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ON THE CLASSIFICATION OF THE NOMOGRAPHIC FUNCTIONS OF FOUR VARIABLES (II)

MARIA MIHOC

Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. In this article we analysed the determinants Massau for some functions of four variables. We constructed also the nomograms in space with coplanar points and compound nomograms consisting of plane nomograms with alignment points.

In [3] the focus is on the study of the functions of four variables and their classification according to the rank of the functions with respect to each variable they depend on.

Further proceeding with this study implies the analysis of the nomograms in space with coplanar points (the nomograms on which the function can be nomographically represented) for some of those function classes.

A lot of authors beginning with R. Soreau, then J. Wojtowicz [5, 6], M. Warmus [4] have dealt with the correct definition of the rank of the functions of three variables with respect to one of its variables (and respectivelly, to all variables). They defined this rank as being equal to the minimum number of linear independent functions from the expression of $F(z_1, z_2, z_3)$. This expression consists of a sum of products where every product term consists of two factors; one of them is a function of one variable (i.e. to one with respect to which we define the rank), the second factor is a function of the other two variables.

We have extended [3] this definition to the case of the functions of four variables $F(z_1, z_2, z_3, z_4)$.

Definition 1. [3] The function of four variables $F(z_1, z_2, z_3, z_4)$ is said to be of rank n with respect to z_1, if there exist the functions $U_i(z_1)$, $V_i(z_2, z_3, z_4)$, $i = \overline{1, n}$,

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so that:

$$F(z_1, z_2, z_3, z_4) \equiv \sum_{i=1}^{n} U_i(z_1) V_i(z_2, z_3, z_4) \tag{1}$$

where n is the greatest possible natural number for which (1) occurs.

The functions $U_i(z_1)$, $i = \overline{1,n}$ are linear independent, $V_i(z_2, z_3, z_4)$, $i = \overline{1,n}$ are linear independent too.

Definition 2. [3] The function $F \equiv F(z_1, z_2, z_3, z_4)$ is called nomographic in space if:

a) the rank of the function F with respect to each of its variables is greater than one,

b) there exist the functions $X_i(z_1)$, $Y_i(z_2)$, $Z_i(z_3)$, $T_i(z_4)$, $i = \overline{1,4}$, so that:

$$F(z_1, z_2, z_3, z_4) \equiv \begin{vmatrix} X_1(z_1) & X_2(z_1) & X_3(z_1) & X_4(z_1) \\ Y_1(z_2) & Y_2(z_2) & Y_3(z_2) & Y_4(z_2) \\ Z_1(z_3) & Z_2(z_3) & Z_3(z_3) & Z_4(z_3) \\ T_1(z_4) & T_2(z_4) & T_3(z_4) & T_4(z_4) \end{vmatrix}. \tag{2}$$

The determinant of type (2) will be called a Massau form (or determinant Massau) of the function F.

Theorem 3. [3] *If the function of four variables $F(z_1, z_2, z_3, z_4)$ is nomographic in space, then it is of rank two, three or four with respect to each of the variables z_i, $i = \overline{1,4}$ i.e. it has one of the forms:*

$$F \equiv X_1 G_1 + X_2 G_2 \tag{3}$$

$$F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3 \tag{4}$$

$$F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3 + X_4 G_4 \tag{5}$$

with respect to variable z_1. The functions G_i, $i = \overline{1,4}$ are the rank one, two or three with respect to their variables.

We have introduced the following abbreviations:

$$F = F(z_1, z_2, z_3, z_4); \qquad X_i = X_i(z_1), \qquad Y_i = Y_i(z_2),$$

$$Z_i = Z_i(z_3), \qquad T_i = T_i(z_4), \qquad G_i = G_i(z_2, z_3, z_4), \qquad i = \overline{1,4}.$$

Definition 4. The nomographic representation of the function F (that has been brought to the form (2)) is equivalent to the nomographic representation of the equation Soreau associated to this function.

The equation Soreau has been obtained by equalisation with zero of the determinant Massau from (2).

The functions F, which have the forms (3)-(5) (or can be brought to these forms) can be nomographically represented by nomograms with coplanar points, because the determinant (2) equated with zero leads to the condition of coplanarity of four points in space, $P_i(x, y, z)$, $i = \overline{1,4}$ (i.e. four points situated in the same plane). The coordinates of these points are (in the system of cartesian coordinates in space $XOYZ$):

$$P_i : \; x = \frac{A_1(z_i)}{A_4(z_i)}, \qquad y = \frac{A_2(z_i)}{A_4(z_i)}, \qquad z = \frac{A_3(z_i)}{A_4(z_i)}, \qquad i = \overline{1,4} \tag{6}$$

and $A_j(z_i)$, $i, j = \overline{1,4}$ successively take the values $X_j(z_1)$, $Y_j(z_2)$, $Z_j(z_3)$, $T_j(z_4)$, $j = \overline{1,4}$.

The formulas (6) are obtained by division of the elements of determinant (2) by those of the fourth column. If at least one element of the last column of the determinant is equal to zero, we can apply an elementary transformation in order to obtain at least one column with all elements different from zero.

Each of point P_i is situated on the curves C_i (of the parameter z_i), where (6) are their parametric equations. By elimination of the parameter z_i from (6) we obtain the equations of two cylindrical surfaces

$$S_1^i(x, y) = 0, \qquad S_2^i(x, z) = 0, \qquad i = \overline{1,4}. \tag{7}$$

The intersection of these surfaces gives exactly the distort curve C_i in space. Therefore, the function of four variables F brought to the form (2) can be nomographically represented by a nomogram in space with coplanar points (see fig. 1). The nomogram consists of four scales (z_i), $i = \overline{1,4}$. These scales are situated on the distort curves in space, C_i, $i = \overline{1,4}$.

The nomogram in figure 1 is employed as follows: We provide the values of the three variables of the equation $F(z_1, z_2, z_3, z_4) = 0$. These values are also the marks of the scales of the variables z_i, situated on the curve C_i, $i = \overline{1,3}$; three points

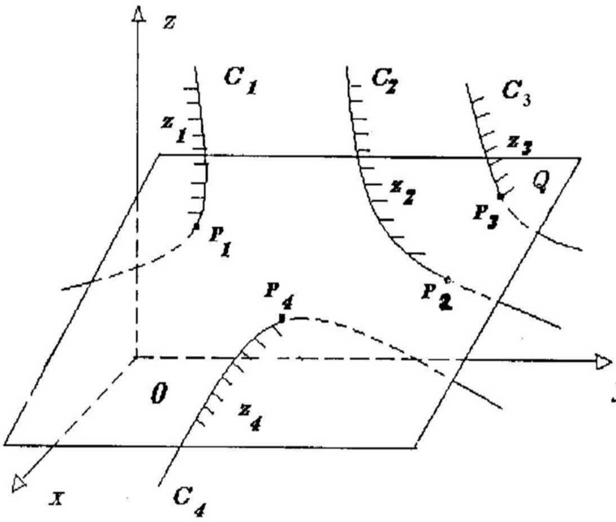

FIGURE 1. The nomogram in space with coplanar points

correspond to them on this curves. These points determine a plane that intersects the fourth curve at another point. The mark of the respective point will give the required value of the variable to arrive at (the fourth variable of the equation).

We proceed to the analysis of some classes of functions $F(z_1, z_2, z_3, z_4)$, which are of rank r_{z_i} with respect to variables z_i, $i = \overline{1,4}$, where $2 \leq r_{z_i} \leq 4$. We will write the forms Massau, to which they can be brought, and we will construct the nomogram by which these functions are represented.

I. The function F is of rank two with respect to each of its variables z_i, $i = \overline{1,4}$ (having form (3))

$$F \equiv X_1 G_1 + X_2 G_2$$

where the functions G_i, $i = 1, 2$ are of rank one each with respect to variable z_2, i.e.

$$G_i(z_2, z_3, z_4) \equiv Y_i(z_2) H_i(z_3, z_4) \tag{8}$$

the functions $H_i(z_3, z_4)$, $i = 1, 2$ are also of rank one each with respect to their variables

$$H_i(z_3, z_4) \equiv Z_i(z_3) T_i(z_4). \tag{9}$$

According to the remarks above, the function F take the form:

$$F \equiv X_1 Y_1 Z_1 T_1 + X_2 Y_2 Z_2 T_2. \tag{10}$$

Six determinants Massau (respectively six equations Soreau) correspond to this function i.e.:

$$a)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & Y_1 & 0 & Y_2 \\ 0 & 0 & Z_2 & -Z_1 \\ T_2 & 0 & T_1 & 0 \end{vmatrix} = 0 \qquad b)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & Y_1 & 0 & Y_2 \\ Z_2 & 0 & Z_1 & 0 \\ 0 & 0 & -T_2 & T_1 \end{vmatrix} = 0$$

$$c)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ Y_2 & 0 & Y_1 & 0 \\ 0 & Z_1 & 0 & Z_2 \\ 0 & 0 & T_2 & -T_1 \end{vmatrix} = 0 \qquad d)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & 0 & -Y_2 & Y_1 \\ 0 & Z_1 & 0 & Z_2 \\ T_2 & 0 & T_1 & 0 \end{vmatrix} = 0 \qquad (11)$$

$$e)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & 0 & Y_2 & -Y_1 \\ Z_2 & 0 & Z_1 & 0 \\ 0 & T_1 & 0 & T_2 \end{vmatrix} = 0 \qquad f)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ Y_2 & 0 & Y_1 & 0 \\ 0 & 0 & -Z_2 & Z_1 \\ 0 & T_1 & 0 & T_2 \end{vmatrix} = 0$$

All these forms Massau are distinctly projective, because there cannot exist a square matrix of fourth order, whose determinant is different from zero, by which one of the forms (11) a)-f) can be brought to any of the remaining forms.

Only the form Massau (and the equation Soreau) (11)a) will be analized below. By multiplying first column of the determinant from the equation (11)a) with the positive factor a, adding it to the last column and then dividing each of its lines by the elements of the last column we obtain the equation Soreau:

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a}\frac{X_2}{X_1} & 0 & 1 \\ 0 & \frac{Y_1}{Y_2} & 0 & 1 \\ 0 & 0 & -\frac{Z_2}{Z_1} & 1 \\ \frac{1}{a} & 0 & \frac{1}{a}\frac{T_1}{T_2} & 1 \end{vmatrix} = 0 \qquad (12)$$

The equation (12) also gives the parametric equations of the curve C_i, $i = \overline{1,4}$; in this case they are the straight lines D_i, i.e. D_i are the supports for the scales of the variables z_i of the nomogram in space.

$$\begin{aligned} &D_1: \quad x=\frac{1}{a} \quad y=\frac{1}{a}\frac{X_2}{X_1} \quad z=0; \quad D_3: \quad x=0 \quad y=0 \quad z=-\frac{Z_2}{Z_1} \\ &D_2: \quad x=0 \quad y=\frac{Y_1}{Y_2} \quad z=0; \quad D_4: \quad x=\frac{1}{a} \quad y=0 \quad z=\frac{1}{a}\frac{T_1}{T_2} \end{aligned} \tag{13}$$

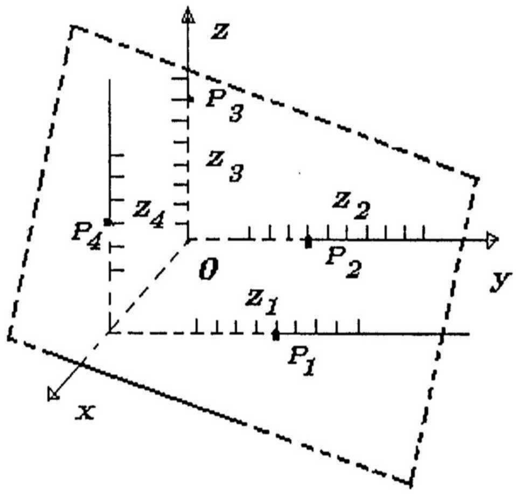

FIGURE 2. The nomogram in space with straight lines

The scales of the variables z_1 and z_2 are situated on the plan XOY and those of the variables z_3 and z_4 on the plan XOZ (see figure 2).

The use of the nomogram is that in the general case of a nomogram in space with coplanar points.

Due to the particular position of the straight lines D_i, $i=\overline{1,4}$, we can also imagine another nomogram for the function (3), brought to the form (11)a), more convenient for the user. This is a compound nomogram consisting of nomograms with alignment points; each constitutive nomogram has three scales in the same plan.

The equation Soreau (11)a) can be decomposed into four equations, i.e.

$$z = 0 \qquad\qquad y = 0$$

$$\begin{vmatrix} w & 0 & 1 \\ \frac{1}{a} & \frac{1}{a}\frac{X_2}{X_1} & 1 \\ 0 & \frac{Y_1}{Y_2} & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} w & 0 & 1 \\ 0 & -\frac{Z_2}{Z_1} & 1 \\ \frac{1}{a} & \frac{1}{a}\frac{T_1}{T_2} & 1 \end{vmatrix} = 0 \tag{14}$$

The first two equations of (14) are the equations of the plans XOY and XOZ; and the others two are the equations Soreau, which represent the conditions of alignment of three points in the respective plans. The first line of each determinant Massau from (14) include the parametric equations of the axis OX, which is the support of a mute scale (a scale without marks). The other lines of the determinants give the parametric equations of the scales z_i, $i = \overline{1,4}$ of the function (10).

Each of the last two equations (14) is nomographically represented by a plan nomogram with alignment points with straight line scales for the variables z_1 and z_2 (respectively z_3 and z_4) and a mute scale.

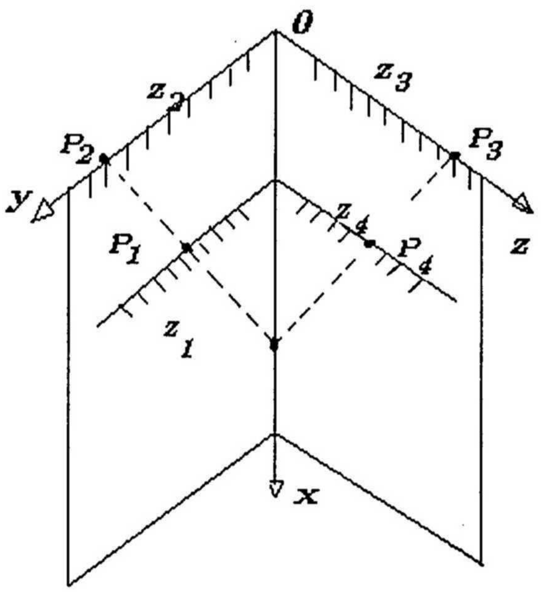

FIGURE 3

The use of the compound nomogram is the following (see figure 3): A straight line that crosses the axis OX (mute scale) in one point is plotted through two of the points of the given mark (which correspond to the given values of variables of (10)). The last point is also joined by a straight line with the third point, of given mark situated on the third scale. The point where the alignment line crosses the fourth scale gives the values of the fourth variable.

We mention the fact that each of four variables of the function (10) can be found if the three other variables are known.

We recall the fact that the genus of one nomogram of the equation of three variables is the number of curve scales of the nomogram consists of. We define now the genus of the nomogram in space.

Definition 5. The genus of a nomogram in space with coplanar points is equal with the number of curve scales the nomogram consists of.

According to this definition the genus of a nomogram in space can be zero, one, two, three or four.

Therefore, the nomograms in space built for the function (10), both the nomogram with coplanar points and the compound nomogram consisting of nomograms with alignment points, are of genus zero; its sales have parallel supports.

The function (10) can be represented and by another nomogram of genus zero if we subject the determinant Massau from (11)a) to the following transformations

$$\begin{vmatrix} \frac{X_1}{aX_1+bX_2} & \frac{X_2}{aX_1+bX_2} & 0 & 1 \\ 0 & \frac{Y_1}{bY_1+dY_2} & 0 & 1 \\ 0 & 0 & \frac{Z_2}{cZ_2-dZ_1} & 1 \\ \frac{T_2}{aT_2+cT_1} & 0 & \frac{T_1}{aT_2+cT_1} & 1 \end{vmatrix} = 0 \qquad (15)$$

and two those nomograms are (see figure 4).

In this case, the function (10) is represented by a nomogram of genus two with straight line scales for the variables z_2 and z_3, and curve scales for the variables z_1 and z_4.

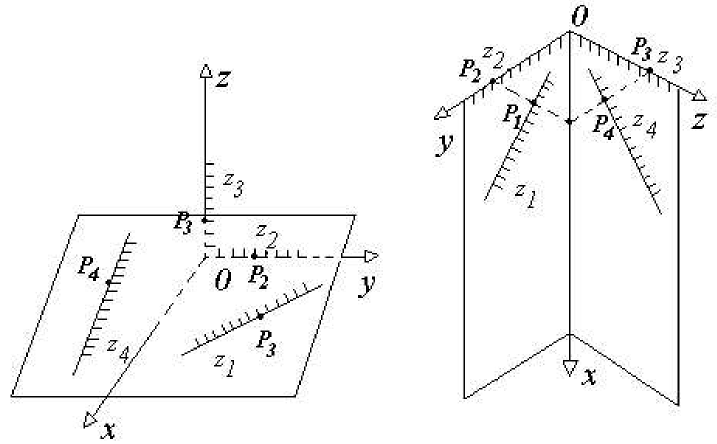

FIGURE 4. The nomogram of genus zero

Other equations (11) can also be represented by similar nomograms with coplanar points, or by a compound nomogram consisting of two plane nomograms with alignment points (like those above). The difference only consisting in the change of the variables of scales of the nomogram, while their supports stay fixed.

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Faculty of Mathematics and Computer Sciences,
Babeş-Bolyai University of Cluj-Napoca, Str. M. Kogălniceanu no. 1,
3400 Cluj-Napoca, Romania
E-mail address: mmihoc@math.ubbcluj.ro