

ON THE MORITA INVARIANCE OF THE HOCHSCHILD HOMOLOGY OF SUPERALGEBRAS

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Abstract. We provide a direct proof that the Hochschild homology of a \mathbb{Z}_2 -graded algebra is Morita invariant.

1. Introduction

The goal of this paper is to show that if R is an arbitrary superalgebra (i.e. \mathbb{Z}_2 -graded algebra) while $M_{p,q}(R)$ is the (super)algebra of (p, q) -supermatrices over R , then the two algebras have the same Hochschild homology (in the \mathbb{Z}_2 -graded sense, see (Kassel, 1986)). This, naturally, suggest the idea of introducing the notion of Morita equivalence between two superrings and of proving that, in general, the Hochschild homology for superalgebras should be Morita invariant. We should discuss this issues at the end of the paper.

2. The Hochschild homology of superalgebras

The Hochschild complex for superalgebras (Kassel, 1986), is very similar to the analogous complex for ungraded case. Namely, the chain groups are, as in the classical case, $C_m(R) = R^{\otimes m+1}$, where, of course, the tensor product should be understood in the graded sense, while the face maps and degeneracies are given by

$$\delta_i^m(a_0 \otimes \cdots \otimes a_m) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_m, \quad \text{if } 0 \leq i < m, \quad (1)$$

$$\delta_m^m(a_0 \otimes \cdots \otimes a_m) = (-1)^{|a_m|(|a_0|+\cdots+|a_{m-1}|)} a_m a + \mathbf{0} \otimes a_1 \otimes \cdots \otimes a_{m-1}, \quad (2)$$

$$s_i^m(a_0 \otimes \cdots \otimes a_m) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_m, \quad 0 \leq i \leq m. \quad (3)$$

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Now the differential is defined in the usual way, meaning $d^m : C_m(R) \rightarrow C_{m-1}(R)$,

$$d^m = \sum_{i=0}^m (-1)^i \delta_i^m. \quad (4)$$

and the Hochschild homology of the superalgebra is just the homology of the complex $(C(R), d)$. In particular, it is easy to see that for any superalgebra R we have

$$H_0(R) = R/\{R, R\}, \quad (5)$$

where $\{R, R\}$ is the subspace generated by the supercommutators of that element.

3. The Morita invariance

We shall simply give the definition of the Morita equivalence here. For a detailed approach, see for, instance, the book of Bass ([1]). The definition is completely analogous to that from the ungraded case.

Definition 1. If A and B are two unital, associative superalgebras over a graded commutative superring R , then A and B are said to be *Morita equivalent* if there exists an $A - B$ -bimodule P and a $B - A$ -bimodule Q such that $P \otimes_B Q \simeq A$ (as $A - A$ -bimodules), while $Q \otimes_A P \simeq B$ (as $B - B$ -bimodules). The tensor products should be taken in the graded sense.

Theorem 1. *Let R be a commutative superring and A and B - two unital R -superalgebras (not necessarily commutative). Let, also, P be an $A - B$ -bimodule which is projective over both rings and Q - an arbitrary $B - A$ -bimodule. Then there is an isomorphism*

$$F_* : H_*(A, P \otimes_B Q) \rightarrow H_*(B, Q \otimes_A P),$$

which is functorial in the 4-tuple $(A, B; P, Q)$.

Before actually proving the theorem, let us, first, prove a technical lemma.

Lemma 1. *Let A be a unital, associative superalgebra over a commutative superring. If M is an arbitrary left A -module, while Q is a projective right A -module, then*

$$H_n(A, M \otimes Q) = \begin{cases} Q \otimes_A M & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

Dually, if N is a right A -module, while P is a projective left A -module, then

$$H_n(A, P \otimes N) = \begin{cases} N \otimes_A P & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

Proof. We shall assume, first, that $Q = A$, which is, clearly, projective, when regarded as right A -module. Moreover, in this case we have $A \otimes_A M \cong M$, so what we have to prove is that

$$H_n(A, M \otimes A) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

It is easily seen, however, that the standard complex for computing the Hochschild homology of A with coefficients in the module $M \otimes A$ is, essentially, the (unnormalized) bar resolution β of the M , which has non-vanishing homology only in degree zero and the zero degree homology is M .

To prove now the general case, take Q an arbitrary projective right A -module. Then the functor $Q \otimes_A -$ is exact and the result follows from the isomorphism $(M \otimes Q) \otimes A^n \cong Q \otimes_A (M \otimes A \otimes A^n)$ established by the maps

$$f : (M \otimes Q) \otimes A^n \rightarrow Q \otimes_A (M \otimes A \otimes A^n),$$

$$f((m \otimes q) \otimes (a_1 \otimes \cdots \otimes a_n)) = (-1)^{|m||q|} q \otimes (m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n)$$

and

$$g : Q \otimes_A (M \otimes A^{n+1}) \rightarrow (M \otimes Q) \otimes A^n,$$

$$g(q \otimes (m \otimes a_0 \otimes \cdots \otimes a_n)) = (-1)^{|m||q|} (m \otimes q) \otimes a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n.$$

The proof of the second part of the lemma is completely similar. \square

Proof of the theorem 1. We consider the following family of modules and maps: $(C_{p,q}, d', d'')$, where

$$C_{m,n} = P \otimes B^n \otimes Q \otimes A^m,$$

where

$$B^n = \underbrace{B \otimes B \otimes \cdots \otimes B}_{n \text{ factors}}$$

and

$$A^m = \underbrace{A \otimes A \otimes \cdots \otimes A}_{m \text{ factors}},$$

and all the tensor products are considered over the ground superring R . Before defining the maps d' and d'' , several remarks are in order.

First of all, it is very clear that

$$C_{m,n} = C_m(A, P \otimes B^n \otimes Q),$$

i.e. $C_{m,n}$ is the group of the Hochschild m -chains of the superalgebra A , with the coefficients in the A -bimodule $P \otimes B^n \otimes Q$. On the other hand, up to a cyclic permutation of the factors in the tensor product, $C_{m,n}$ is, also, the group of the Hochschild n -chains of the superalgebra B with coefficients in a $B - B$ -bimodule. More specifically, we have

$$C_{m,n} = \omega_{m+1,n+1}(C_n(B, Q \otimes A^m \otimes P)),$$

where $\omega_{m+1,n+1} : Q \otimes A^m \otimes P \otimes B^n \rightarrow P \otimes B^n \otimes Q \otimes A^m$ is the cyclic permutation of factors given by

$$\begin{aligned} \omega_{m+1,n+1}(p \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m) &= \\ &= (-1)^{|p|+|q|+\sum_{i=1}^m |a_i|+\sum_{j=1}^n |b_j|} q \otimes a_1 \otimes \cdots \otimes a_m \otimes p \otimes b_1 \otimes \cdots \otimes b_n. \end{aligned}$$

Now we can use the Hochschild differentials to build the maps d' and d'' . Let $m, n \in \mathbb{N}$ two given natural numbers. We define now, for any pair of natural numbers, $m, n \in \mathbb{N}$, $d'_{m,n} : C_{m,n} \rightarrow C_{m-1,n}$ to be the Hochschild differential for A , with coefficients in $P \otimes B^n \otimes Q$. Thus, on the columns we have Hochschild complexes. On the other hand, also for any pair of natural numbers m, n we define the horizontal differentials $d''_{m,n} : C_{m,n} \rightarrow C_{m,n-1}$,

$$d''_{m,n} = (-1)^m b_{m,n} \circ \omega_{m+1,n+1},$$

where $b_{m,n} : C_n(B, Q \otimes A^m \otimes P) \rightarrow C_{n-1}(B, Q \otimes A^m \otimes P)$ is the Hochschild differential. From the construction, it is obvious that both d' and d'' are differentials. We will

prove now that they anticommute. We have

$$\begin{aligned}
 d''d'(p \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m) &= d'' \left(p \otimes b_1 \otimes \cdots \otimes b_n \otimes qa_1 \otimes a_2 \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + \sum_{i=1}^{m-1} (-1)^i p \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + (-1)^{m+|a_m|} \left(|p|+|q| + \sum_{j=1}^{m-1} |a_j| + \sum_{j=1}^n |b_j| \right) a_m p \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a + 1 \otimes \cdots \otimes a_{m-1} \right) = \\
 &= (-1)^m \left[pb_1 \otimes \cdots \otimes b_n \otimes qa_1 \otimes a_2 \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + \sum_{j=1}^{n-1} p \otimes b_1 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_n \otimes qa_1 \otimes a_2 \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + (-1)^{n+|b_n|} \left(|p|+|q| + \sum_{j=1}^m |a_j| + \sum_{j=1}^{n-1} |b_j| \right) p \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n qa_1 \otimes a_2 \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + \sum_{i=1}^{m-1} (-1)^i \left(pb_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_m + \right. \right. \\
 &\quad \left. \left. + (-1)^{n+|b_n|} \left(|p|+|q| + \sum_{j=1}^m |a_j| + \sum_{j=1}^{n-1} |b_j| \right) p \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_m \right) \right. \\
 &\quad \left. + (-1)^{m+|a_m|} \left(|p|+|q| + \sum_{j=1}^{m-1} |a_j| + \sum_{j=1}^n |b_j| \right) \left(a_m p b_1 \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_{m-1} + \right. \right. \\
 &\quad \left. \left. + \sum_{j=1}^{n-1} (-1)^j a_m p \otimes b_1 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_{m-1} + \right. \right. \\
 &\quad \left. \left. + (-1)^{n+|b_n|} \left(|p|+|q| + \sum_{j=1}^m |a_j| + \sum_{j=1}^{n-1} |b_j| \right) a_m p \otimes b_1 \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \cdots \otimes a_{m-1} \right) \right].
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 d'd''(p \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m) &= (-1)^{m-1} d' \left(pb_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + \sum_{i=1}^{n-1} (-1)^i p \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m + \right. \\
 &\quad \left. + (-1)^{n+|b_n|} \left(|p|+|q| + \sum_{j=1}^m |a_j| + \sum_{j=1}^{n-1} |b_j| \right) p \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \cdots \otimes a_m \right) =
 \end{aligned}$$

$$\begin{aligned}
 &= (-1)^{m-1} \left[pb_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes qa_1 \otimes a_2 \otimes \cdots \otimes a_m + \right. \\
 &\quad + \sum_{j=1}^{m-1} (-1)^j pb_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes a \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_m + \\
 &\quad + (-1)^{m+|a_m|} \left(|p|+|q| + \sum_{k=1}^{m-1} |a_k| + \sum_{k=1}^n |b_k| \right) a_m pb_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m + \\
 &\quad + \sum_{i=1}^{n-1} (-1)^i \left(p \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m + \right. \\
 &\quad + \sum_{j=1}^m (-1)^j p \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_m + \\
 &\quad \left. + (-1)^{m+|a_m|} \left(|p|+|q| + \sum_{j=1}^{m-1} |a_j| + \sum_{j=1}^n |b_j| \right) a_m p \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_{m-1} \right) \\
 &\quad + (-1)^{n+|b_n|} \left(|p|+|q| + \sum_{j=1}^m |a_j| + \sum_{j=1}^{n-1} |b_j| \right) \left(p \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n qa_1 \otimes a_2 \otimes \cdots \otimes a_m + \right. \\
 &\quad + \sum_{j=1}^{m-1} (-1)^j p \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_m + \\
 &\quad \left. + (-1)^{m+|a_m|} \left(|p|+|q| + \sum_{j=1}^{m-1} |a_j| + \sum_{j=1}^n |b_j| \right) a_m p \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \cdots \otimes a_{m-1} \right) \Big].
 \end{aligned}$$

An inspection shows immediately that the quantities between the square brackets in the expressions of $d'd''$ and $d''d'$ coincide, while the signs in front of these brackets are opposite, which means that we have

$$d'd'' + d''d' = 0.$$

Thus, as we saw previously that $d'^2 = d''^2 = 0$, it follows that the family of modules and morphisms $(C_{m,n}, d', d'')_{m,n \in \mathbb{N}}$ is a *double complex* of modules. We consider now its total complex, given, for any $n \geq 0$, by

$$Tot_n = \bigoplus_{p+q=n} C_{p,q}$$

and

$$d_n : Tot_n \rightarrow Tot_{n-1}, \quad d_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

As it is well-known (see [3], from where the notations, classical, in fact, are taken), the total complex has two canonical filtrations (a horizontal and a vertical one) and to each of this filtration we can associate a spectral sequence. The two spectral sequences both converge to the homology of the total sequence. We shall show that in our case both spectral sequences collapse at the second step. In fact, the second order terms of the two sequences are

$${}^I E_{p,q}^2 = H'_p H''_{p,q}(C)$$

and

$${}^{II} E_{p,q}^2 = H''_p H'_{q,p}(C).$$

In our particular case, due to the particular form of the vertical and horizontal complexes, we get

$$H''_{p,q}(C) = H_q(B, Q \otimes A^p \otimes P) \quad (6)$$

and

$$H'_{q,p}(C) = H_q(A, P \otimes B^p \otimes Q). \quad (7)$$

As P is a bimodule which is projective at both sides, applying the previous lemma, we can write

$$H''_{p,q}(C) = \begin{cases} P \otimes_B Q \otimes A^p & \text{for } q = 0 \\ 0 & \text{for } q \geq 1 \end{cases}$$

$$H'_{q,p}(C) = \begin{cases} B^p \otimes_A Q \otimes P & \text{for } q = 0 \\ 0 & \text{for } q \geq 1 \end{cases}$$

As a consequence, we obtain for the second terms of the two spectral sequences:

$${}^I E_{p,q}^2 = \begin{cases} H_p(A, P \otimes_B Q) & \text{for } q = 0 \\ 0 & \text{for } q \geq 1 \end{cases}$$

$${}^{II} E_{p,q}^2 = \begin{cases} H_p(B, Q \otimes_A P) & \text{for } q = 0 \\ 0 & \text{for } q \geq 1 \end{cases}$$

Since, as we see, the two spectral sequences collapse, their limits coincide, in fact, with the second terms. Therefore, as they should converge to the same limit (the homology of the total complex), we have, in particular, that, for any $n \geq 0$, we should have

$${}^I E_{n,0}^2 = {}^{II} E_{n,0}^2,$$

i.e.

$$H_n(A, P \otimes_B Q) = H_n(B, Q \otimes_A P)$$

which concludes the proof (the functoriality follows from the way we constructed the double complex). \square

Corollary. *If A and B are Morita equivalent superalgebras, then they have isomorphic Hochschild homologies.*

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