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APPROXIMATE FIXED POINT THEOREMS

MĂDĂLINA BERINDE

Abstract. Two general lemmas are given regarding $\varepsilon-$fixed points of operators on metric spaces. Using these results we prove qualitative and quantitative theorems for various types of well known generalized contractions on metric spaces.

1. **Introduction**

There are plenty of problems in applied mathematics which can be solved by means of fixed point theory. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required, but that of "nearly" fixed points. Another type of practical situations that lead to this approximation is when the conditions that have to be imposed in order to guarantee the existence of fixed points are far too strong for the real problem one has to solve.

It is then natural to introduce the concepts of $\varepsilon-$*fixed point (*or *approximate fixed point)*, which is a "nearly" fixed point, and that of function with the *approximate fixed point property* and to formulate a proper theory regarding them.

In this paper, starting from the article of Tijs, Torre and Branzei [10], we study some well known types of operators on metric spaces, and we give some qualitative and quantitative results regarding $\varepsilon-$fixed points of such operators.

We have to mention that we consider operators on metric spaces, not on complete metric spaces, the usual framework for fixed point problems. Weakening

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the conditions by giving up the completeness of the space we can still guarantee the existence of $\varepsilon-$fixed points for various types of operators.

We begin with two lemmas. The first one is the qualitative result that indicates under which conditions the operator f has the approximate fixed point property. This will be used in order to prove all the results given in the second section. The second lemma is the quantitative result which will be used in order to prove all the results given in the third section.

Let (X, d) be a metric space.

Definition 1.1. *Let $f : X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then x_0 is an $\varepsilon-$**fixed point (approximate fixed point) of** f if*

$$d(f(x_0), x_0) < \varepsilon.$$

Remark 1.1. *As many authors we prefer the terminology with ε, as being more suggestive throughout the paper.*

Remark 1.2. *In this paper we will denote the set of all $\varepsilon-$fixed points of f, for a given ε, by:*

$$F_\varepsilon(f) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } f\}.$$

Definition 1.2. *Let $f : X \to X$. Then f has **the approximate fixed point property (a.f.p.p.)** if*

$$\forall \varepsilon > 0,\ F_\varepsilon(f) \neq \varnothing.$$

The following result guarantees the existence of $\varepsilon-$fixed points for an operator on a metric space.

Lemma 1.1. *Let (X, d) be a metric space, $f : X \to X$ such that f is asymptotic regular, i.e.,*

$$d(f^n(x), f^{n+1}(x)) \to 0 \text{ as } n \to \infty,\ \forall x \in X.$$

Then f has the approximate fixed point property.

Proof. Let $x_0 \in X$. Then:

$$d(f^n(x_0), f^{n+1}(x_0)) \to 0 \text{ as } n \to \infty \ \Leftrightarrow$$

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N}^* \text{ such that } \forall n \geq n_0(\varepsilon),\ d(f^n(x_0), f^{n+1}(x_0)) < \varepsilon\ \Leftrightarrow$$

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N}^* \text{ such that } \forall n \geq n_0(\varepsilon),\ d(f^n(x_0), f(f^n(x_0))) < \varepsilon\ .$$

Denoting

$$y_0 = f^n(x_0),$$

it follows that:

$$\forall \varepsilon > 0,\ \exists y_0 \in X \text{ such that } d(y_0, f(y_0)) < \varepsilon\ ,$$

so for each $\varepsilon > 0$ there exists an $\varepsilon-$fixed point of f in X, namely y_0.

This means exactly that f has the approximate fixed point property. □

Remark 1.3. *The following result (see [5]) gives conditions under which the existence of fixed points for a given mapping is equivalent to that of approximate fixed points.*

Proposition.: *Let A be a closed subset of a metric space (X, d) and $f : A \to X$ a compact map. Then f has a fixed point if and only if it has the approximate fixed point property.*

In the following, by $\delta(A)$ for a set $A \neq \varnothing$ we will understand the diameter of the set A, i.e.,

$$\delta(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Lemma 1.2. *Let (X, d) be a metric space, $f : X \to X$ an operator and $\varepsilon > 0$. We assume that:*

ı): $F_\varepsilon(f) \neq \varnothing$;

ıı): $\forall \eta > 0,\ \exists \varphi(\eta) > 0$ *such that*

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta),\ \forall x, y \in F_\varepsilon(f).$$

Then:

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

Proof. Let $\varepsilon > 0$ and $x, y \in F_\varepsilon(f)$. Then:

$$d(x, f(x)) < \varepsilon,\ d(y, f(y)) < \varepsilon.$$

We can write:

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(y, f(y)) \leq d(f(x), f(y)) + 2\varepsilon$$

$$\Rightarrow d(x, y) - d(f(x), f(y)) \leq 2\varepsilon.$$

Now by (ii) it follows that

$$d(x, y) \leq \varphi(2\varepsilon),$$

so

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

□

Remark 1.4. *Condition (i) in Lemma 1.2 can be replaced by the asymptotic regularity condition, as, by Lemma 1.1, the latter ensures (i). So Lemma 1.2 can be given in the form:*

Lemma 1.3. *Let* (X, d) *be a metric space and* $f : X \to X$ *such that for* $\varepsilon > 0$ *the following hold:*

$\imath$)**:** $d(f^n(x), f^{n+1}(x)) \to 0$ *as* $n \to \infty, \ \forall x \in X$;

$\imath\imath$)**:** $\forall \eta > 0, \ \exists \varphi(\eta) > 0$ *such that*

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta), \ \forall x, y \in F_\varepsilon(f).$$

Then:

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

2. **Qualitative results for operators on metric spaces**

In this section we will formulate and prove, using Lemma 1.1, qualitative results for various types of operators on a metric space, results that establish the conditions under which the mappings considered have the approximate fixed point property.

Let (X, d) be a metric space. Note that the completeness of the space is not required, as in fixed point theorems.

Definition 2.1. *([8]) A mapping* $f : X \to X$ *is an* ***a-contraction*** *if*

$$\exists a \in]0, 1[\text{ such that } d(f(x), f(y) \leq a d(x, y), \ \forall x, y \in X.$$

Theorem 2.1. *Let (X,d) be a metric space and $f: X \to X$ an a-contraction.*

Then:

$$\forall \varepsilon > 0,\ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $\varepsilon > 0$, $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq ad(f^{n-1}(x), f^n(x)) \leq ... \leq a^n d(x, f(x))$$

But $a \in]0,1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty,\ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon(f) \neq \varnothing,\ \forall \varepsilon > 0.$ □

Remark 2.1. *Theorem 2.1 is a result presented and proved, by means of a different method, in [10].*

Any operator satisfying the condition in Definition 2.1 is Lipschitz and implicitly continuous, which means a relatively small class of mappings. Still, the theory of fixed points and consequently $\varepsilon-$fixed points deals also with non-continuous mappings. In 1968, Kannan (see [6],[2]) proved a fixed point theorem for operators which need not be continuous, by considering the following contraction condition.

Definition 2.2. *([6],[8]) A mapping $f: X \to X$ is a* ***Kannan operator*** *if*

$$\exists a \in]0, \frac{1}{2}[\text{ such that } d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \forall x, y \in X.$$

Theorem 2.2. *Let (X,d) be a metric space and $f: X \to X$ a Kannan operator.*

Then:

$$\forall \varepsilon > 0,\ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $\varepsilon > 0$ and $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq a[d(f^{n-1}(x), f(f^{n-1}(x))) + d(f^n(x), f(f^n(x)))] =$$

$$= ad(f^{n-1}(x), f^n(x)) + ad(f^n(x), f^{n+1}(x))$$

$$\Rightarrow (1-a)d(f^n(x), f^{n+1}(x)) \le ad(f^{n-1}(x), f^n(x)) \Rightarrow$$

$$d(f^n(x), f^{n+1}(x)) \le \frac{a}{1-a} d(f^{n-1}(x), f^n(x)) \le ... \le a1 - a^n d(x, f(x))$$

But $a \in]0, \frac{1}{2}[\Rightarrow \dfrac{a}{1-a} \in]0, 1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma1.1 it follows that $F_\varepsilon(f) \neq \varnothing, \forall \varepsilon > 0$. □

In 1972, Chatterjea considered another contraction condition, similar to that of Kannan but independent of this one, and which again does not impose the continuity of the operator.

Definition 2.3. *([4],[8]) A mapping $f : X \to X$ is a **Chatterjea operator** if*

$$\exists a \in]0, \frac{1}{2}[\text{ such that } d(f(x), f(y)) \le a[d(x, f(y)) + d(y, f(x))], \forall x, y \in X.$$

Theorem 2.3. *Let (X, d) be a metric space and $f : X \to X$ a Chatterjea operator. Then:*

$$\forall \varepsilon > 0, \ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $\varepsilon > 0$ and $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \le$$

$$\le a[d(f^{n-1}(x), f(f^n(x))) + d(f^n(x), f(f^{n-1}(x)))] =$$

$$= a[d(f^{n-1}(x), f^{n+1}(x)) + d(f^n(x), f^n(x))] = ad(f^{n-1}(x), f^{n+1}(x))$$

On the other hand

$$d(f^{n-1}(x), f^{n+1}(x)) \le d(f^{n-1}(x), f^n(x)) + d(f^n(x), f^{n+1}(x)) \Rightarrow$$

$$(1-a)d(f^n(x), f^{n+1}(x)) \le ad(f^{n-1}(x), f^n(x)) \Rightarrow$$

$$d(f^n(x), f^{n+1}(x)) \le \frac{a}{1-a} d(f^{n-1}(x), f^n(x)) \le ... \le a1 - a^n d(x, f(x)).$$

But $a \in]0, \frac{1}{2}[\Rightarrow \dfrac{a}{1-a} \in]0, 1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon \neq \varnothing, \forall \varepsilon > 0$. □

In 1972, by combining the three independent (see [7]) contraction conditions above, Zamfirescu (see [11]) obtained another fixed point result for operators which satisfy the following.

Definition 2.4. *([8],[11]) A mapping $f : X \to X$ is a* ***Zamfirescu operator*** *if*

$$\exists a, k, c \in \mathbb{R},\ a \in [0, 1[,\ k \in [0, \frac{1}{2}[,\ c \in [0, \frac{1}{2}[\text{ such that}$$

$$\forall x, y \in X, \text{ at least one of the following is true:}$$

$\imath$): $d(f(x), f(y)) \leq ad(x, y)$;

$\imath\imath$): $d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))]$;

$\imath\imath\imath$): $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

Theorem 2.4. *Let (X, d) be a metric space and $f : X \to X$ a Zamfirescu operator. Then:*

$$\forall \varepsilon > 0,\ F_\varepsilon(f) \neq \varnothing.$$

Proof. First we will try to concentrate the three independent conditions into a single one they all imply, see the proof of Zamfirescu's fixed point theorem given in [3].

Let $x, y \in X$.

Supposing $\imath\imath$) holds, we have that:

$$d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))] \leq$$

$$\leq kd(x, f(x)) + k[d(y, x) + d(x, f(x)) + d(f(x), f(y))] =$$

$$= 2kd(x, f(x)) + kd(x, y) + kd(f(x), f(y)) \Rightarrow$$

$$d(f(x), f(y)) \leq \frac{2k}{1-k} d(x, f(x)) + \frac{k}{1-k} d(x, y). \tag{1}$$

Supposing $\imath\imath\imath$) holds, we have that:

$$d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))] \leq$$

$$\leq c[d(x, y) + d(y, f(y))] + c[d(y, f(y)) + d(f(y), f(x))] =$$

$$= cd(f(x), f(y)) + 2cd(y, f(y)) + cd(x, y) \Rightarrow$$

$$d(f(x), f(y)) \leq \frac{2c}{1-c} d(y, f(y)) + \frac{c}{1-c} d(x, y). \tag{2a}$$

Similarly:

$$d(f(x),f(y)) \leq c[d(x,f(y))+d(y,f(x))] \leq$$

$$\leq c[d(x,f(x))+d(f(x),f(y))]+c[d(y,x)+d(x,f(x))] =$$

$$= cd(f(x),f(y))+2cd(x,f(x))+cd(x,y) \Rightarrow$$

$$d(f(x),f(y)) \leq \frac{2c}{1-c}d(x,f(x))+\frac{c}{1-c}d(x,y). \tag{2b}$$

Now looking at $\imath), (1), (2a), (2b)$ we can denote:

$$\delta = \max\{a, \frac{k}{1-k}, \frac{c}{1-c}\},$$

and it is easy to see that $\delta \in [0,1[$.

For f satisfying at least one of the conditions $\imath), \imath\imath), \imath\imath\imath)$ we have that

$$d(f(x),f(y)) \leq 2\delta d(x,f(x))+\delta d(x,y) \tag{3a}$$

$$\text{and } d(f(x),f(y)) \leq 2\delta d(y,f(y))+\delta d(x,y) \tag{3b}$$

hold.

Using these conditions implied by $\imath)-\imath\imath\imath)$ and taking $x \in X$, we have:

$$d(f^n(x),f^{n+1}(x)) = d(f(f^{n-1}(x)),f(f^n(x))) \overset{(3a)}{\leq}$$

$$\leq 2\delta d(f^{n-1}(x),f(f^{n-1}(x)))+\delta d(f^{n-1}(x),f^n(x)) = 3\delta d(f^{n-1}(x),f^n(x)) \Rightarrow$$

$$d(f^n(x),f^{n+1}(x)) \leq ... \leq (3\delta)^n d(x,f(x)) \Rightarrow$$

$$d(f^n(x),f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon \neq \emptyset, \forall \varepsilon > 0$. □

Remark 2.2. *Theorems 2.1, 2.2, 2.3 are actually contained in Theorem 2.4, as any a-contraction, Kannan operator or Chatterjea operator is also a Zamfirescu operator (see Definitions 2.1, 2.2, 2.3, 2.4.).*

If we go further generalizing, we may consider the contraction condition given in 2004 by V. Berinde, who also formulated a corresponding fixed point theorem, see [2], for example.

Definition 2.5. *A mapping $f : X \to X$ is a **weak contraction** if*

$$\exists a \in]0,1[\text{ and } L \geq 0 \text{ such that } d(f(x), f(y)) \leq ad(x,y) + Ld(y, f(x)), \forall x, y \in X.$$

Theorem 2.5. *Let (X, d) be a metric space and $f : X \to X$ a weak contraction.*

Then:

$$\forall \varepsilon > 0, \ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq ad(f^{n-1}(x), f^n(x)) + Ld(f^n(x), f^n(x)) =$$

$$= ad(f^{n-1}(x), f^n(x)) \leq ... \leq a^n d(x, f(x))$$

But $a \in]0,1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon \neq \varnothing, \forall \varepsilon > 0$.

Remark 2.3. *Theorem 2.5 is even more general than the others above, as any of the above mentioned mappings is also a weak contraction, see Proposition 1 in [2].*

Remark 2.4. *An analogous result could be given for quasi-contractions with* $0 < h < \frac{1}{2}$, *see again [2].*

□

Similar results concerning the existence of $\varepsilon-$fixed points for other classes of operators on metric spaces will be the subject of future papers.

3. Quantitative results for operators on metric spaces

For the same operators we have studied in the previous section, from the qualitative point of view, we will now use Lemma 1.2 in order to obtain quantitative results.

Theorem 3.1. *Let (X, d) be a metric space and $f : X \to X$ an a-contraction.*

Then:

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon}{1-a}, \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. Condition $\imath)$ in Lemma 1.2 is satisfied, as one can see in the proof of Theorem 2.1.

We will show now that $\imath\imath)$ also holds for a-contractions.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$. We also assume that

$$d(x, y) - d(f(x), f(y)) \leq \eta$$

and aim to show that there exists an $\varphi(\eta) > 0$ such that $d(x, y) \leq \varphi(\eta)$.

We have that:

$$d(x, y) \leq d(f(x), f(y)) + \eta \leq ad(x, y) + \eta$$

$$\Rightarrow (1-a)d(x, y) \leq \eta,$$

which implies $d(x, y) \leq \dfrac{\eta}{1-a}$.

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{\eta}{1-a} > 0$ such that

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta).$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon}{1-a}, \forall \varepsilon > 0.$$

□

Theorem 3.2. *Let (X, d) be a metric space and $f : X \to X$ a Kannan operator.*

Then:

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon(1 + a), \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. As in the proof of Theorem 3.1 we only verify that condition $ii)$ in Lemma 1.2 holds.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$ and assume that $d(x, y) - d(f(x), f(y)) \leq \eta$.

Then

$$d(x, y) \leq a[d(x, f(x)) + d(y, f(y))] + \eta.$$

As $x, y \in F_\varepsilon(f)$, we know that $d(x, f(x)) < \varepsilon$ and $d(y, f(y)) < \varepsilon$.

$$\Rightarrow d(x, y) \leq 2a\varepsilon + \eta$$

So $\forall \eta > 0, \exists \varphi(\eta) = \eta + 2a\varepsilon > 0$ such that

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \eta.$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon(1 + a), \forall \varepsilon > 0.$$

□

Theorem 3.3. *Let (X, d) be a metric space and $f : X \to X$ a Chatterjea operator.*

Then:

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon(1 + a)}{1 - 2a}, \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. Again we will only show that condition $ii)$ in Lemma 1.2 holds.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$ and assume that

$$d(x, y) - d(f(x), f(y)) \leq \eta.$$

Then

$$d(x, y) \leq a[d(x, f(y)) + d(y, f(x))] + \eta \leq$$

$$\leq ad(x, f(y)) + ad(y, f(x)) + \eta \leq$$

$$\leq a[d(x, y) + d(y, f(y))] + a[d(y, x) + d(x, f(x))] + \eta.$$

As $x, y \in F_\varepsilon(f)$, it follows that

$$d(x, y) \leq 2ad(x, y) + 2\varepsilon a + \eta.$$

$$\Rightarrow (1 - 2a)d(x, y) \leq 2\varepsilon a + \eta \Rightarrow$$

$$d(x, y) \leq \frac{\eta + 2\varepsilon a}{1 - 2a}$$

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{\eta + 2\varepsilon a}{1 - 2a} > 0$ such that

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta).$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon(1 + a)}{1 - 2a}, \forall \varepsilon > 0.$$

□

Theorem 3.4. *Let (X, d) be a metric space and $f : X \to X$ a Zamfirescu operator. Then*

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon \frac{1 + \rho}{1 - \rho}, \forall \varepsilon > 0,$$

where $\rho = \max\{a, \dfrac{k}{1 - k}, \dfrac{c}{1 - c}\}$ *and a, k, c as in Definition 2.4.*

Proof. In the proof of Theorem 2.4 we have already shown that if f satisfies at least one of the conditions $\imath), \imath\imath)$ or $\imath\imath\imath)$ from Definition 2.4, then

$$d(f(x), f(y)) \leq 2\rho d(x, f(x)) + \rho d(x, y)$$

and

$$d(f(x), f(y)) \leq 2\rho d(y, f(y)) + \rho d(x, y)$$

hold.

Let $\varepsilon > 0$. Again we will only show that condition $\imath\imath)$ in Lemma 1.2 is satisfied, as $\imath)$ holds, see the Proof of Theorem 2.4.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$, and assume that

$$d(x,y) - d(f(x), f(y)) \leq \eta.$$

Then

$$d(x,y) \leq d(f(x), f(y)) + \eta \leq 2\rho d(x, f(x)) + \rho d(x,y) + \eta \Rightarrow$$

$$(1-\rho)d(x,y) \leq 2\rho\varepsilon + \eta \Rightarrow$$

$$d(x,y) \leq \frac{\eta + 2\rho\varepsilon}{1-\rho}.$$

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{\eta + 2\rho\varepsilon}{1-\rho} > 0$ such that

$$d(x,y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x,y) \leq \eta.$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon\frac{1+\rho}{1-\rho}, \forall \varepsilon > 0.$$

□

Remark 3.1. *In the case of weak contractions we have to add a condition, namely* $a + L < 1$*, with the same notations as above, in order to get the result.*

Theorem 3.5. *Let* (X,d) *be a metric space and* $f : X \to X$ *a weak contraction with* $a + L < 1$.

Then

$$\delta(F_\varepsilon(f)) \leq \frac{2+L}{1-a-L}\varepsilon, \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. We show again only that condition (ii) in Lemma 1.2 holds.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$, and assume that

$$d(x,y) - d(f(x), f(y)) \leq \eta.$$

Then

$$d(x,y) \leq d(f(x),f(y)) + \eta \leq ad(x,y) + Ld(y,f(x)) + \eta \leq$$

$$\leq ad(x,y) + Ld(x,y) + Ld(x,f(x)) + \eta \leq (a+L)d(x,y) + L\varepsilon + \eta.$$

$$\Rightarrow (1-a-L)d(x,y) \leq L\varepsilon + \eta \Rightarrow d(x,y) \leq \frac{L\varepsilon + \eta}{1-a-L}$$

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{L\varepsilon + \eta}{1-a-L} > 0$ such that

$$d(x,y) - d(f(x),f(y)) \leq \eta \Rightarrow d(x,y) \leq \eta.$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq \frac{2+L}{1-a-L}\varepsilon, \forall \varepsilon > 0.$$

□

4. **Conclusions**

The theory of $\varepsilon-$fixed points is not less interesting than that of fixed points and many results formulated in the latter can be adapted to a less restrictive framework in order to guarantee the existence of the $\varepsilon-$fixed points and the fact that the diameter of the set containing these points goes to zero when ε goes to zero.

We proved results referring to some types of contractive operators on metric spaces, starting from a result presented in [10] for $a-$contractions, but the study may go further to other classes of operators, which will be the subject of future papers.

5. **Aknowledgements**

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Department of Statistics, Analysis, Forecast and Mathematics,
Faculty of Economic Sciences and Bussiness Administration,
"Babes-Bolyai" University of Cluj-Napoca, 58-60 T. Mihali St., 400591, Cluj-Napoca, Romania

E-mail address: mberinde@math.ubbcluj.ro, madalina_berinde@yahoo.com