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ON AN ENRICHED THEORY OF MODULES (II)

GRIGORE CĂLUGĂREANU

Introduction. Stimulated by the excelent monography [4], the author of the present paper works out the closed and monoidal closed part of the theory of modules over a fixed monoid, a theory for which, in [5], M a c L a n e only worked out the monoidal part.

The reader needs only the first section from [3]-where the basic notions: closed and monoidal monoids and the corresponding morphisms, left and right modules over monoids, are defined, and the basic situations studied, — in order to recover our main definitions.

From section two all the definitions and results following the corrolary 2.7 are needed.

In this way, we shall start this second part of this paper with section three. In what follows, we suppose that $\underline{V}_0$ is a symmetric monoidal closed category with equalizers.

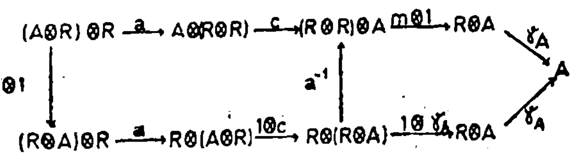
3. The closed and monoidal closed structure of ${}_R\underline{MV}$. LEMMA 3.1. — *The morphism $z_A = \pi(\gamma_A \cdot c_{AR}) : A \to (RA)$, considered for a left R-module (A, α_A, γ_A) over $\underline{V}$, factors through $\{RA\}$. Moreover, denoting by $\mathbf{i}_A : A \to \{RA\}$ the factorization morphism, this is an isomorphism in $\underline{V}_0$.*

Proof. First, we have to check (3) $(\alpha_A, 1) \cdot R^A_{AR} \cdot z_A = (n, 1) \cdot L^R_{AR} \cdot z_A$. Using [II, (3.1), (3.19), (3.22)] we have

$$(n, 1) \cdot L^R_{RA} \cdot z_A = (\pi(m), 1) \cdot L^R_{RA} \cdot z_A = p \cdot (m, 1) \cdot \pi(\gamma_A \cdot c_{AR}) =$$

$$= p \cdot \pi(\gamma_A \cdot c_{AR} \cdot 1 \otimes m) = \pi\pi(\gamma_A \cdot c_{AR} \cdot 1 \otimes m \cdot a) = \pi\pi(\gamma_A \cdot m \otimes 1 \cdot c_{A,R \otimes R} \cdot a).$$

The following commutative diagram



enables us to continue the equalities above

$$= \pi\pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a_{RAR} \cdot c_{AR} \otimes 1) = \pi(\pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a) \cdot c_{AR})$$

using again [II, (3,1)]. At the same time, we have

$$(\alpha_A, 1) \cdot R^R_{AR} \cdot z_A = \pi(M^A_{AR} \cdot c_{(RA),(AA)} \cdot z_A \otimes \alpha_A) = \pi(M^A_{RA} \cdot \alpha_A \otimes z_A \cdot c_{AR}).$$

Thus, the equality (3) is equivalent to the following

$M^A_{RA} \cdot \alpha_A \otimes z_A = \pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a_{RAR})$ or to the one derived from this applying π, which is proved as follows

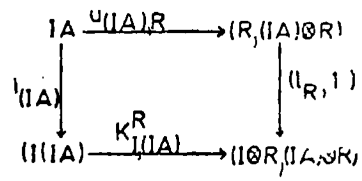
$$\pi(M^A_{RA} \cdot \alpha_A \otimes z_A) = (z_A, 1) \cdot L^R_{AA} \cdot \alpha_A = (\pi(\gamma_A \cdot c_{AR}), 1) \cdot L^R_{AA} \cdot \alpha_A =$$
$$= p \cdot (\gamma_A \cdot c_{AR}, 1) \cdot \alpha_A = p \cdot \pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR})) = \pi\pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a_{RAR})$$

using [II, (3.1), (3.19), (3.22)].

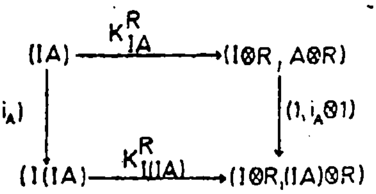
Now, let us prove that $i_A^{-1} : \{RA\} \xrightarrow{equ} (RA) \xrightarrow{(e,1)} (IA) \xrightarrow{i_A^{-1}} A$ is a twosided inverse for i_A, that is, let us check the following two equalities $(e, 1) \cdot z_A = i_A$, $z_A \cdot i_A^{-1} \cdot (e, 1) \cdot equ_{RA} = equ_{RA}$. We simply obtain the first as follows

$$(e, 1) \cdot \pi(\gamma_A \cdot c_{AR}) = \pi(\gamma_A \cdot c_{AR} \cdot 1 \otimes e) = \pi(\gamma_A \cdot e \otimes 1 \cdot c_{AI}) = \pi(l_A \cdot c_{AI}) =$$
$$= \pi(r_A) = i_A.$$

As for the second, we first derive from [II, (7.4)] the following commutative diagram



Using $i_{(IA)} = (1, i_A)$ and the following commutative diagram (by naturality of K)



we have $u_{(IA),R} = (l_R^{-1}, i_A \otimes 1) \cdot K^R_{IA}$. We now prove the second equality required above as follows

$$\gamma_A \cdot i_A^{-1} \cdot (e, 1) \cdot equ_{RA} = \pi(\gamma_A \cdot c_{AR} \cdot i_A^{-1} \otimes 1) \cdot equ_{RA} =$$
$$= (1, \gamma_A \cdot c_{AR} \cdot i_A^{-1} \otimes 1) \cdot u_{(IA),R} \cdot (e, 1) \cdot \mathrm{equ}_{RA} =$$
$$= (1, \gamma_A \cdot c_{AR} \cdot i_A^{-1} \otimes 1) \cdot (l_R^{-1}, i_A \otimes 1) \cdot K^R_{IA} \cdot (e, 1) \cdot \mathrm{equ}_{RA} =$$
$$= (l_R^{-1}, \gamma_A \cdot c_{AR}) \cdot K^R_{IA} \cdot (e, 1) \cdot \mathrm{equ}_{RA} = (l_R^{-1}, \gamma_A \cdot c_{AR}) \cdot (e \otimes 1, 1) \cdot K^R_{RA} \cdot \mathrm{equ}_{RA} =$$
$$= (\mathrm{e} \otimes 1 \cdot l_R^{-1}, \gamma_A \cdot c_{AR}) \cdot (c_{RR}, c_{RA}) \cdot H^R_{RA} \cdot \mathrm{equ}_{RA} =$$
$$= (c_{RR} \cdot e \otimes 1 \cdot l_R^{-1}, 1) \cdot (1, \gamma_A) \cdot H^R_{RA} \cdot \mathrm{equ}_{RA} = (c_{RR} \cdot \mathrm{e} \otimes 1 \cdot l_R^{-1}, 1) \cdot (m, 1) \cdot \mathrm{equ}_{RA} =$$
$$= (1 \otimes e \cdot r_R^{-1} \cdot m, 1) \cdot \mathrm{equ}_{RA} = \mathrm{equ}_{RA},$$

using also [II, (3.4)] and lemma 3.1.

LEMMA 3.2. — *For each left R-module (A, α_A, γ_A) over $\underline{V}$, the morphism* $y_{RA} = M^R_{RA} \cdot c_{(RR),(RA)} \cdot \pi(m \cdot c_{RR}) \otimes \mathrm{equ}_{RA} : R \otimes \{RA\} \to (RA)$ *factors through*

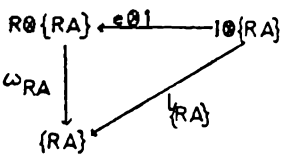
$\{RA\}$. The unique factorization morphism $\omega_{RA}: R\otimes\{RA\} \to (RA)$ provides a left R-module structure over $\underline{V}$ for $\{RA\}$.

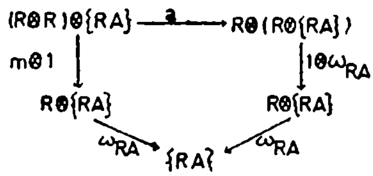
Proof. Using the $\underline{V}$-functoriality of R^A and L^R, the naturality of M, a commutative diagram derived from [III, (4.4)] and (3) from the previous lemma we have

$$(\alpha_A, 1) \cdot R^A_{AR} \cdot M^R_{RA} \cdot c_{(RR),(RA)} \cdot \pi(m \cdot c_{RR})\otimes \text{equ}_{RA} =$$
$$= (n, 1) \cdot L^R_{RA} \cdot M^R_{RA} \cdot c_{(RR),(RA)} \cdot \pi(m \cdot c_{RR})\otimes \text{equ}_{RA},$$

which proves the existence of the required morphism ω_{RA}.

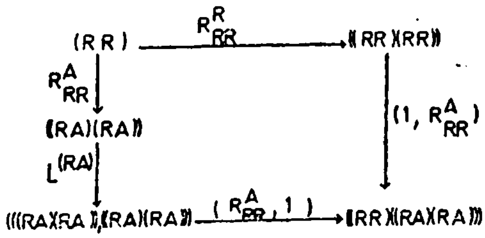
Next, let us check that $(\{RA\}, \omega_{RA})$ actually is a left R-module over $\underline{V}$, that is, the commutativity of the following diagrams





By left composition with equ_{RA}, the first one is equivalent with $y_{RA} \cdot e\otimes 1 = \text{equ}_{RA} \cdot l_{\{RA\}}$ denoting by $y_{RA} = \text{equ}_{RA} \cdot \omega_{RA}$, or, applying π, $\pi(y_{RA}) \cdot e = (1, \text{equ}_{RA}) \cdot j_{\{RA\}}$. Using $\pi(y_{RA}) = (\text{equ}, 1) \cdot R^A_{RR} \cdot \pi(m \cdot c_{RR})$ and $\pi(m \cdot c_{RR}) \cdot e = j_R$ (from proposition 1.10) it is sufficient to verify $R^A_{RR} \cdot j_A = j_{(RA)}$. But this follows easily from [CC2], i.e., $(j_R, 1) \cdot L^R_{RA} = i_{(RA)}$ applying π^{-1} and using [III, (4.4)] for [III, 6.4].

The commutativity of the second diagram is equivalent with $y_{RA} \cdot m\otimes 1 = y_{RA} \cdot 1\otimes\omega_{RA} \cdot a$. We first mention that again from proposition 1.10 we have $\beta \cdot m = M^R_{RR} \cdot \beta\otimes\beta \cdot c_{RR} = M^R_{RR} \cdot c_{(RR),(RR)} \cdot \beta\otimes\beta$, which implies, applying π, $(1, \beta) \cdot \pi(m) = (\beta, 1) \cdot R^R_{RR} \cdot \beta$. Next, using the $\underline{V}$-functoriality of R^A and applying π, we derive an analogous of [CC3]



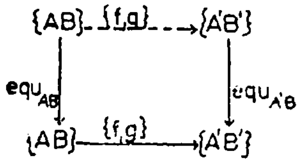
Using all these, the proof is the following

$$\pi\pi(y_{RA} \cdot m\otimes 1) = \pi(\pi(y_{RA}) \cdot m) = (1, \pi(y_{RA})) \cdot \pi(m) =$$
$$= (1, (\text{equ}_{RA}, 1)) \cdot (1, R^A_{RR}) \cdot (1, \beta) \cdot \pi(m) =$$
$$=(1, (\text{equ}_{RA}, 1)) \cdot (1, R^A_{RR}) \cdot (\beta, 1) \cdot R^R_{RR} \cdot \beta =$$
$$= (1, (\text{equ}_{RA}, 1)) \cdot (\beta, 1) \cdot (R^A_{RR}, 1) \cdot L^{(RA)}_{(RA),(RA)} \cdot R^A_{RR} \cdot \beta =$$

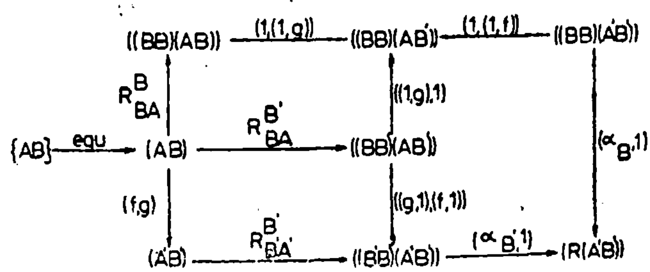
$$= (\beta, 1) \cdot (R^A_{RR}, 1) \cdot ((\mathrm{equ}_{RA}, 1), 1) \cdot L^{\{RA\}}_{\{RA\},(RA)} \cdot R^A_{RR} \cdot \beta =$$
$$= (\pi, (y_{RA}), 1) \cdot L^{\{RA\}}_{\{RA\},(RA)} \cdot R^A_{RR} \cdot \beta = p \cdot (y_{RA}, 1) \cdot R^A_{RR} \cdot \beta =$$
$$= p \cdot \pi(M^R_{RA} \cdot c_{(RR),(RA)} \cdot \pi(m \cdot c_{RR}) \otimes y_{RA}) = p \cdot \pi(y_{RA} \cdot 1 \otimes \omega_{RA}) =$$
$$= \pi\pi(y_{RA} \cdot 1 \otimes \omega_{RA} \cdot a).$$

LEMMA 3.3. — *The construction described in lemma 2.8. defines in a symmetric monoidal closed category $\underline{V}$ with equalizers a bifunctor $\{-,-\}: {}_R\underline{MV}^{op} \times {}_R\underline{MV} \to \underline{V}_0$.*

Proof. If $f: (A', \alpha_{A'}) \to (A, \alpha_A)$ and $g: (B, \alpha_{B'}) \to (B', \alpha_{B'})$ are morphisms of left R-modules over $\underline{V}$, then $\{f, g\}: \{AB\} \to \{A'B'\}$ is the unique morphism of factorization through the equalizers



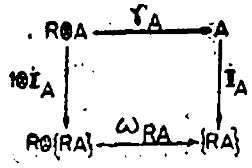
The functoriality is derived from the uniqueness. Thus, in order to prove the existence of the factorization morphism on the above diagram we must check $(\alpha_{B'}, 1) \cdot R^{B'}_{B'A'} \cdot (f, g) \cdot \mathrm{equ}_{AB} = (\alpha_{A'}, 1) \cdot L^{A'}_{A'B'} \cdot (f, g) \cdot \mathrm{equ}_{AB}$. Using the following commutative diagram



the first member of the required equality is $= (1, (f, g)) \cdot (\alpha_B, 1) \cdot R^B_{BA} \cdot \mathrm{equ}_{AB}$. But this is $= (1, (f, g)) \cdot (\alpha_A, 1) \cdot L^A_{AB} \cdot \mathrm{equ}_{AB}$. Finally, a similar diagram for L (like the above one) leads us to the second member of the required equality.

THEOREM 3.4. — *For each left R-module (A, α_A, γ_A) over $\underline{V}$, there is a natural isomorphism in ${}_R\underline{MV}$, $\mathbf{i}_A: A \to \{RA\}$, where $\{RA\}$ has the left R-module structure given by lemma 3.2.*

Proof. We must show that $\mathbf{i}_A$ actually is a morphism of left R-modules over $\underline{V}$, i.e., that the following diagram commutes



By left composition with equ_{RA} it is sufficient to check $z_A \cdot \gamma_A = y_{RA} \cdot 1 \otimes \mathrm{i}_A$. We derive this from the following equivalent equalities $z_A \cdot \gamma_A \cdot c_{AR} = M^R_{RA} \cdot \cdot z_A \otimes z_R$, $(1, z_A) \cdot z_A = (z_R, 1) \cdot L^R_{RA} \cdot z_A$, this last equality being checked analogously, like the one in proposition 1.6.

The naturality (in ${}_R\underline{MV}$!) of the family $\mathrm{i} = \mathrm{i}_A : A \to \{RA\}$ reduces to the naturality of the family $z = z_A$ which is readily checked.

THEOREM 3.5. — $\mathrm{i}_R : R \to \{RR\}$ *is an isomorphism of monoids from the opposite monoid of* R *to the monoid* $\{RR\}$ *of the* R*-endomorphisms of the left* R*-module* R *over itself.*

Proof. Straightforward, using equalities from the proof of proposition 1.10.

Remark. It can be shown, in the usual subjacent way, that if V preserves equalizers and $V \cdot W$ is an epifunctor, then the left R-module (R, n, m) over R is a projective object in ${}_R\underline{MV}$. Analogously one could now define Quasi-Frobenius monoids over $\underline{V}$ in the usual way.

— Let us point out the second bifunctor corresponding to the monoidal structure of ${}_R\underline{MV}$. We assume that $\underline{V}_0$ *has coequalizers* and, for a monoid (R, e, n, m), that (A, γ_A) is a object in ${}_R\underline{MV}$ and (B, δ_B) is an object of $\underline{MV}_R$.

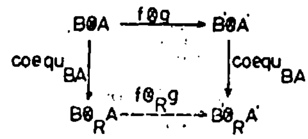
We define the *tensor product* $B \otimes_R A$ as an object in $\underline{V}_0$, namely

$$\mathrm{coequ}\,(((\delta_B \otimes 1_A) \cdot a^{-1}_{BRA},\ 1_B \otimes \gamma_A) : B \otimes (R \otimes A) \to B \otimes A).$$

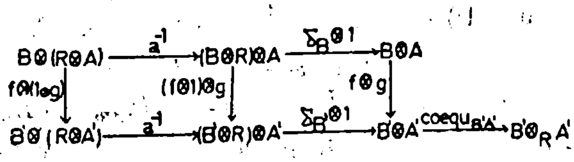
This will be a quotient object of $B \otimes A$. We shall denote by $\mathrm{coequ}_{BA} : B \otimes A \to B \otimes_R A$ the canonic epimorphism to the coequalizer.

PROPOSITION 3.6. *In a monoidal category with coequalizers* $\underline{V}$, *the above construction defines a bifunctor* $\otimes_R : \underline{MV}_R \times {}_R\underline{MV} \to \underline{V}_0$.

Proof. Evidently, $\otimes_R((B, \delta_B), (A, \gamma_A)) = B \otimes_R A$. If $f : (B, \delta_B) \to (B', \delta_{B'})$ is a morphism in $\underline{MV}_R$ and $g : (A, \gamma_A) \to (A', \gamma_{A'})$ is a morphism in ${}_R\underline{MV}$ then $f \otimes_R g : B \otimes_R A \to B' \otimes_R A'$ is the unique morphism of factorization through the coequalizers



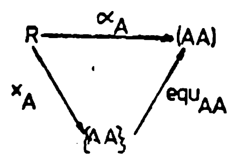
The functioriality is derived from the uniqueness. In order to prove the existence of the factorization morphism on the above diagram we must check $\mathrm{coequ}_{B'A'} \cdot f \otimes g \cdot (\delta_B \otimes 1_A) \cdot a^{-1}_{BRA} = \mathrm{coequ}_{B'A'} \cdot f \otimes g \cdot 1_B \otimes \gamma_A$. But this easily follows using the following commutative diagram



DEFINITION 3.1. *A monoid (R, e, m) over $\underline{V}$ is called commutative if $m \cdot c_{RR} = m$.*

In what follows we suppose the monoid (R,e,n,m) commutative.

LEMMA 3.7. *If (R,e,m) is a commutative monoid over $\underline{V}$ and (A, α_A) is a left R-module then $\alpha_A : R \to (AA)$ factors through $\{AA\}$ like in the following commutative diagram*



Proof. α_A being morphism of monoids over $\underline{V}$ we have

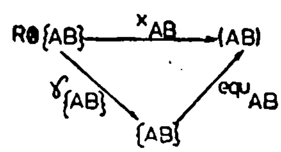
$$M^A_{AA} \cdot \alpha_A \otimes \alpha_A = \alpha_A \cdot m = \alpha_A \cdot m \cdot c_{RR} = M^A_{AA} \cdot \alpha_A \otimes \alpha_A \cdot c_{RR} =$$

$= M^A_{AA} \cdot c_{(AA),(AA)} \cdot \alpha_A \otimes \alpha_A$. Applying π we get $(\alpha_A, 1) \cdot L^A_{AA} \cdot \alpha_A =$

$$= (\alpha_A, 1) \cdot R^A_{AA} \cdot \alpha_A.$$

PROPOSITION 3.8. *If (A, α_A) and (B, α_B) are left R-modules, there is a morphism $\gamma_{\{AB\}} : R \otimes \{AB\} \to \{AB\}$ which gives $\{AB\}$ a structure of left R-module over $\underline{V}$.*

Proof. We consider the morphism $x_{AB} = M^B_{AB} \cdot \alpha_B \otimes \mathrm{equ}_{AB} : R \otimes \{AB\} \to (AB)$ From the previous lemma $\alpha_B = \mathrm{equ}_{BB} \cdot x_B$, so that x_{AB} factors through equ_{AB} using lemma 2.9. Hence a morphism $\gamma_{\{AB\}}$ exists and makes the following diagram commutative



Next, let us show that $(\{AB\}, \gamma_{\{AB\}})$ actually is a left R-module over $\underline{V}$. As usual we get equivalent conditions by left composition with equ_{AB}, namely $x_{AB} \cdot e \otimes 1 = \mathrm{equ}_{AB} \cdot l_{\{AB\}}$, $x_{AB} \cdot m \otimes 1 = x_{AB} \cdot 1 \otimes \gamma_{\{AB\}} \cdot a$. For the first, applying π and [II,(3.1)] we have $\pi(x_{AB}) \cdot e = (\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B \cdot e = (\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot j_B = (\mathrm{equ}_{AB}, 1) \cdot j_{(AB)} = (1, \mathrm{equ}_{AB}) \cdot j_{\{AB\}}$, also using [CC1], the naturality of j and the fact that B is a left R-module over $\underline{V}$.
For the second, we have

$$\pi\pi(x_{AB} \cdot m \otimes 1) = \pi(\pi(x_{AB}) \cdot m) = (1, \pi(x_{AB})) \cdot \pi(m) =$$

$$= (1, (\mathrm{equ}_{AB}, 1)) \cdot (1, L^A_{BB}) \cdot (1, \alpha_B) \cdot \pi(m) = (1, (\mathrm{equ}_{AB}, 1)) \cdot (1, L^A_{BB}) \cdot (\alpha_B, 1) \cdot L^B_{BB} \cdot \alpha_B =$$

$$= (1, (\mathrm{equ}_{AB}, 1)) \cdot (\alpha_B, 1) \cdot (L^A_{BB}, 1) \cdot L^{(AB)}_{(AB),(AB)} \cdot L^A_{BB} \cdot \alpha_B =$$

$$= (\alpha_B, 1) \cdot (L^A_{BB}, 1) \cdot ((\mathrm{equ}_{AB}, 1), 1) \cdot L^{\{AB\}}_{\{AB\},(AB)} \cdot L^A_{BB} \cdot \alpha_B =$$

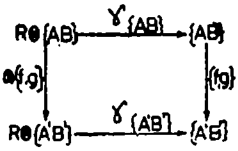
$$= (\pi(x_{AB}), 1) \cdot L^{\{AB\}}_{\{AB\},(AB)} \cdot L^A_{BB} \cdot \alpha_B = p \cdot (x_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B =$$

$$= p \cdot \pi(M^B_{AB} \cdot \alpha_B \otimes x_{AB}) = p \cdot \pi(x_{AB} \cdot 1 \otimes \gamma_{\{AB\}}) = \pi\pi(x_{AB} \cdot 1 \otimes \gamma_{\{AB\}} \cdot a).$$

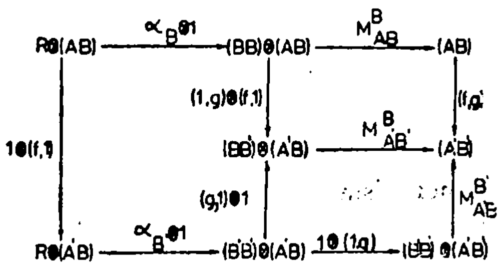
Remark. We must show that in the commutative case the left R-module structures defined on $\{RA\}$, in the lemma 3.2 and in the previous proposition are identical. Using $m \cdot c_{RR} = m$ and applying π^{-1} to the definition of equ_{RA}, one can show that $M^A_{RA} \cdot \alpha_A \otimes \mathrm{equ}_{RA} = M^R_{RA} \cdot c_{(RR),(RA)} \cdot n \otimes \mathrm{equ}_{RA}$, that is, $x_{RA} = y_{RA}$.

THEOREM 3.9. *Lemma 3.3 defines a bifunctor* $\{-,-\}: {}_R\underline{MV}^{op} \times {}_R\underline{MV} \to {}_R\underline{MV}$.

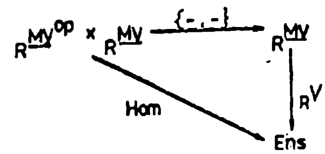
Proof. It only remains to prove that $\{f,g\}: (\{AB\}, \gamma_{\{AB\}}) \to (\{A'B'\}, \gamma_{\{A'B'\}})$ actually is a morphism of left R-modules over $\underline{V}$. The commutativity of the following diagram



follows from the equivalent equality $M^{B'}_{A'B'} \cdot \alpha_B \otimes (f,g) \cdot 1 \otimes \mathrm{equ}_{AB} = (f,g) \cdot M^B_{AB} \cdot \alpha_B \otimes 1 \cdot 1 \otimes \mathrm{equ}_{AB}$, which is true using the following commutative diagram



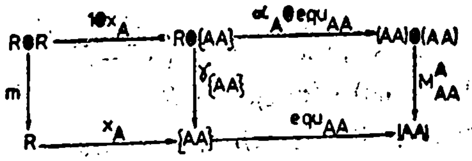
PROPOSITION 3.10. *If* ${}_RV = V \cdot W: {}_R\underline{MV} \to \mathrm{Ens}$ *and* V *preserves equalizers then the following diagram of functors is commutative*



Proof. Straightforward from the remark following lemma 2.8.

PROPOSITION 3.11. *For each left R-module* (A, α_A), *the morphism* $x_A: R \to \{AA\}$ *which appears in lemma 3.7 is a morphism of left R-modules. Further, the family* $\mathbf{j} = \mathbf{j}_{(A,\alpha_A)} = x_A$ *is natural in* ${}_R\underline{MV}$.

Proof. The following commutative diagram shows that x_A is a morphism of left R-modules



The naturality of $\mathbf{j}$ follows from $(1, f) \cdot \alpha_A = (f, 1) \cdot \alpha_{A'}$, true for a morphism of left R-modules $f:(A, \alpha_A) \to (A', \alpha_{A'})$, equ_{AA} being monomorphism.

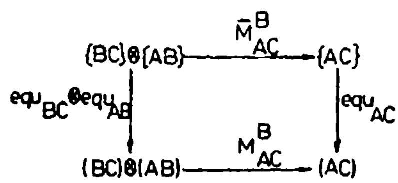
PROPOSITION 3.12. $_RV\ (\mathbf{i}_{\{AA\}})(1_A) = \mathbf{j}_A$.

Proof. By left composition with equ_{AA} one can show that we have to show at the subjacent level that $V(z_{(AA)})(1_A) = \alpha_A$. Finally, one reduces this to $V(u_{(AA),R})(1_A) = j_A \otimes 1_A \cdot l_R^{-1}$ using also the equality $z_{(AA)} = (1_R, \gamma_{(AA)} \cdot c_{(AA),R}) \cdot u_{(AA),R}$ and axiom [CC5].

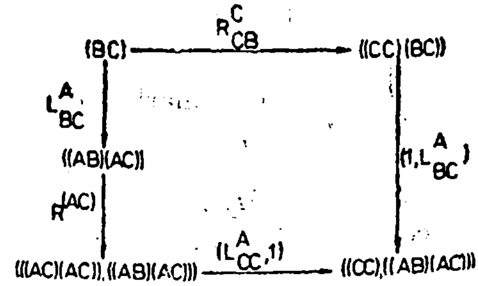
PROPOSITION 3.13. *For each left R-modules* (A, α_A), (B, α_B) *and* (C, α_C) *there is a transformation*

$$\mathbf{L} = \mathbf{L}^{(A,\alpha_A)}_{(B,\alpha_B),(C,\alpha_C)} : \{BC\} \to \{\{AB\}, \{AC\}\} \quad \textit{natural in } {}_R\underline{MV}.$$

Proof. Let us, first, mention the following generalization of lemma 2.9 $(\alpha_C, 1) \cdot R^C_{CA} \cdot M^B_{AC} \cdot \mathrm{equ}_{BC} \otimes \mathrm{equ}_{AB} = (\alpha_A, 1) \cdot L^A_{AC} \cdot M^B_{AC}\ \mathrm{equ}_{BC} \otimes \mathrm{equ}_{AB}$. Hence there is a morphism $\overline{M}^B_{AC}: \{BC\} \otimes \{AB\} \to \{AC\}$ which closes the commutative diagram



Using again π there is a morphism $\overline{L}^A_{BC}: \{BC\} \to (\{AB\}, \{AC\})$ for which $(\mathrm{equ}_{AB}, 1) \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} = (1, \mathrm{equ}_{AC}) \cdot \overline{L}^A_{BC}$ is true. If we want $\overline{L}^A_{BC}$ to give us by factorization a morphism $\mathbf{L}^A_{BC}: \{BC\} \to \{\{AB\}, \{AC\}\}$ we have to check the following equality $(\alpha_{\{AC\}}, 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} : \overline{L}^A_{BC} = (\alpha_{\{AB\}}, 1) \cdot L^{\{AB\}}_{\{AB\},\{AC\}} \cdot \overline{L}^A_{BC}$. In doing so, we may use the definitions of the left R-module structure on $\{AB\}$ and $\{AC\}$, i.e., $(1, \mathrm{equ}_{AB}) \cdot \alpha_{\{AB\}} = (\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B$ and the analogous changing B into C. Again, using the $\underline{V}$-functoriality of L^A, composing to the right with $c_{(BC),(CC)}$ and applying π, we get another analogous of [CC3]:



We shall prove the required equality composing to the left with $(1,(1,\mathrm{equ}_{AC}))$ (which still is a monomorphism, the functors $(X, -)$ being monofunctors). Indeed

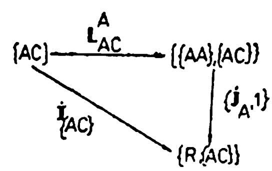
$$(1,(1, \mathrm{equ}_{AC})) \cdot (\alpha_{\{AC\}}, 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \overline{L}^A_{BC} =$$
$$= (\alpha_{\{AC\}}, 1) \cdot ((1, \mathrm{equ}_{AC}), 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \overline{L}^A_{BC} =$$

$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot ((\mathrm{equ}_{AC}, 1), 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \bar{L}^A_{BC} =$$
$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot (1, \mathrm{equ}_{AC}) \cdot \bar{L}^A_{BC} =$$
$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot (\mathrm{equ}_{AB}, 1) \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot (1, (\mathrm{equ}_{AB}, 1)) \cdot R^{\{AC\}}_{\{AC\},(AB)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_C, (\mathrm{equ}_{AB}, 1)) \cdot (1, L^A_{BC}) \cdot R^C_{CB} \cdot \mathrm{equ}_{BC} =$$
$$= (1, (\mathrm{equ}_{AB}, 1) \cdot L^A_{BC}) \cdot (\alpha_B, 1) \cdot L^B_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_B, (\mathrm{equ}_{AB}, 1)) \cdot (L^A_{BB}, 1) \cdot L^{(AB)}_{(AB),(AB)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= ((\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B, 1) \cdot L^{\{AB\}}_{\{AB\},(AC)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_{\{AB\}}, 1) \cdot ((1, \mathrm{equ}_{AB}), 1) \cdot L^{\{AB\}}_{\{AB\},(AC)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_{\{AB\}}, 1) \cdot (1, (1, \mathrm{equ}_{AC})) \cdot L^{\{AB\}}_{\{AB\}\{AC\}} \cdot \bar{L}^A_{BC} =$$
$$= (1, (1, \mathrm{equ}_{AC})) \cdot (\alpha_{\{AB\}}, 1) \cdot L^{\{AB\}}_{\{AB\},\{AC\}} \cdot \bar{L}^A_{BC}.$$

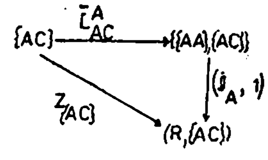
Thus, there is a morphism $\mathbf{L}^A_{BC}: \{BC\} \to \{\{AB\}, \{AC\}\}$ in $\underline{V}_0$. The proof of the naturality in ${}_R\underline{MV}$ of the corresponding family is left to the reader.

THEOREM 3.14. *If $\underline{V}$ is a symmetric monoidal closed category with equalizers, (R,e,n,m) is a commutative monoid over $\underline{V}$ and the subjacency functor $V: \underline{V}_0 \to$ $\to$Ens preserves equalizers, then ${}_R\underline{MV}$, the category of the left R-modules over $\underline{V}$, is a closed category.*

Proof. Using theorem 3.4, the remark following proposition 3.8, theorem 3.9, propositions 3.10, 3.11, 3.12, 3.13, all data for the closed structure of ${}_R\underline{MV}$ are constructed, the „unit'' object being obviously (R,n,m) as a R-module over (R,e,n,m). Proposition 3.12 is axiom $[CC5]$ for ${}_R\underline{MV}$. Hence, one has only to verify the remaining axioms $[CC1-4]$ for ${}_R\underline{MV}$. In what follows we shall prove, for instance, axiom $[CC2]$. We have to check the commutativity of the following diagram



By left composition with $\mathrm{equ}_{R,\{AC\}}$ this is reduced to the following commutativity



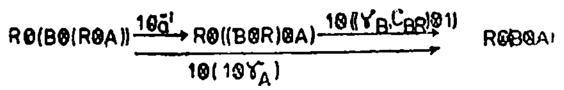
A new composition with $(1, \mathrm{equ}_{AC})$ gives us the required proof:

$$(1, \mathrm{equ}_{AC}) \cdot z_{\{AC\}} = (1, \mathrm{equ}_{AC}) \cdot \pi(\gamma_{\{AC\}} \cdot c_{\{AC\},R}) =$$
$$= \pi(\mathrm{equ}_{AC} \cdot \gamma_{\{AC\}} \cdot c_{\{AC\},R}) = \pi(x_{AC} \cdot c_{\{AC\},R}) = \pi(M^C_{AC} \cdot \alpha_C \otimes \mathrm{equ}_{AC} \cdot c_{\{AC\},R}) =$$
$$= \pi(M^C_{AC} \cdot c_{(AC),(CC)} \cdot \mathrm{equ}_{AC} \otimes \alpha_C) = (\alpha_C, 1) \cdot R^C_{CA} \cdot \mathrm{equ}_{AC} =$$
$$= (\alpha_A, 1) \cdot L^A_{AC} \cdot \mathrm{equ}_{AC} = (\mathbf{j}_A, 1) \cdot (\mathrm{equ}_{AA}, 1) \cdot L^A_{AC} \cdot \mathrm{equ}_{AC} =$$
$$= (\mathbf{j}_A, 1) \cdot (1, \mathrm{equ}_{AC}) \cdot \overline{L}^A_{AC} = (1, \mathrm{equ}_{AC}) \cdot (\mathbf{j}_A, 1) \cdot \overline{L}^A_{AC}.$$

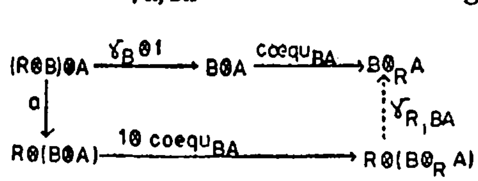
Remark. The astute reader has certainly noticed that we constantly use the following fact: the functors $(X, -) : \underline{V}_0 \to \underline{V}_0$ having left adjoints, namely $- \otimes X : \underline{V}_0 \to \underline{V}_0$, preserve limits (equalizers) and monomorphisms.
— Let us return now to the monoidal structure of ${}_R\underline{MV}$.

THEOREM 3.15. *For a commutative monoid (R, e, m), if the functor $R\otimes - : \underline{V}_0 \to \underline{V}_0$ preserves coequalizers, proposition 3.6 defines a bifunctor $\otimes_R : {}_R\underline{MV} \times {}_R\mathrm{MV} \to {}_R\underline{MV}$.*

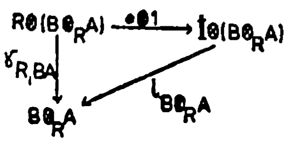
Proof. We first mention that, the basic monoid being commutative, if (A, δ_A) is a right R-module then $(A, \delta_A \cdot c_{RA})$ is a left R-module. Hence, for two left R-modules we shall define $B\otimes_R A = \mathrm{coequ}(((\gamma_B \cdot c_{BR}) \otimes 1_A \cdot a^{-1}_{BRA}, 1_B \otimes \gamma_A) : : B\otimes(R\otimes A) \to B\otimes A)$. In order to get a left R-module structure on $B\otimes_R A$ we prove that the morphism $\tilde{x}_{BA} : R\otimes(B\otimes A) \xrightarrow{a^{-1}} (R\otimes B)\otimes A \xrightarrow{\gamma_B \otimes 1} B\otimes A \xrightarrow{\mathrm{coequ}_{BA}} B\otimes_R A$ coequalizes the following pair

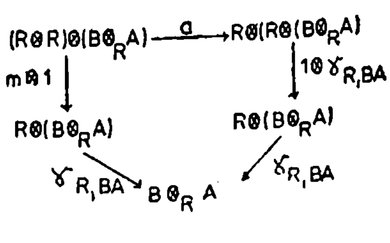


The functor $R\otimes -$ preserving coequalizers, this will prove the existence of a morphism $\gamma_{R,BA} : R\otimes(B\otimes_R A) \to B\otimes_R A$ in V_0 which will provide the left R-module structure on $B\otimes_R A$. We shall avoid this verification which only uses definitions and coherence. So, our $\gamma_{R,BA}$ closes the following commutative diagram



Next, we have to prove that $(B\otimes_R A, \gamma_{R,BA})$ actually is a left R-module, that is, the commutativity of the following diagrams





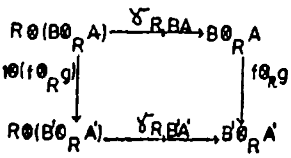
As for the first, we can check the equivalent one obtained by right composition with $1_I \otimes \mathrm{coequ}_{BA}$(this being epimorphism)

$$\gamma_{R,BA} \cdot e\otimes 1 \cdot 1\otimes \mathrm{coequ}_{BA} = \gamma_{R,BA} \cdot 1\otimes \mathrm{coequ}_{BA} \cdot e\otimes 1 =$$
$$= \mathrm{coequ}_{BA} \cdot \gamma_B \otimes 1 \cdot a^{-1} \cdot e\otimes 1 = \mathrm{coequ}_{BA} \cdot \gamma_B\otimes 1 \cdot (e\otimes 1)\otimes 1 \cdot a^{-1} =$$
$$= \mathrm{coequ}_{BA} \cdot l_B \otimes 1 \cdot a^{-1} = \mathrm{coequ}_{BA} \cdot l_{A\otimes B} = l_{B\otimes_R A} \cdot 1\otimes \mathrm{coequ}_{BA}.$$

As for the second, a right composition with $1_{R\otimes R} \otimes \mathrm{coequ}_{BA}$ gives us the required proof

$$\gamma_{R,BA} \cdot m\otimes 1 \cdot 1\otimes \mathrm{coequ}_{BA} = \mathrm{coequ}_{BA} \cdot \gamma_B \otimes 1 \cdot a^{-1} \cdot m\otimes 1 =$$
$$= \mathrm{coequ}_{BA} \cdot \gamma_B\otimes 1 \cdot (1\otimes \gamma_B)\otimes 1 \cdot a\otimes 1 \cdot a^{-1} =$$
$$= \mathrm{coequ}_{BA} \cdot \gamma_B \otimes 1 \cdot a^{-1} \cdot 1\otimes(\gamma_B\otimes 1) \cdot 1\otimes a^{-1} \cdot a =$$
$$= \gamma_{R,BA} \cdot 1\otimes \mathrm{coequ}_{BA} \cdot 1\otimes(\gamma_B\otimes 1) \cdot 1\otimes a^{-1} \cdot a =$$
$$= \gamma_{R,BA} \cdot 1\otimes\gamma_{R,BA} \cdot 1\otimes(1\otimes \mathrm{coequ}_{BA}) \cdot a = \gamma_{R,BA} \cdot 1\otimes \gamma_{R,BA} \cdot a \cdot 1\otimes \mathrm{coequ}_{BA}.$$

Finally, one easily checks that, using notations from proposition 3.6, $f\otimes_R g$ actually is a morphism of left R-modules over $\underline{V}$, i.e., the following diagram commutes



Remark. If we suppose that $\underline{V}_0$ is abelian, $\otimes$ preserves cokernels in both variables and we take cokernels instead of coequalizers we recover the similar MacLane's result.

— Moreover, the following result is true

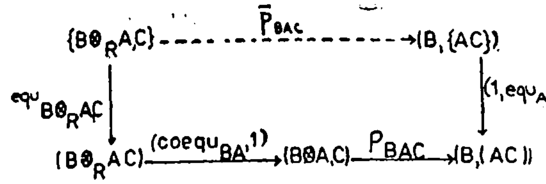
THEOREM 3.16. *If $\underline{V}$ is a symmetric monoidal category with coequalizers, (R, e, m) is a commutative monoid over $\underline{V}$ and $R\otimes -$ preserves coequalizers, then ${}_R\underline{MV}$ is a symmetric monoidal category.*

Proof. Simple generalization of MacLane's result. For instance, if (A, γ_A) is an object in ${}_R\underline{MV}$, we have from the definition 1.9 $\gamma_A \cdot m\otimes 1 = \gamma_A \cdot 1\otimes\gamma_A \cdot a$; that is, γ_A coequalizes the pair $((m \cdot c_{RR})\otimes 1_A \cdot a^{-1}_{RRA},\ 1_R\otimes\gamma)$. Thus, γ_A factors through coequ_{RA} giving one of our natural isomorphisms $l_A: R\otimes_R A \to A$.

We shall end our paper with the principal result which uses all the results obtained above

THEOREM 3.17. *If $\underline{V}$ is a symmetric monoidal closed category, $\underline{V}_0$ has equalizers and coequalizers, V preserves equalizers, (R, e, n, m) is a commutative monoid over $\underline{V}$ and $R\otimes -$ preserves coequalizers, then ${}_R\underline{MV}$, the category of the left R-modules over $\underline{V}$, is a symmetric monoidal closed category.*

Proof. First we must find in $\underline{V}_0$ morphisms $\mathbf{p}_{BAC}: \{B\otimes_R A, C\} \to \{B,\{A,C\}\}$ and prove that this is a natural family of isomorphisms in ${}_R\underline{MV}$. Previously, we prove the existence of morphisms $\bar{p}_{BAC}$ which close commutatively the following diagram



Because $(B, -)$ preserves equalizers it will suffice to check

$$(1,(\alpha_C, 1)) \cdot (1, R^C_{CA}) \cdot \bar{p}_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{B\otimes_R A, C} =$$

$$= (1,(\alpha_A, 1)) \cdot (1, L^A_{AC}) \cdot \bar{p}_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{B\otimes_R A, C}$$

This verification needs the following facts:
(i) denoting by $X = B\otimes_R A$, and applying π to a convenient diagram which expresses the $\underline{V}$-functoriality of R^C, one has $(1, R^C_{CA})\cdot L^A_{XX} = (R^C_{XA}, 1)\cdot R^{(AC)}_{(XC),(CC)}\cdot R^C_{CX}$.
(ii) from [III, (4.4)] applying π, we can find the equality

$$(R^C_{XA}, 1)\cdot R^{(AC)}_{(XC),(XX)} \cdot L^X_{XC} = (R^X_{XA}, 1)\cdot L^{(XX)}_{(AX),(AC)} \cdot L^A_{XC}$$

(iii) the definition of coequ_{BA} gives by a double application of π, the equality $(1, \pi(\mathrm{coequ}_{BA})) \cdot \pi(\gamma_B \cdot c_{BR}) = (\alpha_A, 1)\cdot L^A_{AX} \cdot \pi(\mathrm{coequ}_{BA})$
(iv) we have $(\alpha_X, 1) \cdot R^X_{XA}\cdot \pi(\mathrm{coequ}_{BA}) = (1, \pi(\mathrm{coequ}_{BA})) \cdot \pi(\gamma_B\cdot c_{BR}) = \pi(\pi(\mathrm{coequ}_{BA}) \cdot \gamma_B\cdot c_{BR})$; this follows also using the result of the forthcoming (vii)

Having this in mind the proof goes like this:

$$(1,(\alpha_C, 1))\cdot(1,R^C_{CA})\cdot\bar{p}_{BAC}\cdot(\mathrm{coequ}_{BA}, 1)\cdot\mathrm{equ}_{XC} =$$

$$= (1,(\alpha_C, 1)) \cdot (1, R^C_{CA}) \cdot (\pi(\mathrm{coequ}_{BA}),1) \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} = \qquad \text{(i)}$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (1,(\alpha_C, 1)) \cdot (R^C_{XA}, 1)\cdot R^{(AC)}_{(XC),(CC)}\cdot R^C_{CX}\cdot\mathrm{equ}_{XC} =$$

$$= (\pi(\mathrm{coequ}_{BA}), 1)\cdot (R^C_{XA}, 1) \cdot R^{(AC)}_{(XC),R}\cdot (\alpha_C, 1) \cdot R^C_{CX} \cdot \mathrm{equ}_{XC} = \qquad \text{(ii)}$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (R^C_{XA}, 1) \cdot R^{(AC)}_{(XC),R}\cdot (\alpha_X, 1) \cdot L^X_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (1, (\alpha_X, 1)) \cdot (R^X_{XA}, 1) \cdot L^{(XX)}_{(AX),(AC)} \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} = \qquad \text{(iv)}$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (\mathrm{R}^X_{XA}, 1) \cdot ((\alpha_X, 1), 1) \cdot L^R_{(AX),(AC)}\cdot L^A_{XC} \cdot \mathrm{equ}_{XC} = \qquad \text{(iii)}$$

$$= (\pi(\gamma_B \cdot c_{BR}), 1) \cdot ((1, \pi(\mathrm{coequ}_{BA})), 1)\cdot L^R_{(AX),(AC)}\cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

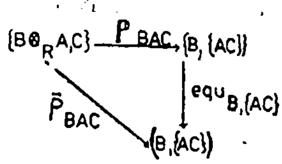
$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (L^A_{AX}, 1) \cdot ((\alpha_A,1),1) \cdot L^R_{(AX),(AC)} \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (1, (\alpha_A, 1)) \cdot (L^A_{AX}, 1) \cdot L^{(AA)}_{(AX),(AC)} \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (1,(\alpha_A, 1)) \cdot (1, L^A_{AC}) \cdot (\pi(\mathrm{coequ}_{BA}), 1) \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (1, (\alpha_A, 1)) \cdot (1, L^A_{AC}) \cdot \bar{p}_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC}.$$

Now, we must show the existence of morphisms $\mathbf{p}_{BAC}$ which commutatively close the diagram



and these will be the required isomorphisms. We have to check the following equality $(\alpha_{\{AC\}}\ ,1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot \bar{p}_{BAC}=(\alpha_B,1)\cdot L^{B}_{B,\{AC\}}\cdot \bar{p}_{BAC}$. Again, we need some preliminary results

(v) from proposition 3.13 we take the equality

$$(1,\mathrm{equ}_{AC})\cdot\alpha_{\{AC\}}=(\mathrm{equ}_{AC}\ ,1)\cdot L^{A}_{CC}\cdot\alpha_C$$

(vi) the following equality holds

$$(L^{A}_{CC}\ ,1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot p_{BAC}=(1,p_{BAC})\cdot R^{C}_{C,B\otimes A};$$

indeed, this follows applying π to axiom $MCC3$, composing to the left with $c_{(B\otimes A,C),(CC)}$ and applying π^{-1}.

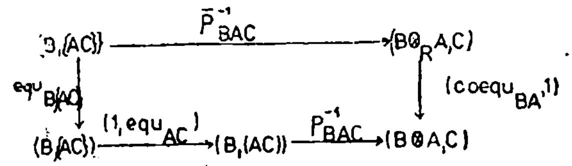
(vii) by a double application of π to the definition of $\gamma_X=\gamma_{R,BA}$ we get $p\cdot(\mathrm{coequ}_{BA},1)\cdot\alpha_X=(1,\pi(\mathrm{coequ}_{BA}))\cdot\alpha_B$

So, the following „enriched diagram chasing" proves the required equality

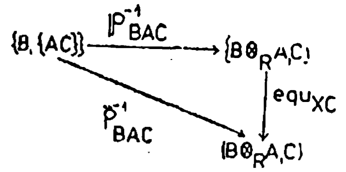
$$(1,(1,\mathrm{equ}_{AC}))\cdot(\alpha_{\{AC\}},1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot\bar{p}_{BAC}=$$
$$=(\alpha_{\{AC\}},1)\cdot((1,\mathrm{equ}_{AC}),1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot\bar{p}_{BAC}=\qquad\text{(v)}$$
$$=(\alpha_C,1)\cdot(L^{A}_{CC},1)\cdot((\mathrm{equ}_{AC},1),1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot\bar{p}_{BAC}=$$
$$=(\alpha_C,1)\cdot(L^{A}_{CC}\ ,1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot(1,\mathrm{equ}_{AC})\cdot\bar{p}_{BAC}=$$
$$=(\alpha_C,1)\cdot(L^{A}_{CC},1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot p_{BAC}\cdot(\mathrm{coequ}_{BA},1)\cdot\mathrm{equ}_{XC}=\qquad\text{(vi)}$$
$$=(\alpha_C,1)\cdot(1,p_{BAC})\cdot R^{C}_{C,B\otimes A}\cdot(\mathrm{coequ}_{BA}\ ,1)\cdot\mathrm{equ}_{XC}=$$
$$=(\alpha_C,1)\cdot(1,p_{BAC})\cdot(1,(\mathrm{coequ}_{BA},1))\cdot R^{C}_{CX}\cdot\mathrm{equ}_{XC}=$$
$$=(1,p_{BAC})\cdot(1,(\mathrm{coequ}_{BA}))\cdot(\alpha_X,1)\cdot L^{X}_{XC}\cdot\mathrm{equ}_{XC}=$$
$$=(\alpha_X,1)\cdot((\mathrm{coequ}_{BA}\ ,1),1)\cdot(1,p_{BAC})\cdot L^{B\otimes A}_{XC}\cdot\mathrm{equ}_{XC}=\qquad MCC3$$
$$=(\alpha_X,1)\cdot((\mathrm{coequ}_{BA},1),1)\cdot(p_{BAC},1)\cdot L^{B}_{(AX),(AC)}\cdot L^{A}_{XC}\cdot\mathrm{equ}_{XC}=\qquad\text{(vii)}$$
$$=(\alpha_B,1)\cdot((1,\pi(\mathrm{coequ}_{BA})),1)\cdot L^{B}_{(AC),(AC)}\cdot L^{A}_{XC}\cdot\mathrm{equ}_{XC}=$$
$$=(\alpha_B,1)\cdot L^{B}_{B,(AC)}\cdot(\pi(\mathrm{coequ}_{BA}),1)\cdot L^{A}_{XC}\cdot\mathrm{equ}_{XC}=$$
$$=(\alpha_B,1)\cdot L^{B}_{B,(AC)}\cdot p_{BAC}\cdot(\mathrm{coequ}_{BA}\ ,1)\cdot\mathrm{equ}_{XC}=$$
$$=(\alpha_B,1)\cdot L^{B}_{B,(AC)}\cdot(1,\mathrm{equ}_{AC})\cdot\bar{p}_{BAC}=(\alpha_B,1)\cdot(1,(1,\mathrm{equ}_{AC}))\cdot L^{B}_{B,\{AC\}}\cdot p_{BAC}=$$
$$=(1,(1,\mathrm{equ}_{AC}))\cdot(\alpha_B,1)\cdot L^{B}_{B,\{AC\}}\cdot\bar{p}_{BAC}$$

We now have, for each three left R-modules (A, α_A, γ_A), (B, α_B, γ_B), (C, α_C, γ_C), a morphism $\mathbf{p}_{BAC}: \{B \otimes A, C\} \to \{B, \{AC\}\}$.

Analogously, we can determine a family of morphisms $\bar{p}^{-1}_{BAC}$ which close the commutative diagrams



and next, a family $\mathbf{p}^{-1}_{BAC}$ which close the commutative triangle



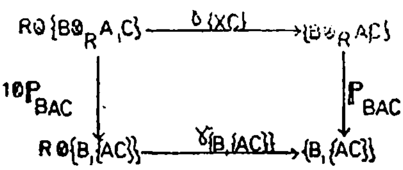
Finally, the following three facts must be checked

(a) $\mathbf{p}_{BAC}$ and $\mathbf{p}^{-1}_{BAC}$ actually are morphisms of left R-modules over $\underline{V}$;

(b) $\mathbf{p}_{BAC}$ and $\mathbf{p}^{-1}_{BAC}$ are mutually inverse;

(c) the family $\mathbf{p} = \mathbf{p}_{BAC}$ is natural in ${}_R\underline{MV}$.

As for (a), we show for instance that $\mathbf{p}_{BAC}$ actually is a morphism of left R-modules over $\underline{V}$. The commutativity of the diagram



reduces by left composition with $\mathrm{equ}_{B, \{AC\}}$ to $M^{\{AC\}}_{B, \{AC\}} \cdot \alpha_{\{AC\}} \otimes \bar{p}_{BAC} = \bar{p}_{BAC} \cdot \gamma_{\{B \otimes_R A, C\}}$ or, applying π, to $(\bar{p}_{BAC}, 1) \cdot L^{B}_{\{AC\}, \{AC\}} \cdot \alpha_{\{AC\}} = (1, \bar{p}_{BAC}) \cdot \alpha_{\{B \otimes_R A, C\}}$, equality which one can verify by left composition with $(1, (1, \mathrm{equ}_{AC}))$ by a new „enriched diagram chasing".

For (b) we choose $\mathbf{p} \cdot \mathbf{p}^{-1} = 1$; indeed we show that

$$(1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B, \{AC\}} \cdot \mathbf{p}_{BAC} \cdot \mathbf{p}^{-1}_{BAC} = (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B, \{AC\}}.$$

We have

$$(1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B, \{AC\}} \cdot \mathbf{p}_{BAC} \cdot \mathbf{p}^{-1}_{BAC} = (1, \mathrm{equ}_{AC}) \cdot \bar{p}_{BAC} \cdot \mathbf{p}^{-1}_{BAC} =$$
$$= p_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC} \cdot \mathbf{p}^{-1}_{BAC} = p_{BAC} (\mathrm{coequ}_{BA}, 1) \cdot \bar{p}^{-1}_{BAC} =$$
$$= p_{BAC} \cdot p^{-1}_{BAC} \cdot (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B, \{AC\}} = (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B, \{AC\}}.$$

For (c) we choose the naturality of $\mathbf{p}_{BAC}$ in (A, α_A, γ_A), that is, the commutativity of the following diagram

$$\begin{array}{ccc}
\{B\otimes_R A,C\} & \xrightarrow{\mathbf{p}_{BAC}} & \{B,\{A,C\}\} \\
\downarrow{\scriptstyle \{1\otimes_R f,1\}} & & \downarrow{\scriptstyle \{1,\{f,1\}\}} \\
\{B\otimes_R A',C\} & \xrightarrow{\mathbf{p}_{BA'C}} & \{B,\{A',C\}\}
\end{array}$$

Again, we prove an equivalent equality

$$(1, \mathrm{equ}_{A'C}) \cdot \mathrm{equ}_{B,\{AC\}} \cdot \mathbf{p}_{BA'C} \cdot \{1\otimes_R f,1\} = (1,\mathrm{equ}_{A'C}) \cdot \bar{p}_{BA'C} \cdot \{1\otimes_R f,1\} =$$
$$= p_{BA'C} \cdot (\mathrm{coequ}_{BA'}, 1) \cdot \mathrm{equ}_{B\otimes_R A', C} \cdot \{1\otimes_R f,1\} =$$
$$= p_{BA'C} \cdot (\mathrm{coequ}_{BA'}, 1) \cdot (1\otimes_R f,1) \cdot \mathrm{equ}_{XC} =$$
$$= p_{BA'C} \cdot (1\otimes f,1) \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC} = (1,(f,1)) \cdot p_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC} =$$
$$= (1,(f,1)) \cdot (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B,\{AC\}} \cdot \mathbf{p}_{BAC} =$$
$$= (1, \mathrm{equ}_{A'C}) \cdot (1,\{f,1\}) \cdot \mathrm{equ}_{B,\{AC\}} \cdot \mathbf{p}_{BAC} =$$
$$= (1, \mathrm{equ}_{A'C}) \cdot \mathrm{equ}_{B,\{A'C\}} \cdot \{1,\{f,1\}\} \cdot \mathbf{p}_{BAC}$$

In this way all the symmetric monoidal closed structure for ${}_R\underline{MV}$ is established. One can complete the proof of our theorem verifying axioms [*MCC*2], [*MCC*3], [*MCC*3'] and [*MCC*4].

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ASUPRA UNEI TEORII ÎMBOGĂŢITE A MODULELOR (II)

(Rezumat)

Utilizînd noţiunile preliminare studiate în partea întîia a aceluiaşi articol, autorul stabileşte rezultatele principale privitoare la partea închisă şi monoidal închisă a teoriei modulelor peste un monoid fixat, rezultate care conduc în final la teorema: dacă $\underline{V}$ este o categorie simetric monoidal închisă, categoria subiacentă V_0 are egalizatori şi coegalizatori, R este un monoid comutativ peste $\underline{V}$, functorul de subiacenţă V păstrează egalizatori şi $R\otimes$ – păstrează coegalizatori, atunci categoria modulelor ${}_R\underline{MV}$ este simetric monoidal închisă.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXV, 3, 1980

CALCULUL PULSAŢIILOR NELINIARE ALE UNEI SFERE DE GAZ ÎN ROTAŢIE

N. LUNGU

I. Introducere. În această lucrare prezentăm metoda de calcul utilizată în pulsaţiile neliniare ale unei sfere în rotaţie uniformă. Acestea apar în cazul cînd pulsează numai un strat, care reprezintă aproximativ 0,9 din raza totală.

Ecuaţiile generale, în cazul simetriei sferice, au fost integrate printr-o metodă numerică care a putut fi uşor adaptată la calculator. În final se prezintă rezultatele numerice.

II. Ecuaţiile mişcării. Fie un mediu fluid sferic simetric, în rotaţie uniformă cu viteza unghiulară constantă ω. În ecuaţiile mişcării folosim masa m_r ca şi coordonată Lagrange. Masa corespunzătoare razei r este [3]

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

unde $\rho(r)$ este densitatea corespunzătoare razei r. Volumul specific este

$$V = \frac{1}{\rho(r)} = 4\pi r^2 \frac{dr}{dm} \tag{1}$$

Ecuaţia lui Newton, considerînd şi rotaţia, are forma [7],

$$\frac{\partial^2 r}{\partial t^2} = -\frac{k^2 m}{r^2} + \frac{2}{3}\omega^2 r - 4\pi r^2 \frac{\partial P}{\partial m_r} \tag{2}$$

unde P este presiunea corespunzătoare masei m_r.

Fluxul caloric se consideră difuzat în întregime prin radiaţie, şi ecuaţia fluxului devine:

$$L_r = -(4\pi r^2)^2 \frac{4a_1}{3k} \frac{d(T^4)}{dm_r} \tag{3}$$

unde a_1 este constanta Ştefan-Boltzmann, iar $k(V,T)$ opacitatea medie. Dacă neglijăm generarea energiei nucleare în învelişul care oscilează, ecuaţia bilanţului caloric este:

$$\frac{\partial E}{\partial t} + P\frac{\partial V}{\partial t} + \frac{dL}{dm_r} = 0 \tag{4}$$

E fiind energia internă pe gram.

Ecuaţiile (1)—(4) formează sistemul diferenţial în care variabile independente sînt t şi m_r. La acest sistem se adaugă funcţiile cunoscute $E = E(V,T)$, $P = P(V,T)$ şi $k = k(V,T)$.

III. **Condiţii la limită.** Pentru integrarea sistemului (1)—(4) formulăm următoarele condiţii la limită:

$$L\Big|_{r=r_0} = L_0, \frac{dr}{dt}\Big|_{r=r_0} = 0 \tag{5}$$

exprimînd faptul că luminozitatea (fluxul total) care intră în înveliş prin partea interioară (la $r = r_0$) este constantă, L_0, şi de asemenea nucleul central nu pulsează (viteza este zero). La limita exterioară a învelişului avem:

$$P|_{r=R_0} = 0, \quad T^4|_{r=R_0} = \frac{1}{2}T_e^4 \tag{6}$$

adică sfera se consideră liberă şi pe suprafaţă are o temperatură efectivă T_e.

IV. **Modelul de echilibru.** Condiţia de echilibru se realizează anulînd derivatele în raport cu timpul. În cazul de faţă modelul de echilibru nu pulsează, dar avem posibilitatea determinării structurii interne a sferei care se va folosi la deducerea soluţiei dinamice. Parametrii fundamentali îi considerăm cei ataşaţi unei stele pulsante de tip R R Lyrae, deduşi prin metode observaţionale, pe care-i utilizăm în calculele numerice. Valorile acestora sînt cunoscute [3], astfel: masa $M_0 = 0{,}75 \cdot 10^{33} g$, $L = 1{,}5 \cdot 10^{35}$ erg/sec, $T_e = 6500\,K$, $R_0 = 3{,}41 \cdot 10^{11}$cm. Pentru ω considerăm o valoare corespunzătoare tipului spectral A_0.

V. **Procedeul de rezolvare.** În scopul integrării sistemului (1)—(4), cu condiţiile la limită (5), (6), este necesar să exprimăm ecuaţiile prin diferenţe. Pentru că se cuplează ecuaţii hidrodinamice şi calorice, este necesar să studiem stabilitatea şi precizia soluţiei. Un studiu similar este făcut de Richtmyer [5], dar problemele se tratează individual. În scopul scrierii prin diferenţe, variabila $r(m, t)$ este reprezentată prin cantitatea discretă r_i^n, unde indicele n (întreg) reprezintă timpul, iar indicele i reprezintă masa m_i interioară razei r_i. Valoarea $i = 1$ reprezintă limita interioară, iar $i = N$ cea exterioară.

$$m_{i-\frac{1}{2}} = m_i - m_{i-1}$$

Volumul specific al elementului de masă $i - \frac{1}{2}$ este

$$V_{i-\frac{1}{2}} = \frac{\frac{4}{3}\pi(r_i^3 - r_{i-1}^3)}{m_{i-\frac{1}{2}}} \tag{7}$$

Presiunea și temperatura corespunzătoare elementului de masă $i - \frac{1}{2}$ sînt $P^n_{i-\frac{1}{2}}$, $T^n_{i-\frac{1}{2}}$. Viteza $\dot{r}_i$ este definită la momentul $t^n + \frac{1}{2}\Delta t$ și o notăm $\dot{r}_i^{n+\frac{1}{2}}$. Atunci ecuația (2) are forma:

$$\ddot{r}_i = \frac{-4\pi r_i^2}{\frac{1}{2}\left(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}\right)}\left(p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}} + Q^{n-\frac{1}{2}}_{i+\frac{1}{2}} - Q^{n-\frac{1}{2}}_{i-\frac{1}{2}}\right) - k\frac{\sum_{j=1}^{i} m_{j-\frac{1}{2}}}{r_i^2} + \frac{2}{3}\omega_i^2 r_i,\ i = \overline{2, N-1} \tag{8}$$

unde

$$Q^{n-\frac{1}{2}}_{i-\frac{1}{2}} = C_Q \frac{(\dot{r}_i - \dot{r}_{i-1})^2}{V^n_{i-\frac{1}{2}} + V^{n-1}_{i-\frac{1}{2}}} \neq 0 \quad \text{pentru } \dot{r}_i - \dot{r}_{i-1} < 0 \tag{9}$$

și este zero în cazul $\dot{r}_i - \dot{r}_{i-1} > 0$. Q se numește „viscozitate artificială" și din definiție rezultă că se introduce numai în compresie, fiind implicată de undele de șoc [5]. Constanta C_Q este aleasă ca un compromis între stabilitate și precizie. Q are ca efect formarea unui front de șoc cu efect de stabilizare.

Ecuația mișcării, pentru $i = \overline{2, N-1}$, are forma:

$$\dot{r}_i^{n+\frac{1}{2}} = \dot{r}_i^{n-1} + \ddot{r}_i^n \Delta t^n \tag{10}$$

unde

$$\Delta t^n = \frac{1}{2}\left(\Delta t^{n-\frac{1}{2}} + \Delta t^{n+\frac{1}{2}}\right) \tag{11}$$

Condițiile la limită sînt: la $i = 1$, $\dot{r}_1^{n+\frac{1}{2}} = 0$; la $i = N$, $P = 0$. Din (10) se poate exprima explicit condiția la limită $\dot{r}_1^{n+\frac{1}{2}} = 0$. Ecuațiile (5) și (6) se scriu prin diferențe și se cuplează, formîndu-se un sistem unic. Sistemul se simplifică dacă notăm $T^4 = W$ și obținem [1].

$$-c_{i+\frac{1}{2}}\,{}^j\delta W^{n+1}_{i-\frac{1}{2}} + b_{i+\frac{1}{2}}\,{}^j\delta W^{n+1}_{i+\frac{1}{2}} - a_{i+\frac{1}{2}}\,{}^j\delta W^{n+1}_{i+\frac{3}{2}} = d_{i+\frac{1}{2}} \tag{12}$$

unde δW se definește prin relația

$${}^{j+1}W^{n+1}_{i+\frac{1}{2}} = {}^jW^{n+1}_{i+\frac{1}{2}} + {}^j\delta W^{n+1}_{i+\frac{1}{2}} \tag{13}$$

Sistemul (12) este tridiagonal și-l rezolvăm prin metoda „Crossing" sau a parcurgerii directe și inverse. Atunci $\delta W_{i-\frac{1}{2}}$ se exprimă în forma:

$$\delta W_{i-\frac{1}{2}} = \delta W_{i+\frac{1}{2}} p_{i-\frac{1}{2}} + q_{i-\frac{1}{2}} \tag{14}$$

Dacă (14) se înlocuiește în (12), obținem

$$\delta W_{i+\frac{1}{2}} = \delta W_{i+\frac{3}{2}} p_{i+\frac{1}{2}} + q_{i+\frac{1}{2}} \tag{15}$$

unde coeficienții $p_{i+\frac{1}{2}}$ și $q_{i+\frac{1}{2}}$ se calculează cu formulele:

$$p_{i+\frac{1}{2}} = \frac{a_{i+\frac{1}{2}}}{b_{i+\frac{1}{2}} - C_{i+\frac{1}{2}} p_{i-\frac{1}{2}}}, \quad q_{i+\frac{1}{2}} = \frac{d_{i+\frac{1}{2}} + C_{i+\frac{1}{2}} q_{i-\frac{1}{2}}}{b_{i+\frac{1}{2}} - C_{i+\frac{1}{2}} p_{i-\frac{1}{2}}} \tag{16}$$

Pentru $i = 1$, din (12), obținem:

$$\delta W_{\frac{3}{2}} = \delta W_{\frac{5}{2}} p_{\frac{3}{2}} + q_{\frac{3}{2}}, \quad p_{\frac{3}{2}} = \frac{a_{3/2}}{b_{3/2}}, \quad q_{3/2} = \frac{d_{3/2} + c_{3/2}\,\delta W_{1/2}}{b_{3/2}} \tag{17}$$

în care $\delta W_{1/2}$ se cunoaște din condiția la limită (5). Se pot calcula astfel coeficienții $p_{i+\frac{1}{2}}$ și $q_{i+\frac{1}{2}}$ prin parcurgere directă. Pentru $i = N$ avem,

$$\delta W_{N+\frac{1}{2}} = p_{N+\frac{1}{2}}\, \delta W_{N+\frac{3}{2}} + q_{N+\frac{1}{2}} \tag{18}$$

în care $\delta W_{N+\frac{3}{2}}$ se cunoaște din (6). Se demonstrează că acest procedeu este stabil și convergent.

VI. **Rezultate numerice.** 1° Procedeul prezentat mai sus a fost programat în FORTRAN IV la calculatorul electronic FELIX C—256. S-a obținut la început modelul de echilibru, considerîndu-se 38 de straturi (noduri) în înveliș. Se determină astfel distribuția funcțiilor M, R, V, T, P de la exterior spre interior (tabelul 1). În figura 1 este dată distribuția temperaturii în înveliș reprezentînd structura fizică a învelișului.

Tabel 1

Modelul de echilibru

Nr. str.	$\Delta M(g)$	$R(cm)$	$V(cm^3)$	$T(K)$	$P(dyn/cm^2)$
38	0,30000E+25	0,34300E+12	0,75959E+09	0,54658E+04	0,39529E+03
34	11520E+26	33245	32322	54749	93049
30	44300	32207	79947E+08	11623E+05	14192E+05
24	33320E+27	30024	16651	73273	52485E+06
18	25000E+28	26379	24623E+07	18879E+06	91443E+07
12	18900E+29	20933	30815E+06	40145	15537E+09
8	83900	15912	61460E+05	72644	14097E+10
4	62900	84264E+11	13109E+04	19303E+07	17562E+12

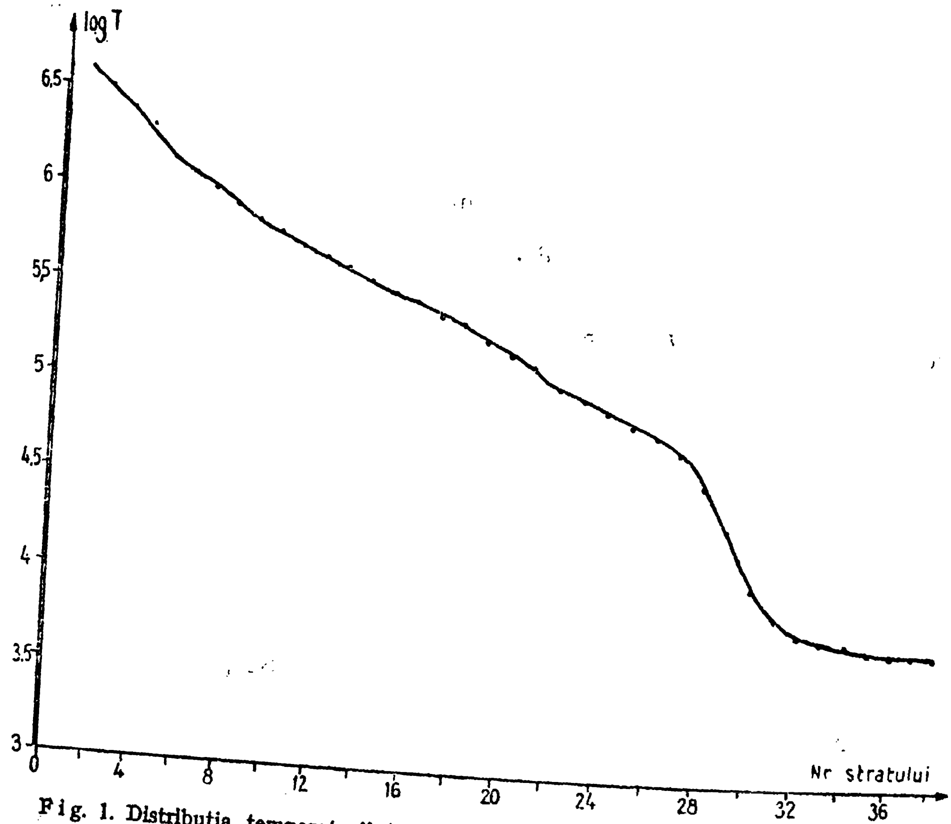

Fig. 1. Distribuţia temperaturii în înveliş (log *T* funcţie de poziţia în înveliş).

Variaţia temperaturii dă informaţii referitoare la pornirea şi întreţinerea pulsaţiilor, observîndu-se un salt rapid în creşterea temperaturii, cauzat de sumul de energie pentru ionizarea *H* şi H_e. Între $6{,}5 \cdot 10^4 K$ şi $2{,}7 \cdot 10^5 K$ se produce o amortizare accentuată a oscilaţiilor.

2° În continuare vom prezenta cîteva rezultate ale modelului dinamic. Deoarece curba vitezei radiale se cunoaşte cu mare precizie din măsurători practice, vom reprezenta grafic viteza radială a straturilor (nodurilor), 35, 28, 26, 20, 10, 4 pe care le vom compara cu practica. În figura 2 se reprezintă $\dot{r}$ în funcţie de timp.

Tabel 2

Dependenţa amplitudinii de poziţia în strat

Stratul nr.	Amplitudinea (*A*)	Stratul nr.	Amplitudinea (*A*)	Stratul nr.	Amplitudinea (*A*)
36	1,550	24	1,370	12	0,850
34	1,685	22	1,220	10	0,960
32	1,550	20	0,920	8	0,720
30	1,500	18	0,920	6	0,520
28	1,440	16	0,905	4	0,500
26	1,440	14	0,930	2	0,460

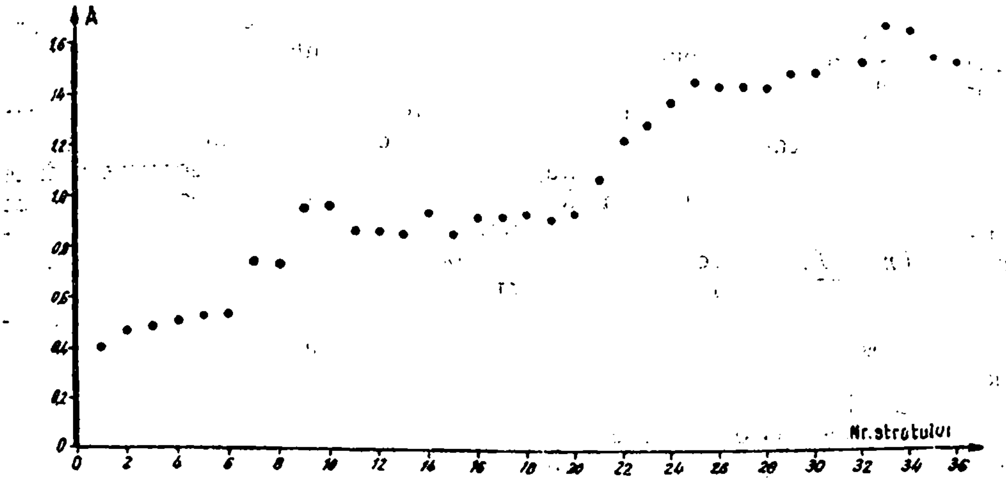

Fig. 2. Vitezele radiale în straturile 35, 28, 26, 20, 10, 4 (în abscisă faza ψ în fracţiuni de perioadă).

Din curbele vitezelor radiale se pot deduce amplitudinile curbelor pe care le dăm în tabelul 2 şi figura 3. Se constată uşor că amplitudinea scade spre interior în cadrul aceleiaşi zone şi în înveliş. De exemplu, la baza învelişului

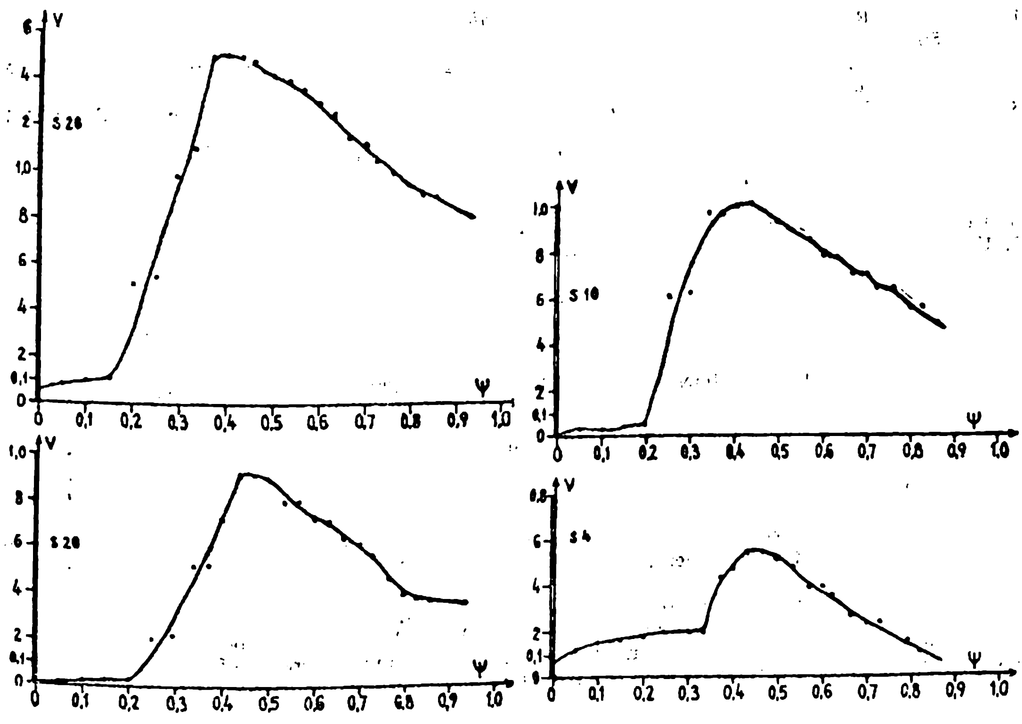

Fig. 3. Dependenţa amplitudinii (A) de poziţia în înveliş.

amplitudinea este de 4 ori mai mică, rezultînd o amortizare a pulsaţiilor spre interiorul sferei. Astfel, putem concluziona că oscilaţiile se găsesc esenţial la suprafaţa sferei.

3° Introducerea termenului care conţine rotaţia are cel puţin trei efecte asupra modelului care oscilează. Un prim efect se manifestă prin turtirea sferei, dar dacă se consideră o rotaţie lentă, atunci oscilaţiile pot fi socotite radiale şi se neglijează turtirea. Al doilea efect se manifestă în micşorarea perioadei de pulsaţie, care poate atinge 30 de procente. Astfel, cu aceiaşi parametri fundamentali (m, L, T_e) în prezenţa rotaţiei perioada de pulsaţie a unei stele R R Lyrae este 0,475 zile, iar în absenţa rotaţiei [3], perioada este 0,680 zile.

Al treilea efect al rotaţiei este desincronizarea pulsaţiilor diverselor straturi. Se observă în figura 2 că straturile nu pulsează în fază şi momentele maximelor nu coincid. Rezultă şi de aici micşorarea perioadei de pulsaţie din cauza rotaţiei.

Se impune aici o concluzie firească, şi anume că rotaţia este diferenţială şi trebuie studiate pulsaţiile neradiale.

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THE CALCULATION OF NONLINEAR PULSATIONS OF THE GAS SPHERE IN ROTATION

(Summary)

In this paper a theoretical method has been developed concernind the calculus of pulsation of RR Lyrae stars.

In this case the equations of the model are the same as those used in paper [3] excepting the equation of the movement. The system is solved by the method of "direct and invers crossing".

We have given here the results of the calculus. In table 1 we give the static model, in figure 1 the temperature (*K*) and in figure 2 the radial velocity for one period.

Our calculus indicates a differential rotation and a gradual increase of ω from the surface towards the interior. Rotation reduces the period of pulsations by up to 30%.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXV, 3, 1980

NOI FORMULE DE CUADRATURĂ CU ELEMENTE FIXE

DUMITRU ACU

1. Fie $F[0,1]$ clasa funcţiilor f definite pe intervalul $[0, 1]$ şi integrabile în sensul lui Lebesgue pe acest interval.

Să presupunem că avem dată formula de cuadratură

$$\int_0^1 f(x)dx = \sum_{i=0}^{n-1} A_i f(x_i) + r_n^{(0)}(f), \tag{1}$$

$$0 \leqslant x_0 < x_1 < \ldots < x_{n-1} \leqslant 1,$$

cu evaluarea exactă pentru rest

$$r_n^{(0)} = \sup_{f \in F[0,1]} |r_n^{(0)}(f)|. \tag{2}$$

Se pune problema găsirii unei formule de cuadratură de tipul

$$\int_0^1 f(x)dx = \sum_{i=0}^{n-1} A_i f(x_i) + \sum_{k=0}^{m-1} B_k f(y_k) + R_m^{(0)}(f) \tag{3}$$

astfel încît ea să fie optimală pe $F[0, 1]$, adică să se determine coeficienţii B_k şi nodurile

$$0 < y_0 < y_1 < \ldots < y_{m-1} < 1$$

astfel ca

$$R_m^{(0)} = \sup_{f \in F[0,1]} |R_m^{(0)}(f)| \tag{4}$$

să fie minim.

Formula de cuadratură astfel obţinută se numeşte *formula optimală ataşată formulei (1)*, pentru clasa de funcţii $F[0, 1]$.

Pentru o formulă de cuadratură (1) dată se pot obţine diverse variante ale problemei puse, dacă se caută formula de cuadratură optimală ataşată formulei (1) printre formulele a căror coeficienţi B_k, $k = \overline{0, m-1}$, şi noduri y_k, $k = \overline{0, m-1}$, nu sînt arbitrare, ci sînt supuse unor legături dinainte date.

O astfel de problemă a fost studiată de M. Levin [3].

În prezenta lucrare obţinem noi formule optimale ataşate unei formule de cuadratură dată, pentru anumite clase de funcţii.

2. Fie $W_0^{(1)}L_2(M; 0, 1)$ mulţimea funcţiilor f, definite pe intervalul $[0, 1]$, absolut continue, şi care satisfac condiţiile $f(0) = 0$, $\|f'\|_{L_2} \leq M$.

Să considerăm ca(1) formula de cuadratură optimală pe clasa de funcţii $W_0^{(1)}L_2(M; 0, 1)$ (vezi [1], [2]);

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{k=0}^{n-1} f\left(\frac{2k+2}{2n+1}\right) + r_n^{(0)}(f), \tag{5}$$

cu

$$r_n^{(0)} = \frac{M}{(2n+1)\sqrt{3}}. \tag{6}$$

În cazul $m = n$, $y_k = (x_{k-1} + x_k)/2$, $k = \overline{0, n-1}$, M. Levin ([3]) găseşte următoarea formulă optimală ataşată formulei (5)

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{k=0}^{n-1} f\left(\frac{2k+2}{2n+1}\right) + \frac{1}{2(2n+1)} f\left(\frac{1}{2n+1}\right) + R_n^{(0)}(f), \tag{7}$$

cu

$$R_n^{(0)} = \frac{M}{(2n+1)\sqrt{3}}\sqrt{1 - \frac{3}{4(2n+1)}} = r_n^{(0)}\sqrt{1 - \frac{3}{4(2n+1)}}. \tag{8}$$

În acest paragraf ne propunem să determinăm formula optimală ataşată formulei (5) cînd $m = n$ iar nodurile $y_0, y_1, \ldots, y_{n-1}$ sînt luate astfel

$$y_0 = r,\ y_1 = \frac{2}{2n+1} + r,\ y_2 = \frac{4}{2n+1} + r,\ \ldots, y_{n-1} = \frac{2n-2}{2n+1} + r, \tag{9}$$

unde r este un număr dat din intervalul $\left(0, \frac{2}{2n+1}\right]$.

Altfel spus, ne propunem să determinăm coeficienţii B_k, $k = \overline{0, n-1}$, în funcţie de r, astfel încît în formula

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + \sum_{k=0}^{n-1} B_k f(y_k) + R_n^{(0)}(f, r) \tag{10}$$

evaluarea exactă a restului, adică

$$R_n^{(0)}(r) = \sup_{W_0^{(1)}L_2(M; 0,1)} \left|R_n^{(0)}(f, r)\right|, \tag{11}$$

să fie minimă.

Utilizînd identitatea

$$f(x) = \int_0^1 f'(x)(x-t)_+ dt, \quad x \in [0, 1],$$

unde

$$u_+ = \begin{cases} 0, & u \leqslant 0 \\ 1, & u > 0, \end{cases}$$

avem

$$R_n^{(0)}(f, r) = \int_0^1 f'(t) K(t) dt, \tag{12}$$

unde

$$K(t) = 1 - t - \frac{2}{2n+1} \sum_{i=0}^{n-1} \left(\frac{2i+2}{2n+1} - t \right)_+ - \sum_{k=0}^{n-1} B_k (y_k - t)_+ .$$

Aplicînd în (12) inegalitatea lui Cauchy-Buniakovski, obţinem

$$R_n^{(0)}(r) \leqslant M \left(\int_0^1 K^2(t) dt \right)^{\frac{1}{2}} . \tag{13}$$

Dacă se ia f_0 definită prin

$$f_0(x) = M \left(\int_0^1 K^2(t) dt \right)^{-\frac{1}{2}} \int_0^x K(t) dt,$$

atunci $f_0 \in W_0^{(1)} L_2(M; 0,1)$ iar în (13) are loc semnul egal. Rezultă că

$$R_n^{(0)}(r) = M \left(\int_0^1 K^2(t) dt \right)^{\frac{1}{2}} . \tag{14}$$

şi problema pusă se reduce la determinarea coeficienţilor B_k, $k = \overline{0, n-1}$, astfel încît integrala

$$I = \int_0^1 K^2(t) dt$$

să fie minimă.

Din $\frac{\partial I}{\partial B_i} = 0$, $i = 0, 1, \ldots, n-1$, rezultă sistemul

$$\sum_{k=0}^{n-1} B_k \int_0^1 (y_k - t)_+ (y_i - t)_+ dt = \int_0^1 (1 - t)(y_i - t)_+ dt -$$

$$- \frac{2}{2n+1} \sum_{i=0}^{n-1} \int_0^1 \left(\frac{2i+2}{2n+1} - t \right)_+ (y_i - t)_+ dt, \quad i = 0, 1, \ldots, n-1,$$

care după calcularea integralelor se scrie sub forma

$$-\sum_{k=0}^{l} B_k y_k + y_l \sum_{k=l+1}^{n-1} B_k = \frac{2l+1}{2n+1} y_l - \frac{y_l^2}{2} - \frac{2(l+1)}{(2n+1)^2}, \quad l = \overline{0, n-1}. \tag{15}$$

Acest sistem este de tip Cramer cu determinantul

$$= \begin{vmatrix} y_0 & y_0 & y_0 & \cdots & y_0 & y_0 \\ y_0 & y_1 & y_1 & \cdots & y_1 & y_1 \\ y_0 & y_1 & y_2 & \cdots & y_2 & y_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ y_0 & y_1 & y_2 & \cdots & y_{n-2} & y_{n-2} \\ y_0 & y_1 & y_2 & \cdots & y_{n-2} & y_{n-1} \end{vmatrix} = y_0(y_1 - y_0)(y_2 - y_1)\cdots(y_{n-1} - y_{n-2}) \neq 0$$

şi, prin urmare, el admite o soluţie unică.

Rezolvînd sistemul (15) şi ţinînd seama de (9), găsim

$$B_0 = \frac{1}{2}\left(\frac{2}{2n+1} - r\right), \tag{16}$$

$$B_1 = B_2 = \ldots = B_{n-1} = 0.$$

Pentru (16) avem

$$I = \int_0^1 K^2(t)dt = \frac{1}{3(2n+1)^2} - \left(\frac{2}{2n+1} - r\right)\left(\frac{1}{2n+1} - \frac{r}{2}\right)\frac{r}{2}. \tag{17}$$

Deci, formula optimală ataşată formulei (7), cu nodurile date de (9), este

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + \frac{1}{2}\left(\frac{2}{2n+1} - r\right)f(r) + R_n^{(0)}(f, r) \tag{18}$$

cu

$$R_n^{(0)}(r) = M\sqrt{\frac{1}{3(2n+1)^2} - \left(\frac{2}{2n+1} - r\right)\left(\frac{1}{2n+1} - \frac{r}{2}\right)\frac{r}{2}}, \quad 0 < r \leqslant \frac{2}{2n+1} \tag{19}$$

Pentru $r = \frac{1}{2n+1}$ din (18) şi (19) rezultă (7) şi (8), adică rezultatele lui M. Levin ([3]).

În continuare ne propunem să determinăm pe $r \in \left(0, \frac{2}{2n+1}\right)$ astfel încît $R_n^{(0)}(r)$ să fie minim.

După (19), aceasta revine la a determina pe $r \in \left[0, \frac{2}{2n+1}\right]$ astfel ca funcţia

$$g(r) = \left(\frac{2}{2n+1} - r\right)\left(\frac{1}{2n+1} - \frac{r}{2}\right)\frac{r}{2}$$

să ia valoarea maximă.

Avem

$$g'(r) = \frac{3r^2}{4} - \frac{2}{2n+1}r + \frac{1}{(2n+1)^2}$$

care se anulează în intervalul $\left[0, \frac{2}{2n+1}\right]$ numai pentru $r = \frac{2}{3}\cdot\frac{1}{2n+1}$. Cum $g''\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right) = -\frac{1}{2n+1} < 0$, rezultă că $r = \frac{2}{3}\cdot\frac{1}{2n+1}$ este un maxim local pentru $g(r)$. Cum $g\left(\frac{2}{2n+1}\right) = 0$ şi $\lim\limits_{r\searrow 0} g(r) = 0$, deducem

$$\max_{r\in\left(0, \frac{2}{2n+1}\right]} g(r) = g\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right) = \frac{8}{27}\cdot\frac{1}{(2n+1)^3}.$$

Deci, dintre toate formulele (18), cea pentru care $\min\limits_{r\in\left(0, \frac{2}{2n+1}\right]} R^{(0)}(r)$ este atins se obţine pentru $r = \frac{2}{3}\cdot\frac{1}{2n+1}$. Această formulă este

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + \frac{2}{3(2n+1)}f\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right) + R_n^{(0)}\left(f, \frac{2}{3}\cdot\frac{1}{2n+1}\right) \tag{20}$$

cu

$$R_n^{(0)}\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right) = \frac{M}{(2n+1)\sqrt{3}}\sqrt{1 - \frac{8}{9}\cdot\frac{1}{2n+1}} = r_n^{(0)}\sqrt{1 - \frac{8}{9}\cdot\frac{2n+1}{1}}. \tag{21}$$

3. Fie $W^{(1)}L_2(M; 0,1)$ mulţimea funcţiilor f definite pe $[0, 1]$, absolut continue şi care satisfac condiţia $\|f'\|_{L_2} \leq M$. Pentru această clasă de funcţii şi pentru formula de cuadratură (1) dată, ne propunem să construim formula optimală ataşată pentru formulele de tipul

$$\int_0^1 f(x)dx = \sum_{i=0}^{n-1} A_i f(x_i) + Cf(0) + \sum_{k=0}^{n-1} B_k f(y_k) + R_n(f). \tag{22}$$

Avem inegalitatea evidentă

$$\sup_{f\in W^{(1)}L_1(M;0,1)} |R_n(f)| \geqslant \sup_{f\in W_0^{(1)}(M;0,1)} |R_n(f)|. \tag{23}$$

Dacă punem condiția ca formula (22) să aibă gradul de exactitate egal cu 0, atunci rezultă

$$C = 1 - \sum_{k=0}^{n-1} (A_k + B_k). \tag{24}$$

Pentru C dat de (24) avem

$$R_n(f(x)) = R_n[f(x) - f(0)],$$

de unde obținem

$$\sup_{f\in W^{(1)}L_1(M;0,1)} |R(f)| \leqslant \sup_{f\in W_0^{(1)}L_1(M;0,1)} |R_n(f)|. \tag{25}$$

Din (23) și (25) rezultă

$$\sup_{f\in W^{(1)}L_1(M;0,1)} |R_n(f)| = \sup_{f\in W_0^{(1)}L_1(M;0,1)} |R_n(f)|.$$

De aici și din cele demonstrate la 2. deducem că dintre formulele de tipul

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + Cf(0) + \sum_{k=0}^{n-1} B_k f(y_k) + R_n(f, r) \tag{26}$$

cu nodurile y_k, $k = \overline{0, n-1}$, date de (9), cea optimală este

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + \frac{r}{2}f(0) + \frac{1}{2}\left(\frac{2}{2n+1} - r\right)f(r) + R_n(f, r) \tag{27}$$

cu

$$R_n(r) = \sup_{f\in W^{(1)}L_1(M;0,1)} |R_n(f, r)| = M\sqrt{\frac{1}{3(2n+1)^2} - \left(\frac{2}{2n+1} - r\right)\left(\frac{1}{2n+1} - \frac{r}{2}\right)\frac{r}{2}}, \tag{28}$$

unde r este un număr dat din intervalul $\left(0, \dfrac{2}{2n+1}\right]$.

Pentru $r = \dfrac{1}{2n+1}$ din (28) și (29) rezultă formula optimală

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + \frac{1}{2(2n+1)}\left[f(0) + f\left(\frac{1}{2n+1}\right)\right] + R_n\left(f, \frac{1}{2n+1}\right)$$

cu

$$R\left(\frac{1}{2n+1}\right) = \frac{M}{(2n+1)\sqrt{3}}\sqrt{1 - \frac{3}{4}\cdot\frac{1}{2n+1}},$$

rezultat obținut de M. Levin în [3].

Folosind cele demonstrate în partea a doua de la 2, deducem că dintre toate formulele (27), cea pentru care $\min\limits_{r\in\left(0,\frac{2}{2n+1}\right]} R_n(r)$ este atins, este cea care se obține din (27) pentru $r = \frac{2}{3}\cdot\frac{1}{2n+1}$;

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{i=0}^{n-1} f\left(\frac{2i+2}{2n+1}\right) + \frac{1}{3(2n+1)}\left[f(0) + 2f\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right)\right] + R_n\left(f, \frac{2}{3}\cdot\frac{1}{2n+1}\right)$$

cu

$$R\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right) = \frac{M}{(2n+1)\sqrt{3}}\sqrt{1 - \frac{8}{9}\cdot\frac{1}{2n+1}}.$$

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NOUVELLES FORMULES DE QUADRATURE A ÉLÉMENTS FIXES

(Résumé)

Dans le présent travail on pose le problème: étant donné la formule de quadrature (1) avec l'évaluation exacte de l'erreur (2), on détermine la formule de quadrature (3), à vrai dire les coefficients B_k et les noeuds y_k, $k = \overline{0, m-1}$, ainsi que le reste $R_m^{(0)}$, soit minime pour la classe de fonctions $F[0, 1]$. La formule de quadrature ainsi obtenue est dite *la formule de quadrature optimale attachée à la formule* (1). On construit ensuite les formules optimales pour deux classes de fonctions.

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REMARKS ON THE NUMERICAL SOLUTION OF A NONLINEAR PARABOLIC EQUATION

ERVIN SCHECHTER

In the present paper we continue the study begun in [5], [6] of a nonlinear degenerate parabolic equation. Our primary aim is to show that the numerical solution yielded by the explicit difference scheme converges to the exact one in $L^p(Q)$ for any $1 \leqq p < +\infty$. This improves the result of [6] where $1 \leqq p < 2$. We also show that the considerations of the above papers, can be extended to the case when the right-hand side of our equation is perturbed by a continuous term.

1. **Preliminaries.** The problem we are dealing with can be formulated as follows:

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) + a(x, t) \quad \text{on} \quad Q = \Omega \times]0,T[\tag{1.1}$$

$$u(x, 0) = u_0(x) \qquad x \in \Omega \tag{1.2}$$

$$u(x, t) = u_1(x, t) \quad \text{on} \quad S = \partial\Omega \times]0, T[. \tag{1.3}$$

Throughout this paper we shall suppose that the functions intervening in (1.1)–(1.3) are subject to the following assumption (A):

(A)

(i) $u_0 \in C(\Omega),\ u_1 \in C(S),\ a \in C(Q)\,;\ u_0,\ u_1,\ a \geqq 0$

(ii) $\varphi \in C(R_+),\ \varphi(u),\ \varphi'(u) > 0$ for $u > 0$;

$$\varphi(0) = \varphi'(0) = 0.$$

The domain $\Omega \subset R^2$, is supposed to be regular, bounded and convex.

DEFINITION. A function $u \in L^\infty(Q)$ is called a weak solution of (1.1)–(1.3) if:

(i) $\dfrac{\partial\varphi(u)}{\partial x_i} \in L^2(Q) \qquad i = 1,2,$

(ii) Conditions (1.2), (1.3) are fulfilled (in the generalized sense),

(iii) $u \geqq 0$ on Q,

(iv) For any $f \in H^1(Q)$ such that $f|_{S_1} = 0$

$$\int_Q \left(u\frac{\partial f}{\partial t} - \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}\right) dx\,dt + \int_\Omega f(x, 0)u_0(x)dx + \int_Q a(x, t)f(x, t)dx\,dt = 0. \tag{1.4}$$

Here $S_1 = S \cup \{(x, T) | x \in \Omega\}$.

As in [5], [6], the approximate solutions are yielded by the following explicit difference scheme:

$$U_{\bar{t}}(k) = \Delta_h\varphi(U(k-1)) + a(k) \qquad \text{on } Q_h \tag{1.5}$$

$$U(0) = u_{0h} \tag{1.6}$$

$$U|_{\Gamma_h} = u_2(x, k\tau), \; x \in \Gamma_h, \; k = 0,1, \ldots, K = \left[\frac{T}{\tau}\right] \tag{1.7}$$

where h is the step in both space directions and τ that of the time direction of the rectangular mesh. Further, Δ_h is the usual "five point" discretization of Δ, u_{0h} is the restriction to the mesh points, of u_0,

$$u_2 : \Gamma_h \times \{k\tau, k = 0, 1, \ldots, K\} \to \mathbf{R}, \; u_2(x, t) = u_1(x^*, t)$$

$x \in \Gamma_h$, where $x^* \in \partial\Omega$, is the nearest point to x (or one of them, but always the same on all levels); Ω_h, Q_h are formed by strictly interior points and Γ_h by the boundary mesh-points.

2. **A maximum principle.** To begin with we introduce the following notation: M is a constant such that $u_0, u_1, a \leqslant M$, $M_0 = (1 + T)M$,

$$\lambda = 4\frac{\tau}{h^2}\max_{[0, M_0]}\varphi'(r).$$

THEOREM 2.1. *Suppose that assumption* (*A*) *holds and* $\lambda \leq 1$. *Then, the solution* U *of the problem* (1.5)—(1.7) *satisfies*:

$$0 \leq U(x, t) \leq M_0, \text{ for } (x, t) \in \bar{Q}_h.$$

Proof. We prove only the right-hand inequality. To this end it suffices to show that $U_{ij}(k) \leqslant (1 + k\tau)M$ for any i, j, k (in Q_h). Suppose that this were not true. Then there would exist a triplet of indices m,n,k with $k \geqslant 1$, such that $U_{mn}(k) > (1 + k\tau)M$ and $U_{ij}(k-1) \leqslant (1 + (k-1)\tau)M$, for any i, j. Hence, if we set $M_1 = (1 + (k-1)\tau)M$,

$$0 > M(1 + k\tau) - U_{mn}(k) = M_1 - U_{mn}(k-1) + \tau\Delta_h(\varphi(M_1) - \varphi(U_{mn}(k-1))) + M\tau - a_{mn}(k-1).$$

The last difference is nonnegative. The sum of the other terms can be written as:

$$\left(1 - 4\frac{\tau}{h^2}\tilde{\varphi}'_{mn}(k-1)\right)(M_1 - U_{mn}(k-1)) + \frac{\tau}{h^2}[\tilde{\varphi}'_{m+1,n}(k-1)(M_1 - U_{m+1,n}(k-1)) + \tilde{\varphi}'_{m-1,n}(k-1)(M_1 - U_{m-1,n}(k-1)) + \tilde{\varphi}'_{m,n+1}(k-1)(M_1 - U_{m,n+1}(k-1)) + \tilde{\varphi}'_{m,n+1}(k-1)(M_1 - U_{m,n-1}(k-1))].$$

The sign $\sim$ indicates an intermediary value from the mean value theorem. This expression is nonnegative. The contradiction we are lead to shows that the inequality is correct.

We mention here that if $a = 0$ then $0 \leqslant U \leqslant M$.

3. **Estimates for the first order differences.** The proofs of the following lemma and theorem are similar to those of the case (see [5], [6]), $a = 0$.

LEMMA 3.1. *Suppose that conditions of Theorem 2.1 are valid. Assume also that the following conditions:*

(i) $u_0 \in C^2(\Omega)$, $\partial^2 u_0/\partial x_1^2,\ \partial^2 u_0/\partial x_2^2 \geqslant 0$ on Ω

(ii) u_1 *and* a *are nondecreasing in* t, *hold. Then,*

$$\tau h^2 \sum_{Q_h} |U_{\bar{t}}(k)| \leqslant C, \text{ for } h < h_0 \tag{3.1}$$

where C is a constant independent of h.

Remark. The lemma remains true if instead of the positivity of the second order derivatives of u_0 we suppose $\Delta\varphi(u_0) \geqslant 0$, or if $\varphi''(u_0) \geqslant 0$ for $f \geqslant 0$ and $\Delta u_0 \geqslant 0$, $\varphi'' \in C(R_+)$.

THEOREM 3.1. *Suppose that:*

(i) *Assumption (A) is valid,* $\varphi \in C^2(\mathrm{R}_+)$, $\varphi''(u) \geqslant 0$ *for* $u \geqslant 0$ *and* $u_0 \in C^2(\overline{\Omega})$

(ii) $\lambda \leqslant 1$

(iii) $\partial u_1/\partial t$ *exists and is bounded on S*

(iv) $\varphi(\infty) = \infty$.

Under these assumptions there are constants h_0, $C > 0$ *such that:*

$$\tau h^2 \sum_{Q_h} |U_{\bar{t}}(k)| < C \quad \text{for} \quad h < h_0. \tag{3.2}$$

COROLLARY 3.1. *Under the hypotheses of the above theorem*

$$\tau h^2 \sum_{Q_h} |\varphi(U(k))_{\bar{t}}| < C \quad \text{for} \quad h < h_0. \tag{3.3}$$

In what follows we shall call rectangular polygon a polygonal domain having the following properties: *a)* The sides are parallel to the axes *b)* The vertices coincide with mesh-points *c)* It is convex in both space directions. For simplicity we use the same notation for the rectangular domains as well as for the nodes they contain.

THEOREM 3.2. *Suppose that U is the solution of the problem (1.5)—(1.7) and that conditions of Theorem 3.1 (or Lemma 3.1) are fulfilled.*

Then, there exists a constant C independent of h such that for any regular, convex domain Ω^*,

$$\tau h^2 \sum_{k=0}^{k} \sum_{\Omega_h^*} (\varphi^2(U(k))_{x_1} + (\varphi^2(U(k))_{x_2}) < C, \tag{3.4}$$

if $h < h_0(\Omega^*)$.

Here Ω_h^* *is the maximal regular polygon contained in* Ω^*.

The proof relies on the same ideea as in the homogeneous case [5]. For simplicity, it was carried out there, when Ω^* was a rectangle R_0. The inequality (3.4) was obtained by constructing a sequence of parallel and concentric rectangles

$$\Omega_h^* = \Omega^* = \mathbf{R}_0 \subset \mathbf{R}_1 \subset \dots \subset \mathbf{R}_N = \Omega$$

and letting $N \to \infty$.

If now Ω_h^* is a rectangular polygonal domain, the sequence of „parallel" domains $\{\mathbf{R}_p\}$ with frontiers $\{S_p\}$, is constructed as follows:

(i) $\mathbf{R}_{p+1} \subset \mathbf{R}_p$.

(ii) Parallel sides are at distance h.

(iii) At the distance h along the side of length h, of a „concave" vertex of S_p, there is a concave vertex o S_{p+1}. Except these vertices S_p and S_{p+1} have no other common points.

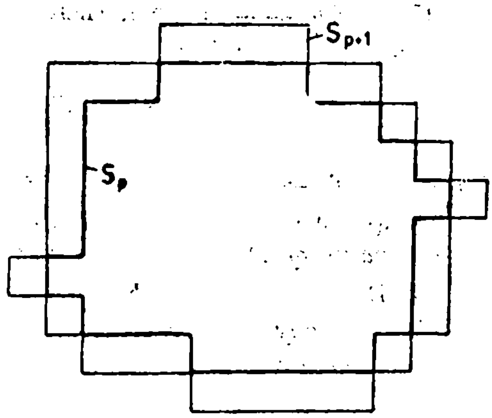

Fig. 1.

We note that, because of the convexity of Ω^*, a concave angle of Ω_h^* must have at least one of its sides of length h.

4. **Auxiliary functional analytic results.** In order to show that the approximating sequences converge to the exact solution of compactness results.

THEOREM 4.1. *Let D be a regular, bounded domain of $\mathbf{R}^n$ and $\{u_j\} \subset L^\infty(D)$ a sequence with the following properties:*

(i) $\operatorname{ess\,sup}_D |u_j(x)| < C, \qquad j = 1, 2, \dots$

C independent of j.

(ii) *There exists $r \in]1, +\infty]$ and $u \in L^r(D)$, such that:*

$$u_j \to u \quad \text{a.e. on } D.$$

Then,

(j) $u \to L^\infty(D)$.

(jj) *There exists a subsebuence $\{u_k\} \subset \{u_j\}$, such that:*

$$u_k \to u \quad \text{in } L^p(D), \text{ for any } p \in [1, +\infty].$$

Proof. First, from our hypotheses it follows that:

$$u_j \to u \quad \text{in} \quad L^r(D) \text{ weakly.}$$

On the other hand, using a diagonalization process, we we can infer from (1) and the reflexivity of L^p, $p \in]1, \infty[$, that $\{u_j\}$ possesses a subsequence $\{u_k\}$ with the property of being weakly convergent for any such p, to u. Since the norms of u are equibounded we have, $u \in L^\infty(D)$.

Using Egorov's theorem, there is a small set $D_0 \subset D$ such that $u_k \to u$ uniformly on $D \setminus D_0$. Hence,

$$\int_D |u_k - u|^p dx = \int_{D\setminus D_0} |u_k - u|^p dx + \int_{D_0} |u_k - u|^p dx,$$

shows that $u_k \to u$ in $L^p(D)$ for $p \in [1, \infty[$.

LEMMA 4.1. *Let $p \in [1, \infty[$, and $K \subset L^p(D)$. Suppose $\{D_j\}$ be a sequence of subdomains of D with the following properties:*

(i) *$D_k \subset D_{k+1}$ for any k.*

(ii) *For any j the restriction of K to D_j is precompact in $L^p(D_j)$.*

(iii) *For any $\varepsilon > 0$, there exists a subscript k such that:*

$$\int_{D\setminus D_k} |u(x)|^p dx < \varepsilon \quad \text{for} \quad u \in K.$$

Then K is precompact in $L^p(D)$.

This lemma was used in [6] (Lemma 2.5).

5. The compactness of the approximating sequence. The notation for the extensions of U_h have the same meaning as in [4], [5], [6]. We also extend U_h on the mesh-points of $\hat{Q}_h = \hat{\Omega}_h \times [0, T[$, where $\hat{\Omega}_h$ is the smallest, closed, rectengular polygon containing Ω, by:

$$U(x, t) = u_1(\hat{x}, t) \quad (x, t) \in \hat{S}_h. \tag{5.1}$$

Here $\hat{S}_h = \partial\Omega_h \times [0, T[$ and $\hat{x} \in \partial\Omega$, is the nearest point to x. Thus, U_h' will have, being continuous functions, well defined traces on S.

LEMMA 5.1. *Suppose that $V_h : \hat{Q}_h \to \mathbf{R}$ satisfy:*

(i) $\tau h^2 \sum_{Q_h} |V_h| < C,$

(ii) $\tau h^2 \sum_{k=1}^{k} \sum_{\bar{\Omega}_h^*} (|V_{x_1}| + |V_{x_2}| + |V_{\bar{t}}|) < C,$

for $h < h_0(\Omega^*)$ and any Ω^* regular and convex such that $\bar{\Omega}^* \subset \Omega$, C independent of h.

Under these hypotheses the set of continuous functions $\{V_h'\}$ is precompact in $L^p(Q)$ for any $1 \leqslant p < 2$.

Moreover, if $\{V_{\bar{h}}\} \subset \{V_h\}$ and one of the sequences $\{V_{\bar{h}}'\}$ or $\{\tilde{V}\}$, is convergent in $L(Q)$ to a function v, the same is true for the other.

Proof. Let Ω' be a regular domain such that $\bar{\Omega}' \subset \Omega$ and h sufficiently small so that for an apropriate Ω^*, $\bar{\Omega}' \subset \Omega^*$. Since (see [4]),

$$\frac{\partial V_h'}{\partial x_i} = (V_{x_i})_{(i)}, \quad i = 1, 2; \qquad \frac{\partial V_h'}{\partial t} = (V_t)_{(0)}$$

on any $q_{(kh)} = \omega_{(kh)} \times [k_0\tau, (k_0+1)\tau] \subset Q_h$,

$$\int_{Q'} \left(\left| \frac{\partial V_h'(x,t)}{\partial x_1} \right| + \left| \frac{\partial V_h'(x,t)}{\partial x_2} \right| + \left| \frac{\partial V_h'(x,t)}{\partial t} \right| \right) dx\, dt \leqslant$$

$$\leqslant \int_{Q_h^*} (-\,,,\,-)\, dx\ dt \leqslant \sum_{(kh)} \int_{q(kh)} \left(\sum_{i=0}^{2} (V_{x_i})_{(i)} \right) dx\ dt < C$$

$(x_0 = t)$. (Here for example $(V_{x_1})_{(1)}$, is on each parallelepiped q, linear in x_0, x_2 and constant in x_1).

Thus, V_h' is bounded in $W_1^1(Q')$, $Q' = \Omega' \times]0,T[$, for any Ω' and consequently precompact in any $L^q(Q')$, $1 \leqslant q < 2$. Now, the first assertion follows from Lemma 4.1. As for the second, its proof is parallel to that of Lemma 3.2, Ch. VI, from [4].

LEMMA 5.2. *Suppose that in Lemma 5.1, instead of condition (i), we have:*

$$\max_{\widehat{Q}_h} |V_h| < C. \tag{5.2}$$

Then there exists a function $v \in L^\infty(Q)$ and a subsequence of V_h; $V_{\bar h}$, such that:

$$V_{\bar h}' \to v \quad in\ L^p(Q),$$

for any $q \in [1, +\infty[$.

Proof. By Lemma 5.1 $\{V_h\}$ is precompact in $L^p(Q)$, $p \in [1,2[$. So we have a subsequence $V_{\bar h} \to v \in L^p(Q)$ a.e., on Q. Our lemma follows from Theorem 4.1.

LEMMA 5.3. *If $V_h : \widehat{Q} \to \mathbf{R}$ and C is a constant such that:*

(i) $$\tau h^2 \sum_K V_h^2 < C,$$

(ii) $$\tau h^2 \sum_{k=1}^{K} \sum_{\Omega_h^*} (V_{x_1}^2 + V_{x_2}^2) < C,$$

for $h < h_0(\Omega^*)$ and any regular Ω^*, $\overline{\Omega}^* \subset \Omega$.

Then, there exists $v \in L^2(Q)$, with $\partial v/\partial x_i \in L^2(Q)$, $i = 1,2$, such that,

$$V_{\bar h}' \to v \ \ and\ \frac{\partial V_h'}{\partial x_i} \to \frac{\partial V}{\partial x_i},\ \ i = 1,2.\ \ weakly\ \ in\ \ L^2(Q).$$

The proof is similar to that of Lemma 5.1 (see also [4]).

THEOREM 5.1. *Let $U_h : \overline{Q}_h \to \mathbf{R}$ be the solution of the difference problem* (4.1)—(4.3) *and conditions of Theorem 3.2 hold.*

Then, there exists a function $u \in L^\infty(Q)$ *and a subsequence* $\{U_{\bar h}\} \subset \{U_h\}$ *such that:*

(i) $$u \geqslant 0$$

(ii) $$\frac{\partial\varphi(u)}{\partial x_i} \in L^2(Q) \qquad i = 1,2.$$

(iii) $$(\varphi(U_{\bar h}))' \to \varphi(u) \text{ in } L^p(Q),\ p \in [1, \infty[$$

(iv) $$(\varphi(U_{\bar h})_{x_i})' \to \frac{\partial\varphi(u)}{\partial x_i},\ i = 1, 2, \text{ weakly in } L^2(Q),$$

(v) $$\tilde U_h \text{ as well as } U'_h \to u \text{ in } L^p(Q),\ p \in [1, +\infty[.$$

Proof. The discrete function $\varphi(U_h)$ satisfies conditions of Lemma 5.2. Consequently there exists $\chi \in L^\infty(Q)$ such that for a subsequence $\varphi(U_{\bar h})$,

$$(\varphi(U_h))' \to \chi \quad \text{in} \quad L^p(Q),\ p \in [1, +\infty[. \tag{5.3}$$

At the same time, by virtue of Lemma 5.3:

$$(\varphi(U_{\bar h})_{x_i})' \to \frac{\partial z}{\partial x},\ i = 1, 2., \text{ weakly in } L^2(Q) \text{ and by Lemma 5.1:}$$

$$\widetilde{\varphi(U_{\bar h})} = \varphi(\tilde U_h) \to \chi \quad \text{in} \quad L(Q).$$

Consequently,

$$\varphi(\tilde U_{\bar h}) \to \chi \quad \text{a.e. on } Q;\ \chi \geqq 0$$

and also:

$$\tilde U_{\bar h} \to \varphi^{-1}(\chi) \quad \text{a.e. on } Q.$$

Denote $u = \varphi^{-1}(\chi)$ and observe that $u \in L^\infty(Q)$. Now, the sequence $\tilde U_{\bar h}$ satisfies conditions of Theorem 4.1, so that:

$$U_h \to u \quad \text{in} \quad L^p(Q) \text{ for } p \in [1, +\infty[.$$

On the other hand because of Lemma 5.1, $U'_h \to u$, in $L(Q)$, which entails (passing to a subsequence, if necessary): $U'_{\bar h} \to u$, a.e. on Q. Again, by Theorem 4.1:

$$U'_h \to u \text{ in } L^p(Q),\ p \in [1, +\infty[,$$

which completes the proof.

THEOREM 5.2. *Suppose that conditions of Theorem 5.1 are valid. Then, the function u is the unique solution of the problem (1.1)—(1.3).*

The proof is similar to that of [5], [6].

Remarks. We have shown that a subsequence of extended numerical solutions converges to the exact one in any L^p, $1 \leq p < +\infty$. The uniqueness, however, assures that the whole sequence of approximations possesses the property.

Using the numerical extension (5.1) it can be easily shown that $U_\lambda^i|_S$ tends uniformly to u_1.

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OBSERVAŢII ASUPRA REZOLVĂRII NUMERICE A UNEI ECUAŢII PARABOLICE NELINIARE

(Rezumat)

Prelungirile soluţiei problemei numerice (1.5)–(1.7) tind, în norma lui $L^p(Q)$, pentru orice $p \in [1, +\infty[$, către soluţia slabă a problemei (1.1)–(1.3).

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ON SOME PRACTICAL QUADRATURE AND CUBATURE FORMULAS

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0. In [2] it was constructed the following cubature formula

$$\iint_D f(x, y)dxdy = -\frac{4hk}{mn}\sum_{i,j=1}^{m,n} f(G_{ij}) + \frac{2h}{m}\sum_{i=1}^{m}\int_{y_0-k}^{y_0+k} f(x_i, y)dy + \frac{2k}{n}\sum_{j=1}^{n}\int_{x_0-h}^{x_0+h} f(x, y_j)dx + R(f) \tag{1}$$

where $D = [x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]$, $x_i = x_0 - h + \frac{2i-1}{m}h$, $y_j = y_0 - k + \frac{2j-1}{n}k$, $G_{ij} = (x_i, y_j)$, $i = 1, \ldots, m;\ j = 1, \ldots, n$ and

$$|R(f)| \leqslant R_{mn} = \frac{h^3k^3}{9m^2n^2}P_{22}, \text{ with } P_{22} = \sup_D |f^{(2,2)}(x, y)| \tag{2}$$

for $f \in C^{2,2}(D)$.

From (1) there can be derived other cubature formulas, applying to the integrals of the right hand member any quadrature formulas.

The purpose of this paper is to obtain from (1) some cubature formulas using suitable optimal quadrature rules.

1. **Optimal quadrature formulas.** Let $W^{(2)}(P_2; a, b)$ be the class of the functions defined on the interval $[a, b]$, for which $\sup_{a\leqslant x\leqslant b}|f''(x)| \leqslant P_2$. Let also $\mathcal{F}$ be the set of the quadrature formulas of the form

$$\int_a^b f(x)dx = \sum_{i=1}^{p} A_i f(x_i) + Bf'(a) + Cf'(b) + R_p(f) \tag{3}$$

where $f \in W^{(2)}(P_2; a, b)$, A_i, $i = 1, \ldots, p$; B, C are arbitrary parameters and $x_i = a + \frac{2i-1}{2}\frac{b-a}{p}$, $i = 1, \ldots, p$ are the nodes, and which has the degree of exactness equal to one.

Let us denote by $\mathfrak{M}_2$ the set of monosplines of the second degree with the nodes x_i, $i = 1, \ldots, p$ which satisfy the conditions

$$M_0(a) = B,\ M_i(x_i) = M_{i+1}(x_i),\ M_p(b) = -C$$

$$M_0'(a) = 0,\ M_i'(x_i) = M_{i+1}'(x_i),\ M_p'(b) = 0,\ i = 0, 1, \ldots, p-1$$

i.e. the monosplines of the form

$$M(x) = \frac{(x-a)^2}{2} + B - \sum_{k=1}^{p} A_k(x - x_k)_+ \qquad (5)$$

where M_i is the restriction of M to the interval (x_i, x_{i+1}).

It follows [1] that the remainder term of (3) has the representation

$$R_p(f) = \int_a^b M(x)f''(x)dx. \qquad (6)$$

Furthermore, between the sets $\mathcal{F}$ and $\mathfrak{M}_2$ there exists a one-to-one correspondence.

Now, from (6) it follows that

$$R_p = \sup_{f \in W^{(2)}(P_2;a,b)} |R_p(f)| = P_2 \int_a^b |M(x)|dx. \qquad (7)$$

The quadrature formula $F \in \mathcal{F}$ for which R_p takes the minimum value is named optimal in the function class $W^{(2)}(P_2; a, b)$.

In the next, there will be constructed optimal quadrature formulas in $W^{(2)}(P_2; a, b)$ for the following two cases: i) B, C — arbitrary and ii) $B = C = 0$.

i) In this case, the optimal quadrature formula corresponds to the monospline M for which

$$J = \int_a^b |M(x)|dx \to \text{minim}$$

in the exactness quadrature conditions.

The integral J which can be written in the form

$$J = \int_a^{x_1} |M_0(x)|dx + \sum_{k=1}^{p-1} \int_{x_k}^{x_{k+1}} |M_k(x)|dx + \int_{x_p}^{b} |M_p(x)|dx \qquad (8)$$

takes the minimum value iff each integral of the right part of (8) takes the minimum value.

By [3] it is known that

$$I_k = \int_{x_k}^{x_{k+1}} |M(x)|dx, \; k = 1, \ldots, p-1$$

is minim iff M_k is on (x_k, x_{k+1}) the Cebisev polynomial $T_{2,k}$;

$$T_{2,k}(x) = \frac{x^2}{2} - ux - \left(\frac{v^2}{8} - \frac{u^2}{2}\right), \; u = \frac{x_k + x_{k+1}}{2}, \; v = \frac{x_{k+1} - x_k}{2}.$$

Using this condition and the fact that the quadrature formula exactness degree is one, we obtain

$$\overline{A}_k = \frac{b-a}{p}, \; k = 1, \ldots, p; \; \overline{B} = -\frac{1}{32}\left(\frac{b-a}{p}\right)^2, \; \overline{C} = -\overline{B}.$$

Also, by (8), we have

$$\overline{J} = \min_{A_k, B, C} J = \frac{p}{32}\left(\frac{b-a}{p}\right)^3.$$

It follows that

$$\overline{R}_p = \frac{p}{32}\left(\frac{b-a}{p}\right)^3 P_2.$$

So, it was proved

THEOREM 1. *The only optimal quadrature formula $\overline{F} \in \mathfrak{F}$ in the function class $W^{(2)}(P_2; a, b)$ is*

$$\int_a^b f(x)dx = \frac{b-a}{p}\sum_{i=1}^{p} f(x_i) + \frac{(b-a)^2}{32p^2}[f'(b) - f'(a)] + \overline{R}_p(f) \tag{9}$$

and

$$\overline{R}_p = \sup_{f \in W^{(2)}(P_2; a, b)} |\overline{R}_p(f)| = \frac{(b-a)^3}{32p^2} P_2. \tag{10}$$

An analoguos theorem can be proved in the same way for the ii) case, namenly:

THEOREM 2. *Let $\mathfrak{F}_0$ be the set of the quadrature formulas of the form* (3) *with $B = C = 0$. Then there exists a unique $\overline{F}_0 \in \mathfrak{F}_0$ which is optimal in the function class $W^{(2)}(P_2; a, b)$, i.e.*

$$\int_a^b f(x)dx = \frac{b-a}{p}\left\{\sum_{i=1}^{p} f(x_i) + \frac{1}{32}[f(x_1) - f(x_2) - f(x_{p-1}) + f(x_p)]\right\} + R(f) \tag{11}$$

and

$$\overline{R}_p^0 = \sup_{\in W^{(2)}(P_2; a, b)} |\overline{R}_p^0(f)| = \left(p - 1 + \frac{11\sqrt{33}}{128}\right)\frac{(b-a)^3}{32p^2} P_2 \tag{12}$$

2. Cubature formulas. Let $W^{(2,2)}(D)$ be the class of the functions which are twice differentiable by each variable on D and

$$\sup_D |f^{(r,s)}(x, y)| \leqslant P_{rs}, \; (r, s) \in \{(2, 0), (0, 2), (2, 2)\}.$$

For $f \in W^{(2,2)}(D)$, applying to each integral of the second member of (1) the quadrature (9)—(10), i.e.

$$\int_{y_0-k}^{y_0+k} f(x_i,y)dy = \frac{2k}{n}\sum_{j=1}^{n} f(x_i, y_j) + \frac{k^2}{8n^2}[f^{(0,1)}(x_i, y_0 + k) - -f^{(0,1)}(x_i, y_0 - k)] + R_{ni}(f), \quad i = 1, \ldots, m \tag{13}$$

with

$$R_{ni} = \sup_{f \in W^{(2,2)}(D)} | R_{ni}(f) | = \frac{k^3}{4n^2} P_{02} \tag{14}$$

and

$$\int_{x_0-h}^{x_0+h} f(x, y_j)dx = \frac{2h}{m}\sum_{i=1}^{m} f(x_i, y_j) + \frac{h^2}{8m^2}[f^{(1,0)}(x_0 + h, y_j) - -f^{(1,0)}(x_0 - h, y_j)] + R_{mj}(f), \quad j = 1, \ldots, n \tag{15}$$

with

$$R_{mj} = \sup_{f \in W^{(2,2)}(D)} | R_{mj}(f) | = \frac{h^3}{4m^2} P_{20}, \tag{16}$$

one obtains

$$\iint_D f(x, y)dxdy = \frac{4hk}{mn}\sum_{i,j=1}^{m,n} f(G_{ij}) + \frac{h^2k}{4m^2n}\sum_{j=1}^{n}[f^{(1,0)}(x_0 + h, y_j) - f^{(1,0)}(x_0 - - h, y_j)] + \frac{hk^2}{4mn^2}\sum_{i=1}^{m}[f^{(0,1)}(x_i, y_0 + k) - f^{(0,1)}(x_i, y_0 - k)] + \overline{R}(f), \tag{17}$$

where

$$| \overline{R}(f) | \leqslant R_{mn} + R_m + R_n.$$

Taking into account that

$$R_m = \frac{2k}{n}\sum_{j=1}^{n} R_{mj} = \frac{h^3k}{2m^2} P_{20}, \quad R_n = \frac{2h}{m}\sum_{i=1}^{m} R_{ni} = \frac{hk^3}{2n^2} P_{02}$$

it follows that

$$| \overline{R}(f) | \leqslant \frac{h^3k^3}{9m^2n^2} P_{22} + \frac{h^3k}{2m^2} P_{20} + \frac{hk^3}{2n^2} P_{02}. \tag{18}$$

In this way, it was obtained the practical cubature formula (17)—(18), i.e. a cubature with all the coefficients and the coordinates of nodes rational numbers, of course if x_0, y_0, h, k are rationals. Futhermore, many of coefficients are equal.

Also, the mentioned cubature formula is in a way optimal, i.e. the quadrature formulas used to derive it from (1) are optimal.

In an analogous way, using the optimal quadrature formula (11)—(12) one derived from (1) the cubature formula

$$\iint_D f(x,y)dxdy = \frac{4hk}{mn}\sum_{i,j=1}^{m,n} f(G_{ij}) + \frac{hk}{8mn}\Big\{\sum_{i=1}^{m}[f(G_{i1}) - f(G_{i2}) - f(G_{i,n-1}) + f(G_{in})] + \sum_{j=1}^{n}[f(G_{1j}) - f(G_{2j}) - f(G_{m-1,j}) + f(G_{mj})]\Big\} + \tilde{R}(f) \tag{19}$$

where

$$|\tilde{R}(f)| \leqslant R_{mn} + R_m + R_n$$

with

$$R_m = \frac{h^3k}{2m^3}\left(m - 1 + \frac{11\sqrt{33}}{128}\right) P_{20},\quad R_n = \frac{hk^3}{2n^3}\left(n - 1 + \frac{11\sqrt{33}}{128}\right) P_{02}$$

It follows that

$$|\tilde{R}(f)| \leqslant \frac{h^3k^3}{9m^2n^2} P_{22} + \frac{h^3k}{2m^2} P_{20} + \frac{hk^3}{2n^2} P_{02}. \tag{20}$$

The advantage of this cubature formula is that it does not use the derivative values of function f.

3. **Fortran procedures.** At the end, there are given two Fortran procedures for the approximation of a double integral using the cubature formulas (17)—(18) respectively (19)—(20), with an absolute error ε. First, it is determined the values of m and n such that the remainder upper border satisfies the condition

$$\frac{h^3k}{2m^2} P_{20} + \frac{hk^3}{2n^2} P_{02} + \frac{h^3k^3}{9m^2n^2} P_{22} \leqslant \varepsilon.$$

The program comments will give us the necessary information to use them correctly.

Procedure Cubature 1

```
      SUBRØUTINE CUBATURE1(AH,AK,XO,YO,EPS,P20,PO2,P22,MS,M,N,API,K)
C
C INPUT VARIABLES: XO,YO,AH,AK,EPS,P20,PO2,P22,MS
C (XO,YO) - THE CENTER ØF RECTANGLE D (THE INTEGRATIØN DØMAIN)
C 2*AH,2*AK - THE LENGTH ØF RECTANGLE EDGES
C EPS- THE ADMISSIBLE ERRØR
C P20,PO2,P22- THE CØRRESPØNGING DERIVATIVES BØRDERS ØN D
C MS- AN UPPER BØUND FØR M
```

```
C
C ØUTPUT VARIABLES: M,N,API,K
C M- THE ABCISSES NUMBER
C N- THE ØRDINATES NUMBER
C API- THE APPRØXIMATIØN ØF THE INTEGRAL
C K- GET THE VALUES -1 IF THE ADMISSIBLE ERRØR IS NØT REACHED AND
C     O ØTHERWISE
C
C PRØGRAM VARIABLES: NL,NU,REST,SM,SN,SMN
C NL- A LØWER BØUND FØR N
C NU- AN UPPER BØUND FØR N
C REST- THE REMAINDER VALUE
C SM,SN- THE CUBATURE SIMPLE SUMS
C SUMN- THE CUBATURE DØUBLE SUM
C
C ØNE DETERMINES THE CØRRESPØNDING VALUES ØF M,N
C
      AH2=AH*AH
      AK2=AK*AK
      DØ 10 M=1,MS
      NL=0.8*AK*M/AH+1
      NU=1.2*AK*M/AH
      DØ 10 N=NL,NU
      AN=N
      M2=M*M
      N2=N*N
      REST=AH*AK*(AH2*P20/2/M2+AK2*PO2/2/N2+AH2*AK2*P22/9/M2/N2)
      IF(REST.LE.EPS) GØ TØ 20
   10 CØNTINUE
      K=-1
      RETURN
C
C ØNE CALCULATES THE APPROXIMATIØN API
C
   20 K=O
      SMN=O
      SM=O
      SN=O
      AI=XO-AH
      BI=YO-AK
      AS=XO+AH
      BS=YO+AK
      DØ 30 I=1,M
      X1=AI+(2*I-1)*AH/M
      SM=SM+DFY(X1,BS)-DFY(X1,BI)
      DØ 30 J=1,N
      Y1=BI+(2*J-1)*AK/N
      SMN=SMN+F(X1,Y1)
      IF(I.EQ.1) SN=SN+DFX(AS,Y1)-DFX(AI,Y1)
   30 CØNTINUE
      API=4*AH*AK*SMN/(M*N)+AH2*AK*SN/(4*M2*N)+AH*AK2*SM/(4*M*N2)
      RETURN
      END
```

Procedure Cubature 2

```
      SUBRØUTINE CUBATURE2(AH,AK,XO,YO,EPS,P20,PO2,P22,MS,M,N,API,K)
      DIMENSIØN X(1000),Y(1000)
```

```
C
C INPUT VARIABLES: XO,YO,AH,AK,EPS,P20,PO2,P22,MS
C (XO,YO)- THE CENTER ØF RECTANGLE D (THE INTEGRATIØN DØMAIN)
C 2*AH,2*AK- THE LENGTH ØF RECTANGLE EDGES
C EPS- THE ADMISSIBLE ERRØR
C P20,PO2,P22- THE CORRESPONDING DERIVATIVES BØRDERS ØN D
C MS- AN UPPER BØUND FØR M
C
C ØUTPUT VARIABLES: M,N,API,K
C M- THE ABCISSES NUMBER
C N- THE ØRDINATES NUMBER
C API- THE APPROXIMATIØN ØF THE INTEGRAL
C K- GET THE VALUE -1 IF THE ADMISSIBLE ERRØR IS NØT REACHED AND
C      O ØTHERWISE
C
C PROGRAM VARIABLES: NL,NU,REST,SM,SN,SMN
C NL- A LØEWR BØUND FØR N
C NU- AN UPPER BØUND FØR N
C REST- THE REMAINDER VALUE
C SM,SN- THE CUBATURE SIMPLE SUMS
C SMN- THE CUBATURE DØUBLE SUM
C
C ØNE DETERMINES THE CORPESPONDING VALUES ØF M,N
C
      AH2=AH*AH
      AK2=AK*AK
      DØ 10 M=1,MS
      NL=0·8*AK*M/AH+1
      NU=1.2*AK*M/AH
      DØ 10 N=NL,NU
      AN=N
      M2=M*M
      N2=N*N
      REST=AH*AK*(AH2*P20/2/M2+AK2*PO2/2/N2+AH2*AK2*P22/9/M2/N2)
      IF(REST.LE.EPS) GØ TØ 20
   10 CØNTINUE
      K=-1
      RETURN
C
C ØNE CALCULATES THE APPROXIMATIØN API
C
   20 K=0
      SMN=0
      SM=0
      SN=0
      DØ 30 I=1,M
      DØ 30 J=1,N
      X(I)=XO-AH+(2*I-1)*AH/M
      Y(J)=YO-AK+(2*J-1)*AK/N
   30 SMN=SMN+F(X(I),Y(J))
      DØ 40 I=1,M
   40 SM=SM+F(X(I),Y(1))-F(X(I),Y(2))-F(X(I),Y(N-1))+F(X(I),Y(N))
      DØ 50 J=1,N
   50 SN=SN+F(X(1),Y(J))-F(X(2),Y(J))-F(X(M-1),Y(J))+F(X(M),Y(J))
      API=4*AH*AK*SMN/M/N+AH*AK*(SM+SN)/8/M/N
      RETURN
      END
```

Example. Let f, D, ε and MS be given respectively by: $f(x, y) = 1/(1 + x + y)$, $D = [0, 1] \times [0, 1]$, $\varepsilon = 0.001$ and $MS = 1000$. Then $AH=0.5$, $AK=0.5$, $XO=0.5$, $YO=0.5$, $P20=P02=2$, $P22=24$. Thus the CALL instruction for the first procedure is:

CALL CUBATURE1(0.5,0.5,0.5,0.5,0.001,2.,2.,24.,1000,M,N,API,K) and the corresponding output data are: $M = 11$, $N = 12$, $API = 0.523195$.

Using the procedure CUBATURE2 one abtains: $M=11$, $N=12$, $API= =0.523161$.

If one takes $\varepsilon = 0.000001$ then the output data for the both procedures are: $M = 326$, $N = 390$, $API = 0.523276$.

We can remark that the approximations obtained by these two procedures are very closed and they are very closed with the exact value of the integral which is $I = 3 \ln 3 - 4 \ln 2 \approx 0.523248$.

A difference exists in the running time. Thus for $\varepsilon = 0.000001$, the running time of the first procedure is $t_1 = 1\,M\,34S$ and this time for the second procedure is $t_2 = 2\,M\,17S$.

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ASUPRA UNOR FORMULE PRACTICE DE CUADRATURĂ ŞI CUBATURĂ

(Rezumat)

În lucrare sînt construite două formule practice optimale de cuadratură, care sînt ulterior folosite la stabilirea de formule practice de cubatură.

În încheiere sînt date două proceduri Fortran pentru realizarea procedeelor de cubatură date.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXV, 3, 1980

SUR LA RÉCURRENCE DE LA MÉTRIQUE DE SCHWARTZSCHILD

P. ENGHIŞ, P. SANDOVICI, M. ŢARINĂ

Dans la théorie de la relativité généralisée la métrique de Schwartzschild est connue

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{1}$$

comme la métrique d'un espace décrit par le champ de gravité avec symmétrie sphérique généré par une masse m en repos.

Nous nous proposons de montrer que cette métrique est récurrente dans un certain sens avec le vecteur de récurrence $\left(-\frac{3}{x^1},\ 0,0,0\right)$.

Un espace riemannian V_n de métrique

$$ds^2 = g_{ij} dx^i dx^j \tag{2}$$

est récurrent, [2] s'il existe un vecteur covariant φ_r tel que

$$R^h_{ijk,r} = \varphi_r R^h_{ijk} \tag{3}$$

où par R^h_{ijk} sont notées les composantes du tenseur de courbure la virgule désignant la dérivée covariante par rapport à la métrique (2).

Si dans (1) nous posons $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$ et nous notons $1 - \frac{2m}{r} = e^{-a}$, la métrique (1) devient

$$ds^2 = -e^a (dx^1)^2 - (x^1)^2[(dx^2)^2 + \sin^2 x^2 (dx^3)^2] + e^{-a}(dx^4)^2 \tag{4}$$

Les symboles de Christoffel de deuxième espèce différents de zéro sont

$$\begin{aligned}
&\left|\begin{matrix}1\\11\end{matrix}\right| = \frac{1}{2}a', \quad \left|\begin{matrix}1\\22\end{matrix}\right| = -x^1 e^{-a}, \quad \left|\begin{matrix}1\\33\end{matrix}\right| = -x^1 e^{-a}\sin^2 x^2\\
&\left|\begin{matrix}1\\44\end{matrix}\right| = -\frac{1}{2}e^{-2a}\cdot a', \quad \left|\begin{matrix}2\\12\end{matrix}\right| = \frac{1}{x^1}, \quad \left|\begin{matrix}2\\33\end{matrix}\right| = -\sin x^2 \cos x^2\\
&\left|\begin{matrix}3\\13\end{matrix}\right| = \frac{1}{x^1}, \quad \left|\begin{matrix}3\\23\end{matrix}\right| = \operatorname{ctg} x^2, \quad \left|\begin{matrix}4\\14\end{matrix}\right| = -\frac{1}{2}a'
\end{aligned} \tag{5}$$

Les composantes différentes de zéro du tenseur de Riemann-Christoffel de deuxième espèce sont

$$\begin{array}{lll} R^1_{212} = \dfrac{-m}{x^1}, & R^1_{313} = \dfrac{-m}{x^1}\sin^2 x^2, & R^1_{414} = -\dfrac{2m(x^1-2m)}{(x^1)^4} \\ R^2_{121} = \dfrac{-m}{(x^1)^2(x^1-2m)}, & R^2_{323} = \dfrac{2m}{x^1}\sin^2 x^2, & R^2_{424} = \dfrac{m(x^1-2m)}{(x^1)^4} \\ R^3_{131} = \dfrac{-m}{(x^1)^2(x^1-2m)}, & R^3_{232} = \dfrac{2m}{x^1}, & R^3_{434} = \dfrac{m(x^1-2m)}{(x^1)^4}. \\ R^4_{141} = \dfrac{2m}{(x^1)^2(x^1-2m)}, & R^4_{242} = \dfrac{-m}{x^1}, & R^4_{343} = \dfrac{-m}{x^1}\sin^2 x^2 \end{array} \qquad (6)$$

De (6) il résulte $R_{ij} = 0$, où par R_{ij} sont notées les composantes du tenseur de Ricci.

Donc :

PROPOSITION 1. *L'espace V_4 avec la métrique* (1) *est un espace Ricci-spécial.*

En calculant pour les composantes non noulles du tenseur de courbure la dérivé covariante par rapport à la métrique (4) on obtient

$$R^h_{ijk,1} = -\frac{3}{x^1} R^h_{ijk}, \quad R^h_{ijk,r} = 0, \quad r = 2,3,4 \qquad (7)$$

donc l'espace est dans ce sens récurrent à vecteur de récurrence :

$$\left(-\frac{3}{x^1}, 0,0,0\right) \qquad (8)$$

Donc on a :

PROPOSITION 2. *L'espace V_4 avec la métrique* (1) *est un espace récurrent dans un certain sens à vecteur de récurrence φ_r donné par* (8).

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ASUPRA RECURENȚEI METRICII LUI SCHWARTZSCHILD

(Rezumat)

În lucrare se arată că metrica (1) este recurentă într-un anumit sens cu vector de recurență (8).

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXV, 3, 1980

QUELQUES REMARQUES SUR LA MÉTRISABILITÉ DES ESPACES A_n Γ-RÉCURRENTS ET T-RÉCURRENTS

P. ENGHIŞ

Soit A_n un espace à connexion affine à torsion. Nous notons, dans un système de coordonnées, avec Γ^i_{jk} les composantes de la connexion affine, avec Γ^h_{ijk} les composantes du tenseur de courbure et avec T^i_{jk} les composantes du tenseur de torsion.

L'espace A_n est nommé de courbure récurrente ou Γ-récurrent, s'il existe un vecteur covariant φ_r tel qu'on ait :

$$\Gamma^h_{ijk,r} = \varphi_r \Gamma^h_{ijk} \tag{1}$$

et de torsion récurrente ou T-récurrent s'il existe un vecteur covariant ψ_r tel qu'on ait :

$$T^i_{jk,r} = \psi_r T^i_{jk} \tag{2}$$

où la virgule désigne la dérivée covariante par rapport à la connexion Γ.

Si on contracte (1) en h et i ou en h et j on constate [6] que les tenseurs contractés du tenseur de courbure dans un espace Γ — récurrent, sont récurrents avec le même vecteur de récurrence. Le tenseur Γ_{ik} est l'analogue du tenseur de Ricci d'un espace riemannien. Si nous notons :

$$E_{ik} = \frac{1}{2}(\Gamma_{ik} + \Gamma_{ki}) \tag{3}$$

le tenseur de Ricci symétrisé, on constate [6] qu'il est aussi récurrent avec le même vecteur de récurrence.

Si on contracte (2) en i et j on obtient [3] que le vecteur de torsion, dans un espace A_n T-récurrent est aussi récurrent avec le vecteur ψ_r. Si on considère dans un espace A_n T-récurrent le tenseur quadratique de torsion

$$T_{kp} = T^i_{jk} T^j_{ip} \tag{5}$$

on constate [3] qu'il est aussi récurrent à vecteur de récurrence $2\psi_r$.

Pour les espaces A_n on définit la longueur d'un vecteur $V(v^i)$ [7] qui a l'origine dans un point régulier (x^k) par une fonction $\lambda(x^i, v^i)$ qui satisfait les conditions :

a) λ est une fonction homogène par rapport à v^i de degré un.

b) λ se conserve par le transport parallèle des vecteurs V d'un point (x^k) dans un point infiniment voisin $(x^k + dx^k)$

c) λ est invariante aux transformations des coordonnées.

Les espaces A_n qui possèdent une telle fonction on les nomme métrisables.

Pour les espaces A_n Γ — récurrents V. Murgescu [7] a montré que si le vecteur de récurrence est un gradient, il est possible de définir une telle fonction par

$$\lambda = \sqrt{e^{-\varphi} E_{ij} v^i v^j} \tag{6}$$

φ étant la fonction dont le gradient est le vecteur de récurrence φ_r. Ces espaces sont nommés E-métrisables.

On sait [8] qu'un espace A_n Γ — récurrent est aussi Riccirécurrent mais la réciproque n'est pas toujours vraie. Mais observons que la propriété de Murgescu peut être généralisée aux espaces Ricci- récurrents à vecteur de récurrence gradient parce que les conditions d'intégrabilité qui résultent de *a), b), c)* sont vérifiées.
Donc on a :

PROPOSITION 1. *Une condition nécessaire et suffisante pour qu'un espace A_n Ricci-récurrent soit E-métrisable est que le vecteur de récurrence soit le gradient d'une fonction scalaire φ.*

Pour les espaces A_n T-récurrents nous avons montré [4] que si le vecteur de T-récurrence ψ_r est le gradient d'une fonction ψ, alors ils sont aussi métrisables avec

$$\lambda = \sqrt{e^{-2\psi} T_{kp} v^k v^p} \tag{7}$$

cette condition étant la condition nécessaire et suffisante pour la métrisabilité. Ces espaces ont été nommés T-métrisables.

On sait [9] que si M est une variété différentiable connexe de dimension n douée avec une connexion linéaire Γ sur laquelle on considère $B(M)$ le fibré principal des repères linéaires, t un champ tensoriel récurrent par rapport à la connexion Γ, Φ_{z_0} le groupe d'holonomie dans $z_0 \in B(M)$ et un homomorphisme déterminé λ de Φ_{z_0} dans le groupe multiplicatif des nombres réels induit par t, alors le champ de covecteurs de récurrence du champ t est le gradient d'une fonction scalaire, si et seulement si λ est constante. En tenant compte de ce résultat et des affirmations antérieures il résulte :

PROPOSITION 2. *Une condition nécessaire et suffisante pour qu'un espace A_n Γ-récurrent ou Ricci-récurrent, ou T-récurrent soit E-métrisable respectivement T-métrisable est que l'homomorphisme $\lambda : \Phi_{z_0} \to R$ induit par le champ tensoriel de courbure ou par le champ tensoriel de Ricci, ou par le champ tensoriel quadratique de torsion, soit constant.*

Parmi les espaces A_n considérons les espaces dont la connexion vérifie la relation

$$T_{i,j} = T_{j,i} \tag{8}$$

où T_i sont les composantes du vecteur de torsion, espaces notés avec $\tilde{A}_n$[2] et nommés par S. Golab [5] espaces à connexion Enghiş.

Si les espaces $\tilde{A}_n$ sont à torsion récurrente nous avons [2] :

$$\psi_i T_j = \psi_j T_i \tag{9}$$

d'où il résulte

$$T_i = \alpha\psi_i \tag{10}$$

où $\alpha = \alpha(x^i)$.

De plus haut et de (10) pour les espaces $\tilde{A}_n$ T-métrisables il résulte:

PROPOSITION 3. *Dans un espace $\tilde{A}_n$ T-métrisable le vecteur de torsion est proportionnel avec le gradient de T-récurrence.*

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CÎTEVA OBSERVAŢII ASUPRA METRIZABILITĂŢII SPAŢIILOR A_n Γ-RECURENTE SAU T-RECURENTE

(Rezumat)

În lucrare se extinde un rezultat a lui V. Murgescu privind metrizabilitatea spaţiilor A_n Γ-recurente la spaţii A_n Ricci-recurente, se dă o nouă condiţie necesară şi suficientă de E-metrizabilitate respectiv T-metrizabilitate a spaţiilor A_n şi se precizează o proprietate a vectorului de torsiune în spaţiile $\tilde{A}_n$ T-metrizabile.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXV, 3, 1980

UN SCHÉMA IMPLICITE AUX DIFFÉRENCES FINIES POUR LE PROBLÈME DE LA COUCHE LIMITE HYDRODYNAMIQUE

DOINA BRĂDEANU

1. Notations et énoncé du problème.

x, y — les coordonnées de la couche limite
u — la vitesse du fluide dans la direction longitudinale Ox
u_∞ — la vitesse du fluide à l'infini
u_1 — la vitesse du fluide sur la frontière extérieure
L — une longueur caractéristique du corps
ν — le coefficient de viscosité cinématique
X, Ψ — les variables adimensionnelles de von Mises: $X = x/L$, $\Psi = \overline{\Psi}\sqrt{\nu L U_\infty}$, $\overline{\Psi}$ étant la fonction de courant donnée par $u = \partial\overline{\Psi}/\partial y$.
$U(X, \Psi)$,
$U_1(X)$ — la vitesse adimensionnelle u/u_∞ et u_1/u_∞
$G(X, \Psi)$ — la fonction adimensionnelle dans $U = \sqrt{U_1^2 - G}$
$\Psi_\infty(X)$ — la fonction de courant sur la frontière extérieure
D — le domaine des variables de von Mises dans la couche limite: $D = (X_0, X_N) \times (0, \Psi_\infty)$

Nous considérons l'opérateur parabolique de von Mises défini par

$$A(G) = \frac{\partial G}{\partial X} - \sqrt{U_1^2 - G}\,\frac{\partial^2 G}{\partial \Psi^2} \tag{1}$$

qui intervient dans l'étude du mouvement plan stationnaire d'un fluide visqueux incompressible dans la couche limite, [4].

La détermination de la fonction inconnue $G(X, \Psi)$ peut se faire en résolvant le problème à la limite suivant:

$$A(G(X, \Psi)) = 0, \quad (X, \Psi) \in D \subset R^2 \tag{2}$$

$$G(X, 0) = U_1^2(X), \; G(X, \Psi_\infty) = 0, \; G(X_0, \Psi) = G^0(\Psi), \; (X, \Psi) \in \partial D \tag{3}$$

où $G^0(\Psi)$ est la valeur initiale connue de la fonction $G(X, \Psi)$.

Conformément aux approximations utilisées dans la théorie de la couche limite et à la condition pour la vitesse sur le corps solide (en accord avec la formule de tension tangentielle-frottement), le domaine de définition $D(A)$ de l'opérateur de von Mises peut être déterminé par

$$D(A) = \Big\{G \in C^{k,m}(D) \cap C^{k,1}(\partial D) \,|\, \lim_{\Psi \to 0} \frac{\partial^p G(X, \Psi)}{\partial \Psi^p} = \infty,$$
$$\forall X \in (X_0, X_N), \; p \geqslant 2, \; G(X, 0) = U_1^2, \; G(X, \Psi_\infty) = 0, \; G(X_0, \Psi) = G^0(\Psi)\Big\}$$

Le problème à la limite (2)—(3) est un problème à une frontière libre parce que la fonction $\Psi_\infty(X)$ est inconnue. En utilisant une méthode numérique pour la résolution du problème (2)—(3), la valeur de la fonction de courant sur la frontière extérieure de la couche limite se détermine par la condition du contact lisse pour la vitesse adimensionnelle $U(X, \Psi)$, condition donnée par l'inégalité

$$\left|\frac{\partial U}{\partial \Psi}\right| \leqslant \varepsilon \quad \text{pour} \quad \Psi = \Psi_\infty \tag{4}$$

où ε est un nombre positif donné suffisamment petit.

L'intégration numérique du problème de von Mises peut être effectuée en tenant compte de la singularité de l'opérateur A pour le cas $\Psi \to 0$.

Le changement de variables

$$\xi = X, \quad \eta = \frac{\Psi}{\Psi_\infty(X)}$$

a l'avantage de transformer le domaine D de la couche limite dans un domaine aux frontières connues de forme rectangulaire, $\widetilde{D} = (X_0, X_N) \times (0, 1)$, mais alors la fonction $\Psi_\infty(X)$ se présente dans l'expression de l'opérateur de von Mises modifié, qui prend l'expression suivante

$$L(G) = \frac{\partial G}{\partial X} - \frac{\Psi'_\infty}{\Psi_\infty}\eta\frac{\partial G}{\partial \eta} - \frac{1}{\Psi^2_\infty}\sqrt{U_1^2 - G}\,\frac{\partial^2 G}{\partial \eta^2} \tag{5}$$

On obtient, alors, de (2)—(3), le problème opératoriel à la limite suivant:

$$L(G(X, \eta)) = 0, \qquad (X, \eta) \in \widetilde{D} \tag{6}$$

$$G(X, 0) = U_1^2(X),\ G(X, 1) = 0,\ \ G(X_0, \eta) = G^0(\eta), \quad (X, \eta) \in \partial\widetilde{D}, \tag{7}$$

où l'opérateur L a le domaine de définition suivant

$$\widetilde{D}(L) = \Big\{G \in C^{k,m}(\widetilde{D}) \cap C^{k,1}(\partial\widetilde{D}) \,|\, \lim_{\eta\to 0}\frac{\partial^p G(X, \eta)}{\partial \eta^p} = \infty,$$

$$\forall X \in (X_0, X_N),\ p \geqslant 2,\ G(X, 0) = U_1^2(X),\ G(X, 1) = 0,\ G(X_0, \eta) = G^0(\eta)\Big\}$$

2. La discrétisation de l'opérateur de von Mises par un schéma implicite aux différences finies. On considère l'équation de von Mises sous la forme (6) et on prend un petit intervalle $[X, X + \Delta X] \subset [X_0, X_N]$ où l'équation peut être écrite sous la forme intégrale suivante

$$\int_X^{X+\Delta X} L(G(X, \eta))dX = 0 \tag{8}$$

où

$$G(X + \Delta X, \eta) = G(X, \eta) + \int_X^{X+\Delta X}\left(\frac{\Psi'_\infty}{\Psi_\infty}\eta G_\eta + \frac{U}{\Psi^2_\infty}G_{\eta\eta}\right)dX$$

Nous introduisons la fonction

$$F(X, \eta, G_\eta, G_{\eta\eta}) = \frac{\Psi'_\infty}{\Psi_\infty} \eta G_\eta + \frac{U}{\Psi^2_\infty} G_{\eta\eta} \tag{9}$$

et nous remplaçons le domaine des variations continues des arguments X et η (le domaine du mouvement dans la couche limite) par un réseau rectangulaire, c'est-à-dire par un ensemble discret de points (noeuds)

$$D_\Delta = \{X_n, \eta_j | 0 \leqslant X_0 \leqslant X_n \leqslant X_N, \ 0 \leqslant \eta \leqslant 1, \ X_n = X_0 + n\Delta X,$$
$$\eta_j = j\Delta\eta; \ n = 0, 1, \ldots, N; \ j = 0, 1, \ldots, J\}$$

En utilisant la formule de quadrature bien connue des trapèzes pour le calcul de l'intégrale de la fonction F, l'équation (8) se transforme en l'équation implicite (aux différences finies sur le réseau D_Δ) suivante

$$g_{n+1,j} = g_{n,j} + \frac{\Delta X}{2} \sum_{k=0}^{1} f_{n+k,j} \tag{10}$$

où

$g_{n,j}$ est la valeur approchée de la fonction G donnée par l'équation d'approximation (10) : $G(X_n, \eta_j) \approx g_{n,j}$;

$$f_{n+k,j} = F\left(X_{n+k}, \eta_j, \frac{\delta_\eta g_{n+k,j}}{2\Delta\eta}, \frac{\delta^2_\eta g_{n+k,j}}{\Delta\eta^2}\right) \equiv f_\Delta(g_{n+k,j}) ; \tag{11}$$

δ_η et δ^2_η sont les opérateurs des différences finies centrales, appliqués sur deux et trois noeuds qui sont d'ordre $(\Delta\eta)^2$.

Par conséquent, nous pouvons attacher au problème à la limite (6)—(7) le schéma implicite aux différences finies suivant

$$a_{n+1,j} g_{n+1,j} - \left(\frac{1}{r} + b_{n+1,j}\right) g_{n+1,j} + c_{n+1,j} g_{n+1,j+1} + s_{n,j} = 0 \tag{12}$$
$$(j = 1, 2, \ldots, J - 1)$$

avec les conditions aux limites

$$g_{n,0} = U^2_1(X_n), \ g_{n,J} = 0, \quad n = 0, 1, 2, \ldots, N \tag{13}$$
$$g_{0,j} = G^0_j, \ j = 0, 1, 2, \ldots, J$$

où

$$a_{n+k,j} = -\frac{j(\Delta\eta)^2}{2}\left(\frac{\Psi'_\infty}{\Psi_\infty}\right)_{n+k} + \frac{U_{n+k,j}}{\Psi^2_{\infty,n+k}}$$

$$b_{n+k,j} = \frac{2U_{n+k,j}}{\Psi^2_{\infty,n+k}} \tag{14}$$

$$c_{n+k,j} = \frac{j(\Delta\eta)^2}{2}\left(\frac{\Psi'_\infty}{\Psi_\infty}\right)_{n+k} + \frac{U_{n+k,j}}{\Psi^2_{\infty,n+k}}$$

$$s_{n,j} = a_{n,j} g_{n,j-1} + \left(\frac{1}{r} - b_{n,j}\right) g_{n,j} + c_{n,j} g_{n,j+1}; \quad r = \frac{\Delta X}{2\Delta\eta^2}. \tag{15}$$

3. L'erreur de troncature du schéma aux différences finies. La consistance. Soit $G_{n,j} \equiv G(X_n, \eta_j)$ la valeur de la solution exacte du problème à la limite (6)—(7) dans le point $(X_n, \eta_j) \in D_\Delta$.
L'erreur de troncature $\tau_{n,j}$ est donnée par la formule

$$\tau_{n,j} = \frac{G_{n+1,j} - G_{n,j}}{\Delta X} - \frac{1}{2}\,[f_\Delta(G_{n+1,j}) + f_\Delta(G_{n,j})] =$$

$$= \frac{1}{\Delta X}\int_{X_n}^{X_{n+1}} F(X, \eta, G_\eta, G_{\eta\eta})dX - \frac{1}{2}\,[f_\Delta(G_{n+1,j}) + f_\Delta(G_{n,j})] \tag{16}$$

La consistance du schéma aux différences (12) est donnée par le théorème suivant:

THÉOREME 1. *Si dans le domaine $\tilde{D}$ les conditions suivantes sont remplies:*

(a) $G \in C^{k,m}(\tilde{D}), \quad \forall k,m \in Z_+$;

(b) $\partial F/\partial G_\eta$ *et* $\partial F/\partial G_{\eta\eta}$ *sont bornées, alors le schéma aux différences finies* (12) *est consistant avec une erreur de troncature d'ordre* 2 *par rapport aux pas* ΔX *et* $\Delta\eta$.

Démonstration. Nous utiliserons les développements en série de Taylor des fonctions de réseau $G_{n+1,j-1}$ et $G_{n+1,j+1}$ au voisinage du point (X_{n+k}, η_j) en posant, pour simplifier l'écriture, $G \equiv G_{n+k,j}$, $k = 0,1$. Alors, les opérateurs des différences centrales δ_η et δ_η^2 appliqués à la fonction G auront les expressions

$$\delta_\eta G = 2G_\eta\Delta\eta + \frac{1}{3}(\Delta\eta)^3 d_\eta^3 G + o(\Delta\eta^5);$$

$$\delta_\eta^2 G = G_{\eta\eta}\Delta\eta^2 + \frac{1}{12}(\Delta\eta)^4 d_\eta^4 G + o(\Delta\eta^6)$$

et l'opérateur f_Δ, de (12), appliqué à la fonction G s'écrira sous la forme

$$f_\Delta(G) = F\Big(X, \eta, G_\eta + \frac{\Delta\eta^2}{6} d_\eta^3 G + o(\Delta\eta^4),\ G_{\eta\eta} + \frac{\Delta\eta^2}{12} d_\eta^4 G + o(\Delta\eta^4)\Big) =$$
$$= F(X, \eta, G_\eta, G_{\eta\eta}) + \frac{\Delta\eta^2}{6}\left(\frac{\partial F}{\partial G_\eta} d_\eta^3 G + \frac{1}{2}\frac{\partial F}{\partial G_{\eta\eta}} d_\eta^4 G\right) + o(\Delta\eta^4) \tag{17}$$

Si dans (16) on applique la formule des trapèzes pour le calcul de l'intégrale et si on y substitue les expressions (17), alors les termes de la forme $F(X, \eta, G_\eta, G_{\eta\eta})$ se réduiront. On déduit maintenant aisément que dans le domaine $\tilde{D}$ dans lequel les conditions (a)—(b) sont remplies, nous avons l'estimation de l'erreur de troncature suivante

$$\tau_{n,j} = o((\Delta X)^2,\ (\Delta\eta)^2).$$

Pour obtenir ce résultat nous avons évidemment tenu compte du fait que la formule des trapèzes introduit une erreur d'ordre 3 par rapport au pas ΔX.

Le schéma aux différences finies (12) est, par conséquent, consistant avec une erreur de troncature d'ordre 2.

4. La stabilité du schéma aux différences finies. Désignons par $g_{n,j}$ la solution exacte de l'équation aux différences finies (12) et par $g^*_{n,j}$ la solution calculée. Alors, l'erreur de stabilité a l'expression $z_{n,j} = g_{n,j} - g^*_{n,j}$.

Nous déterminerons l'équation à laquelle satisfait l'erreur $z_{n,j}$, en supposant que le schéma (12) est stable, ou que cette erreur est petite (on peut donc supposer que les puissances de l'erreur $z_{n,j}$ sont négligeables). À cet effet, en utilisant la série du binôme où on peut se limiter aux deux premiers termes, nous avons

$$U_{n+k,j} = U^*_{n+k,j} - \frac{1}{2}\frac{z_{n+k,j}}{U^*_{n+k,j}},$$

où

$$U^*_{n+k,j} = \sqrt{U^2_{1n+k} - g^*_{n+k,j}}.$$

Nous allons substituer cette expression de $U_{n+k,j}$, calculer tous les termes et négliger les puissances de $z_{n,j}$ dans les équations (12). On arrive à l'équation aux différences finies suivantes pour l'erreur $z_{n,j}$:

$$\sum_{k=0}^{1}\sum_{s=-1,0,1}\frac{1}{\Psi^2_{\infty,n+k}}(\alpha^{(s)}_{n+k,j}z_{n+k,j+s} + \beta^{(s)}_{n+k,j}g^*_{n+k,j+s}) = 0 \qquad (18)$$

où

$$\alpha^{(-1)}_{n+k,j} = U^*_{n+k,j} + \tilde{a}_{n+k,j} = \beta^{(-1)}_{n+k,j};$$

$$\alpha^{(0)}_{n+k,j} = -\left[(-1)^{k+1}\frac{\Psi^2_{\infty,n+k}}{r} + 2U^*_{n+k,j} + \frac{P^*_{n+k,j}}{2U^*_{n+k,j}}\right] = \beta^{(0)}_{n+k,j} - \frac{P^*_{n+k,j}}{2U^*_{n+k,j}};$$

$$\alpha^{(1)}_{n+k,j} = U^*_{n+k,j} - \tilde{a}_{n+k,j} = \beta^{(1)}_{n+k,j}; \qquad (19)$$

$$\tilde{a}_{n+k,j} = -\frac{j(\Delta\eta)^2}{2}(\Psi_\infty\Psi'_\infty)_{n+k};$$

$$P^*_{n+k} = g^*_{n+k,j+1} - 2g^*_{n+k,j} + g^*_{n+k,j-1}.$$

En ce qui concerne la stabilité du schéma (18) nous avons le théorème suivant:

Théorème 2. *Si dans le domaine D_Δ est remplie la condition*

(a) $P^*_{n,j} \geqslant 0,\ \forall n = 0, 1, \ldots, N;\ j = 1, 2, \ldots, J-1$

et si les fonctions qui interviennent à titre de coefficients dans l'expression de l'opérateur différentiel (5) sont des fonctions suffisamment lisses, alors le schéma aux différences (18) est localement stable pour chaque valeur du paramètre r du réseau.

Démonstration. Nous allons utiliser le critère de stabilité de von Neumann, en supposant que les coefficients α/Ψ^2_∞ sont constants et en négligeant le terme

3. L'erreur de troncature du schéma aux différences finies. La consistance. Soit $G_{n,j} \equiv G(X_n, \eta_j)$ la valeur de la solution exacte du problème à la limite (6)–(7) dans le point $(X_n, \eta_j) \in D_\Delta$.
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$$\tau_{n,j} = \frac{G_{n+1,j} - G_{n,j}}{\Delta X} - \frac{1}{2}\left[f_\Delta(G_{n+1,j}) + f_\Delta(G_{n,j})\right] =$$

$$= \frac{1}{\Delta X}\int_{X_n}^{X_{n+1}} F(X, \eta, G_\eta, G_{\eta\eta})dX - \frac{1}{2}\left[f_\Delta(G_{n+1,j}) + f_\Delta(G_{n,j})\right] \tag{16}$$

La consistance du schéma aux différences (12) est donnée par le théorème suivant :

THÉOREME 1. *Si dans le domaine* $\widetilde{D}$ *les conditions suivantes sont remplies*:

(a) $G \in C^{k,m}(\widetilde{D}), \quad \forall k,m \in Z_+$;

(b) $\partial F/\partial G_\eta$ *et* $\partial F/\partial G_{\eta\eta}$ *sont bornées, alors le schéma aux différences finies* (12) *est consistant avec une erreur de troncature d'ordre* 2 *par rapport aux pas* ΔX *et* $\Delta\eta$.

Démonstration. Nous utiliserons les développements en série de Taylor des fonctions de réseau $G_{n+1,j-1}$ et $G_{n+1,j+1}$ au voisinage du point (X_{n+k}, η_j) en posant, pour simplifier l'écriture, $G \equiv G_{n+k,j}$, $k = 0,1$. Alors, les opérateurs des différences centrales δ_η et δ_η^2 appliqués à la fonction G auront les expressions

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$$f_\Delta(G) = F\left(X, \eta, G_\eta + \frac{\Delta\eta^2}{6} d_\eta^3 G + o(\Delta\eta^4), G_{\eta\eta} + \frac{\Delta\eta^2}{12} d_\eta^4 G + o(\Delta\eta^4)\right) =$$
$$= F(X, \eta, G_\eta, G_{\eta\eta}) + \frac{\Delta\eta^2}{6}\left(\frac{\partial F}{\partial G_\eta} d_\eta^3 G + \frac{1}{2}\frac{\partial F}{\partial G_{\eta\eta}} d_\eta^4 G\right) + o(\Delta\eta^4) \tag{17}$$

Si dans (16) on applique la formule des trapèzes pour le calcul de l'intégrale et si on y substitue les expressions (17), alors les termes de la forme $F(X, \eta, G_\eta, G_{\eta\eta})$ se réduiront. On déduit maintenant aisément que dans le domaine D dans lequel les conditions (a)–(b) sont remplies, nous avons l'estimation de l'erreur de troncature suivante

$$\tau_{n,j} = o((\Delta X)^2, (\Delta\eta)^2).$$

Pour obtenir ce résultat nous avons évidemment tenu compte du fait que la formule des trapèzes introduit une erreur d'ordre 3 par rapport au pas ΔX.

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$$\sum_{k=0}^{1} \sum_{s=-1,0,1} \frac{1}{\Psi^2_{\infty,n+k}} (\alpha^{(s)}_{n+k,j} z_{n+k,j+s} + \beta^{(s)}_{n+k,j} g^*_{n+k,j+s}) = 0 \tag{18}$$

où

$$\begin{gathered}
\alpha^{(-1)}_{n+k,j} = U^*_{n+k,j} + \tilde{a}_{n+k,j} = \beta^{(-1)}_{n+k,j}\,; \\
\alpha^{(0)}_{n+k,j} = -\left[(-1)^{k+1}\frac{\Psi^2_{\infty,n+k}}{r} + 2U^*_{n+k,j} + \frac{P^*_{n+k,j}}{2U^*_{n+k,j}}\right] = \beta^{(0)}_{n+k,j} - \frac{P^*_{n+k,j}}{2U^*_{n+k,j}}\,; \\
\alpha^{(1)}_{n+k,j} = U^*_{n+k,j} - \tilde{a}_{n+k,j} = \beta^{(1)}_{n+k,j}\,; \\
\tilde{a}_{n+k,j} = -\frac{j(\Delta\eta)^2}{2}(\Psi_\infty \Psi'_\infty)_{n+k}\,; \\
P^*_{n+k} = g^*_{n+k,j+1} - 2g^*_{n+k,j} + g^*_{n+k,j-1}.
\end{gathered} \tag{19}$$

En ce qui concerne la stabilité du schéma (18) nous avons le théorème suivant :

THÉORÈME 2. *Si dans le domaine D_Δ est remplie la condition*

(a) $\quad P^*_{n,j} \geqslant 0, \ \forall n = 0, 1, \ldots, N\,;\ j = 1, 2, \ldots, J-1$

et si les fonctions qui interviennent à titre de coefficients dans l'expression de l'opérateur différentiel (5) sont des fonctions suffisamment lisses, alors le schéma aux différences (18) est localement stable pour chaque valeur du paramètre r du réseau.

Démonstration. Nous allons utiliser le critère de stabilité de von Neumann, en supposant que les coefficients α/Ψ^2_∞ sont constants et en négligeant le terme libre. Nous cherchons, dans ces conditions, pour le schéma (18) des solutions représentées par la série de Fourier

$$z_{n+k,j+s} = \sum_{m=-\infty}^{\infty} v_{n+k}(m) e^{im(j+s)\Delta\eta}, \quad k = 0\,;\ 1. \tag{20}$$

En substituant (20) dans (18), on obtient la condition suivante

$$v_{n+1}(m) = G(\Delta X, \Delta\eta, m) v_n(m), \quad \forall m \in Z$$

où le facteur d'amplification G a l'expression

$$G(\Delta X, \Delta\eta, m) = -\left(\Psi^2_{\infty,n+1} \sum_{s=-1,0,1} \alpha^{(s)}_{n,j} e^{ims\Delta\eta}\right)\left(\Psi^2_{\infty,n} \sum_{s=-1,0,1} \alpha^{(s)}_{n+1,j} \cdot e^{ims\Delta\eta}\right)^{-1}.$$

Pour le facteur d'amplification on obtient une nouvelle expression par la substitution des coefficients α, donnés par (19) et à la suite de calculs élémentaires. Si, en outre, nous utilisons le fait que les fonctions $U(X, \eta)$ et $\Psi_\infty(X)$ sont lisses, les coefficients de l'opérateur différentiel (5) peuvent être remplacés au voisinage d'un noeud par des valeurs constantes, c'est-à-dire que $U^*_{n+1,j} \approx U^*_{n,j}$ et $\Psi_{\infty,n+1} \approx \Psi_{\infty,n}$. Nous avons donc :

$$G(r, Y) = -\left(a + \frac{\Psi^2_{\infty,n}}{r} + bi\right)\left(a - \frac{\Psi^2_{\infty,n}}{r} + bi\right)^{-1} \tag{21}$$

où

$$Y = m\Delta\eta,$$

$$a = 2U^*_{n,j}\cos Y - \left(2U^*_{n,j} + \frac{P^*_{n,j}}{2U^*_{n,j}}\right),\ U^*_{n,j} > 0\,; \tag{22}$$

$$b = -2\tilde{a}_{n,j}\sin Y$$

La condition de stabilité de von Neumann, [2], exige que pour chaque Y on ait $|G| < 1$. On peut constater aisément, en utilisant (21), que cette inégalité est vérifiée quelle que soit la valeur du paramètre r du réseau, avec Y arbitraire, si

$$a < 0. \tag{23}$$

Or, l'inégalité (a) nous assure que la condition (23) est remplie.

La stabilité locale du schéma aux différences (18) est ainsi complètement démontrée.

Remarques. 1° La condition (a) est remplie dans une région assez vaste située à l'intérieur de la couche limite, [1].

2°. La consistance et la stabilité du schéma aux différences implique, conformément au théorème de Lax, [2], [3], la convergence de ce schéma.

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O SCHEMĂ IMPLICITĂ CU DIFERENŢE FINITE PENTRU PROBLEMA STRATULUI LIMITĂ HIDRODINAMIC

(R e z u m a t)

În lucrare se construieşte o schemă implicită cu diferenţe finite pentru problema stratului limită hidrodinamic, considerată în variabilele lui von Mises (X şi Ψ). Problema la limită discretizată se studiază din punctul de vedere al consistenţei şi al stabilităţii. Se arată că eroarea de trunchiere este de ordinul 2 relativ la paşii reţelei, iar stabilitatea locală a schemei se demonstrează folosind criteriul lui von Neumann.

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THE COMPARISON OF THE MICHAL-BASTIANI AND OF THE CLARKE SUBDIFFERENTIAL

A. B. NÉMETH

0. There are various notions which extend the directional derivative and the subdifferential. The recent paper [3] of J. P. Penot has given a comparative study as well as a wide class of application of them in the optimization theory. However, it isn't given the comparison with the subdifferential introduced by F. H. Clarke [1], a notion that have become in the last years very useful in optimisation. The aim of our note is to complete this gap.

1. The directional derivative and the subdifferential. Let X be a Banach space over the reals and let R^X denote the set of the real-valued functionals defined on X. If for f in R^X and x and v in X the (finite) limit

$$f'(x\,;v):=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}(f(x+\lambda v)-f(x))$$

exists (where λ is a positive real tending monotonically to 0), it is called the *directional derivative of f at x in the direction v*. $f'(x\,;\cdot)$ is called the *directional derivative functional of f at x*.

Let X' denote the space of the linear and continuous functionals on X. Assume that $f'(x\,;.)$ is defined everywhere on X. Then the set

$$\partial f(x):=\{x'\in X':f'(x\,;v)\geqslant\langle x',v\rangle,\ \forall v\in X\}$$

is called the *subdifferential of f at x*.

It is an easy matter to check that even for $X=R$ there exist continuous functions with the directional derivative functions at some points defined only in 0. Accordingly, the condition of the existence of the directional derivative in any direction is a rather restrictive one. However, for the very important family of convex functionals, the directional derivative functional (at each point) is defined everywhere and the subdifferential is a nonempty convex and w^*-compact subset of X' [2].

2. Generalized directional derivatives and generalized subdifferentials. Let f be in R^X. We list below some of the most usual notions of directional derivatives (see [3] and [1]). The (finite or infinite) limits

$$\underline{d}_r f(x\,;v):=\liminf_{\lambda\downarrow 0}\frac{1}{\lambda}(f(x+\lambda v)-f(x))\ \text{ and}$$

$$\overline{d}_r f(x\,;v):=\limsup_{\lambda\downarrow 0}\frac{1}{\lambda}(f(x+\lambda v)-f(x))$$

are called *radial inferior* and respectively *radial superior directional derivatives of f at x in the direction v*.

Denote by $C(x, v)$ the family of continuous functions $c: [0, 1] \to X$, having the properties: $c(0) = x$, $c'_+(0)$ exists and $c'_+(0) = v$. The (finite or infinite) limits

$$\underline{d}_s f(x;v) := \inf_{c \in C(x,v)} \liminf_{\lambda \downarrow 0} \frac{1}{\lambda}(f(c(\lambda)) - f(x)) \text{ and}$$

$$\overline{d}_s f(x;v) := \sup_{c \in C(x,v)} \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(f(c(\lambda)) - f(x))$$

are called the *smooth inferior* and respectively, the *smooth superior directional derivatives of f at x in the direction v*.

The (finite or infinite) limits

$$\underline{d}f(x;v) := \liminf_{\substack{y \to v \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x)) \text{ and}$$

$$\overline{d}f(x;v) := \limsup_{\substack{y \to v \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x))$$

are called the *inferior* and respectively, *the superior Michal-Bastiani directional derivatives of f at x in the direction v.*

The (finite or infinite) limits

$$f_0(x;v) := \liminf_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(y + \lambda v) - f(y)) \text{ and}$$

$$f^0(x;v) := \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(y + \lambda v) - f(y))$$

are called the *inferior* and respectively, the *superior directional derivatives of Clarke of f at x in the direction v.*

Let us denote by $df(x;.)$ one of the above directional derivatives. The set

$$\{x' \in X' : df(x;v) \geqslant \langle x', v\rangle,\ \forall v \in X\}$$

will be called the *subdifferential of f at x with respect to the directional derivative d*. We shall use in the sequel for the subdifferentials with respect to the above introduced directional derivatives, the notations (in order): $\underline{\partial}_r f(x)$, $\overline{\partial}_r f(x)$, $\underline{\partial}_s f(x)$, $\overline{\partial}_s f(x)$, $\underline{\partial} f(x)$, $\overline{\partial} f(x)$, $\partial_0 f(x)$ and $\partial^0 f(x)$.

As an immediate consequence of the above definitions, we have for any v in x

$$(1)\qquad \underline{d}f(x;v) \leqslant \underline{d}_s f(x;v) \leqslant \underline{d}_r f(x;v) \leqslant \overline{d}_r f(x;v) \leqslant \overline{d}_s f(x;v) \leqslant \overline{d}f(x;v)$$

and therefore

$$\underline{\partial} f(x) \subset \underline{\partial}_s f(x) \subset \underline{\partial}_r f(x) \subset \overline{\partial}_r f(x) \subset \overline{\partial}_s f(x) \subset \overline{\partial} f(x).$$

Similarly, for any v in x

$$f_0(x;v) \leqslant f^0(x;v),$$

and hence

$$\partial_0 f(x) \subset \partial^0 f(x).$$

It is a very important fact, that *all the above directional derivatives and subdifferentials coincide with the classical ones,* introduced at 1, *when f is a convex functional.*

3. **Some propreties of the Michal-Bastiani directional derivatives.** Let f be in R^X and $x \in X$. We shall say that f is *Lipschitz at x* if there exist a real number L and a neighbourhood V of x such that

$$|f(v) - f(x)| \leqslant L\|v - x\|$$

for any v in V. The functional f is said to be *locally Lipschitz at x* if there exists a real number L and a neighbourhood V of x such that

$$|f(v) - f(u)| \leqslant L\|v - u\|$$

for any u and v in V.

PROPOSITION 1. *The following assertions are equivalent*

a) $\underline{d}f(x;0)$ *and* $\bar{d}f(x;0)$ *are finite;*

b) $\underline{d}f(x;0) = \bar{d}f(x;0) = 0$;

c) f *is Lipschitz at* x;

d) $\underline{d}f(x;v)$ *and* $\bar{d}f(x;v)$ *are finite for any v in X.*

We need in the proof the following

LEMMA. *The functional f is Lipschitz at x if and only if*

$$\limsup_{\substack{y \to 0 \\ \lambda \downarrow 0}} \frac{1}{\lambda} |f(x + \lambda y) - f(x)| < \infty.$$

Proof. The "only if" part is obvious. To prove the „if" part we assume that the above inequality holds, but f isn't Lipschitz at x. This means that for any natural number n there exists an x_n in X such that $\|x - x_n\| < 1/n^2$ and

$$|f(x_n) - f(x)| > n^2 \|x_n - x\|. \tag{2}$$

Set $y_n = (n\|x_n - x\|)^{-1}(x_n - x)$ and $\lambda_n = n\|x_n - x\|$. Passing if necessary to a subsequence, we can assume that $\lambda_n \downarrow 0$. (The inequality (2) holds also in this case!) We have by definition

$$\lambda_n y_n = x_n - x$$

and from (2) it follows that

$$\frac{1}{\lambda_n} |f(x + \lambda_n y_n) - f(x)| > n^2 \|y_n\| \geqslant n$$

and hence

$$\limsup_{\substack{y\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}|f(x+\lambda y)-f(x)| = \infty.$$

The obtained contradiction proves the lemma. Q.E.D.

The proof of Proposition 1. We begin by proving that $\underline{d}f(x;0)$ and $\bar{d}f(x;0)$ are finite if and only if f is Lipschitz at x (i.e., that a) and c) are equivalent). The „only if" part is again obvious. Let us suppose that $\underline{d}f(x;0)$ and $\bar{d}f(x;0)$ are finite. Then there exists a real number L such that

$$-L < \liminf_{\substack{y\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}(f(x+\lambda y)-f(x)) \leqslant \limsup_{\substack{y\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}(f(x+\lambda y)-f(x)) < L.$$

Hence there exist a neighbourhood V of 0 in x and a neighbourhood U of 0 in R^+ such that

$$-L < \frac{1}{\lambda}(f(x+\lambda y)-f(x)) < L$$

for any y in V and any λ in U, and therefore

$$\limsup_{\substack{y\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}|f(x+\lambda y)-f(x)| < \infty.$$

Apply now the lemma to conclude that f is Lipschitz at x.

If f is Lipschitz at x then $\underline{d}f(x;0) = \bar{d}f(x;0) = 0$ and hence c) $\Rightarrow$ b) $\Rightarrow$ a) $\Rightarrow$ c).

We have obviously d) $\Rightarrow$ a) and hence d) $\Rightarrow$ c). Suppose now that f is Lipschitz at x. For an arbitrary v in X and for W a bounded spherical neighbourhood of v we can get a positive λ_0 such to $x+\lambda y$ be for any λ, $0 < \lambda \leqslant \lambda_0$ and for any y in W in the neighbourhood V of x in which the Lipschitz condition holds, i.e.,

$$-\lambda L\|y\| \leqslant f(x+\lambda y)-f(x) \leqslant \lambda L\|y\|,$$

whereby it follows that $\underline{d}f(x;v)$ and $\bar{d}f(x;v)$ are finite. This proves that c) implies d) Q.E.D.

PROPOSITION 2. *If the functional f is locally Lipschitz at x then there hold*

$$\underline{d}f(x;v) = \underline{d}_s f(x;v) = \underline{d}_r f(x;v)$$

and

$$\bar{d}f(x;v) = \bar{d}_s f(x;v) = \bar{d}_r f(x;v)$$

for any v in X.

Proof. Let v be given and let W its bounded spherical neighbourhood. Suppose that λ_0 is a positive real chosen sufficiently small to $x+\lambda y$ be for any y in W and any λ, $0 < \lambda \leqslant \lambda_0$ in the neighbourhood V of x where f has the locally Lipschitz property. Then

$$-L\|y - v\| \leqslant \frac{1}{\lambda}(f(x + \lambda y) - f(x + \lambda v)) \leqslant L\|y - v\|$$

and hence

$$\liminf_{\substack{y \to v \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x + \lambda v)) = \limsup_{\substack{y \to v \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x + \lambda v)) = 0.$$

We have then

$$\bar{d}f(x;v) = \limsup_{\substack{y \to v \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x)) \leqslant \limsup_{\substack{y \to v \\ \lambda \downarrow 0}} (f(x + \lambda y) -$$

$$-f(x + \lambda v)) + \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(f(x + \lambda v) - f(x)) = \bar{d}_r f(x;v),$$

that is, $\bar{d}f(x;v) \leqslant \bar{d}_r f(x;v)$. The converse of this inequality follows by definition and hence comparing with (1), we deduce the second assertion of the proposition. In a similar way we can verify the first one. Q.E.D.

4. The relation between the Michal-Bastiani and the Clarke subdifferential. Assume that f is in R^X.

PROPOSITION 3. *If the directional derivatives* $\underline{d}f(x;0)$ *and* $\bar{d}f(x;0)$ *are finite, then for any* v *in* x

$$f_0(x;v) \leqslant \underline{d}f(x;v) \leqslant \bar{d}f(x;v) \leqslant f^0(x;v).$$

Proof. We have for any v in X

$$\bar{d}f(x;v) = \limsup_{\substack{y \to v \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x)) =$$

$$= \limsup_{\substack{y \to 0 \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda(v + y)) - f(x)) \leqslant \limsup_{\substack{y \to 0 \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda(v + y)) -$$

$$-f(x + \lambda y)) + \limsup_{\substack{y \to 0 \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x)) \leqslant \limsup_{\substack{z \to 0 \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + z + \lambda v) -$$

$$-f(x + z)) + \limsup_{\substack{y \to 0 \\ \lambda \downarrow 0}} \frac{1}{\lambda}(f(x + \lambda y) - f(x)) = f^0(x;v) + \bar{d}f(x;0).$$

Using now Proposition 1 that asserts that $\bar{d}f(x;0) = 0$, we conclude $\bar{d}f(x;v) \leqslant$ $\leqslant f^0(x;v)$. In a similar way can be verified that $\underline{d}f(x;v) \geqslant f_0(x;v)$. Q.E.D.

Remark. In the above proof we have deduced in fact that

$$\overline{d}f(x;v) \leqslant f^{\circ}(x;v) + \overline{d}f(x;0).$$

It can be shown similarly that

$$f_0(x;v) + \underline{d}f(x;0) \leqslant \underline{d}f(x;v).$$

COROLLARY 1. *If the directional derivatives* $\underline{d}f(x;0)$ *and* $\overline{d}f(x;0)$ *are finite, then*

$$\partial_0 f(x) \subset \underline{\partial}f(x) \subset \underline{\partial}_s f(x) \subset \underline{\partial}_r f(x) \subset \overline{\partial}_r f(x) \subset \overline{\partial}_s f(x) \subset \overline{\partial}f(x) \subset \partial^0 f(x).$$

COROLLARY 2. *If* f *is locally Lipschitz at* x, *then*

$$\partial_0 f(x) \subset \underline{\partial}f(x) = \underline{\partial}_s f(x) = \underline{\partial}_r f(x) \subset \overline{\partial}_r f(x) = \overline{\partial}_s f(x) = \overline{\partial}f(x) \subset \partial^0 f(x).$$

Proof. We observe first that from the condition that f is locally Lipschitz at x and Proposition 1 it follows that we are in the conditions of Corollary 1. We can also use Proposition 2 to deduce the equalities in the above relations. Q.E.D.

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COMPARARE ÎNTRE NOŢIUNILE DE SUBDIFERENŢIALE A LUI MICHAL-BASTIANI ŞI CLARKE

(Rezumat)

Noţiunea de diferenţială şi respectiv de subdiferenţială în diferite spaţii abstracte se defineşte în literatura de specialitate în diferite moduri. În prezenta lucrare se compară două dintre aceste definiţii: definiţia lui Michal-Bastiani şi cea a lui Clarke, în cazul funcţiilor definite pe spaţii Banach cu valori reale. Prin legătura stabilită între aceste două noţiuni se arată că, în anumite condiţii suplimentare, cele două noţiuni coincid.

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GÉNÉRATION D'ÉLÉMENTS SPLINE À L'AIDE DES APPLICATIONS MONOTONES

C. KALIK

On se propose dans ce travail de montrer que certains résultats de l'analyse fonctionnelle qui sert de support à l'étude des problèmes aux limites sont aussi applicables avec profit à l'étude des éléments spline dans un espace Banach quelconque. De cette façon on retrouvera les fonctions spline les plus importantes et en même temps on pourra fournir de nouvelles classes d'éléments spline.

1. Tout comme dans [4] on considère les objets donnés suivants: les espaces Banach X, Y, Z, les applications linéaires et continues $L: X \to Y$, $\pi: X \to Z$ et l'application $T: Y \to 2^{Y^*}$.

Dorénavant on suppose que l'espace Y est séparable.

DÉFINITION 1.1. Un élément $s \in X$ s'appelle (π, L, T)-spline s'il y a un $w^* \in (\mathrm{Ker} L)^{\perp} \cap (\mathrm{Ker}\, \pi)^{\perp}$ de sorte que

$$w^* \in (L^*TL)(s) \tag{1.1}$$

Ici $(\mathrm{Ker} L)^{\perp}$ respectivement $(\mathrm{Ker}\, \pi)^{\perp}$ signifie l'annulateur de $\mathrm{Ker} L$, respectivement de $\mathrm{Ker}\, \pi$.

Soit $z_0 \in Z$ un élément quelconque fixé. On note

$$X(z_0) = \{x \in X \mid \pi x = z_0\}.$$

Nous dirons que $x \in X$ interpole l'élément donné $z_0 \in Z$ si $x \in X(z_0)$.

DÉFINITION 1.2. Un élément $s_0 \in X$ s'appellera (π, L, T)-spline interpolateur de z_0 s'il est un (π, L, T)-spline et s'il interpole l'élément z_0.

Dans la première partie du travail nous établirons un théorème d'existence et un théorème d'unicité pour les (π, L, T)-spline interpolateurs.

Sans restreindre la généralité de nos résultats on peut admettre que l'image de π coincide avec Z, c'est-à-dire que $R(\pi) = Z$. Dans ce cas pour chaque $z_0 \in Z$ nous aurons $X(z_0) \neq \Phi$. Etant donné que π est linéaire et continue, il en résulte que l'ensemble $X(z_0)$ est convexe et fermé. On note

$$Y(z_0) = L(X(z_0)).$$

Nous pouvons observer que $Y(z_0)$ est aussi convexe.

LEMME 1.1. *Si $R(L) = Y$, alors l'ensemble $Y(z_0)$ est fermé.*

Démonstration. Ayant en vue que $Y(z_0)$ s'obtient de $Y(0)$ moyennant une translation, il suffit de prouver que $Y(0)$ est fermé.

On notera par $\overset{\circ}{x}$ l'élément de l'espace $X/\mathrm{Ker} L$ qui contient $x \in X$. Soit $\tilde{L}: X/\mathrm{Ker} L \to Y$ l'application définie par l'égalité $\tilde{L}\overset{\circ}{x} = Lx$. Cette application est linéaire, continue, injective et surjective. Par conséquent, il existe $\tilde{L}^{-1}$, et elle est linéaire et continue.

Soit $\{y_n\} \subset Y(0)$ une suite fondamentale et $x_n \in X(0)$ l'élément pour lequel on a $Lx_n = y_n$. Notons par y la limite de la suite $\{y_n\}$. Il faudra montrer que $y \in Y(0)$.

Soit $x \in X$ un élément pour lequel il y a $Lx = y$. Puisque $Lx_n = \tilde{L}\overset{\circ}{x}_n$ et $Lx = \tilde{L}\overset{\circ}{x}$, on a $\overset{\circ}{x}_n = \tilde{L}^{-1}y$ et $\overset{\circ}{x} = \tilde{L}^{-1}y$. Cependant $\tilde{L}^{-1}$ étant une application continue il s'ensuit que $\tilde{L}^{-1}y_n \to \tilde{L}^{-1}y$, ou que $\overset{0}{x_n} \to \overset{\circ}{x}$. Dans ce cas il existe un $x_n \in \overset{\circ}{x}$ et un $x \in \overset{\circ}{x}$ tels qu'on ait $x_n \to x$. En même temps, tenant compte de ce que l'ensemble $\mathrm{Ker}\,\pi$ est fermé et de $x_n \in \mathrm{Ker}\,\pi$, il en ressort que $x \in \mathrm{Ker}\,\pi$, c'est-à-dire que $y = Lx \in L(\mathrm{Ker}\,\pi) = Y(0)$. Q.E.D.

LEMME 1.2. *Si $R(L) = Y$, alors la condition nécessaire et suffisante pour que*

$$w^* \in (L^*TL)(s), \text{ où } w^* \in (\mathrm{Ker}L)^\perp \cap (\mathrm{Ker}\,\pi)^\perp$$

est qu'il existe $v^* \in Y^\perp(0)$ tel qu'on a $v^* \in T(\sigma)$, où $\sigma = Ls$.

Démonstration. I. Supposons que $w^* \in (L^*TL)(s)$ où $w^* \in (\mathrm{Ker}L)^\perp \cap (\mathrm{Ker}\,\pi)^\perp$. Puisque L est une application linéaire, continue et surjective, il s'ensuit que $\overline{R(L^*)} = R(L^*) = (\mathrm{Ker}L)^\perp$. Par conséquent l'équation $L^*v^* = w^*$ a une solution. Pour cette solution $v^* \in Y^*$, on a $L^*v^* \in (L^*TL)(s)$, donc $v^* \in T(\sigma)$, où $\sigma = L(s)$.

D'autre part, pour chaque $v \in Y(0)$, il existe un $x \in \mathrm{Ker}\pi$ tel que $Lx = v$. On pourra donc écrire que $0 = \langle w^*, x\rangle = \langle L^*v^*, x\rangle = \langle v^*, Lx\rangle = \langle v^*, y\rangle$, ce qui signifie que $v^* \in Y^\perp(0)$.

II. Supposons que pour v^* et σ nous avons $v^* \in Y^\perp(0)$ et $v^* \in T(\sigma)$. Comme $R(L) = Y$, il en résulte l'existence d'un $s \in X$ tel que $Ls = \sigma$. Notons $w^* = L^*v^*$. En ce cas $v^* \in T(\sigma)$ amène $w^* \in (L^*TL)(s)$. D'autre part l'égalité $(\mathrm{Ker}L)^\perp = R(L^*)$ montre que $w^* \in (\mathrm{Ker}L)^\perp$, et l'inclusion $v^* \in Y^\perp(0)$ montre que pour chaque $x \in \mathrm{Ker}\pi$ on aura $\langle w^*, x\rangle = \langle L^*v^*, x\rangle = \langle v^*, Lx\rangle = 0$. Par conséquent $w^* \in (\mathrm{Ker}L)^\perp \cap (\mathrm{Ker}\,\pi)^\perp$. Q.E.D.

Le lemme ci-dessus démontré permet de tirer la conclusion suivante: l'étude de l'existence de (π, L, T)-spline se reduit à l'étude de l'existence des solutions de l'équation multivoque

$$v^* \in T(\sigma),$$

où $v^* \in Y^\perp(0)$.

On pourra maintenant établir le théorème d'existence pour les spline interpolateurs. Donc, soit $z_0 \in Z$ un élément fixé quelconque. On note par y_0 l'élément de $Y(z_0)$ pour lequel on a

$$\|y_0\|^2 = \min\,\{\|y\|^2 | y \in Y(z_0)\}.$$

THÉORÈME 1.1. *Si sont satisfaites les conditions suivantes:*

1. $R(L) = Y$, $R(\pi) = Z$,
2. *la restriction de T sur $Y(z_0)$ est monotone, bornée et continue de façon finie.*
3. *il existe un $R_0 > 0$, tel que $\langle v^*, v - y_0\rangle > 0$ pour $\forall u \in Y(z_0)$ qui satisfait la condition $\|u\| > R_0$ et pour $\forall w^* \in Tu$, alors il existe au moins un (π, L, T)-spline qui interpole $z_0 \in Z$.*

Démonstration. Conformément au lemme 1.2 il suffit de montrer qu'il existe un $\sigma_0 \in Y(z_0)$ et $\dot{v}_0 \in T(\sigma_0)$ tels que $\langle v^*, y\rangle = 0$ pour chaque $y \in Y(0)$. Mais le théorème 2.1 du travail [4] montre que les conditions 2. et 3. du théorème sont suffisantes pour qu'il existe un σ_0 et un v^* ayant les propriétés demandées. Q.E.D.

THÉORÈME 1.2. *Si sont satisfaites les conditions suivantes:*

1. $R(L) = Y,\ R(\pi) = Z,$
2. $\mathrm{Ker}L \cap \mathrm{Ker}\pi = \{0\},$
3. *T est strictement monotone, alors il existe tout au plus un (π, L, T)-spline qui interpole $z_0 \in Z$.*

Démonstration. Nous allons nous convaincre, pour commencer, que la condition 3. attire l'existence de tout au plus un élément $\sigma_0 \in Y(z_0)$ tel qu'il y a un $\dot{v}_0$ pour lequel $\dot{v}_0 \in T\sigma_0$ et $\dot{v}_0 \in Y^{\perp}(0)$. Supposons le contraire, c'est-à-dire que $\exists\sigma_i \in Y(z_0)$, $\exists \dot{v}_i \in T\sigma_i$ de sorte que $\dot{v}_i \in Y^{\perp}(0)$, $i = 1,2$ et $\sigma_1 \neq \sigma_2$. Alors 3. donne $\langle \dot{v}_1 - \dot{v}_2, \sigma_1 - \sigma_2\rangle > 0$. Mais $\sigma_1 - \sigma_2 \in Y(0)$ et $\dot{v}_1 - \dot{v}_2 \in Y^{\perp}(0)$, ce qui contredit l'inégalité stricte de ci-dessus. Cela signifie qu'il existe tout au plus un $\sigma_0 \in Y(z_0)$ pour lequel on aura un $\dot{v}_0 \in T\sigma_0$ tel que $\dot{v}_0 \in Y^{\perp}(0)$.

Afin que la démonstration du théorème soit complète, il suffit de montrer que l'équation $Ls_i = \sigma_0$ a tout au plus une solution. En effet, si on a $Ls_i = \sigma_0$, $i = 1,2$, alors $s_1 - s_2 \in X(0) = \mathrm{Ker}\ \pi$, et en même temps, $s_1 - s_2 \in \mathrm{Ker}\ L$. Donc grâce a 2. on aura $s_1 - s_2 = 0$. Q.E.D.

2. Un cas particulier important est celui où $T = \partial f$, dans laquelle $f: Y \to (-\infty, +\infty]$ est une fonctionnelle propre, convexe et inférieurement demi-continue (ábréviation i.d.c.).

THÉORÈME 2.1. *Si s_0 est un $(\pi, L, \partial f)$ — spline qui interpole z_0, alors on aura l'égalité*

$$f(Ls_0) = \min\{f(Lu)\,|\,u \in X(z_0)\} \tag{2.1}$$

Démonstration. Soit $w^* \in (L^*\partial f L)(s_0)$ et $L^*v^* = w^*$. Par suite on aura $v^* \in \partial f(Ls_0)$, ce qui permet d'écrire

$$f(Lu) - f(Ls_0) \geqslant \langle v^*, Lu - Ls_0\rangle,\ \forall u \in X(z_0).$$

Compte tenu de $v^* \in Y^{\perp}(0)$ et vu $Lu - Ls_0 \in Y(0)$, $\forall u \in X(z_0)$, il en résulte $< v^*, Lu - Ls_0 > = 0$, $\forall u \in X(z_0)$. De là, en s'appuyant sur l'inégalité précédente, on obtient l'égalité (2.1). Q.E.D.

Dans ce cadre on peut donner une légère extension de la notion de l'élément spline:

DÉFINITION 2.1. L'élément $s_0 \in X$ s'appelle (π, L, f) — spline interpolateur de $z_0 \in Z$, si

$$f(Ls_0) = \min\{f(Lu)\,|\,u \in X(z_0)\}. \tag{2.2}$$

THÉORÈME 2.2. *Si sont satisfaites les conditions suivantes:*

1. $R(L) = Y,\ R(\pi) = Z,$
2. *la restriction de f sur $Y(z_0)$ est propre, convexe et i.d.c,*
3. $\lim f(v) = +\infty$, si $\|v\| \to +\infty$,

alors il existe un (π, L, f)-spline interpolateur de z_0.

Ce théorème est la conséquence immédiate de la propriété bien connue de minimum des fonctionnelles convexes. Il est tout aussi facile d'établir le théorème suivant :

THÉORÈME 2.3. *Si sont satisfaites les conditions suivantes :*

1. $R(L) = Y,\ R(\pi) = Z,$

2. $\mathrm{Ker} L \cap \mathrm{Ker}\,\pi = \{0\},$

3. *la restriction de la fonctionnelle f sur $Y(z_0)$ est strictement convexe, alors il existe tout au plus un (π, L, f) — spline, interpolateur de z_0.*

Il vaut la peine de signaler le cas où

$$f(u) = G(\|u\|),$$

Ici $G: [0, +\infty) \to (-\infty, +\infty]$ est une fonction monotone croissante. Dans ce cas l'égalité (2.1) équivaut à

$$\|Ls_0\| = \min \{\|Lu\| \mid u \in X(z_0)\}. \tag{2.3}$$

3. Nous allons présenter ici un exemple concernant les résultats ci-dessus mentionnés.

Soient (a, b) un intervalle borné ou non et les espaces Sobolev $W^{p,k}(a, b)$ respectivement $\mathring{W}^{p,k}(a, b)$ [6]. Notons par V un sousespace de $W^{p,k}(a, b)$ pour lequel on aura $\mathring{W}^{p,k}(a, b) \subset V \subset W^{p,k}(a,b)$. Soit $X = V$ et $Lu = \{u, u', \ldots \ldots, u^{(k)}\}$, $u \in V$. L'application

$$L : V \to [L^p(a, b)]^{k+1}$$

est linéaire, continue et $\overline{R(L)} = R(L)$. Soit aussi $Y = R(L)$.

Sur l'intervalle (a, b) on choisit un nombre fini de noeuds : $a < x_1 < x_2 < \ldots < x_n < b$. Pour définir l'application π, introduisons d'abord de nouvelles notations. Soit $\alpha_{i,j} \in \{0, 1\}$, $\alpha_i = \{\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,k-1}\}$. Notons par u une fonction quelconque du V. Soit

$$u^{\alpha_{i,j}} = \begin{cases} \Phi, & \text{si } \alpha_{i,j} = 0 \\ u^{(j)}(x_i), & \text{si } \alpha_{i,j} = 1, \end{cases}$$

et

$$\pi_i u = \{u^{\alpha_{i,0}}, u^{\alpha_{i,1}}, \ldots, u^{\alpha_{i,k}}\}.$$

Enfin, on note

$$\pi u = \{\pi_1 u, \pi_2 u, \ldots, \pi_n u\} \tag{3.1}$$

Si l'on désigne $\sum_{i=1}^{n} \sum_{j=0}^{k-1} \alpha_{i,j} = N$, alors $\pi u \in \mathbf{R}^N$. Par conséquent, dans le cas présent $Z = \mathbf{R}^N$ et l'application linéaire, continue et surjective $\pi : X \to Z$ est donnée par la formule (3.1).

Choisissons une application univoque $T : Y \to Y^*$. Soit $A_i : (a, b) \times \mathbf{R}^{k+1} \to \mathbf{R}$ des fonctions données qui satisfont aux conditions C_1, C_2, C_3 et C_4 :

C_1. pour presque chaque $x \in (a, b)$ fixé, A_i, comme fonction de $\xi = (\xi_0, \xi_1, \ldots, \xi_k) \in \mathbf{R}^{k+1}$, soit continue. Pour chaque $\xi \in \mathbf{R}^{k+1}$ fixé, A_i comme fonction de x, soit mesurable,

C_2. il existe un nombre entier $p > 1$ de sorte qu'on ait une constante $C > 0$ et la fonction $a_i \in L^q(a, b)$, où $p^{-1} + q^{-1} = 1$, et pour lesquelles ait lieu l'inégalité

$$|A_i(x, \xi)| \leqslant a_i(x) + C|\xi|^{p-1}, \qquad i = \overline{0, k},$$

quel que soit $\xi \in \mathbf{R}^{k+1}$ et pour presque chaque $x \in (a, b)$,

C_3. il y ait l'inégalité

$$\sum_{i=0}^{k} [A_i(x, \xi) - A_i(x, \xi')](\xi_i - \xi_i') \geqslant 0, \qquad \forall \xi, \xi' \in \mathbf{R}^{k+1}.$$

C_4. il existe deux constantes $C_0 > 0$ et $C_1 \in$ R telles qu'on ait

$$\sum_{i=0}^{k} A_i(x, \xi)\xi_i \geqslant C_0|\xi|^p - C_1, \qquad \forall \xi \in \mathbf{R}^{k+1}.$$

Pour chaque $v = (v_0, v_1, \ldots, v_k) \in Y$ on notera $A_i(x, v) = A_i(x, v_0, v_1, \ldots, v_k)$ et

$$Tv = \{A_0(x, v), A_1(x, v), \ldots, A_k(x, v)\}. \qquad (3.2).$$

THÉORÈME 3.2. *Si les fonctions A_i, satifont aux conditions $C_1 - C_4$, alors l'égalité (3.2) définit une application de Y a Y^* qui est continue, bornée, monotone et telle que*

$$\lim_{\|v\| \to \infty} < Tv, v - y_0 > = +\infty$$

pour $\forall y_0 \in Y$.

Démonstration. Considérons l'application T étendue sur $[L^p(a, b)]^{k+1}$ et ayant des valeurs dans $[L^q(a, b)]^{k+1}$. Les conditions C_1 et C_2 assurent la continuité de T[5]. Et même plus que cela. On a

$$\|Tv\|^q = \int_a^b \sum_{i=0}^{k} |A_i(x, v)|^q dx \leqslant C_1 \left\{ \int_a^b [|a_i(x)|^q + \sum_{i=0}^{k} |v_i|^{(p-1)q}] dx \leqslant C_2 + C_3 \|v\|^p.\right.$$

C'est-à-dire qu'a lieu une délimitation de forme

$$\|Tv\| \leqslant C_4 + C_5\|v\|^{p-1},$$

ce qui montre que T est bornée.

La monotonie de T est une conséquence directe de la condition C_3.

Finalement il resulte de C_4 que

$$\langle Tv, v - y_0\rangle = \langle Tv, v\rangle - \langle Tv, y_0\rangle \geqslant \langle Tv, v\rangle - \|Tv\| \cdot \|y_0\| =$$

$$= \int_a^b \sum_{i=0}^k A_i(x, v) \cdot v_i dx - \|Tv\| \cdot \|y_0\| \geqslant C_0 \int_a^b \sum_{i=0}^k |v_i|^p dx - C_1(b-a) -$$

$$- \|Tv\| \cdot \|y_0\| \geqslant C_0\|v\|^p - C_2(b-a) - C_4 - C_5\|v\|^{p-1},$$

ce qui montre que $\lim\limits_{\|v\|\to+\infty} \langle Tv, v - y_0\rangle = +\infty$. Q.E.D.

Ce théorème prouve que les conditions du théorème 1.1 et même celles du théorème 1.3 sont valables. Cela revient à dire que dans le cas présent l'existence et même l'unicité de (π, L, T)-spline interpolateur sont assurées.

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GENERAREA ELEMENTELOR SPLINE CU AJUTORUL APLICAŢIILOR MONOTONE

(Rezumat)

În prima parte a lucrării se stabileşte o teoremă de existenţă şi o teoremă de unicitate a elementelor spline în spaţii Banach, în cadrul aplicaţiilor monotone. Pe urmă se arată că teoremele stabilite, în cazul funcţionalelor convexe, conduc la existenţa, respectiv unicitatea punctelor de minim ale funcţionalelor considerate pe anumite mulţimi convexe. În sfîrşit, se arată că anumiţi operatori diferenţiali liniari sau neliniari satisfac condiţiile teoremelor de existenţă şi de unicitate.

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ÉQUATIONS A QUATRE VARIABLES REPRÉSENTABLES PAR UN NOMOGRAMME COMPOSÉ AVEC DEUX ÉCHELLES PROJECTIVES SUR DES CONIQUES

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Étant donné deux échelles projectives chacune d'elle située sur une conique et une droite D, on se propose de trouver une équation à quatre variables $F(k_1, k_2,k_3,k_4) = 0$ qui soit résolue au moyen d'un nomogramme composé de la structure suivante:

soient k_1, k_2 les variables des échelles qui se trouvent sur la première conique, et k_3, k_4 celles des échelles trouvées sur la deuxième. Étant connues trois des variables k_1,k_2,k_3,k_4 (par exemple k_1,k_2,k_3) on déterminera la quatrième variable de la manière suivante: la droite qui unit les points de cotes k_1 et k_2 situés sur la première conique rencontre la droite D en un point. Si l'on unit ce point d'intersection au point de cote k_3 on obtient une droite qui rencontre la deuxième conique la seconde fois dans le point de cote k_4.

Nous construisons les échelles projectives sur les deux coniques, comme il suit [1]. Soit

$$v^2 - uw = 0 \qquad v'^2 - u'w' = 0,$$

où

$$\begin{aligned} u &= a_1x + b_1y + c_1 & u' &= a_1'x + b_1'y + c_1' \\ v &= a_2x + b_2y + c_2 & v' &= a_2'x + b_2'y + c_2' \\ w &= a_3x + b_3y + c_3 & w' &= a_3'x + b_3'y + c_3'. \end{aligned} \tag{1}$$

les deux coniques qui portent chacune d'elles les échelles projectives construites selon les formules

$$k = -\frac{v}{u} \qquad k' = -\frac{v'}{u'}.$$

On démontre également [1] que si une droite qui passe par le point des coordonnées (x_0, y_0) rencontre les coniques dans les points de cotes k_1, k_2 respectivement k_3, k_4, on a les relations suivantes entre ces variables

$$u_0k_1k_2 + v_0(k_1 + k_2) + w_0 = 0 \qquad u_0'k_3k_4 + v_0'(k_3 + k_4) + w_0' = 0 \tag{2}$$

où $u_0,v_0,w_0,u_0',v_0',w_0'$ sont les expressions linéaires (1) dans lesquelles on a mis $x = x_0$, $y = y_0$.

Supposons maintenant que le point (x_0, y_0) appartient à la droite D. En exprimant cette condition analytiquement nous en obtenons la relation correspondante entre les variables k_i. Soit $Ax + By + C = 0$ l'équation de la droite D. Il en résulte

$$\begin{array}{lll} Ax_0 + & By_0 + & C = 0 \\ (a_1p + a_2s + a_3)x_0 + & (b_1p + b_2s + b_3)y_0 + & c_1p + c_2s + c_3 = 0 \\ (a'_1p' + a'_2s' + a'_3)x_0 + & (b'_1p' + b'_2s' + b'_3)y_0 + & c'_1p' + c'_2s' + c'_3 = 0 \end{array} \qquad (3)$$

où l'on obtient les deux dernières équations des équations (2) en les ordonnant d'après x_0, y_0 et où l'on introduit pour simplifier l'écriture les notations

$$p = k_1k_2 \qquad s = k_1 + k_2 \qquad p' = k_3k_4 \qquad s' = k_3 + k_4.$$

En écrivant la compatibilité du système (3) on obtient la relation cherchée entre les k_i, qui se résout à l'aide d'un nomogramme du type indiqué

$$\begin{vmatrix} A & B & C \\ a_1p + a_2s + a_3 & b_1p + b_2s + b_3 & c_1p + c_2s + c_3 \\ a'_1p' + a'_2s' + a'_3 & b'_1p' + b'_2s' + b'_3 & c'_1p' + c'_2s' + c'_3 \end{vmatrix} = 0.$$

En développant le déterminant du premier membre de cette formule, on obtient :

$$\begin{aligned} &(\bar{N}\bar{r}_1\bar{r}'_1)pp' + (\bar{N}\bar{r}_2\bar{r}'_1)sp' + (\bar{N}\bar{r}_3\bar{r}'_1)p' + \\ &+(\bar{N}\bar{r}_1\bar{r}'_2)ps' + (\bar{N}\bar{r}_2\bar{r}'_2)ss' + (\bar{N}\bar{r}_3\bar{r}'_2)s' + \\ &+ (\bar{N}\bar{r}_1\bar{r}'_3)p + (\bar{N}\bar{r}_2\bar{r}'_3)s + (\bar{N}\bar{r}_3\bar{r}'_3) = 0 \end{aligned} \qquad (4)$$

où $\bar{N}, \bar{r}_i, \bar{r}'_j$ $(i, j = 1, 2, 3)$ sont des vecteurs avec des composantes scalaires (A, B, C), (a_i, b_i, c_i) respectivement (a'_j, b'_j, c'_j) et $(\bar{N}\bar{r}_i\bar{r}'_j)$ représente le produit mixte des vecteurs entre parenthèses. Par conséquent le nomogramme indiqué résout l'équation (4).

Inversement, considérons une équation présentant la même structure que l'équation (4)

$$A_1pp' + A_2sp' + A_3p' + B_1ps' + B_2ss' + B_3s' + C_1p + C_2s + C_3 = 0 \qquad (5)$$

et construisons un nomogramme du type indiqué qui peut résoudre l'équation (5). Cela signifie qu'il faut déterminer une droite $Ax + By + C = 0$ et les expressions linéaires (1) de sorte que l'équation donné (5) devienne identique à (4). En identifiant les deux équations on arrive au système suivant

$$\begin{array}{lll} (\bar{N}\bar{r}_1\bar{r}'_1) = A_1 & (\bar{N}\bar{r}_2\bar{r}'_1) = A_2 & (\bar{N}\bar{r}_3\bar{r}'_1) = A_3 \\ (\bar{N}\bar{r}_1\bar{r}'_2) = B_1 & (\bar{N}\bar{r}_2\bar{r}'_2) = B_2 & (\bar{N}\bar{r}_3\bar{r}'_2) = B_3 \\ (\bar{N}\bar{r}_1\bar{r}'_3) = C_1 & (\bar{N}\bar{r}_2\bar{r}'_3) = C_2 & (\bar{N}\bar{r}_3\bar{r}'_3) = C_3. \end{array} \qquad (6)$$

Les nombres A_i, B_i, C_i $(i = 1, 2, 3)$ étant donnés, les vecteurs $\bar{N}, \bar{r}_i, \bar{r}'_j (i, j = 1, 2, 3)$ qui déterminent le nomogramme du type indiqué qui correspond à l'équation (5) doit vérifier ce système d'équations. Par conséquent, si l'on donne une équation de la forme (5) et si l'on veut construire un nomogramme du type indiqué il faut résoudre le système (6) par rapport aux inconnues $\bar{N}, \bar{r}_i, \bar{r}'_j$.

Mais un autre problème survient : toute équation (5) peut-elle être représentée avec un nomogramme du type indiqué ou ce n'est que qulques-unes qui puissent l'être. Pour répondre à cette question il faut voir si le système (6) a des solutions ou non.

Supposons d'abord que le système (6) a une solution de telle sorte que les vecteurs $\bar{r}_1, \bar{r}_2, \bar{r}_3$ respectivement $\bar{r}'_1, \bar{r}'_2, \bar{r}'_3$ soit linéairement indépendants. Dans ce cas en introduisant les notations

$$\bar{a} = \bar{N} \times \bar{r}_1 \qquad \bar{b} = \bar{N} \times \bar{r}_2 \qquad \bar{c} = \bar{N} \times \bar{r}_3 \tag{7}$$

on peut écrire le système (6) comme il s'ensuit:

$$\begin{array}{lll} \bar{a} \cdot \bar{r}'_1 = A_1 & \bar{b} \cdot \bar{r}'_1 = A_2 & \bar{c} \cdot \bar{r}'_1 = A_3 \\ \bar{a} \cdot \bar{r}'_2 = B_1 & \bar{b} \cdot \bar{r}'_2 = B_2 & \bar{c} \cdot \bar{r}'_2 = B_3 \\ \bar{a} \cdot \bar{r}'_3 = C_1 & \bar{b} \cdot \bar{r}'_3 = C_2 & \bar{c} \cdot \bar{r}'_3 = C_3. \end{array} \tag{8}$$

On peut observer que les vecteurs $\bar{a}, \bar{b}, \bar{c}$ étant perpendiculaires tous sur un seul et même vecteur $\bar{N}$, ils seront coplanaires. Il y a donc une relation linéaire $\lambda\bar{a} + \mu\bar{b} + \nu\bar{c} = 0$ de sorte que les nombres λ, μ, ν ne soient pas tous nuls.

Les équations de la première ligne du système (8) peuvent être considérées comme un système pour les inconnus a'_1, b'_1, c'_1. D'une manière analogue les équations de la deuxième respectivement de la troisième ligne peuvent être considérées comme un système pour les inconnus a'_2, b'_2, c'_2 respectivement a'_3, b'_3, c'_3. A cause de la symétrie nous pouvons étudier un de ces systèmes, par exemple le premier

$$\bar{a} \cdot \bar{r}'_1 = A_1 \qquad \bar{b} \cdot \bar{r}'_1 = A_2 \qquad \bar{c} \cdot \bar{r}'_1 = A_3. \tag{9}$$

Ce système conformément à l'hypothèse est compatible. Vu qu'il y a une relation linéaire entre les premiers membres des équations (9), la même relation doit exister entre les deuxièmes membres des celles-ci

$$\lambda A_1 + \mu A_2 + \nu A_3 = 0.$$

D'une manière analogue on obtient

$$\lambda B_1 + \mu B_2 + \nu B_3 = 0$$
$$\lambda C_1 + \mu C_2 + \nu C_3 = 0.$$

Le système formé des trois dernières équations a une solution non triviale, donc

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = 0. \tag{10}$$

On tire ainsi la conclusion suivante : si l'équation (5) peut être représentée avec un nomogramme du type indiqué, les coefficients de cette équation doivent vérifier la condition (10).

Inversement, on suppose que la condition (10) est vérifiée et l'on démontre, que dans cette hypothèse l'équation (5) est représentable par un nomogramme du type indiqué. On démontrera cette affirmation en construisant effectivement les échelles des nomogrammes. C'est pourquoi il faudra résoudre le système (6).

Considérons le système

$$\lambda A_1 + \mu A_2 + \nu A_3 = 0$$
$$\lambda B_1 + \mu B_2 + \nu B_3 = 0$$
$$\lambda C_1 + \mu C_2 + \nu C_3 = 0.$$

Celui-ci — conformément à la condition (10) — aura une solution nontriviale. On peut supposer que le rang de la matrice des coefficients des inconnus du système considéré est 2, car contrairement l'équation (5) serait réduite à deux équations qui contiendraient chacune deux variables. Ce cas ne présente aucun intérêt. Pour fixer les idées on suppose que les deux premières équations sont les équations principales du système. Dans ce cas on aura

$$\lambda = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \quad \mu = \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix} \quad \nu = \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \tag{11}$$

faisant abstraction d'un facteur de proportionnalité.

Soient par la suite $\bar{r}_1, \bar{r}_2, \bar{r}_3$ trois vecteurs arbitraires, linéairement indépendants et N un vecteur inconnu de composantes scalaires A, B, C. Ces composantes seront déterminées de telle manière qu'entre les vecteurs complanaires (7) il y a la relation

$$\lambda\bar{a} + \mu\bar{b} + \nu\bar{c} = 0 \tag{12}$$

où λ, μ, ν sont donnés par (11). (On a vu plus haut que cette condition est nécessaire pour la compatibilité du système (9)). On calcule les composantes scalaires des vecteurs a, b, c

$$\bar{a} = \bar{N} \times \bar{r}_1 : \begin{vmatrix} B & C \\ b_1 & c_1 \end{vmatrix}, \quad \begin{vmatrix} C & A \\ c_1 & a_1 \end{vmatrix}, \quad \begin{vmatrix} A & B \\ a_1 & b_1 \end{vmatrix}$$

$$\bar{b} = \bar{N} \times \bar{r}_2 : \begin{vmatrix} B & C \\ b_2 & c_2 \end{vmatrix}, \quad \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix}, \quad \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix}$$

$$\bar{c} = \bar{N} \times \bar{r}_3 \begin{vmatrix} B & C \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} C & A \\ c_3 & a_3 \end{vmatrix}, \quad \begin{vmatrix} A & B \\ a_3 & b_3 \end{vmatrix}.$$

Au moyen de ces composantes et à l'aide des formules (10) on écrit les relations scalaires équivalentes à la relation (12)

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ c_1 & c_2 & c_3 \end{vmatrix} B - \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ b_1 & b_2 & b_3 \end{vmatrix} C = 0$$

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ a_1 & a_2 & a_3 \end{vmatrix} C - \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ c_1 & c_2 & c_3 \end{vmatrix} A = 0$$

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ b_1 & b_2 & b_3 \end{vmatrix} A - \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ a_1 & a_2 & a_3 \end{vmatrix} B = 0.$$

Ce système linéaire, homogène par rapport aux inconnus A, B, C a une solution nontriviale, puisque le déterminant du système est un déterminant antisymétrique de l'ordre 3. Ce système fournit donc les composantes scalaires du vecteur $\bar{N}$

$$A = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ a_1 & a_2 & a_3 \end{vmatrix}, \ B = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \ C = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad (13)$$

où l'on a fait de nouveau abstraction d'un facteur de proportionnalité. On a déterminé ainsi le vecteur $\bar{N}$ à l'aide duquel on peut former les vecteurs $\bar{a}, \bar{b}, \bar{c}$ et avec ceux-ci le système (9). On doit démontrer que ce système est compatible et présente une simple infinité de solutions.

Le rang de la matrice du système (9) est deux. En effet, le déterminant du système étant $(\bar{a}\ \bar{b}\ \bar{c})$ celui-ci est zéro, puisque $\bar{a}, \bar{b}, \bar{c}$ sont linéairement dépendants. Si tous les mineurs de l'ordre deux de ce déterminant étaient nuls, cela signifierait que tous les vecteurs $\bar{a} \times \bar{b}$, $\bar{b} \times \bar{c}$, $\bar{c} \times \bar{a}$ seraient nuls, donc tous les vecteurs seraient colinéaires. Cela est impossible, puisque les vecteurs $\bar{r}_i$ ne sont pas linéairement dépendants.

Conformément à la construction, entre les vecteurs $\bar{a}, \bar{b}, \bar{c}$ il existe la même relation linéaire qu'entre les triplets A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3. Ainsi la matrice élargie du système aura le même rang que la matrice du système, ce qui signifie que notre système a une simple infinité de solutions.

Ainsi l'on obtient le procédé suivant pour la détermination des échelles du nomogramme pour l'équation (5) qui vérifiera la condition (10) : On choisit trois vecteurs arbitraires, linéairement indépendants $\bar{r}_1, \bar{r}_2, \bar{r}_3$, ensuite on calcule les composantes du vecteur $\bar{N}$, avec les formules (13). Par la suite on écrit les équations du système (6) d'où l'on détermine les vecteurs $\bar{r}'_1, \bar{r}'_2, \bar{r}'_3$.

On considère l'exemple suivant : construisons un nomogramme du type indiqué pour l'équation

$$p's - ps' + s' - s = 0.$$

La condition (10) est vérifiée, parce qu'on a

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 0.$$

Appliquons le procédé décrit plus haut. On choisira les trois droites suivantes

$$u = -\sqrt{q^2 - r^2}\,x + ry = 0$$
$$v = qy - q^2 + r^2 = 0$$
$$w = \sqrt{q^2 - r^2}\,x + ry = 0$$

où $q > r$. Cela signifie que le support de la première échelle projective est la conique $x^2 + y^2 - 2qy + q^2 - r^2 = 0$, c'est-à-dire un cercle. Ces droites déterminent les vecteurs $\bar{r}_1\left(-\sqrt{q^2 - r^2}, r, 0\right)$, $\bar{r}_2(0, q, r^2 - q^2)$, $\bar{r}_3\left(\sqrt{q^2 - r^2}, r, 0\right)$. On calcule les composantes du vecteur $\bar{N}$ à l'aide des formules (13)

$$A = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ -\sqrt{q^2 - r^2} & 0 & \sqrt{q^2 - r^2} \end{vmatrix} = 0, \quad B = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ r & q & r \end{vmatrix} = 2r,$$

$$C = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & r^2 - q^2 & 0 \end{vmatrix} = 0.$$

Prenant en considération que A, B, C sont déterminés faisant abstraction d'un facteur de proportionnalité on écrit $A = C = 0$, $B = 1$. (Cela signifie que la droite D sera l'axe Ox). On écrit par la suite le système (6)

$$\begin{vmatrix} a_1' & b_1' & c_1' \\ 0 & 1 & 0 \\ -\sqrt{q^2 - r^2} & r & 0 \end{vmatrix} = 0 \quad \begin{vmatrix} a_1' & b_1' & c_1' \\ 0 & 1 & 0 \\ 0 & q & r^2 - q^2 \end{vmatrix} = -1$$

$$\begin{vmatrix} a_2' & b_2' & c_2' \\ 0 & 1 & 0 \\ -\sqrt{q^2 - r^2} & r & 0 \end{vmatrix} = 1 \quad \begin{vmatrix} a_2' & b_2' & c_2' \\ 0 & 1 & 0 \\ 0 & q & r^2 - q^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_3' & b_3' & c_3' \\ 0 & 1 & 0 \\ -\sqrt{q^2 - r^2} & r & 0 \end{vmatrix} = 0 \quad \begin{vmatrix} a_3' & b_3' & c_3' \\ 0 & 1 & 0 \\ 0 & q & r^2 - q^2 \end{vmatrix} = 1.$$

(Les équations de la dernière colonne n'ont pas été écrites, puisqu'elles sont des conséquences des deux premières colonnes). En résolvant le système écrit on obtient

$$a_1' = \frac{1}{q^2 - r^2} \qquad c_1' = 0$$
$$a_2' = 0 \qquad c_2' = \frac{1}{\sqrt{q^2 - r^2}}$$
$$a_3' = \frac{-1}{q^2 - r^2} \qquad c_3' = 0$$

b'_1, b'_2, b'_3 restant indéterminés. On écrit les équations des droites u', v', w', ensuite celle de la conique qui est le support de l'échelle projective

$$u' = \frac{x}{q^2 - r^2} + b'_1 y = 0$$

$$v' = b'_2 y + \frac{1}{\sqrt{q^2 - r^2}} = 0$$

$$w' = \frac{-x}{q^2 - r^2} + b'_3 y = 0 \tag{14}$$

$$v'^2 - u'w' = \left(b'_2 y + \frac{1}{\sqrt{q^2 - r^2}}\right)^2 - \left(\frac{-x}{q^2 - r^2} + b'_3 y\right)\left(\frac{x}{q^2 - r^2} + b'_1 y\right) = 0.$$

Essayons maintenant de déterminer les coefficients b'_1, b'_2, b'_3 de sonte que la deuxième conique soit aussi un cercle. Pour cela il est nécessaire et suffisant que b'_1, b'_2, b'_3, vérifient le système

$$b_2'^2 - b'_1 b'_3 = \frac{1}{(q^2 - r^2)^2} \qquad b'_1 = b'_3$$

Posons $b'_2 = \frac{q}{(q^2 - r^2)^{3/2}}$. Il en résulte

$$b'_1 = b'_3 = \frac{r}{(q^2 - r^2)^{3/2}}$$

Remplaçant ces valeurs en (14) on obtient, après avoir multiplié ces équations par $(q^2 - r^2)^{3/2}$

$$u' = x\sqrt{q^2 - r^2} + ry \quad v' = qy + (q^2 - r^2) \quad w' = -x\sqrt{q^2 - r^2} + ry$$

$$v'^2 - u'w' = (x^2 + y^2)(q^2 - r^2) + 2q(q^2 - r^2)y + (q^2 - r^2)^2 = 0.$$

Donc la deuxième conique sera aussi un cercle, symétriquement placé par rapport au premier, relatif à l'axe Ox.

(Manuscrit reçu la 20 mars 1980)

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ECUAŢII CU PATRU VARIABILE REPREZENTABILE CU O NOMOGRAMĂ COMPUSĂ CU DOUĂ SCĂRI PROIECTIVE SITUATE PE CONICE

(Rezumat)

Pentru ecuaţia (5) coeficienţii căreia verifică condiţia (10) se construieşte o nomogramă compusă cu două scări proiective situate pe conice. Se consideră şi un exemplu.

RECENZII

C a r l d e B o o r, **A Practical Guide to Spline** (Applied Mathematical Sciences 27). Springer Verlag New-York — Heidelberg — Berlin, I.S.B.N., 1978, XXIV + 392 pp.

Cartea conţine o retrospectivă a autorului asupra teoriei şi aplicaţiilor funcţiilor spline, îndeosebi din punctul de vedere al avantajelor acestora în practica calculului. Pe lîngă proprietăţile fundamentale ale funcţiilor spline polinomiale centrul de greutate a cărţii îl constituie programele Fortrain scrise efectiv de către autor pentru numeroase aplicaţii practice ale funcţiilor spline.

Autorul însuşi avînd o contribuţie însemnată la dezvoltarea teoriei actuale a funcţiilor spline a selectat din această vastă problematică numai acele aplicaţii a căror tratare prin spline prezintă avantaje certe.

Cartea conţine 16 capitole, fiecare fiind însoţit de programele Fortrain corespunzătoare.

Capitolele 1—2 conţin o scurtă recapitulare a materialului de analiză numerică necesare capitolelor ce urmează. Astfel, este prezentată pe scurt teoria diferenţelor divizate şi teoria interpolării polinomiale.

Următoarele patru capitole, urmînd oarecum dezvoltarea istorică a problemelor, conţin interpolarea şi aproximarea funcţiilor prin funcţii segmentar polinomiale, iar capitolele 7 şi 8 tratează aspectul calculativ al funcţiilor segmentar polinomiale de interpolare.

Noţiunea propriu-zisă de funcţie spline este definită în capitolul 9, ca o combinaţie liniară de funcţii B-spline. Capitolele 10 şi 11 cuprind un studiu detaliat al funcţiilor B-spline.

Următoarele capitole cuprind toate diferite aplicaţii ale funcţiilor spline, bazate îndeosebi pe proprietăţile esenţiale ale acestor funcţii de bază numite B-spline. Astfel, se studiază din punct de vedere practic interpolarea şi aproximarea funcţiilor prin funcţii spline, rezolvarea aproximativă a unor clase de ecuaţii diferenţiale cu condiţii date cu ajutorul funcţiilor spline polinomiale, aproximarea curbelor plane etc. Ultimul capitol se ocupă cu extinderea şi generalizarea noţiunii de spline la cazul mai multor variabile.

Fiecare capitol este înzestrat cu probleme interesante şi material auxiliar ilustrativ, care oferă o posibilitate cititorului de a sesiza avantajele utilizării funcţiilor spline în analiza numerică.

Prin conţinutul său, prin varietatea problemelor tratate şi mai ales prin rezolvarea efectivă a acestora (incluzînd şi programele Fortrain), cartea este foarte utilă matematicienilor, fizicienilor, inginerilor şi tuturor celor ce utilizează funcţiile spline în preocupările lor.

GH. MICULA

C h r u s t o p h e r T. H. B a k e r, **The Numerical Treatment of Integral Equations**, Oxford Univ. Press, Oxford, 1977, XIV + 1034 pp.

Cartea conţine o tratare aproape exhaustivă a metodelor numerice pentru ecuaţiile integrale, acoperind toate tipurile importante de ecuaţii integrale cum sînt ecuaţiile integrale de tip Fredholm şi de tip Volterra, de speţa întîia şi de speţa a doua, ecuaţiile integrale singulare de tip Cauchy, ecuaţii integrale neliniare de tip Fredholm etc.

Volumul deosebit de mare al cărţii face posibilă discutarea amănunţită a celor mai multe metode numerice pentru ecuaţiile integrale, atît din punct de vedere al teoriei acestora, cît mai ales aplicarea lor la numeroase exemple concrete. Pe lîngă algoritmul metodelor, autorul studiază pe un spaţiu larg estimarea erorii şi convergenţa acestor metode.

Capitolul 1 conţine o introducere generală în teoria ecuaţiilor integrale, clasificarea lor, rezultatele principale teoretice şi numeroase exemple interesante. El conţine de asemenea cîteva noţiuni şi rezultate importante din analiza funcţională.

Capitolul 2 trece în revistă cunoştinţele de analiză numerică ce vor fi utilizate în următoarele capitole, cum ar fi interpolarea, aproximarea funcţiilor, integrarea numerică şi tratarea numerică a algebrei lineare.

Capitolul 3 cuprinde problema valorilor proprii ale operatorilor integrali Fredholm cu nucleu neted. În primele 12 paragrafe sînt prezentate numeroase metode numerice cum ar fi metoda cuadraturilor, metoda proiecţiilor (Ritz-Galerkin, colocaţie), metoda nucleului degenerat, metoda celor mici pătrate, metoda Rayleigh-Ritz etc. În paragrafele următoare se studiază problema convergenţei şi estimarea erorii acestor metode.

Mult spaţiu acordă autorul studiului operatorilor integrali hermitici, precum şi exemplelor ce ilustrează metodele aplicate operatorilor integrali definiţi cu ajutorul funcţiei Green, precum şi metodele de extrapolare bazate pe formula de cuadratură a lui Gregory.

Capitolul 4 continuă tematica din capitolul 3, prezentată însă pentru ecuaţii integrale de tip Fredholm neomogene cu nucleu neted. Toate metodele importante sînt prezentate şi ilustrate cu exemple, studiindu-se algoritmii şi convergenţa acestor metode, folosind cît mai puţin posibil analiza funcţională. Un aparat matematic mai abstract este utilizat însă în prezentarea teoriei lui Anselone asupra aproximării operatorilor colectiv compacţi.

Capitolul 5 conţine un material bogat asupra unor clase de ecuaţii integrale ce nu au fost cuprinse în capitolele 3 şi 4, majoritatea fiind ecuaţii integrale cu nucleu cu singularităţi cum ar fi: ecuaţii integrale singulare de tip Cauchy, ecuaţii Wiener-Hopf, ecuaţii integrale Fredholm de speţă a întîia, ecuaţii integrale Fredholm neliniare etc.

Capitolul 6, ultimul capitol, este consacrat metodelor numerice pentru ecuaţiile integrale de tip Volterra. Autorul dă un ansamblu complet de metode numerice pentru ecuaţii Volterra de speţa a doua şi de speţa a întîia, incluzînd şi ecuaţiile integrale cu nucleu cu singularităţi. Se studiază în profunzime şi chestiunea stabilităţii metodelor, iar exemplele concrete oferă posibilitate comparării metodelor.

Cartea conţine, pe lîngă o sinteză a variatelor metode numerice clasice şi moderne pentru ecuaţii integrale, şi un număr însemnat de rezultate originale ale autorului şi o bibliografie foarte bogată, ceea ce o recomandă tuturor celor interesaţi în acest domeniu important al matematicii.

GH. MICULA

F. Singer. **Programmierung mit COBOL**, 3., überarbeitete Auflage, B. G. Teubner Verlag, Stuttgart, 1978. — Teubner Studienskripten 55: Datenverarbeitung.

Cartea, care în 1978 a ajuns la ediţia a treiai prezintă principalele structuri ale limbajulu, de programare COBOL, exemplificate cu programe. Cele şapte capitole se ocupă cu următoarele: 1. Bazele prelucrării electronice a datelor; 2. Introducere în limbajul de programare COBOL; 3. Programare în COBOL — partea I — structuri simple (descrierea diviziunilor şi a principalelor instrucţiuni); 4. Tratarea datelor (structuri de date, fişiere, intrare-ieşire); 5. Programare în COBOL — partea II — structuri complexe (expresii aritmetice, tabele, condiţii etc.); 6. ANSI COBOL (Acest capitol prezintă printre altele problemele legate de sortare, editare de rapoarte, segmentare, biblioteci); 7. Anexă — cuprinzînd istoricul limbajului COBOL, cuvintele rezervate şi bibliografia.

Tratarea este sistematică şi îngrijită, permite o asimilare treptată a cunoştinţelor prezentate.

Z. KÁSA

M. M. Richter, **Logikkalkule**, B. G. Teubner, Stuttgart, 1978.

În cele peste 230 de pagini, cartea tratează aspectele mai importante ale logicii (clasice şi intuiţioniste), a propoziţiilor şi predicatelor de ordinul I, atingînd şi chestiuni din cercetarea actuală în acest domeniu.

Autorul are în vedere următoarele trei aspecte: formalizarea noţiunii de adevăr, formalizarea noţiunii de demonstrabilitate şi considerente privind studiul demonstraţiei. Pentru elucidarea acestora, sînt prezentate şi utilizate chestiuni de bază din teoria algebrelor universale, congruenţe, algebre Boole precum şi calculele de tip Hilbert sau calculul secvenţial al lui Gentzen.

Deşi conţine o mare cantitate de informaţie-într-un text sistematic concentrat, cartea posedă un mare grad de independenţă, în sensul că nu pretinde cititorului (cu o pregătire matematică preliminară) decît puţine noţiuni colaterale.

Celor dornici să aprofundeze anumite probleme dintre cele tratate le stă la dispoziţie o bogată listă bibliografică (compusă din 51 titluri).

Această carte poate fi folositoare tuturor celor interesaţi atît în studiul cît şi în aplicarea la nivel actual a logicii matematice.

N. BOTH

W. J. Paul, **Komplexitätstheorie**, B. G. Teubner, Stuttgart, 1978, 248 pag.

Teoria complexităţii este acea ramură, relativ tînără, a informaticii teoretice care studiază aspectele cantitative ale proceselor de calcul. Prezenta carte îşi propune să prezinte cititorului problemele actuale ale acestei teorii. Pornind de la teoria funcţiilor recursive, care studiază posibilităţile de calcul ale funcţiilor cu ajutorul algoritmilor, autorul abordează problema volumului de calcul necesar evaluării funcţiilor calculabile cu maşini Turing. Sînt prezentate în continuare toate aspectele esenţiale şi actuale, inclusiv problemele deschise, ale acestei teorii.

Cartea are următoarele şapte capitole: 1. Calculabilitate; 2. Funcţii necalculabile; 3. Complexitate realistă; 4. Complexitatea maşinilor Turing; 5. Nedeterminism; 6. Probleme complete; 7. Teoria abstractă a complexităţii.

Materialul prezentat în această carte reprezintă conţinutul cursului de un semestru pe care autorul l-a ţinut în anul 1976/77 la Universitatea din Bielefeld. Cartea fiind destinată în primul rînd studenţilor, materialul este redactat într-o formă accesibilă, este însoţit de numeroase exemple şi nu cere din partea cititorului cunoştinţe prealabile în acest domeniu.

C. TARŢIA

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