studiate jocurile cu doi parteneri fără sumă nulă. În ultima parte a lucrării se prezintă teoria lui Neumann şi Morgenstern referitoare la jocurile cu cooperaţie cu n parteneri. Se defineşte funcţia caracteristică a unui astfel de joc şi se demonstrează proprietăţile acestor funcţii caracteristice. Soluţia jocului se prezintă ca o mulţime de imputaţii. De asemenea se defineşte funcţia Shapley şi se dau diferite proprietăţi ale acesteia.

M. RĂDULESCU

Intreprinderea Poligrafică Cluj, Municipiul Cluj-Napoca cda. 3076/1981

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