STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 1, March 2004

DATA DEPENDENCE OF THE FIXED POINTS SET OF WEAKLY PICARD OPERATORS IN GENERALIZED METRIC SPACES

CLAUDIA BACOŢIU

Abstract. In this paper we will extend the results concerning the data dependence of the fixed points set of weakly Picard operators to a generalized metric space (X, d) with $d(x, y) \in \mathbb{R}^n$, $n \in \mathbb{N}^*$.

1. Introduction

Definition 1. *Let $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$. We will consider, by definition:*
- $x \leq y \Leftrightarrow x_i \leq y_i \ \forall i = \overline{1, n}$;
- $| x |= (| x_1 |, | x_2 |, ..., | x_n |)$;
- $max(x, y) = (max(x_1, y_1), max(x_2, y_2), ..., max(x_n, y_n))$.

Definition 2. *Let X be a nonempty set; an application $d : X \times X \to \mathbb{R}^n_+$ is called generalized metric on X iff:*
(i) $d(x, y) \geq 0 \ \forall x, y \in X;\ d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x) \ \forall x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$.
In this case, (X, d) is said to be a generalized metric space (g.m.s. on short).

The related definitions of completeness, weakly Picard operators, the operator f^∞ in a g.m.s. are the same as in the standard metric spaces.

Definition 3. *If (X, d) is a g.m.s., we will consider the Pompeiu-Hausdorff functional*
$H : P(X) \times P(X) \to (\mathbb{R}_+ \bigcup\{+\infty\})^n, \quad H = (H_1, H_2, ..., H_n)$
$H_i(A, B) := max\{\sup_{a\in A} \inf_{b\in B} d_i(a, b), \sup_{b\in b} \inf_{a\in A} d_i(a, b)\} \ \forall A, B \in P(X) \ \forall i = \overline{1, n}.$

Definition 4. *If (X, d) is a g.m.s., an operator $f : X \to X$ is called C-weakly Picard iff f is weakly Picard and there exists $C \in \mathcal{M}_{n,n}(\mathbb{R})$ such that*
$d(x, f^\infty(x)) \leq Cd(x, f(x)) \ \forall x \in X.$

Received by the editors: 17.09.2003.
2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. fixed point, Picard operator, generalized metric.

2. **Main results**

Theorem 1. *Let (X, d) be a complete g.m.s. and $f, g : X \to X$ two operators. We suppose that:*
(i) there exist $C, D \in \mathcal{M}_{n,n}(\mathbb{R})$ such that f is C-weakly Picard and g is D-weakly Picard;
(ii) there exists $\eta \in \mathbb{R}^n_+$ such that $d(f(x), g(x)) \leq \eta \ \forall x \in X$.
Then $H(F_f, F_g) \leq \max\{C\eta, D\eta\}$.

To prove this theorem we will use the next Lemma:

Lemma 1. *If (X, d) is a g.m.s. and A, $B \in P(X)$; $\eta, \zeta \in \mathbb{R}^n_+$ such that:*
$\forall a \in A \ \exists b \in B : \ d(a, b) \leq \eta$;
$\forall b \in B \ \exists a \in A : \ d(a, b) \leq \zeta$.
Then $H(A, B) \leq \max\{\eta, \zeta\}$.

Proof-Theorem 1:
Let $x \in F_g$; then:
$d(x, f^\infty(x)) \leq Cd(x, f(x)) = Cd(g(x), f(x)) \leq C\eta$.
By a similar argument, we have that $d(x, g^\infty(x)) \leq D\eta \ \forall x \in F_f$.
It follows from Lemma 1 that $H(F_f, F_g) \leq \max\{C\eta, D\eta\}$.□
If in Theorem 1 we take f, g A-orbitally contractions, we have:

Theorem 2. *Let (X, d) be a complete g.m.s. and $f, g : X \to X$ two orbitally continuous operators. We suppose that:*
(i) $\exists A \in \mathcal{M}_{n,n}(\mathbb{R})$, $A^k \xrightarrow[k \to \infty]{} 0$ (i.e. the matrix A converges to zero) such that
$d(f^2(x), f(x)) \leq Ad(f(x), x) \ \forall x \in X$ and
$d(g^2(x), g(x)) \leq Ad(g(x), x) \ \forall x \in X$;
(ii) there exists $\eta \in \mathbb{R}^n_+$ such that $d(f(x), g(x)) \leq \eta \ \forall x \in X$.
Then:
a) $F_f \neq \emptyset$ and $F_g \neq \emptyset$;
b) $H(F_f, F_g) \leq (I - A)^{-1}\eta$.

3. **Applications**

We will consider the following systems of integral equations with deviating argument:

$$x(t) = x(a) + \int_a^b K(t, s, x(s))ds \quad \forall t \in [a, b] \tag{1}$$

$$x(t) = x(a) + \int_a^b N(t, s, x(s))ds \quad \forall t \in [a, b] \tag{2}$$

where $K, N \in C([a, b] \times [a, b] \times \mathbb{R}^n, \mathbb{R}^n)$.
By Theorem 2 we have:

Theorem 3. *We suppose that:*
(i) $K(a, s, u) = 0 \in \mathbb{R}^n$ and $N(a, s, u) = 0 \in \mathbb{R}^n \ \forall s \in \mathbb{R} \ \forall u \in \mathbb{R}^n$;

(ii) there exists $\eta \in \mathbb{R}^n_+$ *such that*
$|K(t,s,u) - N(t,s,u)| \le \eta \ \forall t,s \in [a,b] \ \forall u \in \mathbb{R}^n$;
(iii) there exists $L \in \mathcal{M}_{n,n}(\mathbb{R})$ *such that*
$|K(t,s,u) - K(t,s,v)| \le L|u-v| \ \forall t,s \in [a,b] \ \forall u,v \in \mathbb{R}^n$ *and*
$|N(t,s,u) - N(t,s,v)| \le L|u-v| \ \forall t,s \in [a,b] \ \forall u,v \in \mathbb{R}^n$;
(iv) the matrix $(b-a)L$ *converges to zero.*
If S_1 *and* S_2 *are the solutions sets of the systems (1) and (2) in* $C([a,b],\mathbb{R}^n)$ *then:*
a) $S_1 \neq \emptyset$ *and* $S_2 \neq \emptyset$;
b) $H_{\|\cdot\|}(S_1,S_2) \le [I - (b-a)L](b-a)\eta$;
where we consider the space $C([a,b],\mathbb{R}^n)$ *with the generalized metric induced by the Tchebychev norm* $\|y\| := (\|y_1\|_{C[a,b]}, \|y_2\|_{C[a,b]}, ..., \|y_n\|_{C[a,b]}) \ \forall y \in C([a,b],\mathbb{R}^n)$ *and* $H_{\|\cdot\|}$ *is the related Pompeiu-Hausdorff functional.*

Proof. We consider the operators
$f,g : C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ defined by
$f(x)(t) = x(a) + \int_a^b K(t,s,x(s))ds \ \forall t \in [a,b] \ \forall x \in C([a,b]$;
$g(x)(t) = x(a) + \int_a^b N(t,s,x(s))ds \ \forall t \in [a,b] \ \forall x \in C([a,b]$
and we will apply the Theorem 2.
We have $f^2(x)(t) = x(a) + \int_a^b K(t,s,f(x(s))ds \ \forall t \in [a,b] \ \forall x \in C([a,b]$,
so $|f^2(x)(t) - f(x)(t)| \le (b-a)L\|f(x) - x\| \Rightarrow$
$\|f^2(x) - f(x)\| \le \underbrace{(b-a)L}_{converges\ to\ 0} \|f(x) - x\|$, so $F_f = S_1 \neq \emptyset$.
By a similar argument, $\|g^2(x) - g(x)\| \le (b-a)L\|f(x) - x\|$, so $F_g = S_2 \neq \emptyset$.
We also have $\|f(x) - g(x)\| \le (b-a)\eta \ \forall x \in C([a,b])$.
We are in the conditions of the Theorem 2 $\Rightarrow H_{\|\cdot\|}(F_f,F_g) \le [I-(b-a)L](b-a)\eta$,
i.e. $H_{\|\cdot\|}(S_1,S_2) \le [I-(b-a)L](b-a)\eta$.□

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"Samuel Brassai" High School, Cluj-Napoca, Romania
E-mail address: Claudia.Bacotiu@clujnapoca.ro