STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

STARLIKE FUNCTIONS WITH REGULAR REFRACTION PROPERTY

PETRU T. MOCANU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Let $C: z = z(t), t \in [a, b]$, be a smooth Jordan curve of the class C^2 and let f be a complex univalent function of the class C^1 in a domain which contains the curve C together with its interior. Suppose that the origin lies inside of C and $f(0) = 0$. Let $\Gamma = f(C)$ and suppose that Γ is starlike with respect to the origin. Let consider the radius vector $\overrightarrow{R}$ from 0 to a point $w \in \Gamma$ and let $\overrightarrow{N}$ be the outer normal to Γ at the point $w = f[z(t)]$. Let denote by $\omega = (\overrightarrow{N}, \overrightarrow{R})$ the angle between $\overrightarrow{N}$ and $\overrightarrow{R}$ and consider the vector $\overrightarrow{V}$ starting from w, such that $\sin\Psi = \gamma \sin\omega$, where $\Psi = (\overrightarrow{N}, \overrightarrow{V})$ and γ is a positive number. We say that the starlike curve $\Gamma = f(C)$ has the regular refraction property, with index γ, iff the argument of the vector $\overrightarrow{V}$ is an increasing function of $t \in [a, b]$. The concept of regular refraction property was introduced in [2] and developed in [3], [4], [5], [6] and [7]. We mention that this concept is closed to the concept of α-convexity introduced in [1]. In this paper we continue to study this geometric property by introducing the concept of regular refraction interval of a given function. We also give a significant example.

1. **Preliminaries**

Let f an analytic and univalent function in a domain D and let $C: z = z(t)$, $t \in [a, b]$, be a smooth Jordan curve of the class C^2. Suppose that D contains the curve C together with its interior and that the origin lies inside of C and $f(0) = 0$. Let $\Gamma = f(C)$ and suppose that Γ is starlike with respect to 0.

Received by the editors: 02.03.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Starlike functions, regular refraction.

Let $\overrightarrow{R}$ be the radius vector from 0 to a point $w \in \Gamma$ and let $\overrightarrow{N}$ be the outer normal to Γ at the point $w = f(z(t))$. Let denote by $\omega = (\overrightarrow{N}, \overrightarrow{R})$ the angle between $\overrightarrow{N}$ and $\overrightarrow{R}$ and let consider the vector $\overrightarrow{V}$ starting from w, such that

$$\sin\Psi = \gamma \sin\omega, \tag{1.1}$$

where $\Psi = (\overrightarrow{N}, \overrightarrow{V})$ and γ is a positive number.

From the optical point of view, we remark that if Γ separates two media of different refraction indices and if $\overrightarrow{R}$ and $\overrightarrow{V}$ are the trajectories of the light in these media (starling from the origin), then (1) is the well -known refraction law.

Definition 1.1. We say that the curve $\Gamma = f(C)$ has the *regular refraction property* with index γ, iff the argument of the vector $\overrightarrow{V} = \overrightarrow{V}(t)$, defined by (1) is an increasing function of $t \in [a, b]$, i.e.

$$\frac{d}{dt} \arg \overrightarrow{V}(t) \geq 0, t \in [a, b]. \tag{1.2}$$

We also say, in this case, that the function f has the *regular refraction property* on $C : z = z(t)$.

Sometimes we are interesting to study the property of regular refraction only on some arcs of the curve C.

2. Main results

If we let $\varphi = \arg f(z)$ and $\chi = \arg \overrightarrow{V}$, then we have

$$\chi = \varphi + \omega - \psi.$$

If $z = z(t)$ and if we denote $\dot{z}, \dot{\chi}, ...$ the derivatives with respect to t, then we have

$$\dot{\chi} = \dot{\varphi} + F\dot{\omega},$$

where

$$F = 1 - \frac{\gamma \cos\omega}{[1 - \gamma^2 + \gamma^2 \cos^2\omega]^{\frac{1}{2}}} = 1 - \frac{\gamma}{\sqrt{1 + (1 - \gamma^2)\tan^2\omega}}$$

and $\omega = \arg P$, with $|\sin\omega| \leq \frac{1}{\gamma}$ with

$$P = \frac{\dot{z} f'(z)}{i f(z)}, z = z(t). \tag{2.1}$$

The condition (1.2) becomes

$$\Im\Big[iP + F\frac{\dot{P}}{P}\Big] \geq 0, t \in [a, b], \tag{2.2}$$

where

$$F = 1 - \frac{\gamma \Re P}{[(1-\gamma^2)|P|^2 + \gamma^2(\Re P)^2]^{\frac{1}{2}}}, |\sin\omega| \leq \frac{1}{\gamma}, \tag{2.3}$$

with P given by (2.1).

Hence we deduce the following result.

Theorem 2.1. *The function f has the regular refraction property, with index γ, on the curve $C : z = z(t), t \in [a, b]$, if and only if the inequality (2.2) holds for all $t \in [a, b]$.*

If we let $f(z) \equiv z$, then we have $P = i\frac{\dot{z}}{z}$,

$$F = 1 - \frac{\gamma \Im \frac{\dot{z}}{z}}{[(1-\gamma^2)|\frac{\dot{z}}{z}|^2 + \gamma^2(\Im\frac{\dot{z}}{z})^2]^{\frac{1}{2}}} \tag{2.4}$$

and (2.2) becomes

$$(1-F)\Im\frac{\dot{z}}{z} + F\Im\frac{\ddot{z}}{\dot{z}} \geq 0, z = z(t) \tag{2.5}$$

where F is given by (2.4), with $|\sin\omega| \leq \frac{1}{\gamma}$.

Since the curvature of the curve C at the point $z = z(t)$ is given by

$$k = k(t) = \frac{1}{|\dot{z}|}\Im\frac{\ddot{z}}{\dot{z}},$$

the condition (2.5) can be rewritten as

$$\gamma\Big(\Im\frac{\dot{z}}{z}\Big)^2 + \Big\{\Big[(1-\gamma^2)\Big|\frac{\dot{z}}{z}\Big|^2 + \gamma^2\Big(\Im\frac{\dot{z}}{z}\Big)^2\Big]^{\frac{1}{2}} - \gamma\Im\frac{\dot{z}}{z}\Big\}|\dot{z}|k \geq 0 \tag{2.6}$$

and we deduce

Theorem 2.2. *The curve $C : z = z(t), t \in [a, b]$ has the regular refraction property of index $\gamma \geq 0$ if and only if the inequality (2.6) holds for all $t \in [a, b]$.*

If C is convex then $k \geq 0$ and we deduce the following interesting result.

Corollary 2.3. *If the smooth curve C is convex, then it has the regular refraction property of any index $\gamma \in [0, 1]$.*

If we let

$$\Delta = (1-\gamma^2)\Big|\frac{\dot z}{z}\Big|^2 + \gamma^2\Big(\Im\frac{\dot z}{z}\Big)^2,$$

then Theorem 2.2 can be rewritten as

Theorem 2.4. *The curve $C : z = z(t), t \in [a,b]$ has the regular refraction property of index γ if and only if the following inequalities hold for all $t \in [a,b]$:*

(i) $\Delta \leq 0$;

(ii) $\gamma\Big(\Im\frac{\dot z}{z}\Big)^2 + \Big[\sqrt{\Delta} - \gamma\Im\frac{\dot z}{z}\Big]\Im\frac{\ddot z}{\dot z} \geq 0.$

Let f be analytic and univalent in the closed unit disc $\overline{U}$, with $f(0) = 0$ and $f'(0) = 1$. If $C = C_r : re^{it}, t \in [0, 2\pi], 0 < r \leq 1$, then we have

$$P = p(z) = \frac{zf'(z)}{f(z)}.$$

and Theorem 2.1 becomes

Theorem 2.5. *The function f has the regular refraction property of index γ on the circle C_r if and only if*

$$\Re\Big[p(z) + F(z,\gamma)\frac{zp'(z)}{p(z)}\Big] \geq 0, \text{ for } |z| = r, \tag{2.7}$$

where

$$p(z) = \frac{zf'(z)}{f(z)} \tag{2.8}$$

$$F(z,\gamma) = 1 - \frac{\gamma\Re p(z)}{[(1-\gamma^2)|p(z)|^2 + \gamma^2(\Re p(z))^2]^{\frac{1}{2}}} \tag{2.9}$$

and

$$(1-\gamma^2)|p(z)|^2 + \gamma^2(\Re p(z))^2 \geq 0. \tag{2.10}$$

Definition 2.6. We say that the normalized analytic and univalent function f in the unit disc belongs to the class $\mathcal{RP}(\gamma)$, of functions with *regular refraction property* of index γ iff

$$\Re J(f; z, \gamma) \geq 0, \text{ for all } z \in U, \tag{2.11}$$

$$J(f; z, \gamma) = p(z) + F(z,\gamma)\frac{zp'(z)}{p(z)}, \tag{2.12}$$

with p and F given by (2.8), and (2.9) respectively, with condition (2.10).

Let S^* and K be respectively the class of starlike and convex functions in the unit disc.

Also, let $M(\alpha)$ be the class of α-convex functions in U.

It is easy to prove the following main result:

Theorem 2.7. *If* $f \in \mathcal{RP}(\gamma), 0 \le \gamma \le 1$ *then* $f \in S^*$.

Moreover

$$K \subset \mathcal{RP}(\gamma_1) \subset \mathcal{RP}(\gamma_2) \subset S^*, \text{ for } 0 < \gamma_1 < \gamma_2 < 1$$

and

$$K \subset \mathcal{RP}(1-\alpha) \subset M(\alpha), \text{ for } 0 < \alpha < 1.$$

We also have

$$\mathcal{RP}(\gamma_2) \subset \mathcal{RP}(\gamma_1) \subset S^*, \text{ for } 1 < \gamma_1 < \gamma_2.$$

An interesting extremal problem suggested by Theorem 2.7 is the following:

Given the function f, find the largest interval $[\gamma_0, \gamma_1]$, with $\gamma_0 \le 1 \le \gamma_1$, such that $f \in \mathcal{RP}(\gamma)$, for all $\gamma \in [\gamma_0, \gamma_1]$. We shall call this interval as *the regular refraction interval* of the function f.

We illustrate this last problem by the following.

Example 2.8. Let

$$f(z) = z \exp\left(\frac{z^n}{2n}\right), z \in \overline{U}.$$

In this case we have

$$p(z) = \frac{1}{2}(2 + z^n) \text{ and } \frac{zp'(z)}{p(z)} = \frac{nz^n}{2 + z^n}.$$

If $z = e^{it}$, then we have

$$\cos nt = x - 1, \text{ with } 0 \le x \le 2$$

and

$$|p(z)|^2 = \frac{1}{4}(1 + 4x), \Re p(z) = \frac{1}{2}(1 + x), \Re\frac{zp'(z)}{p(z)} = n\frac{2x - 1}{1 + 4x}.$$

Hence

$$F(z, \gamma) = 1 - \frac{\gamma(1 + x)}{\sqrt{E(x, \gamma)}},$$

where

$$E(x,\gamma) = 1 + 2(2-\gamma^2)x + \gamma^2x^2.$$

Hence the inequality (2.7) becomes

$$\frac{1}{2}(1+x) + n\Big(1 - \frac{\gamma(1+x)}{\sqrt{E(x,\gamma)}}\Big)\frac{2x-1}{1+4x} \le 0, \text{ for } 0 \le x \le 2. \tag{2.13}$$

We remark that for $\gamma \le 2$ we have

$$E(x,\gamma) \ge 0, \text{ for } x \in [0,2].$$

For $x = 0$ we have $\frac{2n-1}{2n} \le \gamma < 2$, and for $x = 2$ we have $\gamma < 1 + \frac{9}{2n}$.

From (2.13) we deduce

$$\frac{1}{\gamma^2} \ge \frac{1}{1+4x}\Big\{x(2-x) + \Big[\frac{2n(2x+x-1)}{4x^2+(4n+5)x+1-2n}\Big]^2\Big\} \equiv \Phi_n(x),$$

with $\frac{1}{2} < x \le 2$.

For $n = 1$ we have

$$\max_{x\in\left[\frac{1}{2},2\right]} \Phi_1(x) = 0.25059...$$

and we deduce that the regular refraction interval of the function

$$f(z) = z\exp\left(\frac{z}{2}\right)$$

is given by $\left[\frac{1}{2}, 1.9976\cdots\right]$.

For $n = 2$ we have

$$\max_{x\in\left[\frac{1}{2},2\right]} \Phi_2(x) = 0.2934...$$

and we deduce that the regular refraction interval of the function

$$f(z) = z\exp\left(\frac{z^2}{4}\right)$$

is given by $\left[\frac{3}{4}, 1.9123\cdots\right]$.

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Babeş-Bolyai University
Faculty of Mathematics and Computer Science
400084 Cluj-Napoca, Romania
E-mail address: `pmocanu@math.ubbcluj.ro`