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ANALYSIS OF AN INTEGRAL EQUATION WITH MODIFIED ARGUMENT

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Abstract. This paper contains a study of the Fredholm integral equation with modified argument

$$(1)\qquad x(t)=\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds+f(t),\ \ t\in[a,b],$$

concerning:

- the existence and uniqueness of the solution using Schauder's theorem and Contractions Principle;

- continuous dependence on data of the solution using data dependence general theorem;

- approximation of the solution using successive approximations method with two quadrature formula: the trapezoidal rule and the rectangle quadrature formula.

1. **Notations and preliminaries**

Let X be a nonempty set, $A : X \to X$ an operator and we shall use the following notation:

$$F_A := \{x \in X|\ A(x) = x\} \text{ - the fixed point set of } A.$$

We consider the Banach space $X = C\,[a, b]$ endowed with the Chebyshev norm $\|\cdot\|$.

In the section 2 we need the following results (see [2], [8], [9], [10] and [12]).

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Theorem 1.1. (*Schauder*). Let X be a Banach space and $Y \subset X$ a nonempty, bounded, convex and closed set. If $A : Y \to Y$ is a completely continuous operator, then A has at least one fixed point.

Theorem 1.2. (*Contractions Principle*). Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction ($\alpha < 1$). In these conditions we have:

(i) $F_A = \{x^*\}$;

(ii) $A^n(x_0) \to x^*$, as $n \to \infty$;

(iii) $d(x^*, A^n(x_0)) \leq \dfrac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

In the section 3 we need the following result (see [2], [8], [9], [10] and [12]).

Theorem 1.3. (*Dependence on data).* Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that:

(i) A is an α-contraction ($\alpha < 1$) and $F_A = \{x^*\}$;

(ii) $x_B^* \in F_B$;

(iii) there exist $\eta > 0$ such that $d(A(x), B(x)) < \eta$ for all $x \in X$.

In these conditions we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha} .$$

In the section 4 we need the following results (see [2], [7], [8], [9] and [12]).

We will use for the calculus of the integrals of the succesive approximations sequence, two quadrature formulae:

1) The trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] + R^T(f) , \tag{1}$$

with a very sharp division of the interval $[a, b]$ through the points $a = x_0 < x_1 < \ldots < x_n = b$ and $f \in C^2[a, b]$. We use for the rest of the formula $R^T(f) = \sum_{i=1}^{n} R_i^T(f)$ the following estimation:

$$\left|R^T(f)\right| \leq M^T \frac{(b-a)^3}{12n^2} . \tag{2}$$

2) The rectangle quadrature formula

(a) If we consider the intermediary points of the division of the interval $[a, b]$ at the left terminal point of the partial intervals $[x_i,\, x_{i+1}]$, $\xi_i = x_i$, we will have the following formula:

$$\int_a^b f(x)dx = \frac{b-a}{n}\left[f(a) + \sum_{i=1}^{n-1} f(x_i)\right] + R^D(f)\,, \tag{3}$$

or

(b) If we consider the intermediary points of the division of the interval $[a, b]$ at the right terminal point of the partial intervals $[x_i,\, x_{i+1}]$, $\xi_i = x_{i+1}$, we will have the following formula:

$$\int_a^b f(x)dx = \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b)\right] + R^D(f)\ , \tag{4}$$

with a very sharp division of the interval $[a, b]$ through the points $a = x_0 < x_1 < \ldots < x_n = b$ and $f \in C^1[a, b]$. We use for the rest of the formula $R^D(f) = \sum_{i=1}^{n} R_i^D(f)$ the following estimation:

$$\left|R^D(f)\right| \le M^D \frac{(b-a)^2}{n}\ . \tag{5}$$

2. **Existence of the solution**

Theorems of existence of the solution for several type of integral equations with modified argument have been presented in the papers [1], [2], [5], [9], [10], [11], [12] **.**

In what follows we will establish theorems of existence of the solution of the integral equation (1) in $C[a, b]$ and in the $\overline{B}(f; R)$ sphere.

A. Existence of the solution in $C[a, b]$

Let us consider the Fredholm integral equation with modified argument (1) and assume that the following conditions are satisfied:

(a_1) $K \in C([a, b] \times [a, b] \times \mathbb{R}^4)$;

(a_2) $f \in C[a, b]$;

(a_3) $g \in C\,([a, b],\, [a, b])$.

Theorem 2.1. Suppose (a_1)-(a_3) are satisfied. In addition suppose

(a_4) there exist $M_K > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4)| \le M_K\,, \quad for\ \ all\ \ t\in[a,b],\ u_1,u_2,u_3,u_4\in\mathbb{R}\,.$$

Then the integral equation (1) has at least one solution $x^* \in C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : C[a,b] \to C[a,b]$, defined by

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \tag{6}$$

for all $t\in[a,b]$.

We have

$$\|A(x)\|_{C[a,b]} \le \|f\|_{C[a,b]} + M_K(b-a)\,, \quad for\ \ all\ \ x\in C[a,b]\ .$$

Let $Y \subset C[a,b]$ be a nonempty, bounded subset. Then $A(Y)$ is also a bounded subset. From the uniform continuity of K with respect to t, it follows that the operator A is continuous and that the subset $A(Y)$ is equicontinuous. Therefore $\overline{A(Y)}$ is a compact subset.

Let be $Y = \overline{conv}A(C[a,b])$ and now Y is a nonempty, bounded, convex and closed subset. We consider the operator $A : Y \to Y$ also noted with A and defined by same relation (7). Y is an invariant subset by A.

On the other hand, by *Arzela-Ascoli theorem*, A is completely continuous.

The conditions of the *Schauder's theorem* are satisfied. □

We have the following theorem of existence and uniqueness of the solution of the integral equation (1) in $C[a,b]$:

Theorem 2.2. Suppose (a_1)-(a_3) are satisfied. In addition suppose

(a_5) there exist $L > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| \le$$

$$\le L\left(|u_1-v_1| + |u_2-v_2| + |u_3-v_3| + |u_4-v_4|\right),$$

for all $t,s\in[a,b]$, $u_i,\ v_i\in\mathbb{R}$, $i=\overline{1,4}$;

(a_6) $4L(b-a) < 1$.

Then the integral equation (1) has a unique solution $x^* \in C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : C[a,b] \to C[a,b]$, defined by the relation (7). The set of the solutions of the integral equation (1) coincide with the set of fixed points of the operator A. By (a_5) and using the Chebyshev norm, we have

$$\|A(x_1) - A(x_2)\|_{C[a,b]} \leq 4L\,(b-a)\,\|x_1 - x_2\|_{C[a,b]}$$

and therefore, by (a_6) it result that the operator A is an α-contraction with the coefficient $\alpha = 4L\,(b-a)$. The conclusion result from the *Contractions Principle.* □

B. Existence of the solution in the $\overline{B}(f;R)$ sphere

We suppose the following conditions are satisfied:

(a_1') $K \in C([a,b] \times [a,b] \times J^4)$, $J \subset \mathbb{R}$ closed interval;

and (a_2), (a_3).

In addition, we denote M_K a positive constant such that, for the restriction $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \mathbb{R}$ compact, we have

$$|K(t,s,u_1,u_2,u_3,u_4)| \leq M_K\,, \quad for\ all\ \ t \in [a,b],\ u_1,u_2,u_3,u_4 \in J\,. \tag{7}$$

We have the following theorem of existence of the solution of the integral equation (1) in $\overline{B}(f;R) \subset C[a,b]$:

Theorem 2.3. Suppose (a_1'), (a_2), (a_3) are satisfied. In addition suppose

(b_1) $M_K(b-a) \leq R$ (the invariability condition of the $\overline{B}(f;R)$ sphere).

Then the integral equation (1) has at least one solution $x^* \in \overline{B}(f;R) \subset C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : \overline{B}(f;R) \to C[a,b]$, defined by the relation (7), where R is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;R)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}]$$

and we suppose that there exist at least one number R with this property.

We establish under what conditions, the $\overline{B}\,(f;R)$ sphere is an invariant set for the operator A . We have

$$|A(x)(t)-f(t)|=\left|\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds\right|\leq$$

$$\leq\int_a^b |K(t,s,x(s),x(g(s)),x(a),x(b))|\,ds$$

and by (8) we have

$$|A(x)(t)-f(t)|\leq M_K(b-a),\;\; for\;\; all\;\; t\in[a,b]\,,$$

and then by (b_1) it result that the $\overline{B}(f;R)$ sphere is an invariant set for the operator A. Now we have the operator $A:\overline{B}(f;R)\to\overline{B}(f;R)$, also noted with A, defined by same relation, where $\overline{B}(f;R)$ is a closed subset of the Banach space $C[a,b]$.

Next we assure the conditions of the *Schauder's theorem.*

We have

$$\|A(x)\|_{C[a,b]}\leq\|f\|_{C[a,b]}+R\,,\;\; for\;\; all\;\; x\in\overline{B}(f;R)$$

and it follows that the subset $A(\overline{B}(f;R))$ is bounded. From the uniform continuity of K with respect to t, it follows that the subset $A(\overline{B}(f;R))$ is equicontinuous. Now it result that $A(\overline{B}(f;R))$ is a compact subset.

Also, from the uniform continuity of K with respect to t, it follows that the operator A is continuous. On the other hand, by *Arzela-Ascoli theorem*, A is completely continuous. The proof follows the *Schauder's theorem.* □

Theorem 2.4. Suppose the conditions (a_1'), (a_2), (a_3), (b_1) and (a_6) are satisfied. In addition suppose

(b_2) there exist $L>0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4)-K(t,s,v_1,v_2,v_3,v_4)|\leq$$

$$\leq L\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|\right),$$

for all $t,s\in[a,b]$, $u_i,\,v_i\in J$, $i=\overline{1,4}$;

Then the integral equation (1) has a unique solution $x^*\in\overline{B}(f;R)\subset C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : \overline{B}(f;R) \to C[a,b]$, defined by the relation (7), where R is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;R)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}]$$

and we suppose that there exist at least one number R with this property.

If we use a reasoning as the one used in the proof of theorem 2.3, we will obtain that the $\overline{B}(f;R)$ sphere is an invariant set for the operator A, and the invariability condition (b_1), of the $\overline{B}(f;R)$ sphere is hold.

Now we have the operator $A : \overline{B}(f;R) \to \overline{B}(f;R)$, also noted with A, defined by same relation, where $\overline{B}(f;R)$ is a closed subset of the Banach space $C[a,b]$. The set of the solutions of the integral equation (1) coincide with the set of fixed points of the operator A.

By a similar reasoning as in the proof of theorem 2.2 and using the condition (b_2) it result that the operator A is an α-contraction with the coefficient $\alpha = 4L\,(b-a)$.

Now the proof result from the *Contractions Principle.* □

3. **Dependence on data**

Theorems of dependence on data for several type of integral equations with modified argument have been presented in the papers [5], [6], [9], [12].

In what follows we consider the integral equation (1) and we will study the dependence of the solution of the integral equation (1) with respect to K and f .

Now we consider the perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t)\,, \quad t \in [a,b] \tag{8}$$

and we have the following theorem of dependence on data of the solution of the integral equation (1):

Theorem 3.1. Suppose

(i) the conditions of the theorem 2.2 are satisfied and denote x^* the unique solution of the integral equation (1).

(ii) $H \in C([a,b] \times [a,b] \times \mathbb{R}^4)$ and $h \in C[a,b]$;

(iii) there exist $\eta_1,\ \eta_2 > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4)| \le \eta_1\ ,$$

$for\ \ all\ \ t,s \in [a,b],\ u_1,u_2,u_3,u_4 \in \mathbb{R}$ and

$$|f(t) - h(t)| \le \eta_2 \quad for\ \ all\ \ t \in [a,b]\,.$$

In these conditions, if y^* is a solution of the integral equation (9), then we have:

$$\|x^* - y^*\| \le \frac{\eta_1(b-a) + \eta_2}{1 - 4L(b-a)}\,.$$

Proof. We consider the operator A which appear in the proof of the theorem 2.2.

Let $B : C[a,b] \to C[a,b]$ be an operator defined by

$$B(y)(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t)\,, \quad t \in [a,b]\ .$$

By the condition (iii) we have

$$\|A(x) - B(x)\| \le \eta_1(b-a) + \eta_2\ .$$

The proof result from *data dependence general theorem.* □

4. **Approximation of the solution**

Approximative methods for various type of integral equations with modified argument have been presented in the papers [1], [2], [3], [4], [7], [8], [9] **.**

We will determine as follows, a method for the approximation of the solution of the integral equation (1).

We suppose that the conditions of one of the two existence and uniqueness theorems from section 2 are satisfied. In order to lay down the ideas we consider the case of the integral equation (1) with a unique solution in the sphere $\overline{B}(f;R) \subset C[a,b]$

(theorem 2.4), called x^* and established using the successive approximation method. We have the sequence of the successive approximations:

$$\begin{aligned} x_0(t) &= f(t) \\ x_1(t) &= \int_a^b K(t,s,x_0(s),x_0(g(s)),x_0(a),x_0(b))ds + f(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m(t) &= \int_a^b K(t,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))ds + f(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and we consider a division of the interval $[a,b]$ through the points $a = x_0 < x_1 < \ldots < x_n = b$.

A. Approximation of the solution using the trapezoidal rule

We suppose that:
(h_{11}) $K \in C^2([a,b]\times[a,b]\times J^4)$, $J \subset \mathbb{R}$ closed interval ;
(h_{12}) $f \in C^2[a,b]$;
(h_{13}) $g \in C^2([a,b],[a,b])$

and we will approximate the terms of the successive approximations sequence using the trapezoidal rule (2) with the rest from (3). Generally, for the term $x_m(t_k)$ we have

$$\begin{aligned} x_m(t_k) = \frac{b-a}{2n}&[K(t_k,a,x_{m-1}(a),x_{m-1}(g(a)),x_{m-1}(a),x_{m-1}(b))+ \\ &+2\sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(g(t_i)),x_{m-1}(a),x_{m-1}(b))+ \\ &+K(t_k,b,x_{m-1}(b),x_{m-1}(g(b)),x_{m-1}(a),x_{m-1}(b))] + f(t_k) + R^T_{m,k}\,, \end{aligned} \tag{9}$$

$k = \overline{0,n}$, $m \in \mathbb{N}$, with the estimation of the rest

$$\left|R^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2}\cdot \max_{s\in[a,b]} \left|[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))]''_s\right| .$$

According to (h_{11}) it result that the derivative of the function K from the expression of the rest $R^T_{m,k}$ exist and has the following form:

$$[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))]''_s = \frac{\partial^2 K}{\partial s^2} +$$

$$+2\frac{\partial^2 K}{\partial s \partial x_{m-1}} \cdot x'_{m-1}(s) + 2\frac{\partial^2 K}{\partial s \partial x_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot g'(s) +$$

$$+\frac{\partial^2 K}{\partial x^2_{m-1}} \left(x'_{m-1}(s)\right)^2 + 2\frac{\partial^2 K}{\partial x^2_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot x'_{m-1}(s) \cdot g'(s) +$$

$$+\frac{\partial K}{\partial x_{m-1}} \cdot x''_{m-1}(s) + \frac{\partial^2 K}{\partial x^2_{m-1}} \cdot \left(\frac{\partial x_{m-1}}{\partial g}\right)^2 \cdot \left(g'(s)\right)^2 +$$

$$+\frac{\partial K}{\partial x_{m-1}} \cdot \frac{\partial^2 x_{m-1}}{\partial g \partial s} \cdot x'_{m-1}(s) \cdot g'(s) + \frac{\partial K}{\partial x_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot g''(s)\,,$$

where

$$x^{(\alpha)}_{m-1}(t) = \int_a^b \frac{\partial^{(\alpha)} K(t, s, x_{m-2}(s), x_{m-2}(g(s)), x_{m-2}(a), x_{m-2}(b))}{\partial t^{(\alpha)}} ds + f^{(\alpha)}(t)\,,$$

$\alpha = 1, 2$.

If we denote

$$M^T_1 = \max_{|\alpha| \le 2, t, s \in [a,b]} \left| \frac{\partial^{|\alpha|} K}{\partial t^{\alpha_1} \partial s^{\alpha_2} \partial u_1^{\alpha_3} \partial u_2^{\alpha_4} \partial u_3^{\alpha_5} \partial u_4^{\alpha_6}} \right| ,$$

$$M^T_2 = \max_{\alpha \le 2, t \in [a,b]} \left| f^{(\alpha)}(t) \right| , \quad M^T_3 = \max_{\alpha \le 2, t \in [a,b]} \left| g^{(\alpha)}(t) \right| ,$$

then we obtain for $x_{m-1}(t)$ and its derivative, the following estimations:

$$\left| x^{(\alpha)}_{m-1}(t) \right| \le M^T_1 (b-a) + M^T_2 \ , \qquad \alpha = \overline{0,2}$$

while for the derivative of function K, we have

$$[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))]''_s \le M^T_1 \left\{1 + 3\left[M^T_1(b-a) + \right.\right.$$

$$\left.\left. + M^T_2\right]\left(1 + M^T_3\right) + \left[M^T_1(b-a) + M^T_2\right]^2 \left[1 + 3M^T_3 + \left(M^T_3\right)^2\right]\right\} = M^T_0 \ .$$

It is obvious that M^T_0 doesn't depend on m and k, so the estimation of the rest is

$$\left| R^T_{m,k} \right| \le M^T_0 \cdot \frac{(b-a)^3}{12n^2}\,, \qquad (10)$$

where $M_0^T = M_0^T\left(K, D^{(\alpha)}K, f, D^{(\alpha)}f, g, D^{(\alpha)}g\right)$, $|\alpha| \le 2$, and we obtain a formula for the approximative calculus of the integrals of the successive approximations sequence. Using the method of successive approximations and the formula (10) with the estimation of the rest resulted from (11), we suggest further on an algorithm in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and through induction we obtain

$$\begin{aligned} x_m(t_k) &= \frac{b-a}{2n}\left[K(t_k, a, \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(g(a)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b)) + \right. \\ &\quad + 2\sum_{i=1}^{n-1} K(t_k, t_i, \widetilde{x}_{m-1}(t_i), \widetilde{x}_{m-1}(g(t_i)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b)) + \\ &\quad \left. + K(t_k, b, \widetilde{x}_{m-1}(b), \widetilde{x}_{m-1}(g(b)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b))\right] + \\ &\quad + f(t_k) + \widetilde{R}^T_{m,k} = \widetilde{x}_m(t_k) + \widetilde{R}^T_{m,k} \ , \quad k = \overline{0,n} \end{aligned}$$

where

$$\left|\widetilde{R}^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2} M_0^T \left[4^{m-1}L^{m-1}(b-a)^{m-1} + \ldots + 1\right] , \quad k = \overline{0,n} \ .$$

Since the conditions of theorem 2.4 are satisfied we have $4L(b-a) < 1$, and it result the estimation:

$$\left|\widetilde{R}^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2\left[1 - 4L(b-a)\right]} M_0^T \ .$$

We have thus obtained the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in\mathbb{N}}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in\mathbb{N}}$ using a division of the interval $[a,b]$, $a = x_0 < x_1 < \ldots < x_n = b$, with the following error in calculus:

$$|x_m(t_k) - \widetilde{x}_m(t_k)| \le \frac{(b-a)^3}{12n^2\left[1 - 4L(b-a)\right]} M_0^T \ .$$

B. Approximation of the solution using the rectangle quadrature formula

We suppose that:

(h_{21}) $K \in C^1([a,b] \times [a,b] \times J^4)$, $J \subset \mathbb{R}$ closed interval ;

(h_{22}) $f \in C^1[a,b]$;

(h_{23}) $g \in C^1([a,b],[a,b])$

and we will approximate the terms of the successive approximations sequence using the rectangle quadrature formula (4) with the rest from (5). Generally, for the term $x_m(t_k)$ we have

$$x_m(t_k) = \frac{b-a}{2n}\left[K(t_k,a,x_{m-1}(a),x_{m-1}(g(a)),x_{m-1}(a),x_{m-1}(b)) + \right. \\ \left. + \sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(g(t_i)),x_{m-1}(a),x_{m-1}(b))\right] + \\ + f(t_k) + R^D_{m,k}\,, \quad k=\overline{0,n},\ m \in \mathbb{N} \tag{11}$$

with the estimation of the rest

$$\left|R^D_{m,k}\right| \le \frac{(b-a)^2}{n} \cdot \max_{s\in[a,b]} \left|[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))]^{'}_s\right| .$$

According to (h_{21}) it result that the derivative of the function K from the expression of the rest $R^D_{m,k}$ exist and has the following form:

$$[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))]^{'}_s = \frac{\partial K}{\partial s} + \\ + \frac{\partial K}{\partial x_{m-1}} \cdot x^{'}_{m-1}(s) + \frac{\partial K}{\partial x_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot g^{'}(s)\ ,$$

where

$$x^{'}_{m-1}(t) = \int_a^b \frac{\partial K(t,s,x_{m-2}(s),x_{m-2}(g(s)),x_{m-2}(a),x_{m-2}(b))}{\partial t} ds + f^{'}(t)\ .$$

If we denote

$$M^D_1 = \max_{|\alpha|\le 1, t,s\in[a,b]} \left|\frac{\partial^{|\alpha|}K}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right| ,$$

$$M^D_2 = \max_{\alpha\le 1, t\in[a,b]} \left|f^{(\alpha)}(t)\right| ,\quad M^D_3 = \max_{\alpha\le 1, t\in[a,b]} \left|g^{(\alpha)}(t)\right|$$

then we obtain for $x^{'}_{m-1}(t)$ the following estimation:

$$\left|x^{'}_{m-1}(t)\right| \le M^D_1(b-a) + M^D_2\ ,$$

while for the derivative of function K, we have

$$[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))]^{'}_s \le$$

$$\leq M_1^D \left\{1 + \left[M_1^D (b-a) + M_2^D\right] \left(1 + M_3^D\right)\right\} = M_0^D \ .$$

It is obvious that M_0^D doesn't depend on m and k, so the estimation of the rest is

$$\left|R_{m,k}^D\right| \leq M_0^D \cdot \frac{(b-a)^2}{n} \ , \tag{12}$$

where $M_0^D = M_0^D \left(K, D^{(\alpha)}K, f, D^{(\alpha)}f, g, D^{(\alpha)}g\right)$, $\alpha = 1$, and we obtain a formula for the approximative calculus of the integrals of the successive approximations sequence. Using the method of successive approximations and the formula (12) with the estimation of the rest resulted from (13), we suggest further on an algorithm in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and through induction we obtain

$$\begin{aligned} x_m(t_k) \quad = \quad & \frac{b-a}{n} \Bigg[K(t_k, a, \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(g(a)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b)) + \\ & + \sum_{i=1}^{n-1} K(t_k, t_i, \widetilde{x}_{m-1}(t_i), \widetilde{x}_{m-1}(g(t_i)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b))\Bigg] + \\ & + f(t_k) + \widetilde{R}_{m,k}^D = \widetilde{x}_m(t_k) + \widetilde{R}_{m,k}^D \ , \quad k = \overline{0,n} \quad , \end{aligned}$$

where

$$\left|\widetilde{R}_{m,k}^D\right| \leq \frac{(b-a)^2}{n} M_0^D \left[4^{m-1} L^{m-1} (b-a)^{m-1} + \ \dots \ + 1\right] , \quad k = \overline{0,n} \ .$$

Since the conditions of theorem 2.4 are satisfied we have $4L(b-a) < 1$, and it result the estimation:

$$\left|\widetilde{R}_{m,k}^D\right| \leq \frac{(b-a)^2}{n\left[1 - 4L(b-a)\right]} M_0^D \ ,$$

and we have thus obtained the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in\mathbb{N}}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in\mathbb{N}}$ using a division of the interval $[a, b]$, $a = x_0 < x_1 < \ldots < x_n = b$, with the following error in calculus:

$$|x_m(t_k) - \widetilde{x}_m(t_k)| \leq \frac{(b-a)^2}{n\left[1 - 4L(b-a)\right]} M_0^D \ .$$

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