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STARLIKENESS CONDITIONS FOR THE BERNARDI OPERATOR

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. Let U be the unit disc of the complex plane: $U=\{z\in C, |z|<1\}$ and $A_n=\left\{f\in H\left(U\right), f\left(z\right)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+..., z\in U\right\}$, and the class of starlike functions in U, $S^*(\alpha)=\left\{f\in A, \operatorname{Re}\dfrac{zf'\left(z\right)}{f\left(z\right)}>\alpha, z\in U\right\}$ the class of starlike functions of order α. We consider the integral operator $F\left(z\right)=\dfrac{1+\gamma}{z^{\gamma}}\int\limits_0^z f\left(t\right)t^{\gamma-1}dt$ and we study its starlikeness properties.

1. **Introduction**

In this paper a α order starlikeness condition for Bernardi operator is obtained. This condition is on extension of the results of Gh. Oros, see [1], which is obtained from our result for $\alpha=1$.

Lemma A. [2] *Let q the univalent function in U and let θ and ϕ be analytic functions in the domain $D\subset q\left(U\right)$ with $\phi\left(w\right)\neq 0$, when $w\in q\left(U\right)$.*

Set

$$Q\left(z\right)=nzq'\left(z\right)\phi\left[q\left(z\right)\right]$$

$$h\left(z\right)=\theta\left[q\left(z\right)\right]+Q\left(z\right)$$

and suppose that:

i) Q is starlike

and

ii) $\operatorname{Re}\dfrac{zh'\left(z\right)}{Q\left(z\right)}=\operatorname{Re}\left[\dfrac{\theta'\left[q\left(z\right)\right]}{\phi\left[q\left(z\right)\right]}+\dfrac{zQ'\left(z\right)}{Q\left(z\right)}\right]>0.$

If p is analytic in U, with

$$p\left(0\right)=q\left(0\right), p'\left(0\right)=...=p^{(n-1)}\left(0\right)=0, p\left(U\right)\subset D$$

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and

$$\theta\left[p\left(z\right)\right]+zp'\left(z\right)\phi\left[p\left(z\right)\right]\prec\theta\left[q\left(z\right)\right]+zq'\left(z\right)\phi\left[q\left(z\right)\right]$$

then $p\prec q$*, and* q *is the best dominant.*

2. Main results

Theorem 1. *Let* $\gamma\geq 0,\alpha>0$ *and*

$$h\left(z\right)=\frac{1}{1-\alpha z}+\frac{n\alpha z}{\left(1-\alpha z\right)\left(1+\gamma-\alpha\gamma z\right)} \qquad (1)$$

If $f\in A_n$ *and*

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec h\left(z\right)$$

then

$$Re\frac{zF'\left(z\right)}{F\left(z\right)}>\frac{1}{1+\alpha}$$

where

$$F\left(z\right)=\frac{1+\gamma}{z^{\gamma}}\int_{0}^{z}f\left(t\right)t^{\gamma-1}dt \qquad (2)$$

Proof. From 2 we deduce

$$\gamma F\left(z\right)+zF'\left(z\right)=\left(\gamma+1\right)f\left(z\right) \qquad (3)$$

If we consider

$$p\left(z\right)=\frac{zF'\left(z\right)}{F\left(z\right)}$$

then (3) becomes

$$\frac{zp'\left(z\right)}{p\left(z\right)+\gamma}+p\left(z\right)=\frac{zf'\left(z\right)}{f\left(z\right)}$$

But

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec h\left(z\right)$$

implies

$$\frac{zp'\left(z\right)}{p\left(z\right)+\gamma}+p\left(z\right)\prec h\left(z\right)$$

We apply Lemma 1 to prove that:

$$Re\frac{zF'\left(z\right)}{F\left(z\right)}>\frac{1}{1+\alpha}$$

We have:

$$q\left(z\right)=\frac{1}{1-\alpha z}$$

$$\theta(w) = w$$

$$\phi(w) = \frac{1}{w+\gamma}$$

$$\theta[q(z)] = \frac{1}{1-\alpha z}$$

$$\phi[q(z)] = \frac{1-\alpha z}{1+\gamma-\alpha\gamma z}$$

$$Q(z) = nzq'(z)\,\phi[q(z)] = \frac{n\alpha z}{(1-\alpha z)(1+\gamma-\alpha\gamma z)}.$$

$$h(z) = \theta[q(z)] + Q(z) = \frac{1}{1-\alpha z} + \frac{n\alpha z}{(1-\alpha z)(1+\gamma-\alpha\gamma z)}$$

Because Q is starlike and Re $\phi[q(z)] > 0$,from Lemma 1 we deduce

$$p \prec q \Leftrightarrow \frac{zF'(z)}{F(z)} \prec \frac{1}{1+\alpha z} \Rightarrow Re\frac{zF'(z)}{F(z)} > \frac{1}{1+\alpha}$$

The last relation is equivalent to

$$F \in S^*\left(\frac{1}{1+\alpha}\right)$$

Remark. For $\alpha = 1$ we obtain the result of Gh. Oros [1].

Corollary 1. *Let*

$$h(z) = \frac{1}{1-z} + \frac{n\alpha z}{(1-z)(2-z)}$$

If $f \in A$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)} > \frac{1}{2}$$

where

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt$$

Proof. In Theorem 1 we put $\alpha = 1, \gamma = 1, n = 1$.

Corollary 2. *Let*

$$h(z) = \frac{1}{1-2z} + \frac{n\alpha z}{(1-2z)(1+\gamma-2\gamma z)}$$

If $f \in A_n$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)} > \frac{1}{3}$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)\, t^{\gamma-1} dt$$

Proof. In Theorem 1 we put $\alpha = 2$.

Theorem 2. *Let* $\gamma \geq 0, \alpha > 0$ *and*

$$h(z) = \frac{1+\alpha z}{1-\alpha z} + \frac{2n\alpha z}{(1-\alpha z)(1+\gamma-(1-\gamma)\alpha z)} \tag{4}$$

If $f \in A_n$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)} > \frac{1-\alpha}{1+\alpha}$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)\, t^{\gamma-1} dt \tag{5}$$

Proof. From (5) we deduce:

$$\gamma F(z) + zF'(z) = (\gamma+1) f(z) \tag{6}$$

Let

$$p(z) = \frac{zF'(z)}{F(z)}$$

Then (3) becomes

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = \frac{zf'(z)}{f(z)}$$

But

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

implies

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) \prec h(z)$$

We use Lemma 1 to prove that:

$$Re\frac{zF'(z)}{F(z)} > \frac{1-\alpha}{1+\alpha}$$

We have:

$$q(z)=\frac{1+\alpha z}{1-\alpha z}$$

$$\theta(w)=w$$

$$\phi(w)=\frac{1}{w+\gamma}$$

$$\theta[q(z)]=\frac{1+\alpha z}{1-\alpha z}$$

$$\phi[q(z)]=\frac{1-\alpha z}{1+\gamma-(1-\gamma)\alpha z}$$

$$Q(z)=nzq'(z)\phi[q(z)]=\frac{2n\alpha z}{(1-\alpha z)(1+\gamma-(1-\gamma)\alpha z)}$$

$$h(z)=\theta[q(z)]+Q(z)=\frac{1+\alpha z}{1-\alpha z}+\frac{2n\alpha z}{(1-\alpha z)(1+\gamma-(1-\gamma)\alpha z)}$$

Because Q is starlike and Re $\phi[q(z)]>0$ from Lemma 1 we deduce :

$$p\prec q\Leftrightarrow\frac{zF'(z)}{F(z)}\prec\frac{1+\alpha z}{1-\alpha z}\Rightarrow Re\frac{zF'(z)}{F(z)}>\frac{1-\alpha}{1+\alpha}$$

The last relation is equivalent to

$$F\in S^*\left(\frac{1-\alpha}{1+\alpha}\right)$$

Remark. For $\alpha=1$ we obtain the result of Gh. Oros [1].

Corollary 3. *Let*

$$h(z)=\frac{1+2z}{1-z}$$

If $f\in A$ *and*

$$\frac{zf'(z)}{f(z)}\prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)}>0$$

where

$$F(z)=\frac{2}{z}\int_0^z f(t)\,dt$$

Proof. In Theorem 2 we put $\alpha=1,\gamma=1,n=1$.

Corollary 2. *Let*

$$h(z)=\frac{1+2z}{1-2z}+\frac{4nz}{(1-2z)(1+\gamma+2(1-\gamma)z)}$$

If $f \in A_n$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)} > -\frac{1}{3}$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)\, t^{\gamma-1} dt$$

Proof. In Theorem 2 we put $\alpha = 2$.

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