Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 3, 331–350

Inverse theorem for the iterates of modified Bernstein type polynomials

T.A.K. Sinha, P.N. Agrawal and K.K. Singh

Abstract. Gupta and Maheshwari [12] introduced a new sequence of Durrmeyer type linear positive operators P_n to approximate p^{th} Lebesgue integrable functions on $[0,1]$. It is observed that these operators are saturated with $O(n^{-1})$. In order to improve this slow rate of convergence, following Agrawal et al [2], we [3] applied the technique of an iterative combination to the above operators P_n and estimated the error in the L_p- approximation in terms of the higher order integral modulus of smoothness using some properties of the Steklov mean. The present paper is in continuation of this work. Here we have discussed the corresponding inverse result for the above iterative combination $T_{n,k}$ of the operators P_n.

Mathematics Subject Classification (2010): 41A25, 41A27, 41A36.

Keywords: Linear positive operators, iterative combination, integral modulus of smoothness, Steklov mean, inverse theorem.

1. Introduction

Motivated by the definition of Phillips operators (cf. [1] and [15]), Gupta and Maheshwari [12] proposed modified Bernstein type polynomials P_n to approximate functions in $L_p[0,1]$ as follows:

For $f \in L_p[0,1], 1 \le p < \infty$,

$$P_n(f;x) = \int_0^1 W_n(x,t)f(t)\,dt,\ x \in [0,1],$$

where $W_n(x,t) = n\sum_{\nu=1}^{n} p_{n,\nu}(x)p_{n-1,\nu-1}(t) + (1-x)^n\delta(t)$,

$$p_{n,\nu}(t) = \binom{n}{\nu} t^{\nu}(1-t)^{n-\nu},\ \ 0 \le t \le 1,$$

and $\delta(t)$ being the Dirac-delta function, is the kernel of the operators P_n.

Since the order of approximation by the operators P_n is, at best, $O(n^{-1})$, however smooth the function may be, following [3], the iterative combination $T_{n,k} : L_p[0,1] \to C^{\infty}[0,1]$ of these operators is defined as

$$T_{n,k}(f;x) = \left(I - (I - P_n)^k\right)(f;x) = \sum_{m=1}^{k} (-1)^{m+1} \binom{k}{m} P_n^m(f;x), \; k \in \mathbb{N},$$

where $P_n^0 \equiv I$ and $P_n^m \equiv P_n(P_n^{m-1})$ for $m \in \mathbb{N}$.

In order to improve the rate of convergence, Micchelli [16] introduced an iterative combination for Bernstein polynomials and obtained some direct and saturation results. Gonska and Zhou [11] showed that the iterative combinations can be regarded as iterated Boolean sums and obtained global direct and inverse results in the sup-norm. The iterated Boolean sums have also been studied by several other authors (e.g. [4],[8],[17],[18] and [21]) wherein they have obtained direct and saturation results. Ding and Cao [7] discussed direct and inverse theorems in the sup- norm for iterated Boolean sums of the multivariate Bernstein polynomials using the technique of K-functionals. Sinha et al [19] proved an inverse theorem in the L_p- norm for the Micchelli combination of Bernstein-Durrmeyer polynomials.

Gonska and Zhou [11] obtained the results in the sup- norm using the Ditzian Totik modulus of smoothness and $K-$ functional. Ding and Cao [7] also obtained the results in sup- norm using $K-$ functional. Sevy ([17] and [18]) considered the limits of the linear combinations of iterates of Bernstein and Durrmeyer polynomials in the sup- norm by keeping the degree n of the approximants as a constant while the order of iteration becomes infinite and showed that they converge to the Lagrange interpolation polynomial and the least square approximating polynomial on $[0,1]$ respectively. The more general results have been obtained in [21].

Motivated by the work of Sinha et al [19], Agrawal et al [3] considered the Micchelli combinations for the operator proposed by Gupta and Maheshwari [12] and obtained some direct results in L_p- norm. In the present paper, we continue the work done in [19] by proving a corresponding local inverse theorem in the L_p- norm.

The iterates are defined as

$$P_n^{m+1}(f;x) = \int_0^1 W_n(x,t) P_n^m(f;t)\, dt, \; x \in [0,1].$$

At every stage it uses the entire previous operator value. The analysis in L_p- case, therefore, differs from the study of operators in [10] and linear combinations of operators in [8]. The proof of the theorem is carried out by using the properties of Steklov means. Due to the presence of the Dirac- delta term in the kernel of these operators, the analysis of the proof is quite different. It uses the multinomial theorem, Hölder's inequality and the Fubini's theorem repeatedly.

Throughout the present paper, we assume that $I = [0,1]$, $I_j = [a_j, b_j], j = 1,2,$ where $0 < a_1 < a_2 < b_2 < b_1 < 1$ and by C we mean a positive constant not necessarily the same at each occurrence.

In [3], we obtained the following direct theorem:

Theorem 1.1. *Let $f \in L_p(I), p \geq 1$. Then, for sufficiently large values of n there holds*

$$\|T_{n,k}(f;x) - f(x)\|_{L_p(I_2)} \leq C\left(\omega_{2k}\Big(f, \frac{1}{\sqrt{n}}, p, I_1\Big) + n^{-k}\|f\|_{L_p(I)}\right),$$

where C is a constant independent of f and n.

Remark 1.2. From the above theorem, it follows that if $\omega_{2k}(f,\tau,p,I_1) = O(\tau^\alpha)$, as $\tau \to 0$ then $\|T_{n,k}(f;x) - f(x)\|_{L_p(I_2)} = O(n^{-\alpha/2})$, as $n \to \infty$, where $0 < \alpha < 2k$.

The aim of this paper is to characterize the class of functions for which

$$\|T_{n,k}(f;x) - f(x)\|_{L_p(I_2)} = O(n^{-\alpha/2}), \text{ as } n \to \infty, \text{ where } 0 < \alpha < 2k.$$

Thus, we prove the following theorem (*inverse theorem*):

Theorem 1.3. *Let $f \in L_p(I), p \geq 1$. Let $0 < \alpha < 2k$ and*

$$\|T_{n,k}(f;x) - f(x)\|_{L_p(I_1)} = O(n^{-\alpha/2}), \text{ as } n \to \infty.$$

Then, $\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha)$, as $\tau \to 0$.

Remark 1.4. We observe that without any loss of generality we may assume that $f(0) = 0$. To prove it, let $f_1(t) = f(t) - f(0)$. By definition,

$$T_{n,k}(f_1;x) = \sum_{m=1}^{k} (-1)^{m+1}\binom{k}{m} P_n^m(f_1;x).$$

Further, using linearity,

$$P_n^m(f_1;x) = P_n^m(f;x) - f(0)P_n^m(1;x) = P_n^m(f;x) - f(0).$$

This implies that $T_{n,k}(f_1;x) = T_{n,k}(f;x) - f(0)$. This entails that

$$T_{n,k}(f_1;x) - f_1(x) = T_{n,k}(f;x) - f(x),$$

where $f_1(0) = 0$.

Since $f(0) = 0$ (in view of the above remark), it follows that $P_n f(0) = 0$. Consequently, $P_n^m f(0) = 0, \forall m \in \mathbb{N}$.

2. Preliminaries

In this section, we mention some definitions and prove auxiliary results which we need in establishing our main theorem.

Lemma 2.1. *Let $r > 0$ and ν be an integer such that $0 \leq \nu \leq n$. Then for every ν there holds*

$$\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n} - t\right|^r dt = O\left(\frac{1}{n^{\frac{r}{2}+1}}\right), \text{ as } n \to \infty.$$

Proof. Let i be an integer such that $2i > r$. An application of Hölder's inequality in integral gives

$$\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^r dt \leq \left(\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^{2i} dt\right)^{\frac{r}{2i}}\left(\int_0^1 p_{n,\nu}(t)\,dt\right)^{1-\frac{r}{2i}}. \tag{2.1}$$

It follows that

$$\int_0^1 t^j p_{n,\nu}(t)\,dt = \binom{n}{\nu}B(\nu+j+1, n-\nu+1) = \frac{(\nu+1)(\nu+2)...(\nu+j)}{(n+1)(n+2)...(n+j+1)}.$$

Hence, by binomial expansion

$$\begin{aligned}\int_0^1 p_{n,\nu}(t)\left(\frac{\nu}{n}-t\right)^{2i} dt &= \sum_{j=0}^{2i}\binom{2i}{j}(-1)^j\left(\frac{\nu}{n}\right)^{2i-j}\frac{(\nu+1)(\nu+2)...(\nu+j)}{(n+1)(n+2)...(n+j+1)} \\ &= \frac{1}{(n+1)n^{2i}}\left\{\nu^{2i}-\binom{2i}{1}\nu^{2i-1}(\nu+1)\left(1+\frac{2}{n}\right)^{-1}\right. \\ &+\binom{2i}{2}\nu^{2i-2}(\nu+1)(\nu+2)\left(1+\frac{2}{n}\right)^{-1}\left(1+\frac{3}{n}\right)^{-1} \\ &\left.+...+(\nu+1)(\nu+2)...(\nu+2i)\prod_{s=2}^{2i+1}\left(1+\frac{s}{n}\right)^{-1}\right\}.\end{aligned} \tag{2.2}$$

Now,

$$\prod_{s=2}^{j+1}\left(1+\frac{s}{n}\right)^{-1} = 1+\frac{p_1(j)}{n}+\frac{p_2(j)}{n^2}+\frac{p_3(j)}{n^3}+..., \tag{2.3}$$

where $p_1(j)$ is a second degree polynomial in j, $p_2(j)$ is a fourth degree polynomial in j and so on.

Similarly,

$$(\nu+1)(\nu+2)...(\nu+j) = \nu^j + q_1(j)\nu^{j-1} + q_2(j)\nu^{j-2} + ... + j!, \tag{2.4}$$

where $q_1(j)$ is a second degree polynomial in j, $q_2(j)$ is a fourth degree polynomial in j and so on.

Thus from (2.2)- (2.4), we have

$$\int_0^1 p_{n,\nu}(t)\left(\frac{\nu}{n}-t\right)^{2i}dt$$

$$= \frac{1}{(n+1)n^{2i}}\left\{\sum_{j=0}^{2i}\binom{2i}{j}(-1)^j\nu^{2i-j}(\nu^j+q_1(j)\nu^{j-1}+q_2(j)\nu^{j-2}+...)\times\right.$$
$$\left.\left(1+\frac{p_1(j)}{n}+\frac{p_2(j)}{n^2}+...\right)\right\}$$
$$= \frac{1}{(n+1)n^{2i}}\left\{\sum_{j=0}^{2i}\binom{2i}{j}(-1)^j(\nu^{2i}+q_1(j)\nu^{2i-1}+q_2(j)\nu^{2i-2}+...)\times\right.$$
$$\left.\left(1+\frac{p_1(j)}{n}+\frac{p_2(j)}{n^2}+...\right)\right\}$$
$$= O\left(\frac{1}{n^{i+1}}\right), \text{ as } n\to\infty. \tag{2.5}$$

This holds for every ν, where $0\le\nu\le n$ and in view of the following identity:

$$\sum_{j=0}^{2i}(-1)^j\binom{2i}{j}j^m = \begin{cases} 0, & m=0,1,...,2i-1 \\ (2i)!, & m=2i. \end{cases}$$

Now, on combining (2.1), (2.5) and in view of $\int_0^1 p_{n,\nu}(t)dt = \frac{1}{n+1}$, we obtain

$$\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^r dt \le C\left(\frac{1}{n^{i+1}}\right)^{\frac{r}{2i}}\left(\frac{1}{n+1}\right)^{1-\frac{r}{2i}} = O\left(\frac{1}{n^{\frac{r}{2}+1}}\right). \qquad \square$$

For $m\in\mathbb{N}$, the m^{th} order moment for P_n is defined as

$$\mu_{n,m}(x) = P_n\left((t-x)^m;x\right).$$

Lemma 2.2. [2] *The elementary moments are* $\mu_{n,0}(x)=1, \mu_{n,1}(x)=\frac{(-x)}{(n+1)}$ *and for* $m\ge 1$ *there holds the recurrence relation*

$$(n+m+1)\mu_{n,m+1}(x) = x(1-x)\left\{\mu'_{n,m}(x)+2m\,\mu_{n,m-1}(x)\right\}+(m(1-2x)-x)\mu_{n,m}(x).$$

Consequently,

(i) $\mu_{n,m}(x)$ *is a polynomial in* x *of degree* m*;*

(ii) $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, *as* $n\to\infty$, *uniformly in* $x\in I$, *where* $[\beta]$ *is the integer part of* β.

Corollary 2.3. *There holds for* $r>0$

$$P_n(|t-x|^r;x) = O\left(n^{-r/2}\right), \text{ as } n\to\infty, \text{ uniformly in } x\in I.$$

Proof. Let s be an even integer $> r$. An application of Hölder's inequality in integral and Lemma 2.2 in the next step gives

$$P_n(|t-x|^r;x) = \int_0^1 W_n(x,t)|t-x|^r\,dt$$

$$\leq \left(\int_0^1 W_n(x,t)|t-x|^s\,dt\right)^{\frac{r}{s}} \left(\int_0^1 W_n(x,t)\,dt\right)^{1-\frac{r}{s}} \leq C(n^{-s/2})^{r/s} = Cn^{-r/2}. \quad \square$$

Lemma 2.4. [3] *There holds for* $l \in \mathbb{N}$

$$x^l(1-x)^l D^l\left(p_{n,\nu}(x)\right) = \sum_{\substack{2i+j\leq l\\ i,j\geq 0}} n^i(\nu-nx)^j q_{i,j,l}(x)p_{n,\nu}(x),$$

where $D \equiv \frac{d}{dx}$ *and* $q_{i,j,l}(x)$ *are certain polynomials in* x *independent of* n *and* ν.

Lemma 2.5. [3] *There holds for* $k,l \in \mathbb{N}$

$$T_{n,k}\left((t-x)^l;x\right) = O(n^{-k}),\ \text{as } n\to\infty,\ \text{uniformly in } x \in I.$$

Lemma 2.6. *Let* $r > 0$ *and* $V_n(x,t) =: n\sum_{\nu=1}^{n} p_{n,\nu}(x)p_{n-1,\nu-1}(t)$, *then*

$$\int_0^1 V_n(x,t)|x-t|^r\,dx = O(n^{-r/2}),\ \text{as } n\to\infty,$$

uniformly for all t *in* $[0,1]$.

Proof. Let $J =: \int_0^1 V_n(x,t)|x-t|^r\,dx$ and s be an even integer $> r$. Then, proceeding along the lines of the proof of the Corollary 2.3 and in view of

$$\int_0^1 p_{n,\nu}(x)dx = \frac{1}{n+1}$$

we have

$$J \leq \left(\int_0^1 V_n(x,t)(x-t)^s dx\right)^{\frac{r}{s}} \left(\frac{n}{n+1}\right)^{1-\frac{r}{s}}.$$

We may write

$$\int_0^1 V_n(x,t)(x-t)^s\,dx \quad = \quad (-1)^s.n\sum_{i=0}^{s}\binom{s}{i}t^{s-i}(-1)^i\sum_{\nu=1}^{n} p_{n-1,\nu-1}(t)\int_0^1 p_{n,\nu}(x)x^i\,dx.$$

Since

$$\int_0^1 p_{n,\nu}(x)x^i\,dx = \frac{(\nu+1)...(\nu+i)}{(n+1)...(n+i+1)},$$

it follows that

$$\int_0^1 V_n(x,t)(x-t)^s\,dx = (-1)^s.n\sum_{i=0}^{s}\binom{s}{i}t^{s-i}(-1)^i\sum_{\nu=0}^{n-1}p_{n-1,\nu}(t)\frac{(\nu+2)...(\nu+i+1)}{(n+1)...(n+i+1)}. \tag{2.6}$$

Now, $(\nu+2)...(\nu+i+1) = \nu^i + p_1(i)\nu^{i-1} + p_2(i)\nu^{i-2} + ...$, where $p_j(i)$ is a polynomial in i of degree $2j$. Moreover,

$$\nu^i = q_0(i)\nu^{(i)} + q_1(i)\nu^{(i-1)} + q_2(i)\nu^{(i-2)} + ... + q_{i-1}(i)\nu^{(1)}, \tag{2.7}$$

where $q_0(i) = q_{i-1}(i) = 1, \nu^{(j)} = \nu(\nu-1)(\nu-2)...(\nu-j+1), j = 0,1,2...,i$ and $q_j(i)$ is a polynomial in i of degree $2j$.

Utilizing (2.7) in (2.6) and using the properties of binomial coefficients, we get the required order. □

Definition 2.7. *Let $f \in L_p(I)$, $p \geq 1$. Then for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m^{th} order is defined as follows:*

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\left(f(t) + (-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i}f(t)\right)dt_1...dt_m,$$

where $t \in I_1$ and Δ_h^m is m^{th} order forward difference operator of step length h.

Lemma 2.8. *The function $f_{\eta,m}$ satisfies the following properties*

(a) $f_{\eta,m}$ has derivatives up to order m over I_1, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e. and belongs to $L_p(I_1)$;
(b) $\|f_{\eta,m}^{(r)}\|_{L_p(I_2)} \leq C_r\,\eta^{-r}\omega_r(f,\eta,p,I_1), r = 1,2,...,m$;
(c) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leq C_{m+1}\,\omega_m(f,\eta,p,I_1)$;
(d) $\|f_{\eta,m}\|_{L_p(I_2)} \leq C_{m+2}\|f\|_{L_p(I_1)}$;
(e) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \leq C_{m+3}\,\eta^{-m}\|f\|_{L_p(I_1)}$,

where $C_i's$ are certain constants that depend on i but are independent of f and η.

The proof follows from Theorem 18.17 ([13]) and ([20], Exercise 3.12, pp.165-166).

Lemma 2.9. *Let $f \in L_p(I)$, $p \geq 1$ and $r,m \in \mathbb{N}$. Then there holds*

$$\|P_n^m(f(t)(t-x)^r;x)\|_{L_p(I)} \leq C\,n^{-r/2}\|f\|_{L_p(I)}.$$

Proof. Using Remark 1.4

$$P_n^m(f(t)(t-x)^r;x) = \int_0^1\int_0^1...\int_0^1 V_n(x,t_1)V_n(t_1,t_2)...V_n(t_{m-1},t_m)(t_m-x)^r f(t_m)dt_m...dt_1.$$

A repeated use of Hölder's inequality and in view of $\int\limits_0^1 V_n(x,t)dt = O(1)$ makes

$$|P_n^m(f(t)(t-x)^r;x)|^p$$
$$\leq \int\limits_0^1\int\limits_0^1 ... \int\limits_0^1 V_n(x,t_1)V_n(t_1,t_2)...V_n(t_{m-1},t_m)|t_m-x|^{rp}\,|f(t_m)|^p\;dt_m...dt_1.$$

We now consider integration on both sides. On the right side by virtue of Fubini's theorem, the integration is done with respect to x followed by $t_1, t_2, ..., t_m$ respectively. Thus

$$\int\limits_0^1 |P_n^m(f(t)(t-x)^r;x)|^p\;dx \leq \int\limits_0^1 ... \int\limits_0^1 \left(\int\limits_0^1 V_n(x,t_1)|t_m-x|^{rp}dx\right)\times \qquad (2.8)$$
$$V_n(t_1,t_2)...V_n(t_{m-1},t_m)\,|f(t_m)|^p\;dt_1...dt_m.$$

Let $s > rp$ be an integer. Then, using Hölder's inequality and in view of

$$\int\limits_0^1 p_{n,\nu}(x)dx = \frac{1}{n+1}$$

we have

$$\int\limits_0^1 V_n(x,t_1)|t_m-x|^{rp}\,dx \leq \left(\int\limits_0^1 V_n(x,t_1)|t_m-x|^s dx\right)^{\frac{rp}{s}}\left(\frac{n}{n+1}\right)^{1-\frac{rp}{s}}. \qquad (2.9)$$

By multinomial expansion

$$|t_m-x|^s \leq (|t_m-t_{m-1}|+|t_{m-1}-t_{m-2}|+...+|t_1-x|)^s$$
$$\leq \sum_{\substack{r_1+r_2+...+r_m=s,\\ r_k\geq 0,\,\forall 1\leq k\leq m}} \binom{s}{r_1,r_2,..,r_m}|t_m-t_{m-1}|^{r_m}...|t_1-x|^{r_1}. \qquad (2.10)$$

Now, we combine (2.8)-(2.10), resort Lemma 2.6 m times and Hölder's inequality $(m-1)$ times to reach

$$\int\limits_0^1 |P_n^m(f(t)(t-x)^r;x)|^p\;dx$$
$$\leq C\left(\sum_{\substack{r_1+r_2+...+r_m=s,\\ r_k\geq 0,\,\forall 1\leq k\leq m}} \binom{s}{r_1,r_2,..,r_m} n^{-\frac{r_1+r_2+...+r_m}{2}}\right)^{\frac{rp}{s}} \int\limits_0^1 |f(t_m)|^p\;dt_m$$
$$\leq Cn^{-\frac{rp}{2}}m^{rp}\|f\|^p_{L_p(I)},$$

using bound of multinomial coefficients. Taking p^{th} root on both sides we complete the proof of lemma. □

Lemma 2.10. *Let $m, s \in \mathbb{N}$ and $f \in L_p(I),\ p \geq 1$ have a compact support in $[a,b] \subset (0,1)$. Then there holds*

$$\left\| \frac{d^{2s}}{dx^{2s}} P_n^m(f;x) \right\|_{L_p[a,b]} \leq C\, n^s \|f\|_{L_p[a,b]}.$$

Proof. An application of Lemma 2.4 enables us to express

$$\begin{aligned} \frac{d^{2s}}{dx^{2s}} P_n^m(f;x) &= \frac{d^{2s}}{dx^{2s}} \int_0^1 W_n(x,v) P_n^{m-1}(f;v)\, dv \\ &= n \sum_{\nu=1}^{n} p_{n,\nu}(x) \sum_{\substack{2i+j \leq 2s \\ i,j \geq 0}} n^i \frac{(\nu - nx)^j q_{i,j,s}(x)}{(x(1-x))^{2s}} \times \\ &\quad \int_0^1 p_{n-1,\nu-1}(v) P_n^{m-1}(f;v)\, dv. \end{aligned} \tag{2.11}$$

When $p > 1$, applying Hölder's inequality twice, first for summation and then for integration, we obtain

$$\begin{aligned} &\left| \sum_{\nu=1}^{n} (\nu - nx)^j p_{n,\nu}(x)\, n \int_0^1 p_{n-1,\nu-1}(v) P_n^{m-1}(f;v)\, dv \right|^p \\ &\qquad \leq \sum_{\nu=1}^{n} |\nu - nx|^{jp} p_{n,\nu}(x)\, n \int_0^1 p_{n-1,\nu-1}(v) \left| P_n^{m-1}(f;v) \right|^p dv. \end{aligned} \tag{2.12}$$

The above inequality is true for $p = 1$, as well. Now, we integrate both sides of (2.12) with respect to x and take help of Lemma 2.1 in next step to obtain

$$\begin{aligned} &\int_a^b \left| \sum_{\nu=1}^{n} (\nu - nx)^j p_{n,\nu}(x)\, n \int_0^1 p_{n-1,\nu-1}(v) P_n^{m-1}(f;v)\, dv \right|^p dx \\ &\qquad \leq \sum_{\nu=1}^{n} \left(\int_a^b p_{n,\nu}(x) |\nu - nx|^{jp}\, dx \right) n \int_0^1 p_{n-1,\nu-1}(v) \left| P_n^{m-1}(f;v) \right|^p dv \\ &\qquad \leq \frac{C_1 n^{jp/2}}{n} . n \int_0^1 \left(\sum_{\nu=1}^{n} p_{n-1,\nu-1}(v) \right) \left| P_n^{m-1}(f;v) \right|^p dv \\ &\qquad \leq C_1 n^{jp/2} \| P_n^{m-1}(f;.) \|_{L_p(I)}^p. \end{aligned} \tag{2.13}$$

Let $C_2 =: \sup\limits_{x \in [a,b]} \sup\limits_{\substack{2i+j \leq 2s \\ i,j \geq 0}} \dfrac{|q_{i,j,s}(x)|}{(x(1-x))^{2s}}$.

We now combine (2.11) and (2.13) and conclude that

$$\begin{aligned}\left\|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right\|_{L_p[a,b]} &\leq C_1^{1/p}\,C_2\left(\sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i n^{j/2}\right)\|P_n^{m-1}(f;.)\|_{L_p(I)}\\ &\leq C\,n^s\|f\|_{L_p(I)} = C\,n^s\|f\|_{L_p[a,b]}.\end{aligned}$$

Hence, the required result follows. □

Lemma 2.11. *Let $m,s\in\mathbb{N}$ and $f\in L_p[0,1],\ p\geq 1$ have a compact support in $[a,b]\subset(0,1)$. Moreover, let $f^{(2s-1)}\in AC[a,b]$ and $f^{(2s)}\in L_p[a,b]$, then*

$$\left\|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right\|_{L_p[a,b]}\leq C\|f^{(2s)}\|_{L_p[a,b]}.$$

Proof. Since P_n^m maps polynomials into polynomials of the same degree, using Lemma 2.4 we have

$$\begin{aligned}\frac{d^{2s}}{dx^{2s}}P_n^m(f;x) &= \frac{1}{(2s-1)!}\int_0^1\dots\int_0^1\frac{d^{2s}}{dx^{2s}}(W_n(x,t_1))W_n(t_1,t_2)\dots\times\\ &\quad W_n(t_{m-1},t_m)\int_x^{t_m}(t_m-w)^{2s-1}f^{(2s)}(w)\,dw\,dt_m\dots dt_1\\ &= \frac{1}{(2s-1)!}\sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i\sum_{\nu=1}^{n}(\nu-nx)^j\,\frac{q_{i,j,s}(x)}{(x(1-x))^{2s}}\times\\ &\quad p_{n,\nu}(x)\int_0^1\dots\int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)\dots W_n(t_{m-1},t_m)\times\\ &\quad \int_x^{t_m}(t_m-w)^{2s-1}f^{(2s)}(w)\,dw\,dt_m\dots dt_1.\end{aligned}$$

Let us define $W_n(x,t)=0,\, t\notin[0,1]$. Now, we break the interval of integration in t_m in the following way:

There exists for each n an integer $r(n)$ such that

$$\frac{r}{\sqrt{n}}\leq\max\{1-a,b\}\leq\frac{r+1}{\sqrt{n}}.$$

Let $C = \sup_{x\in[a,b]} \sup_{\substack{2i+j\leq 2s\\ i,j\geq 0}} \frac{|q_{i,j,s}(x)|}{(x(1-x))^{2s}}$. Then

$$\left|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right| \leq C \sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i \sum_{\nu=1}^{n} |\nu - nx|^j \, p_{n,\nu}(x)\times \tag{2.14}$$
$$\int_0^1 ... \int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)...W_n(t_{m-2},t_{m-1})\times$$
$$\left\{\int_0^1 W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\left|\int_x^{t_m}|f^{(2s)}(w)|dw\right| dt_m\right\} dt_{m-1}...dt_1.$$

The expression inside the curly bracket in (2.14), however is bounded by

$$\leq \sum_{l=0}^{r}\left\{\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{l+1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\int_x^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m\right.$$
$$\left.+\int_{x-\frac{l+1}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\int_{x-\frac{l+1}{\sqrt{n}}}^{x}|f^{(2s)}(w)|dw\,dt_m\right\}$$
$$\leq \sum_{l=1}^{r}\left\{\frac{n^2}{l^4}\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{l+1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s+3}\int_x^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m\right. \tag{2.15}$$
$$\left.+\frac{n^2}{l^4}\int_{x-\frac{l+1}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s+3}\int_{x-\frac{l+1}{\sqrt{n}}}^{x}|f^{(2s)}(w)|dw\,dt_m\right\}$$
$$+\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m.$$

Using (2.15) in (2.14)

$$\left|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right| \leq C \sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i \sum_{\nu=1}^{n} |\nu - nx|^j \, p_{n,\nu}(x)\times$$

$$\int_0^1 ... \int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)...W_n(t_{m-2},t_{m-1})\times$$

$$\Big\{\sum_{l=1}^{r}\Big(\frac{n^2}{l^4}\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{l+1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m-x|^{2s+3}\int_{x}^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m$$

$$+\frac{n^2}{l^4}\int_{x-\frac{l+1}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m-x|^{2s+3}\int_{x-\frac{l+1}{\sqrt{n}}}^{x}|f^{(2s)}(w)|dw\,dt_m\Big)$$

$$+\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m-x|^{2s-1}\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m\Big\}dt_{m-1}...dt_1,$$

$$= J_1+J_2+J_3, \text{ say.}$$

In order to estimate J_1, J_2 and J_3, we use multinomial expansion

$$|t_m-x|^{2s+3} \le \sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_k\ge 0,\,\forall 1\le k\le m}} \binom{2s+3}{r_1,r_2,..,r_m}\times |t_m-t_{m-1}|^{r_m}|t_{m-1}-t_{m-2}|^{r_{m-1}}...|t_1-x|^{r_1}.$$

Thus

$$J_1\le C\sum_{\substack{2i+j\le 2s\\ i,j\ge 0}} n^i\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\left(\int_{x}^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right)\right)\sum_{\nu=1}^{n}|\nu-nx|^j\,p_{n,\nu}(x)\times$$

$$\sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_i\ge 0,\,\forall 1\le i\le m}} \binom{2s+3}{r_1,r_2,..,r_m}\int_0^1...\int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)...W_n(t_{m-1},t_m)\times$$

$$|t_m-t_{m-1}|^{r_m}|t_{m-1}-t_{m-2}|^{r_{m-1}}...|t_1-x|^{r_1}\,dt_m\,dt_{m-1}...dt_1.$$

A repeated application of Corollary 2.3 makes

$$J_1\le C\sum_{\substack{2i+j\le 2s\\ i,j\ge 0}} n^i\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\left(\int_{x}^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right)\right)\times \tag{2.16}$$

$$\sum_{\nu=1}^{n}|\nu-nx|^j p_{n,\nu}(x)\Big\{\sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_k\ge 0,\,\forall 1\le k\le m}}\binom{2s+3}{r_1,r_2,..,r_m}\frac{1}{n^{(r_m+\ldots+r_2)/2}}\Big\}\times$$

$$\left(\int_0^1 np_{n-1,\nu-1}(t_1)|t_1-x|^{r_1}dt_1\right).$$

In order to obtain a bound for J_1 in (2.16) we require an estimate of

$$\int_0^1 np_{n-1,\nu-1}(t_1)|t_1-x|^{r_1}dt_1.$$

This is accomplished with the help of Lemma 2.1 and moments of Bernstein polynomials ([14], Theorem 1.5.1).

$$\int_0^1 np_{n-1,\nu-1}(t_1)|t_1-x|^{r_1}dt_1 \le \int_0^1 np_{n-1,\nu-1}(t_1)\left(\left|t_1-\frac{\nu-1}{n-1}\right|+\left|\frac{\nu-1}{n-1}-x\right|\right)^{r_1} dt_1$$

$$= \sum_{i_1=0}^{r_1}\binom{r_1}{i_1}\left|\frac{(\nu-nx)-(1-x)}{n-1}\right|^{i_1}\int_0^1 np_{n-1,\nu-1}(t_1)\left|t_1-\frac{\nu-1}{n-1}\right|^{r_1-i_1} dt_1$$

$$\le C\sum_{i_1=0}^{r_1}\binom{r_1}{i_1}n^{-(r_1-i_1)/2}\left|(\nu-nx)-(1-x)\right|^{i_1}.$$

Therefore,

$$J_1 \le C\sum_{\substack{2i+j\le 2s\\ i,j\ge 0}} n^i\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\left(\int_x^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right)\right)\sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_k\ge 0,\,\forall 1\le k\le m}}\binom{2s+3}{r_1,r_2,..,r_m}\times$$
$$\left\{\sum_{\nu=1}^{n}|\nu-nx|^j p_{n,\nu}(x)\left(\sum_{i_1=0}^{r_1}\binom{r_1}{i_1}n^{i_1/2}\left|\frac{(\nu-nx)-(1-x)}{n-1}\right|^{i_1}\right)\right\}\frac{1}{n^{(2s+3)/2}}$$
$$\le Cm^{2s+3}\frac{n^s}{n^{(2s+3)/2}}\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\int_x^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right). \tag{2.17}$$

We now take p norm in x in above. Let p,q be the conjugate exponents such that

$$\frac{1}{p}+\frac{1}{q}=1$$

and $\psi_l(x,.)$ denote the characteristic function of the interval $[x,x+\frac{l+1}{\sqrt{n}}]$. By using Hölder's inequality and Fubini's theorem

$$\begin{aligned}
\int_a^b \left(\int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)| dw \right)^p dx &\le \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \int_a^b \int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)|^p dw\, dx \\
&= \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \int_a^b \int_0^1 \psi_l(x,w)|f^{(2s)}(w)|^p dw\, dx \\
&= \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \int_0^1 \left(\int_a^b \psi_l(x,w)\, dx \right) |f^{(2s)}(w)|^p\, dw \\
&\le \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \left(\frac{l+1}{\sqrt{n}}\right) \int_0^1 |f^{(2s)}(w)|^p\, dw \\
&= \left(\frac{l+1}{\sqrt{n}}\right)^{p} \|f^{(2s)}\|^p_{L_p[0,1]}.
\end{aligned}$$

Hence,

$$\left\| \int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)| dw \right\|_{L_p[a,b]} \le \left(\frac{l+1}{\sqrt{n}}\right) \|f^{(2s)}\|_{L_p(I)}.$$

This implies by (2.17), that

$$\|J_1\|_{L_p[a,b]} \le Cm^{2s+3}\|f^{(2s)}\|_{L_p(I)} = Cm^{2s+3}\|f^{(2s)}\|_{L_p[a,b]}.$$

In order to find estimates J_2 and J_3 we proceed in a similar manner and obtain the required order. Combining the estimates of J_1, J_2 and J_3, we complete the proof. □

3. Proof of Inverse Theorem

Proof. We choose numbers x_i and $y_i, i = 1, 2, 3$ that satisfy $0 < a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1 < 1$.

We choose a function $g \in C_0^{2k}$ such that $\operatorname{supp} g \subset (x_2, y_2)$ with $g(x) = 1$ on $[x_3, y_3]$ and $\bar{f} = fg$.

Now, for all values of $\gamma \le \tau$ we have

$$\begin{aligned}
\left\|\Delta_\gamma^{2k} \bar{f}(x)\right\|_{L_p[x_2,y_2]} &\leqslant \left\|\Delta_\gamma^{2k}(\bar{f}(x) - T_{n,k}(\bar{f};x))\right\|_{L_p[x_2,y_2]} + \left\|\Delta_\gamma^{2k} T_{n,k}(\bar{f};x)\right\|_{L_p[x_2,y_2]} \\
&= \Sigma_1 + \Sigma_2, \text{ say.} \qquad (3.1)
\end{aligned}$$

Let $\bar{f}_{\eta,2k}(x)$ denote the Steklov mean for the function $\bar{f}(x)$. Then, Lemmas 2.10 and 2.11 entail

$$
\begin{aligned}
\Sigma_2 &= \left\|\Delta_\gamma^{2k} T_{n,k}(\bar{f};x)\right\|_{L_p[x_2,y_2]} \\
&\le \gamma^{2k}\left\{ \left\|T_{n,k}^{(2k)}(\bar{f}-\bar{f}_{\eta,2k};x)\right\|_{L_p[x_1,y_1]} + \left\|T_{n,k}^{(2k)}(\bar{f}_{\eta,2k};x)\right\|_{L_p[x_1,y_1]} \right\} \\
&\le C\gamma^{2k}\left\{ n^k \|\bar{f}-\bar{f}_{\eta,2k}\|_{L_p[x_2,y_2]} + \|\bar{f}_{\eta,2k}^{(2k)}\|_{L_p[x_2,y_2]} \right\} \\
&\le C\gamma^{2k}\left(n^k + \frac{1}{\eta^{2k}}\right)\omega_{2k}(\bar{f},\eta,p,[x_2,y_2]), \qquad (3.2)
\end{aligned}
$$

for sufficiently small values of γ and η.

It follows from the hypothesis that a component of Σ_1 is bounded as

$$
\begin{aligned}
&\left\|\bar{f}(x) - T_{n,k}(\bar{f};x)\right\|_{L_p[x_2,y_2]} \\
&\quad \le \|g(x)(f(x) - T_{n,k}(f;x))\|_{L_p[x_2,y_2]} + \|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]} \\
&\quad \le \frac{C}{n^{\alpha/2}} + \|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]}. \qquad (3.3)
\end{aligned}
$$

We now establish that

$$
\|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]} = O(n^{-\alpha/2}). \qquad (3.4)
$$

This is a major point in the proof of our theorem. Assuming (3.4)to be true, it follows from (3.1)-(3.4) that

$$
\left\|\Delta_\gamma^{2k}\bar{f}(x)\right\|_{L_p[x_2,y_2]} \le C_1\left\{ \frac{1}{n^{\alpha/2}} + \gamma^{2k}\left(n^k + \frac{1}{\eta^{2k}}\right)\omega_{2k}(\bar{f},\eta,p,[x_2,y_2])\right\}. \qquad (3.5)
$$

We choose $\eta = n^{-1/2}$ and take $\sup_{\gamma\le\tau}$ in (3.5) to obtain

$$
\omega_{2k}(\bar{f},\tau,p,[x_2,y_2]) \le C\left\{\eta^\alpha + \left(\frac{\tau}{\eta}\right)^{2k}\omega_{2k}(\bar{f},\eta,p,[x_2,y_2])\right\}.
$$

Now, making use of the Lemma ([6], p.696), we get

$$
\omega_{2k}(\bar{f},\tau,p,[x_2,y_2]) \le C\,\tau^\alpha
$$

and therefore

$$
\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha), \text{ as } \tau \to 0.
$$

The proof of (3.4) is accomplished by induction on α. When $\alpha \le 1$, by mean value theorem and Lemma 2.9

$$
\begin{aligned}
\|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]} &= \|T_{n,k}(f(t)(t-x)g'(\xi);x)\|_{L_p[x_2,y_2]} \\
&\le \frac{C}{n^{1/2}}\|f\|_{L_p(I)},
\end{aligned}
$$

where ξ lies between t and x. This proves (3.4) when $\alpha \le 1$.

We next assume that (3.4) is true when α lies in $[r-1,r)$ and prove that it is true for $\alpha \in [r,r+1)$. Let $f_{\eta,2k}(t)$ denote the Steklov mean. We express

$$
\begin{aligned}
f(t)(g(t)-g(x)) \\
&= \{(f(t)-f_{\eta,2k}(t))+(f_{\eta,2k}(t)-f_{\eta,2k}(x))+f_{\eta,2k}(x)\}(g(t)-g(x)) \\
&= (f(t)-f_{\eta,2k}(t))(g(t)-g(x))+(f_{\eta,2k}(t)-f_{\eta,2k}(x))(g(t)-g(x)) \\
&+ f_{\eta,2k}(x)(g(t)-g(x)) \\
&= \Sigma_3+\Sigma_4+\Sigma_5, \text{ say.} \qquad (3.6)
\end{aligned}
$$

Let $\psi(t)$ denote the characteristic function of $[x_2-\delta_0, y_2+\delta_0]$. This entails that $\|P_n^m((f(t)-f_{\eta,2k}(t))(g(t)-g(x));x)\|_{L_p[x_2,y_2]}$

$$
\begin{aligned}
&\leq \|P_n^m((f(t)-f_{\eta,2k}(t))(t-x)g'(\xi)\psi(t);x)\|_{L_p[x_2,y_2]} \\
&+ \|P_n^m((f(t)-f_{\eta,2k}(t))(t-x)g'(\xi)(1-\psi(t));x)\|_{L_p[x_2,y_2]} \\
&\leq \|g'\|_{C(I)}\, \|P_n^m(|f(t)-f_{\eta,2k}(t)|\,|t-x|\psi(t);x)\|_{L_p[x_2,y_2]} \\
&+ \|g'\|_{C(I)}\delta_0^{-(2k-1)}\, \left\|P_n^m(|f(t)-f_{\eta,2k}(t)|\,|t-x|^{2k}(1-\psi(t));x)\right\|_{L_p[x_2,y_2]} \\
&\leq Cn^{-1/2}\|(f-f_{\eta,2k})\psi(t)\|_{L_p[x_2,y_2]}+Cn^{-k}\|f-f_{\eta,2k}\|_{L_p(I)} \\
&\leq Cn^{-1/2}\omega_{2k}(f,\eta,f,p,[x_1,y_1]) \\
&+ Cn^{-k}\|f\|_{L_p(I)} \text{ (in view of Lemmas 2.9 and 2.8).} \qquad (3.7)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|T_{n,k}(\Sigma_3;x)\| &= \|\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}P_n^m(\Sigma_3;x)\| \\
&\leq Cn^{-1/2}\omega_{2k}(f,\eta,f,p,[x_1,y_1])+Cn^{-k}\|f\|_{L_p(I)}. \qquad (3.8)
\end{aligned}
$$

To obtain an estimate of Σ_5, we note that $g(t)$ is a very smooth function and hence

$$
\begin{aligned}
T_{n,k}(g(t)-g(x);x) &= \sum_{j=1}^{2k-1}\frac{g^{(j)}(x)}{j!}T_{n,k}((t-x)^j;x) \\
&+ \frac{1}{(2k)!}T_{n,k}(g^{(2k)}(\xi)(t-x)^{2k};x), \qquad (3.9)
\end{aligned}
$$

where ξ lies between t and x.

Now, applying Lemmas 2.5 and 2.9 on the right hand side of (3.9) respectively, we have

$$
\|T_{n,k}(\Sigma_5;x)\|_{L_p[x_2,y_2]} \leq Cn^{-k}\|f_{\eta,2k}\|_{L_p[x_2,y_2]} \leq Cn^{-k}\|f\|_{L_p(I)}. \qquad (3.10)
$$

Since, by virtue of Lemma 2.8, $f_{\eta,2k}$ is $2k$ times differentiable, there follows

$$\Sigma_4 = \left\{ \sum_{i=1}^{2k-1} \frac{(t-x)^i}{i!} f_{\eta,2k}^{(i)}(x) + \frac{1}{(2k-1)!} \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)dw \right\} \times$$

$$\left\{ \sum_{j=1}^{2k-2} \frac{(t-x)^j}{j!} g^{(j)}(x) + \frac{(t-x)^{2k-1}}{(2k-1)!} g^{(2k-1)}(\xi) \right\}$$

$$= \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-2} \frac{(t-x)^{i+j}}{i!j!} f_{\eta,2k}^{(i)}(x) g^{(j)}(x)$$

$$+ \frac{g^{(2k-1)}(\xi)}{(2k-1)!} \sum_{i=1}^{2k-1} \frac{(t-x)^{2k+i-1}}{i!} f_{\eta,2k}^{(i)}(x)$$

$$+ \frac{1}{(2k-1)!} \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)dw \times$$

$$\left\{ \sum_{j=1}^{2k-2} \frac{(t-x)^j}{j!} g^{(j)}(x) + \frac{(t-x)^{2k-1}}{(2k-1)!} g^{(2k-1)}(\xi) \right\}$$

$$= \Sigma_6 + \Sigma_7 + \Sigma_8, \text{ say,} \tag{3.11}$$

where ξ lies between t and x.

By Lemma 2.5 and Theorem 3.1 ([9], p.5)

$$\|T_{n,k}(\Sigma_6; x)\|_{L_p[x_2,y_2]} \le Cn^{-k} \left(\sum_{i=1}^{2k-1} \|f_{\eta,2k}^{(i)}(x)\|_{L_p[x_2,y_2]} \right)$$

$$\le Cn^{-k} \left(\|f_{\eta,2k}\|_{L_p[x_2,y_2]} + \|f_{\eta,2k}^{(2k-1)}(x)\|_{L_p[x_2,y_2]} \right). \tag{3.12}$$

Similarly,

$$\|T_{n,k}(\Sigma_7; x)\|_{L_p[x_2,y_2]} \le Cn^{-k} \left(\|f_{\eta,2k}\|_{L_p[x_2,y_2]} + \|f_{\eta,2k}^{(2k-1)}(x)\|_{L_p[x_2,y_2]} \right). \tag{3.13}$$

We now examine a typical term of $T_{n,k}(\Sigma_8; x)$ expressed as

$$P_n^m \left((t-x)^i \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)dw; x \right)$$

$$= P_n^m \left((t-x)^i \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)\psi(w)dw; x \right)$$

$$+ P_n^m \left((t-x)^i \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)(1-\psi(w))dw; x \right)$$

$$= \Sigma_9 + \Sigma_{10}, \text{ say.} \tag{3.14}$$

We may write

$$|\Sigma_9| \leq \int_0^1 \ldots \int_0^1 W_n(x,t_1)W_n(t_1,t_2)\ldots W_n(t_{m-1},t_m)|t_m-x|^{2k+i-1}\times \tag{3.15}$$

$$\left|\int_x^{t_m} |f^{(2k)}_{\eta,2k}(w)|\psi(w)dw\right| dt_m dt_{m-1}\ldots dt_2 dt_1.$$

Now, proceeding along the lines of the proof of Lemma 2.11, we obtain

$$\|\Sigma_9\|_{L_p[x_2,y_2]} \leq \frac{C}{n^{(2k+i)/2}}\|f^{(2k)}_{\eta,2k}\|_{L_p[x_2-\delta,y_2-\delta]}. \tag{3.16}$$

The presence of $(1-\psi(w))$ in Σ_{10} implies that $|w-x|>\delta_0$. Therefore

$$|\Sigma_{10}| \leq \int_0^1 \ldots \int_0^1 W_n(x,t_1)W_n(t_1,t_2)\ldots W_n(t_{m-1},t_m)\times \tag{3.17}$$

$$|t_m-x|^{2k+i-1+2k}\left(\delta_0^{-2k}\left|\int_x^{t_m}|f^{(2k)}_{\eta,2k}(w)|(1-\psi(w))dw\right| dt_m dt_{m-1}\ldots dt_2 dt_1\right).$$

Proceeding along the lines of the proof of Lemma 2.11 again yields

$$\|\Sigma_{10}\|_{L_p[x_2,y_2]} \leq \frac{C}{n^{(4k+i)/2}}\|f^{(2k)}_{\eta,2k}\|_{L_p(I)}. \tag{3.18}$$

Combining (3.14), (3.16) and (3.18), we get

$$\|T_{n,k}(\Sigma_8;x)\|_{L_p[x_2,y_2]}$$
$$\leq C\left\{\frac{1}{n^{(2k+i)/2}}\|f^{(2k)}_{\eta,2k}\|_{L_p[x_2-\delta_0,y_2+\delta_0]}+\frac{1}{n^{(4k+i)/2}}\|f^{(2k)}_{\eta,2k}\|_{L_p(I)}\right\}. \tag{3.19}$$

Utilizing (3.6)-(3.19), we are led to

$$\|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]}$$
$$\leq C\Bigg\{\frac{1}{n^{1/2}}\omega_{2k}(f,\eta,p,[x_1,y_1])+\frac{1}{n^k}\|f\|_{L_p(I)}+\frac{1}{n^k}\Bigg(\|f_{\eta,2k}\|_{L_p[x_2,y_2]}$$
$$+\|f^{(2k-1)}_{\eta,2k}\|_{L_p[x_2,y_2]}\Bigg)+\frac{1}{n^{(2k+1)/2}}\|f^{(2k)}_{\eta,2k}\|_{L_p[x_2-\delta_0,y_2-\delta_0]}+\frac{1}{n^{(4k+1)/2}}\|f^{(2k)}_{\eta,2k}\|_{L_p(I)}\Bigg\}.$$

This is further simplified by Lemma 2.8 by taking $\eta=n^{-1/2}$ for large values of n as

$$\|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]}$$
$$\leq C\Bigg\{\frac{1}{n^{1/2}}\omega_{2k}(f,n^{-1/2},p,[x_1,y_1])+\frac{1}{n^k}\|f\|_{L_p(I)}$$
$$+\frac{1}{n^{1/2}}\omega_{2k-1}(f,n^{-1/2},p,[x_1,y_1])\Bigg\}. \tag{3.20}$$

The induction hypothesis implies that for $[c,d]\subset(a_1,b_1)$

$$\omega_{2k}(f,n^{-1/2},p,[c,d])=O(n^{-\alpha/2}),\ n\to\infty. \tag{3.21}$$

This induces, by (Theorem 6.1.2, [20]),

$$\omega_{2k-1}(f, n^{-1/2}, p, [c, d]) = O(n^{-\alpha/2}),\ n \to \infty. \tag{3.22}$$

Incorporating (3.21) and (3.22) in (3.20), we obtain

$$\|T_{n,k}(f(t)(g(t) - g(x)); x)\|_{L_p[x_2,y_2]} = O(n^{-(\alpha+1)/2}),\ n \to \infty.$$

This proves (3.4) and hence the proof of the theorem follows. □

Acknowledgements. The last author is thankful to the "Councill of Scientific and Industrial research, New Delhi, India" for providing financial support to carry out the above work. The authors are extremely thankful to the referee for a very careful reading of the manuscript and making valuable comments.

References

[1] Agrawal, P.N., Gupta, V., *On the iterative combination of Phillips operators*, Bull. Inst. Math. Acad. Sinica, **18**(1990), no. 4, 361-368.

[2] Agrawal, P.N., Gupta V., Gairola, A.R., *On iterative combination of modified Bernstein-type polynomials*, Georgian Math. J., **15**(2008), no. 4, 591-600.

[3] Agrawal, P.N., Singh, K.K., Gairola, A.R., *L_p-approximation by iterates of Bernstein-Durrmeyer type polynomials*, Int. J. Math. Anal., (Ruse), **4**(2010), no. 9-12, 469-479.

[4] Agrawal, P.N., Kasana, H.S., *On the iterative combinations of Bernstein polynomials*, Demonstratio Math., **17**(1984), no. 3, 777-783.

[5] Becker, M., Nessel, R.J., *An elementary approach to inverse approximation theorems*, J. Approx. Theory, **23**(1978), no. 2, 99-103.

[6] Berens, H., Lorentz, G.G., *Inverse theorems for Bernstein polynomials*, Indiana Univ. Math. J., **21**(1971/72), 693–708.

[7] Ding, C., Cao, F., *K-functionals and multivariate Bernstein polynomials*, J. Approx. Theory, **155**(2008), no. 2, 125-135.

[8] Gawronski, W., Stadtmüller, U., *Linear combinations of iterated generalized Bernstein functions with an application to density estimation*, Acta Sci. Math., (Szeged), **47**(1984), no. 1-2, 205-221.

[9] Goldberg, S., Meir, A., *Minimum moduli of ordinary differential operators*, Proc. London Math. Soc., **3**(1971), no. 23, 1- 15.

[10] Gonska, H.H., Zhou, X.L., *A global inverse theorem on simultaneous approximation by Bernstein-Durrmeyer operators*, J. Approx. Theory, **67**(1991), no. 3, 284-302.

[11] Gonska, H.H., Zhou, X.L., *Approximation theorems for the iterated Boolean sums of Bernstein operators*, J. Comput. Appl. Math., **53**(1994), no. 1, 21-31.

[12] Gupta, V., Maheshwari, P., *Bezier variant of a new Durrmeyer type operators*, Riv. Mat. Univ. Parma, (7), **2**(2003), 9-21.

[13] Hewitt, E., Stromberg, K., *Real and Abstract Analysis*, Narosa Publishing House, New Delhi, India, 1978.

[14] Lorentz, G.G., *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.

[15] May, C.P., *On Phillips operator*, J. Approximation Theory, **20**(1977), no. 4, 315-332.

[16] Micchelli, C.A., *The saturation class and iterates of the Bernstein polynomials*, J. Approximation Theory, **8**(1973), 1-18.

[17] Sevy, J.C., *Convergence of iterated Boolean sums of simultaneous approximants*, Calcolo **30**(1993), no. 1, 41-68.

[18] Sevy, J.C., *Lagrange and least-squares polynomials as limits of linear combinations of iterates of Bernstein and Durrmeyer polynomials*, J. Approx. Theory, **80**(1995), no. 2, 267-271.

[19] Sinha, T.A.K. et al., *Inverse theorem for an iterative combination of Bernstein-Durrmeyer polynomials*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 4, 153-165.

[20] Timan, A.F., *Theory of Approximation of Functions of a Real Variable* (English Translation), Dover Publications Inc., New York, 1994.

[21] Wenz, H.J., *On the limits of (linear combinations of) iterates of linear operators*, J. Approx. Theory, **89**(1997), no. 2, 219–237.

T.A.K. Sinha
S. M. D. College, Department of Mathematics
Poonpoon, Patna-803213 (Bihar), India
e-mail: thakurashok1212@gmail.com

P.N. Agrawal
Indian Institute of Technology Roorkee
Department of Mathematics
Roorkee-247667 (Uttarakhand), India
e-mail: pna_iitr@yahoo.co.in

K.K. Singh
I.C.F.A.I. University, Faculty of Science & Technology
Dehradun-248197 (Uttarakhand), India
e-mail: kksiitr.singh@gmail.com