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OPTIMAL QUADRATURE FORMULAS BASED ON THE φ-FUNCTION METHOD

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Abstract. In this survey paper it is studied the optimality in sense of Nikolski for some classes of quadrature formulas, using the method of φ-function. It is presented the one-to-one correspondence between φ-functions and the quadrature formulas. Also, there are given some examples of quadrature formulas which are optimal in sense of Nikolski with regard to the error.

1. **Introduction**

Let H be a linear space of real-valued functions, defined and integrable on a finite interval $[a,b] \subset \mathbb{R}$, and $S : H \to \mathbb{R}$ be the integration operator defined by

$$S(f) = \int_a^b f(x)dx.$$

Let

$$\Lambda = \{\lambda_i \,|\lambda_i : H \to \mathbb{R},\, i = 1, ..., n\}$$

be a set of linear functionals. For $f \in H$, one considers the quadrature formula

$$S(f) = Q_n(f) + R_n(f), \tag{1}$$

where

$$Q_n(f) = \sum_{i=1}^{n} A_i \lambda_i(f)$$

and $R_n(f)$ denotes the remainder term.

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Remark 1. *Usually,* $\lambda_i(f)$, $i = 1, .., n$ *are the values of the function* f *or of certain of its derivatives on the quadrature nodes from* $[a, b]$.

An important problem regarding the quadrature formulas is the optimality problem with respect to the error. In this paper it is studied the optimality in sense of Nikolski for some classes of quadrature formulas, using the one-to-one corespondence between φ-functions and quadrature formulas.

Definition 2. *The quadrature formula (1) is called optimal in the sense of Nikolski, in the space* H, *if*

$$F_n(H, A, X) = \sup_{f \in H} |R_n(f)|,$$

attains the minimum value with regard to A *and* X, *where* $A = (A_1, ..., A_n)$ *are the coefficients and* $X = (x_1, ..., x_n)$ *are the quadrature nodes.*

2. **The method of $\varphi-$ function**

Suppose that $f \in C^r[a, b]$ and for some given $n \in \mathbb{N}$ consider the nodes $a = x_0 < \ldots < x_n = b$. On each interval $[x_{k-1}, x_k]$, $k = 1, ..., n$, it is considered a function φ_k, $k = 1, ..., n$, with the property that

$$\varphi_k^{(r)} = 1, \quad k = 1, ..., n. \tag{2}$$

One defines the function φ as follows:

$$\varphi|_{[x_{k-1}, x_k]} = \varphi_k, \quad k = 1, ..., n, \tag{3}$$

i.e., the restriction of the function φ to the interval $[x_{k-1}, x_k]$ is φ_k. Based on the additivity property of the defined integral and on the relations (2), we have

$$S(f) := \int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^{(r)}(x) f(x) dx.$$

Using the integration by parts, one obtains

$$S(f) = \sum_{k=1}^{n} \left\{ \left[\varphi_k^{(r-1)}(x) f(x) - \varphi_k^{(r-2)}(x) f'(x) + ... + (-1)^{r-1} \varphi_k(x) f^{(r-1)}(x) \right] \Big|_{x_{k-1}}^{x_k} \right. \tag{4}$$

$$\left. + (-1)^r \int_{x_{k-1}}^{x_k} \varphi_k(x) f^{(r)}(x) dx \right\}$$

$$= -\varphi_1^{(r-1)}(x_0) f(x_0) + \left[\varphi_1^{(r-1)}(x_1) - \varphi_2^{(r-1)}(x_1) \right] f(x_1) + ... +$$

$$+ \left[\varphi_{n-1}^{(r-1)}(x_{n-1}) - \varphi_n^{(r-1)}(x_{n-1}) \right] f(x_{n-1}) + \varphi_n^{(r-1)}(x_n) f(x_n) -$$

$$- \left\{ -\varphi_1^{(r-2)}(x_0) f'(x_0) + \left[\varphi_1^{(r-2)}(x_1) - \varphi_2^{(r-2)}(x_1) \right] f'(x_1) + ... + \right.$$

$$\left. + \left[\varphi_{n-1}^{(r-2)}(x_{n-1}) - \varphi_n^{(r-2)}(x_{n-1}) \right] f'(x_{n-1}) + \varphi_n^{(r-2)}(x_n) f'(x_n) \right\} +$$

$$+ ... +$$

$$+ (-1)^{r-1} \left\{ -\varphi_1(x_0) f^{(r-1)}(x_0) + \left[\varphi_1(x_1) - \varphi_2(x_1) \right] f^{(r-1)}(x_1) + ... + \right.$$

$$\left. + \left[\varphi_{n-1}(x_{n-1}) - \varphi_n(x_{n-1}) \right] f^{(r-1)}(x_{n-1}) + \varphi_n(x_n) f^{(r-1)}(x_n) \right\}$$

$$+ (-1)^r \int_a^b \varphi(x) f^{(r)}(x) dx.$$

For

$$A_{0j} = (-1)^{j+1} \varphi_1^{(r-j-1)}(x_0), \tag{5}$$

$$A_{kj} = (-1)^j (\varphi_k - \varphi_{k+1})^{(r-j-1)}(x_k), \quad k = 1, ..., n-1,$$

$$A_{nj} = (-1)^j \varphi_n^{(r-j-1)}(x_n), \quad j = 0, 1, ..., r-1,$$

relation (4) becomes

$$\int_a^b f(x) dx = \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_{kj} f^{(j)}(x_k) + R_n(f), \tag{6}$$

with

$$R_n(f) = (-1)^r \int_a^b \varphi(x) f^{(r)}(x) dx. \tag{7}$$

Remark 3. *Knowing the function φ, one can find the coefficients A_{kj}, $k = 0, ..., n$, $j = 0, ..., r-1$, and the nodes x_k, $k = 1, ..., n-1$, based on the relations (5). This method of constructing the quadrature formulas is called the φ-function method* [10].

Remark 4. *From (7) it follows that the degree of exactness of the quadrature formula (6) is at least* $r-1$.

3. The one-to-one corespondence between $\varphi-$ functions and quadrature formulas

First of all, one remarks that to a function φ, which satisfies (3) and (2), corresponds the quadrature formula (6).

Conversely, let us consider the quadrature formula (6), which has the degree of exactness $r-1$. By Peano's theorem it follows that

$$R_n(f) = \int_a^b R_n^t \left[\frac{(t-x)_+^{r-1}}{(r-1)!} \right] f^{(r)}(x)dx,$$

where

$$R_n^t \left[\frac{(t-x)_+^{r-1}}{(r-1)!} \right] = \frac{(x_n-x)_+^r}{r!} - \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_{kj} \frac{(x_k-x)_+^{r-j-1}}{(r-j-1)!}.$$

So,

$$(-1)^r R_n^t \left[\frac{(t-x)_+^{r-1}}{(r-1)!} \right] = \frac{(x-x_n)_+^r}{r!} + (-1)^{r+1} \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_{kj} \frac{(x_k-x)_+^{r-j-1}}{(r-j-1)!},$$

i.e,

$$(-1)^r R_n^t \left[\frac{(t-x)_+^{r-1}}{(r-1)!} \right] = \varphi(x).$$

If

$$\varphi_i = \varphi|_{[x_{i-1},x_i]}, \quad i = 1, ..., n,$$

then

$$\varphi_i(x) = \frac{(x-x_n)_+^r}{r!} + (-1)^{r+1} \sum_{k=i}^{n} \sum_{j=0}^{r-1} A_{kj} \frac{(x_k-x)_+^{r-j-1}}{(r-j-1)!},$$

$$\varphi_{i+1}(x) = \frac{(x-x_n)_+^r}{r!} + (-1)^{r+1} \sum_{k=i+1}^{n} \sum_{j=0}^{r-1} A_{kj} \frac{(x_k-x)_+^{r-j-1}}{(r-j-1)!},$$

and we get that

$$(\varphi_i - \varphi_{i+1})(x) = (-1)^{r+1} \sum_{j=0}^{r-1} A_{ij} \frac{(x_i-x)_+^{r-j-1}}{(r-j-1)!}.$$

Further,

$$(\varphi_i - \varphi_{i+1})^{(r-\nu-1)}(x) = (-1)^\nu \sum_{j=0}^{r-1} A_{ij} \frac{(x_i-x)_+^{(\nu-j)}}{(\nu-j)!},$$
$$(\varphi_i - \varphi_{i+1})^{(r-\nu-1)}(x_i) = (-1)^\nu A_{i\nu}.$$

It follows that

$$A_{0\nu} = (-1)^{\nu+1}\varphi_1^{(r-\nu-1)}(x_0),$$
$$A_{i\nu} = (-1)^\nu (\varphi_i - \varphi_{i+1})^{(r-\nu-1)}(x_i), \quad i = 1, ..., n-1,$$
$$A_{n\nu} = \varphi_n^{(r-\nu-1)}(x_n), \quad \nu = 0, 1, ..., r-1.$$

So, the correspondence is proved.

4. **The optimality problem**

We consider $H^{m,2}[a, b]$, $m \in \mathbb{N}$, the space of functions f in C^{m-1}, with the $m-1$th derivative absolute continuous on $[a, b]$ and with f^m in $L^2[a, b]$. Suppose that $f \in H^{m,2}[a, b]$, $m \in \mathbb{N}$. From (7) one obtains

$$|R_n(f)| \leq \left\| f^{(m)} \right\|_2 \left(\int_a^b \varphi^2(x)dx \right)^{1/2}.$$

So, the optimal quadrature formula of the form (6) is determined by the parameters A and X for which

$$F(A, X) = \int_a^b \varphi^2(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx$$

attains the minimum value.

Remark 5. *Taking into account the property of minimal $L_w^2[a, b]-$norm, (w is a weight function), of the orthogonal polynomials, the function $F(A, X)$ takes the minimal value when φ_k is the orthogonal polynomial on $[x_{k-1}, x_k]$, $k = 1, \ldots, n$, with regard to the weight w.*

For example, if $w = 1$ the corresponding orthogonal polynomial on $[a, b]$ is the Legendre polynomial

$$l_r(x) = \frac{d^r}{dx^r}\left[(x-a)^r(y-b)^r\right].$$

It means that the parameters of the optimal quadrature formula can be obtained by identifying the functions $\varphi_k = \varphi|_{[x_{k-1},x_k]}$ with the corresponding orthogonal polynomials on $[x_{k-1}, x_k]$, $k = 1, ..., n$.

Example 6. *One considers the quadrature formula*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{8}$$

obtained from (6) for $r = 1$, *with*

$$R_n(f) = \int_0^1 \varphi(x)f'(x)dx.$$

Theorem 7. *For* $f \in H^{1,2}[0, 1]$, *the quadrature formula of the form (8), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2n}\left[f(0) + 2\sum_{i=1}^{n-1} f(\frac{i}{n}) + f(1)\right] + R_n^*(f),$$

with

$$|R_n^*(f)| \leq \frac{1}{2n\sqrt{3}}\left\|f'\right\|_2.$$

Proof. Relations (5) become

$$\begin{aligned} &A_0 = -\varphi_1(0), \\ &A_k = \varphi_k(x_k) - \varphi_{k+1}(x_k), \quad k = 1, ..., n-1, \\ &A_n = \varphi_n(1), \end{aligned} \tag{9}$$

and from (2) we get

$$\varphi_k' = 1, \;\; k = 1, ..., n. \tag{10}$$

From (9) and (10) it follows

$$\varphi_1(x) = x - A_0,$$

$$\varphi_2(x) = x - A_0 - A_1,$$

$$...$$

$$\varphi_k(x) = x - A_0 - A_1 - ... - A_{k-1},$$

$$...$$

$$\varphi_n(x) = x - A_0 - A_1 - ... - A_{n-2} - A_{n-1}.$$

As the quadrature formula (8) has the degree of exactness zero, i.e., $R_n(e_0) = 0$ $(e_0(x) = 1)$ we have

$$A_0 + ... + A_n = 1.$$

It follows that for φ_n we have

$$\varphi_n(x) = x - 1 + A_n.$$

Now, the optimal coefficients A_k, $k = 0, ..., n$ and the optimal nodes x_k, $k = 1, ..., n-1$ are obtained by minimizing the functions

$$F_1(A, X) = \int_0^1 \varphi^2(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx.$$

But, $\int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx$ takes its minimum value for $\varphi_k \equiv l_1$, the Legendre polynomial of degree one, on the interval $[x_{k-1}, x_k]$, i.e.,

$$\varphi_k(x) = x - \frac{x_{k-1} + x_k}{2}$$

and

$$\int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx = \frac{(x_k - x_{k-1})^3}{12}.$$

It follows that

$$\sum_{i=0}^{k-1} A_i = \frac{x_{k-1} + x_k}{2} \tag{11}$$

and

$$\int_0^1 \varphi^2(x)dx = \frac{1}{12} \sum_{k=1}^{n} (x_k - x_{k-1})^3.$$

Hence,

$$\bar{F}_1(X) := \min_A F_1(A, X) = \frac{1}{12}\sum_{k=1}^{n}(x_k - x_{k-1})^3.$$

As

$$\frac{\partial \bar{F}_1(X)}{\partial x_k} = \frac{1}{4}\left[(x_k - x_{k-1})^2 - (x_{k+1} - x_k)^2\right],$$

the optimal nodes constitute the solution of the system

$$x_k - x_{k-1} = x_{k+1} - x_k, \quad k = 1, ..., n-1,$$

with

$$x_0 = 0, \quad x_n = 1,$$

i.e.,

$$x_k^* = \frac{k}{n}, \quad k = 0, ..., n \tag{12}$$

and

$$\bar{F}_1(X^*) = \frac{1}{12n^2}.$$

From (11) and (12) one obtains the optimal coefficients

$$A_0^* = \frac{1}{2n}$$
$$A_1^* = ... = A_{n-1}^* = \frac{1}{n}$$
$$A_n^* = \frac{1}{2n}.$$

Finally, we have

$$F_1(A^*, X^*) := \min_{A,X} F_1(A, X) = \frac{1}{12n^2},$$

and the proof follows. □

Example 8. *For $f \in H^{2,2}[0,1]$ one considers the quadrature formula of the form*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{13}$$

with $0 = x_0 < x_1 < ... < x_n = 1$.

Theorem 9. *For $f \in H^{2,2}[0,1]$, the quadrature formula of the form (13), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k^* f(x_k^*) + R_n^*(f),$$

with

$$\begin{aligned}
&A_0^* = A_n^* = \frac{3}{4}\mu,\\
&A_1^* = A_{n-1}^* = \frac{5+2\sqrt{6}}{4}\mu,\\
&A_k^* = \sqrt{6}\mu, \quad k = 2, ..., n-2,\\
&x_k^* = [2 + (k-1)\sqrt{6}]\mu, \quad k = 1, ..., n-1,
\end{aligned}$$

and

$$|R_n^*(f)| \le \frac{\mu^2}{2\sqrt{5}} \|f''\|_2 ,$$

where

$$\mu = \frac{1}{4 + (n-2)\sqrt{6}}.$$

Proof. For $r = 2$ relation (4) becomes

$$\begin{aligned}
\int_0^1 f(x)dx = &- \varphi_1'(0)f(0) + \sum_{k=1}^{n-1} (\varphi_k' - \varphi_{k+1}')(x_k)f(x_k) + \varphi_n'(1)f(1) \qquad (14)\\
&+ \varphi_1(0)f'(0) - \sum_{k=1}^{n-1} (\varphi_k - \varphi_{k+1})(x_k)f'(x_k) - \varphi_n(1)f'(1)\\
&+ \int_0^1 \varphi(x)f''(x)dx.
\end{aligned}$$

Taking into account (13), we have

$$\begin{aligned}
&A_0 = -\varphi_1'(0),\\
&A_k = (\varphi_k' - \varphi_{k+1}')(x_k), \quad k = 1, ...n-1,\\
&A_n = \varphi_n'(1),
\end{aligned}$$

and

$$\varphi_1(0) = 0, \tag{15}$$
$$(\varphi_k - \varphi_{k+1})(x_k) = 0, \quad k = 1, ...n-1,$$
$$\varphi_n(1) = 0,$$

respectively,

$$R_n(f) = \int_0^1 \varphi(x) f''(x) dx. \tag{16}$$

Relation (2) becomes

$$\varphi_k'' = 1, \qquad k = 1, ..., n. \tag{17}$$

From (15) and (17) it follows that

$$\begin{aligned} \varphi_1(x) &= \frac{x^2}{2} - A_0 x, \\ \varphi_k(x) &= \frac{x^2}{2} - \sum_{j=0}^{k-1} A_k (x - x_k), \quad k = 2, ...n-1, \\ \varphi_n(x) &= \frac{(1-x)^2}{2} - A_n (1-x). \end{aligned} \tag{18}$$

By (16) one obtains

$$|R_n(f)| \leq \left(\int_0^1 \varphi^2(x) dx \right)^{1/2} \|f''\|_2 .$$

Next, the problem is to minimize the function

$$\begin{aligned} F_2(A, X) &= \int_0^1 \varphi^2(x) dx \\ &= \int_0^{x_1} \varphi_1^2(x) dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx + \int_{x_{n-1}}^1 \varphi_n^2(x) dx \end{aligned}$$

with regard to the parameters $A = (A_0, ..., A_n)$ and $X = (x_1, ..., x_{n-1})$.

By (18) it follows that the integrals

$$\int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx, \quad k = 2, ..., n-1,$$

attain the minimum values for

$$\varphi_k \equiv \frac{1}{2} \tilde{l}_{2,k}, \quad k = 2, ..., n-1, \tag{19}$$

where $\tilde{l}_{2,k}$ is the two degree Legendre polynomial on the interval $[x_{k-1}, x_k]$,

$$\tilde{l}_{2,k}(x) = x^2 - (x_{k-1} + x_k)x + \frac{1}{6}(x_{k-1}^2 + 4x_{k-1}x_k + x_k^2).$$

We have

$$\int_{x_{k-1}}^{x_k} \tilde{l}_{2,k}^2(x)dx = \frac{4}{45}\left(\frac{x_k - x_{k-1}}{2}\right)^5.$$

From (18) and (19), one obtains

$$\sum_{i=0}^{k-1} A_i = \frac{x_k + x_{k-1}}{2}, \quad k = 2, ..., n-1, \tag{20}$$

and, also, from

$$\frac{d}{dA_0}\left[\int_0^{x_1} (\frac{x^2}{2} - A_0 x)^2 dx\right] = 0,$$
$$\frac{d}{dA_n}\left\{\int_{x_{n-1}}^1 \left[\frac{(1-x)^2}{2} - A_n(1-x)\right]^2 dx\right\} = 0$$

it follows

$$A_0 = \frac{3}{8}x_1, \quad A_n = \frac{3}{8}(1 - x_{n-1}), \tag{21}$$

respectively,

$$\int_0^{x_1} (\frac{x^2}{2} - \frac{3}{8}x_1 x)^2 dx = \frac{1}{32}x_1^5$$
$$\int_{x_{n-1}}^1 \left[\frac{(1-x)^2}{2} - \frac{3}{8}(1-x_n)(1-x)\right]^2 dx = \frac{1}{320}(1-x_{n-1})^5.$$

So,

$$\bar{F}_2(X) := \min_A F_2(A, X) = \frac{1}{32}x_1^5 + \frac{1}{720}\sum_{k=2}^{n-1}(x_k - x_{k-1})^5 + \frac{1}{320}(1 - x_{n-1})^5. \tag{22}$$

Now, from

$$\frac{\partial}{\partial x_k}\sum_{i=2}^{n-1}(x_i - x_{i-1})^5 = 5[(x_k - x_{k-1})^4 - (x_{k+1} - x_k)^4] = 0, \quad k = 2, ..., n-1$$

one obtains

$$x_k - x_{k-1} = \frac{x_n - x_1}{n-2}, \quad k = 2, ..., n-1. \tag{23}$$

For

$$\tilde{F}_2(x_1, x_{n-1}) = \min_{x_2, \ldots, x_{n-2}} \bar{F}_2(X)$$

we have

$$\tilde{F}_2(x_1, x_{n-1}) = \frac{1}{32}x_1^5 + \frac{(x_{n-1} - x_1)^5}{720(n-2)^4} + \frac{1}{320}(1 - x_{n-1})^5.$$

From the following system

$$\begin{cases} \frac{\partial \tilde{F}_2(x_1, x_{n-1})}{\partial x_1} = 0 \\ \frac{\partial \tilde{F}_2(x_1, x_{n-1})}{\partial x_{n-1}} = 0 \end{cases}$$

one obtains

$$x_1^* = 1 - x_{n-1}^* = 2\mu \tag{24}$$

and

$$\tilde{F}_2(x_1^*, x_{n-1}^*) = \frac{1}{20}\mu^4. \tag{25}$$

Finally, the proof follows from (20)–(25). □

Theorem 10. *For a function $f \in H^{2,2}[0,1]$, the quadrature formula of the form*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + B_0 f'(0) + B_1 f'(1) + R_n(f), \tag{26}$$

is optimal with regard to the error for

$$\begin{aligned} A_0 &= A_n = \frac{1}{2n}, \\ A_k &= \frac{1}{n}, \quad k = 1, ..., n-1, \\ B_0 &= \frac{1}{12n^2}, \\ B_1 &= -B_0, \end{aligned}$$

$$x_0 = 0, \quad x_k = \frac{k}{n}, \quad k = 1, ..., n-1, \quad x_n = 1$$

and

$$|R_n(f)| \leq \frac{1}{12n^2\sqrt{5}} \left\| f'' \right\|_2.$$

Proof. From (14), we get

$$A_0 = -\varphi_1'(0),$$
$$A_k = (\varphi_k' - \varphi_{k+1}')(x_k), \quad k = 1, ...n-1,$$
$$A_n = \varphi_n'(1),$$
$$B_0 = \varphi_1(0),$$
$$B_1 = -\varphi_n(1),$$

and

$$(\varphi_k - \varphi_{k+1})(x_k) = 0, \quad k = 1, ...n-1.$$

It follows that

$$\varphi_k(x) = \frac{x^2}{2} - \sum_{i=0}^{k-1} A_i(x - x_i) + B_0, \quad k = 1, ..., n.$$

As the integral

$$\int_0^1 \varphi^2(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx$$

attains the minimum value for

$$\varphi_k \equiv \frac{1}{2}\tilde{l}_{2,k}, \quad k = 1, ..., n,$$

from these last identities, using the fact that the degree of exactness of the quadrature formula is one, the proof follows. □

Remark 11. *In an analogous way, for $f \in H^{2,1}[0,1]$ one can prove that the quadrature formula of the form (26), optimal with regard to the error, has the coefficients:*

$$A_0^* = A_n^* = \frac{1}{2n},$$
$$A_k^* = \frac{1}{n}, \quad k = 1, ..., n-1,$$
$$B_0^* = \frac{3}{32n^2},$$
$$B_1^* = -B_0^*,$$

the nodes

$$x_0^* = 0,$$
$$x_k^* = \frac{k}{n}, \quad k = 1, ..., n-1,$$
$$x_n^* = 1,$$

and

$$|R_n^*(f)| \leq \frac{1}{32n^2} \|f''\|_1 .$$

It is important in the proof that the functions $\frac{1}{2}\varphi_k$, $k = 1, ..., n$, *are identified with the Cebyshev polynomials of the second kind.*

Now, let us consider the general case, i.e., the quadrature formula (6), with the remainder term given by (7), for $r \geq 1$ and for $f \in H^{r,p}[0, 1]$. The problem is to find the values of the parameters A_{kj} and x_k, $k = 0, ..., n$, $j = 0, ..., r-1$ for which

$$F(A, X) := \int_0^1 |\varphi(x)|^p \, dx$$

attains the minimum value. We have

$$F(A, X) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\varphi_i(x)|^p \, dx,$$

where

$$\varphi_i(x) = \frac{x^r}{r!} + \sum_{k=0}^{i-1} \sum_{j=0}^{r-1} A_{kj} \frac{(x - x_k)^j}{j!}, \quad x \in [x_{i-1}, x_i].$$

As the polynomials φ_i are independent, the function $F(A, X)$ can be minimized, first with regard to the coefficients A_{kj}, $k = 0, ..., n$, $j = 0, ..., r-1$, considering the nodes fixed, and then, with regard to the nodes $x_1, ..., x_{n-1}$.

Using the notation $\frac{A_{kj}}{j!} = \frac{B_{kj}}{r!}$ one obtains

$$\varphi_i = \frac{1}{r!} \psi_i,$$

with

$$\psi_i(x) = x^r + \sum_{k=0}^{i-1} \sum_{j=0}^{r-1} B_{kj} (x - x_k)^j$$

and

$$F(A, X) = \frac{1}{(r!)^p} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\psi_i(x)|^p \, dx.$$

Using the minimum norm property of the orthogonal polynomials, the integrals

$$I_i = \int_{x_{i-1}}^{x_i} |\psi_i(x)|^p \, dx, \quad i = 1, ..., n,$$

can be minimized by identifying the polynomials ψ_i with the corresponding orthogonal polynomials, say θ_i, for different values of p. One obtains

$$\tilde{F}(x_1, ..., x_{n-1}) = \tfrac{1}{(r!)^p} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\theta_i(x)|^p \, dx,$$

that is further minimized with regard to x_i, $i = 1, ..., n-1$.

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