NEW INVERSE INTERPOLATION METHODS

ALEXANDRA OPRIŞAN

Abstract. The goal of this paper is to give some numerical methods for the solution of nonlinear equations, generated by inverse interpolation of Abel Goncharov type and a particular case of Lidstone inverse interpolation.

1. Preliminars

Let $\Omega \subset \mathbf{R}$ and $f : \Omega \to \mathbf{R}$. Consider the equation

$$f(x) = 0, \quad x \in \Omega, \tag{1}$$

and attach to it a mapping

$$F : D \to D, \quad D \subset \Omega^n.$$

Let $x_0, ..., x_{n-1} \in D$. Using the mapping F and the numbers $x_0, ..., x_{n-1}$ we construct iteratively the sequence

$$x_0, x_1, ..., x_{n-1}, x_n, ... \tag{2}$$

where

$$x_i = F(x_{i-n}, ..., x_{i-1}), \quad i = n, ... \tag{3}$$

The problem is to choose F and the numbers $x_0, ..., x_{n-1} \in D$ such that sequence (2) converges to a solution of equation (1).

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Definition 1. *The method of approximating a solution of equation (1) by the elements of sequence (2), computed as in (3) is called F - method attached to equation (1) and to the values $x_0, ..., x_{n-1}$. Numbers $x_0, ..., x_{n-1}$ are called starting values, and the pth element of sequence (2) is called pth order approximation of the solution. If the set of the starting values consists of a single element, the corresponding F - method is called one step method, otherwise it is called multi-step method.*

Definition 2. *If sequence (2) converges to a solution of equation (1), F - method is said to be convergent, otherwise is divergent.*

Definition 3. *Let $x^* \in \Omega$ be a solution of equation (1) and let $x_0, ..., x_n, ...$ be a sequence generated by a given F - method. Number $p = p(F)$ having the property*

$$\lim_{x_i \to x^*} \frac{x^* - F(x_{i-n+1}, ..., x_i)}{(x^* - x_i)^p} = C \neq 0, \tag{4}$$

is called order of the F - method, and constant C is the asymptotical error.

Let $x^* \in \Omega$ be a solution of the equation (1) and $V(x^*)$ a neighborhood of x^*. Assume that f has inverse on $V(x^*)$ and denote $q = f^{-1}$. Since $f(x^*) = 0$, it follows that $x^* = g(0)$. This way, the approximation of the solution x^* is reduced to the approximation of the $g(0)$. The approximation of the inverse g by means of a certain interpolating method, and x^* by the value of the interpolating element at point zero is called inverse interpolation procedure. This approach generates a large number of approximation methods for the solution of an equation (thus for the zeros of a function), according to the employed interpolation method.

Such examples of methods, based on Taylor, Lagrange and Hermite inverse interpolation are:

Let x^* be a solution of $f(x) = 0$, $V(x^*)$ a neighbourhood of x^*, $f \in C^m[V(x^*)], f'(x) \neq 0$ for $x \in V(x^*)$ and $x_i \in V(x^*)$. Using Taylor polynomial of the degree $m-1$, that interpolates the function $g = f^{-1}$, one obtains the one step method [2]:

$$F_m^T(x_i) = x_i + \sum_{k=1}^{m-1} \frac{(-1)^k}{k!}[f(x_i)]^k g^{(k)}(f(x_i)). \tag{5}$$

Also, if $g^{(m)}(0) \neq 0$, we have $ord(F_m^T) = m$.

Based on Lagrange interpolation, it follows the multistep method [2]

$$F_m^L(x_0, ..., x_m) = \sum_{k=0}^{m} \frac{f_0 ... f_{k-1} f_{k+1} ... f_m}{(f_0 - f_k) .../...(f_m - f_k)} x_k \tag{6}$$

where $f_k = f(x_k)$, is a multistep method based on inverse Lagrange interpolation.

The order of this method is the solution of equation:

$$t^{m+1} - t^m - ... - t - 1 = 0.$$

More general methods are generated by Hermite and Birkhoff interpolation [2], [5]. Such, let x^* be a solution of the equation (1), $V(x^*)$ a neighbourhood of x^* and $x_0, x_1 ..., x_m \in V(x^*)$.For $n = r_0 + ... + r_m + m$, where r_k represents the multiplicity order of the point x_k, $k = 0, ..., m$, if $f \in C^{n+1}(V(x^*))$ and $f'(x) \neq 0$ for $x \in V(x^*)$, we have the following Hermite approximation method:

$$F_n^H(x_0, ..., x_m) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} \sum_{\upsilon=0}^{r_k - j} \frac{(-1)^{j+\upsilon}}{j!\upsilon!} f_k^{j+\upsilon} v_k(0) (\frac{1}{v_k(y)})^{(\upsilon)}_{y=f_k} g^{(j)}(f_k) \tag{7}$$

where $f_k = f(x_k), k = 0, ..., m, g = f^{-1}$, and

$$v_k(y) = (y - f_0)^{r_0+1} ... (y - f_{k-1})^{r_{k-1}+1} (y - f_{k+1})^{r_{k+1}+1} ... (y - f_m)^{r_m+1}$$

The order of F_n^H, is [5] the unique real positive root of the equation:

$$t^{m+1} - r_m t^m - r_{m-1} t^{m-1} - .... - r_1 t - r_0 = 0. \tag{8}$$

where $r_0, ..., r_m$ are permutation of the multiplicity orders of the nodes x_k, $k = 0, ..., m$ satisfying the conditions:

$$(1) \quad r_0 + r_1 + ... + r_m > 1$$

$$(2) \quad r_m \geq r_{m-1} \geq ... \geq r_1 \geq r_0,$$

respectively of the equation:

$$t^{m+1} - (r+1)\sum_{j=0}^{m} t^j = 0. \tag{9}$$

if $r_0 = ... = r_m$.

2. Abel-Goncharov inverse interpolation method

On the base of Abel-Goncharov interpolation, we have the following method for the solution of equation $f(x) = 0$:

Theorem 4. *Let $n \in N$; $a, b \in R$; $a < b$; $f : [a,b] \to R$ be a function having n derivatives $f^{(i)}, i = 1, 2, ..., n$. The values $x_i \in [a,b]$, $i = 0, ..., n$ and $f^{(i)}(x_i)$, $i = 0, ..., n$, with $x_i \neq x_j$ for $i \neq j$ are given. Let x^* be the solution of the equation $f(x) = 0$ and $V(x^*)$ a neighborhood of x^*. If $f \in C^{n+1}(V(x^*))$ and $f^{(i)}(x_i) \neq 0$, $i = 0, ..., n$ then we have the following method of Abel-Gonciarov type:*

$$F_n^{AG}(x_0, ..., x_n) = q(y_0) - y_0.q'(y_1) - \sum_{k=2}^{n} \frac{q^{(k)}(y_k)}{k!} \left(\sum_{j=0}^{k-1} g_j(0) \binom{k}{j} y_j^{k-1} \right) \tag{10}$$

Proof. Suppose that $\exists\ q = f^{-1}$. Then

$$q = P_n q + R_n q$$

with

$$(P_n q)(y) = \sum_{k=0}^{n} g_k(y)\, q^{(k)}(y_k)$$

and

$$g_0(y) = 1$$

$$g_1(y) = y - y_0$$

$$g_k(y) = \frac{1}{k!}\left[y^k - \sum_{j=0}^{k-1} g_j(y) \binom{k}{j} y_j^{k-1} \right]$$

Because $x^* = q(0)$, $q \simeq P_n q \Longrightarrow x^* \simeq (P_n q)(0)$

$$(P_n q)(0) = \sum_{k=0}^{n} g_k(0)\, q^{(k)}(y_k)$$

$$(P_n q)(0) = q(y_0) - y_0.q'(y_1) - \sum_{k=2}^{n} \frac{q^{(k)}(y_k)}{k!} \left(\sum_{j=0}^{k-1} g_j(0) \binom{k}{j} y_j^{k-1} \right)$$

$$\Longrightarrow x^* \simeq q(y_0) - y_0.q'(y_1) - \sum_{k=2}^{n} \frac{q^{(k)}(y_k)}{k!} \left(\sum_{j=0}^{k-1} g_j(0) \binom{k}{j} y_j^{k-1} \right) :=$$

$$:= F_n^{AG}(x_0, ..., x_n).$$

□

Particular cases.

1). $n = 1$ (nodes x_0, x_1 and $f(x_0), f'(x_1)$ given)

$$F_1^{AG}(x_0, x_1) = q(y_0) - y_0.q'(y_1)$$

$$F_1^{AG}(x_0, x_1) = q(y_0) - y_0 \frac{1}{f'(x_1)}$$

$$\Longrightarrow F_1^{AG}(x_0, x_1) = x_0 - \frac{f(x_0)}{f'(x_1)} \tag{11}$$

$\Longrightarrow F_1^{AG}(x_0, x_1) = F_1^B(x_0, x_1)$ and the method F_1^{AG} coincide with the method F_1^B generated by the Birkhoff inverse interpolation.

Remark 5. *If $x_0 = x_1 := x_i$ (the nodes coincide), then:*
$F_1^{AG}(x_i) = x_i - \dfrac{f(xi)}{f'(x_i)} \Longrightarrow$
$F_1^{AG}(x_i) = F_2^T(x_i)$ *and the method coincide with the method F_2^T generated by inverse interpolation Taylor for two nodes.*

The order of this method is the solution of the equation:

$$t^2 - t - 1 = 0$$

so

$$ord(F_1^{AG}) = \frac{1+\sqrt{5}}{2}$$

2). $n = 2$. $(x_0, f(x_0), x_1, f'(x_1), x_2, f''(x_2)$ given)

$$g_0(0) = 1$$

$$g_1(0) = -y_0$$

$$g_2(0) = \frac{1}{2}[2y_0y_1 - y_0^2]$$

$$\Longrightarrow (P_2q)(0) = q(y_0) - y_0.q'(y_1) - \tfrac{1}{2}[2y_0y_1 - y_0^2].q''(y_2) =$$
$$= x_0 - \frac{f(x_0)}{f'(x_1)} - \frac{1}{2}\frac{f''(x_2)}{[f'(x_2)]^3}[2f(x_0)f(x_1) - f(x_0)^2] \Longrightarrow$$

$$F_2^{AG}(x_0, x_1, x_2) = x_0 - \frac{f(x_0)}{f'(x_1)} - \frac{1}{2}\frac{f''(x_2)}{[f'(x_2)]^3}[2f(x_0)f(x_1) - f(x_0)^2]. \qquad (12)$$

Remark 6. *For $x_0 = x_1 = x_2 := x_i$, the method coincide with the method generated by Taylor inverse interpolation, for $n = 3$.*

$$F_3^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{1}{2}\left[\frac{f(x_i)}{f'(x_i)}\right]^2 \frac{f''(x_i)}{f'(x_i)}.$$

The order of this method is the solution of the equation:

$$t^3 - t^2 - t - 1 = 0$$

so

$$ord(F_2^{AG}) = 1.839$$

3. **Lidstone inverse interpolation method**

For the particular case of Lidstone interpolation, on $[x_0, x_1], x_0 \neq x_1, i = \overline{0,1}, m = 2$, and

$$\begin{cases} L_{2i+1}f = f^{(2i)}(x_0) \\ L_{2i+2}f = f^{(2i)}(x_1) \end{cases}$$

it follows that

$$\left(L_2^{\Delta}f\right)|_{[x_0,x_1]}(x) = \sum_{k=0}^{1}\left[\Lambda_k\left(\frac{x_1 - x}{h}\right)f^{(2k)}(x_0) + \Lambda_k\left(\frac{x - x_0}{h}\right)f^{(2k)}(x_1)h^{2k}\right]$$

where

$$\begin{cases} \Lambda_0(x) = x \\ \Lambda_1''(x) = \Lambda_0(x) = x \\ \Lambda_1(0) = \Lambda_1(1) = 0 \end{cases}$$

The interpolation polynomial is:

$$\left(L_2^{\Lambda}f\right)(x) = \sum_{i=0}^{1}\sum_{j=0}^{1} r_{m,i,j}(x) f^{(2j)}(x_i)$$

$\Longrightarrow \left(L_2^\Lambda f\right)(x) = r_{2,0,0}(x) f(x_0) + r_{2,0,1}(x) f''(x_0) + r_{2,1,0}(x) f(x_1) + r_{2,1,1}(x) f''(x_1)$ where

$$r_{2,0,j}(x) = \Lambda_j \left(\frac{x_1 - x}{h}\right) h^{2j}, 0 \leq x \leq x_1; i = 0$$

$$r_{2,1,j}(x) = \Lambda_j \left(\frac{x - x_0}{h}\right) h^{2j}, x_0 \leq x \leq x_1; i = 1$$

$r_{2,0,0}(x) = \Lambda_0 \left(\frac{x_1 - x}{h}\right) h = x_1 - x$
$r_{2,0,1}(x) = \Lambda_1 \left(\frac{x_1 - x}{h}\right) h^2$
$r_{2,1,0}(x) = \Lambda_0 \left(\frac{x - x_0}{h}\right) h = x - x_0$
$r_{2,1,1}(x) = \Lambda_1 \left(\frac{x - x_0}{h}\right) h^2$ but

$$\Lambda_1(x) = \int_0^1 g_1(x, s)\, sds = \int_0^x (x-1) s^2 ds + \int_x^1 (s-1) xssds = \frac{x^3 - x}{6} + c$$

$\Lambda_1(0) = \Lambda_1(1) = 0 \Longrightarrow c = 0$

and

$$r_{2,0,1}(x) = \Lambda_1 \left(\frac{x_1 - x}{h}\right) h^2 = \frac{1}{6h}(x_1 - x)(x_1 - x - h)(x_1 - x + h)$$

$$r_{2,1,1}(x) = \Lambda_1 \left(\frac{x - x_0}{h}\right) h^2 = \frac{1}{6h}(x - x_0)(x - x_0 - h)(x - x_0 + h)$$

We know that for $g = f^{-1}$,

$$g = L_2^\Lambda g + R_2^\Lambda g$$

and $x^* = g(0), g \simeq L_2^\Lambda g \Longrightarrow x^* \simeq L_2^\Lambda g(0)$.

$$L_2^\Lambda g(0) = x_1 g(x_0) + \frac{x_1}{6h}\left(x_1^2 - h^2\right) g''(x_0) - x_0 g(x_1) + \frac{x_0}{6h}\left(h^2 - x_0^2\right) g''(x_1)$$

$$\Longrightarrow x^* = x_1 g(x_0) + \frac{x_1}{6h}\left(x_1^2 - h^2\right) g''(x_0) - x_0 g(x_1) + \frac{x_0}{6h}\left(h^2 - x_0^2\right) g''(x_1)$$

and so we have the following method:

$$F_2^\Lambda(x_0, x_1) = x_1 g(x_0) + \frac{x_1}{6h}\left(x_1^2 - h^2\right) g''(x_0) - x_0 g(x_1) + \frac{x_0}{6h}\left(h^2 - x_0^2\right) g''(x_1)$$

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Babeş-Bolyai University, Kogălniceanu 1, Cluj-Napoca, Romania

E-mail address: sachaoprisan@yahoo.com