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INJECTIVITY CRITERIA FOR C^1 FUNCTIONS DEFINED IN NON-CONVEX DOMAINS

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In the present paper we obtain sufficient conditions for the injectivity of functions of class C^1 defined in type φ convex domains. In particular, we obtain some injectivity criteria for functions of class C^1 defined in some simply and doubly connected domains, and we derive as a corollary the well-known Ozaki-Nunokawa-Krzyz univalence criterion.

1. **Preliminaries**

We denote by $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ the open disk centered at $z_0 \in \mathbb{C}$ of radius $r > 0$ and by $U = B(0,1)$ the unit disk in $\mathbb{C}$.

In [4], the authors introduced the *convexity constant* $K(D)$ of a planar domain $D \subset \mathbb{C}$, as follows:

Definition 1.1 ([4])**.** For a domain $D \subset \mathbb{C}$, we define the convexity constant of the domain D by

$$K(D) = \inf_{\substack{a,b \in D \\ a \neq b}} \sup_{\gamma \in \Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)},$$

where $\Gamma(a, b; D)$ is the family of all rectifiable arcs $\gamma \subset D$ with distinct endpoints a and b, and $l(\gamma)$ denotes the length of γ.

The authors showed that in the class of simply connected domains, the convexity constant $K(D)$ characterizes the convexity of the domain D, in the following sense:

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Theorem 1.2 ([4])**.** *The simply connected domain $D \subset \mathbb{C}$ is convex if and only if $K(D) = 1$.*

Given two domains $\Omega \subset D \subset \mathbb{C}$, denote by D_Ω the domain

$$D_\Omega = D - \overline{\Omega} = \left\{ z \in \mathbb{C} : z \in D,\ z \notin \overline{\Omega} \right\} \tag{1.1}$$

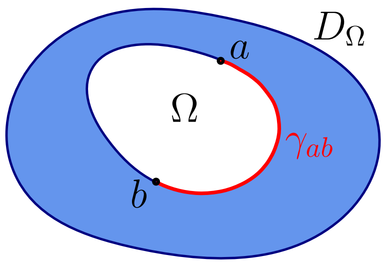

FIGURE 1. The domain $D_\Omega = D - \overline{\Omega}$.

In [4], the authors proposed the following conjecture:

Conjecture 1.3. *If D and Ω are convex domains with $\overline{\Omega} \subset D$, the convexity constant of the domain $D_\Omega = D - \overline{\Omega}$ is given by*

$$K(D_\Omega) = \min_{\substack{a,b \in \partial\Omega \\ a \neq b}} \frac{|a-b|}{l(\gamma_{ab})},$$

where γ_{ab} denotes the shorter of the two arcs of the boundary $\partial\Omega$ with endpoints a and b.

They proved the validity of the above conjecture in the following cases:

1. If $D \subset \mathbb{C}$ is a domain and $\gamma \subset D$ is a Jordan arc which joins two points $z_0 \in D$ and $w_0 \in \partial D$, then $K(D_\gamma) = 0$.
2. If $D \subset \mathbb{C}$ is a convex domain, $z_0 \in D$ and $r > 0$ are chosen such that $\overline{B(z_0, r)} \subset D$, then $K(D_{B(z_0,r)}) = \frac{2}{\pi}$.
3. If D is a convex domain and $z_0 \in D$ and $r > 0$ are chosen such that $\overline{S(z_0, r)} \subset D$, then $K(D_{S(z_0,r)}) = \frac{1}{2}$, where

$$S(z_0, r) = \left\{ z \in \mathbb{C} : |\mathrm{Re}(z - z_0)| < \frac{r}{2}, |\mathrm{Im}(z - z_0)| < \frac{r}{2} \right\}$$

denotes the interior of the square having z_0 as center of symmetry and sides parallel to the coordinate axes, of length equal to r.

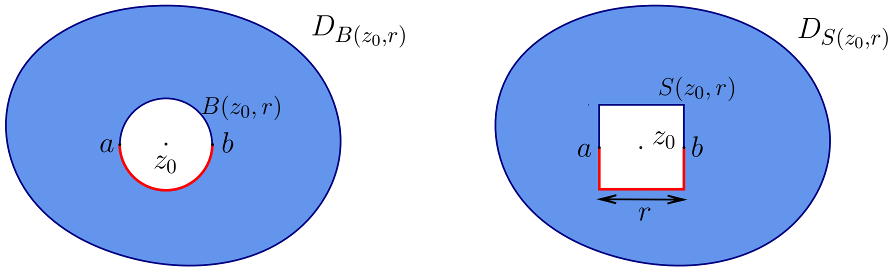

FIGURE 2. The domains $D_{B(z_0 r)} = D - \overline{B\,(z_0 r)}$ and $D_{S(z_0 r)} = D - \overline{S\,(z_0 r)}$.

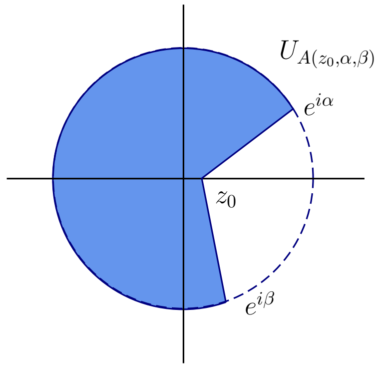

FIGURE 3. The domain $U_{A(z_0,\alpha,\beta)} = U - \overline{A\,(z_0,\alpha,\beta)}$.

4. The convexity constant of the domain $U_{A(z_0,\alpha,\beta)}$ is given by

$$K(U_{A(z_0,\alpha,\beta)}) = \begin{cases} 1, & \text{if } z_0 \in \left[\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, 1\right) \\ \sin\frac{\arg\left(e^{i\alpha}-z_0\right)+\arg\left(e^{i\beta}-z_0\right)}{2}, & \text{if } z_0 \in \left(-1, \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right) \end{cases},$$

where

$$A(z_0,\alpha,\beta) = \{z_0 + re^{i\theta} : r > 0, \ -\arg\left(e^{i\beta}-z_0\right) < \theta < \arg\left(e^{i\alpha}-z_0\right)\}$$

represents the angular region with vertex z_0 and opening angles α and β.

M. O. Reade ([5]) generalized the class of convex planar domains as follows:

Definition 1.4 ([5])**.** Let $\varphi \in [0, \pi)$ be a real number. We say that the domain $D \subset \mathbb{C}$ is a type φ convex domain if for any distinct points $a, b \in D$ there exists $c \in D$ such that the line segments $[a, c], [c, b] \subset D$ and

$$\left|\arg \frac{b-c}{c-a}\right| \leq \varphi. \tag{1.2}$$

The family of type φ convex domains is denoted by C_φ.

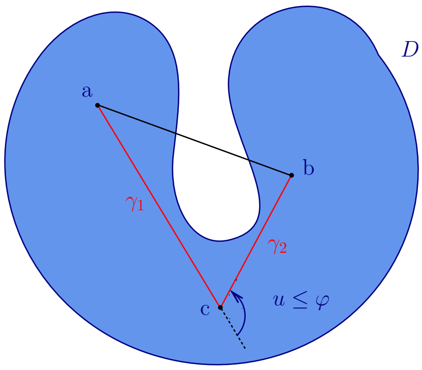

FIGURE 4. A type φ convex domain D, $\varphi \in [0, \pi)$.

Remark 1.5. Geometrically, condition (1.2) shows that the angle $u = \pi - \widehat{acb}$ is less than or equal to φ (see Figure 4).

It can be shown (see [4]) the following connection between type φ convex domains and the convexity constant:

Lemma 1.6. *If $D \in C_\varphi$ is a type φ convex domain for some $\varphi \in [0, \pi)$, then $K(D) \geq \cos \frac{\varphi}{2}$.*

Remark 1.7. The above lemma shows that if D is a type φ convex domain, the convexity constant of D cannot be too small. In particular, if $D \subset \mathbb{C}$ is a convex domain then it is also a type φ convex domain for $\varphi = 0$, and therefore from the above lemma it follows that $K(D) = 1$.

2. Univalence criteria for functions of class $C^1(D)$

P. T. Mocanu ([2], p. 137) obtained the following univalence criterion for C^1 functions defined in type φ domains:

Theorem 2.1. *Let $D \in C_\varphi$, $\varphi \in [0, \pi)$. If the function $f \in C^1(D)$ satisfies one of the two equivalent conditions*

i) $|\arg f'_\theta(z)| < \frac{\pi-\varphi}{2}$, *$z \in D$, for any $\theta \in [0, 2\pi)$*

ii) $\operatorname{Re} \frac{\partial f(z)}{\partial z} - \left|\operatorname{Im} \frac{\partial f(z)}{\partial z}\right| \tan \frac{\varphi}{2} > \frac{1}{\cos \frac{\varphi}{2}} \left|\frac{\partial f(z)}{\partial \bar{z}}\right|$, *$z \in D$,*

then the function f is injective in D and $Jf(z) > 0$, $z \in D$.

Using the convexity constant of a domain, we can obtain a similar result as follows:

Theorem 2.2. *Let $f : D \subset U \to \mathbb{C}$ be a C^1 function in the domain $D \in C_\varphi$ for some $\varphi \in [0, \pi)$. If*

$$\left|D_\theta \left(\frac{1}{f(z)} - \frac{1}{z}\right)\right| \leq \cos \frac{\varphi}{2}, \qquad z \in D, \tag{2.1}$$

for all $\theta \in [0, 2\pi)$, where D_θ is the operator defined on C^1 functions by

$$D_\theta f = \frac{\partial f}{\partial z} + e^{-2i\theta} \frac{\partial f}{\partial \bar{z}},$$

then the function f is injective in D.

Proof. Let $a, b \in D$, $a \neq b$ be arbitrarily fixed distinct points.

Since $D \in C_\varphi$, by definition, there exists $c \in D$ such that $\gamma = [a, c] \cup [c, b] \subset D$. Let $\gamma_1(t) = a + t(c - a)$, $t \in [0, 1]$ and $\gamma_2(t) = c + t(b - c)$, $t \in [0, 1]$, be two parametrizations of the line segments $[a, c]$, respectively $[c, b]$.

We have:

$$\begin{aligned}
&\frac{1}{f(c)}-\frac{1}{f(a)}-\left(\frac{1}{c}-\frac{1}{a}\right)=\\
&=\int_0^1\frac{d}{dt}\left(\frac{1}{f(\gamma_1(t))}-\frac{1}{\gamma_1(t)}\right)dt\\
&=\int_0^1\frac{\partial}{\partial z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\frac{d\gamma_1(t)}{dt}+\frac{\partial}{\partial\bar z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\frac{d\overline{\gamma_1(t)}}{dt}dt\\
&=\int_0^1(c-a)\frac{\partial}{\partial z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))+\overline{(c-a)}\frac{\partial}{\partial\bar z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\,dt\\
&=(c-a)\int_0^1 D_{\theta_1}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\,dt,
\end{aligned}$$

where $\theta_1=\arg(c-a)$, and similarly

$$\frac{1}{f(b)}-\frac{1}{f(c)}-\left(\frac{1}{b}-\frac{1}{c}\right)=(b-c)\int_0^1 D_{\theta_2}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_2(t))\,dt,$$

where $\theta_2=\arg(b-c)$.

We obtain

$$\begin{aligned}
\frac{1}{f(b)}-\frac{1}{f(a)}-\left(\frac{1}{b}-\frac{1}{a}\right)&=\frac{1}{f(b)}-\frac{1}{f(c)}-\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{1}{f(c)}-\frac{1}{f(a)}-\left(\frac{1}{c}-\frac{1}{a}\right)\\
&=(c-a)\int_0^1 D_{\theta_1}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\,dt+\\
&\quad(c-b)\int_0^1 D_{\theta_2}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_2(t))\,dt,
\end{aligned}$$

and therefore using the hypothesis we have

$$\begin{aligned}
\left|\frac{1}{f(b)}-\frac{1}{f(a)}-\left(\frac{1}{b}-\frac{1}{a}\right)\right|&\le|c-a|\int_0^1\left|D_{\theta_1}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\right|dt+\\
&\quad|b-c|\int_0^1\left|D_{\theta_2}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_2(t))\right|dt\\
&\le|c-a|\cos\frac{\varphi}{2}+|b-c|\cos\frac{\varphi}{2}\\
&=l(\gamma)\cos\frac{\varphi}{2}.
\end{aligned}$$

If $f(a)=f(b)$, from the above inequality we obtain equivalent

$$\frac{|b-a|}{l(\gamma)}\le|ab|\cos\frac{\varphi}{2},$$

where $\gamma = [a, c] \cup [c, b]$.

Approximating now an arbitrary curve $\gamma \in \Gamma(a, b; D)$ by a polygonal path $\gamma_n = [a_0, c_1] \cup \ldots \cup [c_n, b] \subset D$ and using an argument similar to the previous proof, by passing to the limit we obtain:

$$\frac{|b-a|}{l(\gamma)} \leq |ab| \cos \frac{\varphi}{2},$$

for any $\gamma \in \Gamma(a, b; D)$, and therefore

$$\sup_{\gamma \in \Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)} \leq |ab| \cos \frac{\varphi}{2},$$

which shows that

$$K(D) = \inf_{a,b \in D} \sup_{\gamma \in \Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)} \leq \sup_{\gamma \in \Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)} \leq |ab| \cos \frac{\varphi}{2}.$$

Since from Lemma 1.6 we have $K(D) \geq \cos \frac{\varphi}{2} > 0$, we obtain

$$K(D) \leq |ab| \, K(D),$$

which contradicts the hypothesis $a, b \in D \subset U$ (and therefore $|ab| < 1$).

The contradiction shows that the hypothesis $f(a) = f(b)$ is false, and therefore we must have $f(a) \neq f(b)$ for all $a, b \in D$ distinct, which shows that f is injective in D, concluding the proof. □

Following the proof of the above theorem it can be seen that we can replace the right side of (2.1) by the larger constant $K(D)$, thus obtaining the following more general result:

Theorem 2.3. *Let $f : D \subset B(0, R) \to \mathbb{C}$ be a C^1 function in the domain $D \in C_\varphi$ for some $\varphi \in [0, \pi)$. If*

$$\left| D_\theta \left(\frac{1}{f(z)} - \frac{1}{z} \right) \right| \leq \frac{K(D)}{R^2}, \qquad z \in D, \tag{2.2}$$

for all $\theta \in [0, 2\pi)$, where D_θ is the operator defined on C^1 functions by

$$D_\theta f = f_z + e^{-2i\theta} f_{\bar{z}},$$

then the function f is injective in D.

Remark 2.4. Using the values of the convexity constants of the domains D_{Ω} presented in Section 1, from the above theorem we obtain as corollaries sufficient conditions for univalence for functions of class C^1 defined in some simply and doubly connected domains.

Remark 2.5. In the particular case $D = U$, we have $K(U) = 1$, and Theorems 2.2 and 2.3 above become (in the case when $f : U \to \mathbb{C}$ is a normalized analytic function in U) the well-known Ozaki-Nunokawa-Krzyz univalence criterion (see [1], [3]).

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