

NUMERICAL SOLUTION OF THE KORTEWEG - DE VRIES BURGERS EQUATION BY USING QUINTIC SPLINE METHOD

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Abstract. In this work we will discuss the solution of the modified Burgers equation by using the collocation method with quintic splines. The test problem will be obtained discuss the accuracy of this problem. We make a comparison between the numerical and exact solution of the modified Burgers equation. The last section to discuss the stability analysis of this method.

1. Introduction

In this paper we will introduce a numerical solution for the Korteweg -de Vries Burgers equation (KdVB) which is a non-linear partial differential equation which involves both damping and dispersion take the following form

$$u_t + \epsilon uu_x - \nu u_{xx} + \mu u_{xxx} = 0 \quad (1)$$

This equation was derived by Su and Gardner [1] for a wide class of nonlinear system in the weak non-linearity and long wavelength approximation. The steady state solution of the KdVB equation has been shown to model [2] weak plasma shocks propagation perpendicularly to a magnetic field. When diffusion dominates dispersion the steady state solutions of the KdVB equation are monotonic shocks, and when dispersion dominates, the shocks are oscillatory. The KdVB equation has been obtained when including electron inertia effects in the description of weak nonlinear plasma waves [3]. The KdVB equation has also been used in a study of wave propagation through liquid field elastic tube [4] and for a description of shallow water waves on viscous fluid [5]. Canosa and Gazdag [6], who discussed the evolution of non-analytic initial data into a monotonic shock, have given brief details of a numerical solution for the KdVB equation using the accurate space derivative method. In this chapter we will use the finite element method with Quintic Spline interpolation function, and we will show

the state of solution in variant times. Grad and Hu [3] showed that the dissipation effects dominate over dispersive effect when:

$$4\mu \leq \nu^2 \tag{2}$$

In this case the solution of (1) is a shock decreasing monotonically from the upstream to the downstream value of u. if

$$\nu^2 < 4\mu \tag{3}$$

The dispersive effects dominate over the dispative effects; in this case the shock becomes oscillatory upstream and monotonic downstream. In this work we introduce the Quintic Spline with finite element method to solve the KDVB equation, and discuss the stability and the accuracy of this solution comparing with the exact solution [7] with some initial and boundary conditions.

2. Exact Solution of the KdVB Equation

In this section we will introduce the exact solution of the KdvB equation which appeared at the first time for the two dimensional KdVB equation at [7]. We modify the solution to take the form:

$$\frac{12\nu^2}{\epsilon\mu} \left[1 - \frac{e^{\frac{2\nu}{\epsilon\mu}(x-\omega t)}}{(e^{\frac{\nu}{\epsilon\mu}(x-\omega t)} + E)^2} \right] \tag{4}$$

where E, is a positive constant, $\omega = \frac{12\nu^2}{25\mu}$, ϵ is the coefficient of the nonlinear term, ν is the viscosity coefficient and μ is the coefficient of the dispersive term. We note that the accuracy of the numerical solution depend on E.

3. Numerical Solution of the KdVB Equation with Collocation Quintic Spline Method

Consider the KdVB equation (1), where the ϵ is a positive parameter and the subscripts x, and t indicate to the differentiation with respect to x and t. The boundary conditions are chosen from:

$$u(a, t) = 1, u(b, t) = 0$$

$$u_x(a, t) = 0 = u_x(b, t)$$

$$u_{xx}(a, t) = 0 = u_{xx}(b, t)$$

Consider $x_i = a + ih, h = \frac{b-a}{N}, i = -3, -2, \dots, N + 3$. Then $\Pi := a = x_0 < x_1 < \dots < x_n = b$ is an equal distance partition of the interval $[a, b]$ by the knots x_i . Define the quintic B-spline function as

$$\phi_i(x) = \frac{1}{h^5} \begin{cases} (x - x_{i-3})^5 & x \in [x_{i-3}, x_{i-2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 & x \in [x_{i-1}, x_i] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_i)^5 & x \in [x_i, x_{i+1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_i)^5 + 15(x - x_{i+1})^5 & x \in [x_{i+1}, x_{i+2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_i)^5 + 15(x - x_{i+1})^5 - (x - x_{i+2})^5 & x \in [x_{i+2}, x_{i+3}] \\ 0 & \text{otherwise.} \end{cases}$$

and let $\phi_i(x)$, be those quintic splines, for $i = 0, 1, \dots, N$. Let

$$x_n = \text{span}\{\phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \dots, \phi_{N+1}, \phi_{N+2}\}$$

form a basis for the function defined over $[a, b]$, where the values of the quintic splines $\phi_i(x)$, and all its first, and second derivatives vanishes outside the interval $[x_{i-3}, x_{i+3}]$. We establish the value of $\phi_i(x)$ and its derivatives in the following table:

Table 1

x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
ϕ	0	1	26	66	26	1	0
ϕ'	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$-\frac{50}{h}$	$-\frac{5}{h}$	0
ϕ''	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$-\frac{120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
ϕ'''	0	$\frac{60}{h^3}$	$-\frac{120}{h^3}$	0	$\frac{120}{h^3}$	$\frac{60}{h^3}$	0

Our task is to find an approximate solution $u_N(x, t)$ to the solution $u(x, t)$ in the form:

$$u_N(x, t) = \sum_{i=-2}^{N+2} \phi_i(x_j) \delta_i(t) \quad (5)$$

Where δ_i are unknowns dependent on time to be determined. Substitute from the values of $\phi_i(x)$ and its derivatives into (1), and suppose that δ_i are linearly

interpolated between two levels n and $n + 1$ by

$$\delta_i = \theta \delta_i^{n+1} + (1 - \theta) \delta_i^n$$

Where $0 \leq \theta \leq 1$ is a parameter at the time $n\Delta t$. The time derivative descriptive using the finite difference formula

$$\frac{d\delta}{dt} = \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t}$$

We get

$$\begin{aligned} & \sum_{i=-2}^{N+2} \left(\phi_i + \frac{\theta \epsilon \Delta t}{h} \phi_i' - \nu \frac{\theta \Delta t}{h^2} \phi_i'' + \mu \frac{\theta \Delta t}{h^3} \phi_i''' \right) \delta_i^{n+1} = \\ & = \sum_{i=-2}^{N+2} \left(\phi_i + \frac{(1-\theta) \epsilon \Delta t}{h} \phi_i' - \nu \frac{(1-\theta) \Delta t}{h^2} \phi_i'' + \mu \frac{(1-\theta) \Delta t}{h^3} \phi_i''' \right) \delta_i^n \end{aligned} \quad (6)$$

Giving the parameter θ the value $1/2$ we get the Crank-Nicolson formula which implies the recurrence relation

$$\sum_{i=-2}^{N+2} \left(\phi_i + \frac{\epsilon \Delta t}{2h} \phi_i' - \nu \frac{\Delta t}{2h^2} \phi_i'' + \mu \frac{\Delta t}{2h^3} \phi_i''' \right) \delta_i^{n+1} = \sum_{i=-2}^{N+2} \left(\phi_i + \frac{\epsilon \Delta t}{2h} \phi_i' - \nu \frac{\Delta t}{2h^2} \phi_i'' + \mu \frac{\Delta t}{2h^3} \phi_i''' \right) \delta_i^n \quad (7)$$

Applying the boundary condition we can eliminate δ_{-2} , δ_{-1} , δ_{N+1} and δ_{N+2} to get the following system of non linear equations:

$$a_i \delta_{i-2}^{n+1} + b_i \delta_{i-1}^{n+1} + c_i \delta_i^{n+1} + d_i \delta_{i+1}^{n+1} + e_i \delta_{i+2}^{n+1} = a_i' \delta_{i-2}^n + b_i' \delta_{i-1}^n + c_i' \delta_i^n + d_i' \delta_{i+1}^n + e_i' \delta_{i+2}^n \quad (8)$$

We can write this system of equations in the form

$$A [\delta] \delta^{n+1} = B [\delta] \delta^n \quad (9)$$

where the matrices and are Penta-diagonal matrices. The elements of the matrices and are given by:

$$\begin{aligned} a_i &= 1 - r_1 z_{i-2} - r_2 + r_3, a_i' = 1 + r_1 z_{i-2} + r_2 + r_3 \\ b_i &= 26 - 10r_1 z_{i-2} - 2r_2 + 2r_3, b_i' = 26 + 10r_1 z_{i-2} + 2r_2 - 2r_3 \\ c_i &= 66 + 6r_2, c_i' = 66 - 6r_2 \\ d_i &= 26 + 10r_1 z_{i-2} - 2r_2 - 2r_3, d_i' = 26 - 10r_1 z_{i-2} + 2r_2 + 2r_3 \\ e_i &= 1 + r_1 z_{i-2} - r_2 + r_3, e_i' = 1 - r_1 z_{i-2} + r_2 - r_3 \end{aligned} \quad (10)$$

where

$$r_1 = \frac{5\epsilon\delta t}{2h}, r_2 = \frac{10\nu\delta t}{h^2},$$

$$r_3 = \frac{30\mu\delta t}{h^3},$$

$$z_{i-2} = \delta_{i-2} + 26\delta_{i-1} + 66\delta_i + 26\delta_{i+1} + \delta_{i+2}$$

To solve this system we apply at first the initial condition to determine

$$\delta_{-2}^0, \delta_{-1}^0, \delta_0^0, \dots, \delta_N^0, \delta_{N+1}^0, \delta_{N+2}^0$$

When $t = 0$, equation (6) takes the formula

$$u_N^0(x, t) = \sum_{i=-2}^{N+2} \phi_i(x_j) \delta_i(t)^0 \quad (11)$$

The approximate solution must satisfy the following:

- (a). It must agree with the initial condition $u(x, 0)$ at the knots and
- (b). The first, second, and third derivatives of the approximate initial condition agree with those of the exact initial conditions at both ends of the range. So we get the system:

$$A\delta^0 = u_0(x) \quad (12)$$

Where A is $(N + 5) \times (N + 5)$ square matrix which can be restored by the Penta-diagonal algorithm to $(N + 1) \times 5$. In the following we will give an illustration to point out how to as an example to compute the element of the matrix A . substitute from (8),(10) and (11) in (12) we have,

$$a'_0\delta_{-2}^0 + b'_0\delta_{-1}^0 + c'_0\delta_1^0 + d'_0\delta_1^0 + e'_0\delta_2^0 = u_0(x_0)$$

i.e.

$$\begin{aligned} & (1 + r_1z_{-2} + r_2 + r_3)\delta_{-2}^0 + (26 + 10r_1z_{-2} + 2r_2 - 2r_3)\delta_{-1}^0 + (66 - 6r_2)\delta_0^0 + \\ & + (26 - 10r_1z_{-2} + 2r_2 + 2r_3)\delta_1^0 + (1 - r_1z_{-2} + r_2 - r_3)\delta_2^0 = u_0(x_0) \end{aligned} \quad (13)$$

Substitute the values of r_1, r_2 , and z_{-2} in (13) and use the boundary conditions to eliminate δ_{-2} and δ_{-1} to get the first row in the matrix A and so on.

$$A = \begin{bmatrix} 54 & 60 & 6 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 25.25 & 67.5 & 26.25 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & 0 \\ \vdots & & & & & & & & & & & \\ 0 & \cdots & 6 & 60 & 54 \end{bmatrix},$$

By solving the system (13) we get $\{\delta_{-2}^0, \delta_{-1}^0, \delta_0^0, \dots, \delta_N^0, \delta_{N+1}^0, \delta_{N+2}^0\}$. So we iterate by using the Pascal program, and hence the solution of equation (1) written as:

$$u(x, t) = \delta_{i-2} + 26\delta_{i-1} + 66\delta_i + 26\delta_{i+1} + \delta_{i+2} \tag{14}$$

4. Stability Analysis

We apply the Von-Neumann stability for equation (9) so we must linearize this equation and put the nonlinear term as $z_{i-2} = d + 26d + 66d + 26d + d = 120d$, according to the Von-Neuman we have

$$\delta_j^n = \epsilon^n e^{ikx_j} \tag{15}$$

Hence after dividing by at both sides of equation (9) with the help of equation (16) we get

$$g = \frac{A + iB}{A - iB} \tag{16}$$

where

$$g = \frac{\epsilon^{n+1}}{\epsilon^n},$$

$$A = 4 \cos^2(kh) + 26 \cos^2\left(\frac{kh}{2}\right) + 3 + r_2(\cos(kh) + 2)(\cos(kh) - 1)$$

$$A_1 = 4 \cos^2(kh) + 26 \cos^2\left(\frac{kh}{2}\right) + 3 - r_2(\cos(kh) + 2)(\cos(kh) - 1)$$

$$B = 4r_2 \sin(kh)(\cos^2(\frac{kh}{2}) + 5) + 4r_3 \sin(kh)(\cos(kh) - 1) \tag{17}$$

We note that $A_1 < A_2$, so

$$|g| = \left| \frac{A^2 + B^2}{A_1^2 + B^2} \right| \leq 1$$

Which means that the Quintic Splines method is unconditionally stable.

5. Test Problem

Canoza and Gazdag [4] have shown that the steady state solution for the KdvB equation with boundary conditions $u(a, t) = 1$ and $u(b, t) = 0$, exhibits different Behaviour depending on the relative values of ν and μ : (a) it is a shock wave decreasing monotonically from upstream to downstream if

$$\nu^2 \geq 4\mu$$

(b) it is a shock wave which becomes oscillatory upstream and monotonic downstream if

$$\nu^2 < 4\mu$$

These observations are confirmed in the following simulations take the initial condition as the step function

$$u(x, t) = \begin{cases} 1 & \text{if } 0 \leq x \leq 150, \\ 0 & \text{if } x > 150. \end{cases} \quad (18)$$

With μ and like Canoza and Gazdag [4] when $\nu = 6.0$, and 0.1 and 0.05. So we take the boundary conditions as:

$$\begin{aligned} u(0, t) &= 1, u(220, t) = 0, \\ u_x(0, t) &= 0 = u_x(220, t) \end{aligned} \quad (19)$$

Now we make some comparison between the exact solution (4) with $\epsilon = 2$ and the parameter $E = 1000$. Note. The value of the constant E is large to be in the neighborhood of the boundary conditions.

6. Graphics

In this section we plot some graphics to note the behaviour of our numerical solution at some various values of the viscosity and dispersive coefficients, as follows.

Figures(1.a-1.f) show the behavior of the computed solution with, $\nu = 5, \mu = 6$ it means that ν^2 near to 4μ and at time step $\Delta t = 0.02$, and $\Delta x = 0.55$ It is confirmed that when the viscosity value is large ($\nu=5$) then numerical solution of the KdvB equation is a shock wave decreasing monotonically from the upstream to the downstream value of the solution [5]. Similar shock wave solutions have been obtained for Burgers' equation [6,7].

Figures (2a-2f) show the behavior of the solution from $t=0\text{sec}$ to $t=50\text{sec}$, with $\nu = 1$, $\mu = 20$, $\Delta t = 0.02$, and $\Delta x = 0.55$. We see that oscillations is increasing with respect to the time, but it is still stable.

Figures (3.a - 3.f) show the behavior of the numerical solution at $\nu = 2$ and $\mu = 4$, which means that $\nu^2 \equiv 4\mu$, which give a very smooth solution, and we will discuss the errors later.

Figures (4.a - 4.f), show the behaviour of the computed solution for $\nu = 0.05$ at times from $t = 0$ to $t = 50$. When viscosity value is small the numerical solution of the KdVB equation is a shock wave which becomes oscillatory upstream and monotonic downstream confirming the theoretical treatment [8,9,10]. These graphs also show that as n is decreased further the computed solutions become more oscillatory. The results are consistent with graphs presented by Vliegthart for KdV equation, where for identical initial conditions similar behavior is observed [8].

7. Computational Results

In this section we compare between the numerical and exact solution for the KdVB equation and the errors of the Collocation method at each time step

Table 2

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\nu = 5, \mu = 6, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$

Time(Sec)	10	20	30	40	50	60
L_2x10^3	0.0010	0.0012	0.0013	0.0012	0.0012	0.0011
$L_\infty x10^3$	0.0002	0.0002	0.0003	0.0003	0.0003	0.0003

As we note from table 2 the numerical results are very close to the exact results at $\nu = 5, \mu = 6, t = 10sec,$ and $\Delta t = 0.02sec, \Delta x = 0.73cm.$ For the time increases the results are still close to the exact one which means that the method is very accurate. The errors are given in the following table.

Table 3

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines $\nu = 1, \mu = 2, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to

Time (Sec)	10	20	30	40	50	60
L_2x10^3	0.0005	0.0006	0.0008	0.0010	0.0012	0.0016
$L_\infty x10^3$	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001

Table 4

gives the relation between the numerical and exact solution of KdV-Burgers equation by using Collocation method at $\nu = 0.05\mu = 1, t = 10sec, \Delta t = 0.02sec$ and

Time (Sec)	10	20	30	40	50	60
L_2x10^3	1.1723	1.2886	1.3618	1.4161	1.4598	1.4964
$L_\infty x10^3$	0.6	0.6	0.6	0.6	0.6	0.6

As we said before the numerical solution to the KdVB equation by using the Collocation method depends on the ratio $\frac{\nu^2}{4\mu} \ll 1,$ and when the ratio is very closed the solution is more accurate and the method is very good to examine

Table 5

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\mu = 10, \nu = 1, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time(Sec)	10	20	30	40	50	60
L_2x10^3	60.534	66.394	70.084	72.821	75.013	76.847
$L_\infty x10^3$	23.865	23.866	23.867	23.868	23.869	23.870

Table 6

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\mu = 1, \nu = 2, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time (Sec)	10	20	30	40	50	60
$L_2 \times 10^3$	0.0032	0.0032	0.0031	0.0031	0.0031	0.0031
$L_\infty \times 10^3$	0.0013	0.0014	0.0014	0.0014	0.0014	0.00014

Table 7

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\mu = 0.1, \nu = 0.005, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time (Sec)	10	20	30	40	50	60
$L_2 \times 10^3$	0.0128	0.0144	0.0154	0.0160	0.0167	0.0171
$L_\infty \times 10^3$	0.0059	0.0060	0.0063	0.0060	0.0065	0.0059

Table 8

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\mu = 0.1, \nu = 0.005, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time (Sec)	10	20	30	40	50	60
$L_2 \times 10^3$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$
$L_\infty \times 10^3$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$

8. Conclusion

The finite element method with the quintic spline is capable of producing an accurate and stable numerical solution for the Korteweg-de Vries-Burgers' equation even while the values of the viscosity coefficient are small [11]. The linear stability

analysis shows that the numerical scheme is unconditionally stable. This is the first trail to compute numerically, the solution of the KDVB equation. So, this work compares the numerical solution of the KDVB equation with the exact one. But, there is no available other numerical example in the literatures to compare with.

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