Babeş-Bolyai University
Faculty of Mathematics and Computer Science

GRADUATION EXAM
Written Test - June-July 2026
Mathematics Computer Science Study Programme

SUBJECT I. Algebra

1. **(3 points)** Let $V = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$.
   a) Show that V is a subring with a multiplicative identity in the ring $(M_2(\mathbb{R}), +, \cdot)$.
   b) One defines $\psi : V \to \mathbb{R}$, $\psi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = a$. Prove that ψ is a ring homomorphism.
2. **(6 points)** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x + y, 2x + y + z, 2x + 2z)$.
   a) Show that f is a linear map from the $\mathbb{R}$-vector space ${}_{\mathbb{R}}\mathbb{R}^3$ into itself.
   b) Write the matrix of f in the standard (canonical) basis of ${}_{\mathbb{R}}\mathbb{R}^3$.
   c) Determine a basis and the dimension for $\mathrm{Ker} f$ and $\mathrm{Im} f$.
   d) Is f an $\mathbb{R}$-isomorphism? Motivate your answer.

SUBJECT II. Calculus

1. **(3 points)** Determine the general form of the partial sum of rank n, and then investigate the sum and the nature of the series:
$$\sum_{n \geq 1} \ln\left(\frac{n(n+2)^2}{(n+1)^3}\right).$$
2. **(3 points)** Write the Taylor's polynomial of random rank n, attached to function f and the point $a = 0$, for $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sin x \cos(2x)$, $\forall x \in \mathbb{R}$.
   Note: to determine the n-th order derivative, you can use either Leibniz'z formula or the trigonometric identity $2 \sin a \cos b = \sin(a + b) + \sin(a - b)$.
3. **(3 points)** Determine $\int \frac{x}{\sqrt{x^2-6x+10}} dx$, $\forall x \in \mathbb{R}$.

SUBJECT III. Geometry

1. **(5 points)** In triangle ABC, the coordinates of the points $A(3, 4)$ and $B(4, -3)$ and of the center of the circumcircle $O(0, 0)$ are known. It is known that the third vertex C lies in the third quadrant and belongs to the line $d : x - y = 1$.
   a) Write the equation of the circumcircle of triangle ABC.
   b) Determine the coordinates of the point C.
   c) Write the equations of the lines AB and BC.
   d) Calculate the area of triangle ABC.
2. **(4 points)** Consider the hyperbola
$$\mathcal{H} : \frac{x^2}{9} - \frac{y^2}{16} = 1.$$
   We denote its foci by F_1 and F_2.
   a) Determine the coordinates of the foci and the equations of the asymptotes of the hyperbola.
   b) Let the family of lines be $d_t : \ x = t$, where $t > 3$. Determine, in terms of the fixed parameter $t > 3$, the points of intersection between $\mathcal{H}$ and d_t.
   c) Determine $t > 3$ such that on the line $d_t : x = t$ there are two points M_1, M_2 with the property that $|M_iF_1 - M_iF_2| = 6$ for every $i \in \{1, 2\}$, and $M_1M_2 = 6$.

SUBJECT IV. Computer Science

One of the following programming languages C++, Python, Java or C# can be used to solve problems 1 and 2. Indicate the programming language used.
Existing libraries (from Python, C++, Java, C#) can be used in the provided solutions.

1. **(2p)** Write a program that:
   a) Implements a class ***Event*** with the following protected attributes:
      - ***title*** of type string,
      - ***duration*** of type integer (duration in minutes).

      Add to the class:
      - a ***constructor*** with parameters,
      - ***get/set*** methods for all attributes,
      - ***toString*** method that returns a string containing the title and the duration of the event separated by space.

   b) Derive the class ***CriticalEvent*** from class ***Event*** having all attributes of class ***Event*** and add a new private attribute ***severity*** of type string (e.g.: "low", "medium", "high"). Write the methods ***get/set*** for the new attribute. The ***toString*** method for the ***CriticalEvent*** class will return the content of the method ***toString*** from class ***Event*** followed by a space and the severity level.

2. (**2p**) Create a vector with two ***Event*** objects and one ***CriticalEvent*** object. Write a function that takes the created vector as a parameter and returns the total duration of all events in the vector.

3. (**2p**) Write the missing lines of code in the following function so that it returns the title of the event with the maximum duration from the vector. If there are multiple events with the same maximum duration, the title of the first one found will be returned.

```
string maxDuration(const vector<Event*>& events){
        ...
}
```

4. (**2p**) Write the result of running the following code sequence.

```
vector<Event*> v = {new Event("Conference", 60), new CriticalEvent("Earthquake", 120, "high"),
new CriticalEvent("Flood", 30, "medium"), new Event("Seminar", 45)};
for (int i = 0; i < v.size(); i++) {
        Event ev = *v[i];
        if (ev.getDuration() > 50)
           cout << ev.toString() << endl;
}
```

5. (**1p**) What is the time complexity of the QuickSort algorithm in the average case and the best case?

NOTE.
All subjects are compulsory and full solutions are requested.
An initial score of **1 point** is awarded to each subject. The minimum passing grade is 5,00.
The working time is 3 hours.

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Grading scheme

SUBJECT I. Algebra

Default 1p

1. (a) One can check, using the subring characterization theorem, that V is a subring in $(M_2(\mathbb{R}), +, \cdot)$

   (i) $O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in V$ 0.5p

   (ii) $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} = \begin{pmatrix} a-a' & b-b' \\ 0 & a-a' \end{pmatrix} \in V,\ \forall a, b, a', b' \in \mathbb{R}$ 0.5p

   (iii) $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} = \begin{pmatrix} aa' & ab'+ba' \\ 0 & aa' \end{pmatrix} \in V,\ \forall a, b, a', b' \in \mathbb{R}$ 0.5p

   The multiplicative identity of $M_2(\mathbb{R})$ is the matrix $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in V$ 0.5p

   (b) ψ is a ring homomorphism since for any $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} \in V$ we have

   $\psi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}\right) = \psi\left(\begin{pmatrix} a+a' & b+b' \\ 0 & a+a' \end{pmatrix}\right) = a + a' = \psi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) + \psi\left(\begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}\right)$ .. 0.5p

   $\psi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}\right) = \psi\left(\begin{pmatrix} aa' & ab'+ba' \\ 0 & aa' \end{pmatrix}\right) = aa' = \psi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) \psi\left(\begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}\right)$ 0.5p

2. (a) The fact that f is a linear map results as follows:

   f is aditive 0.5p

   since for any $(x, y, z), (x', y', z') \in \mathbb{R}^3$

$$\begin{aligned} f((x,y,z) + (x',y',z')) &= f(x+x', y+y', z+z') \\ &= (x+x'+y+y', 2(x+x')+y+y'+z+z', 2(x+x')+2(z+z')) \\ &= (x+y, 2x+y+z, 2x+2z) + (x'+y', 2x'+y'+z', 2x'+2z') = f(x,y,z) + f(x',y',z') \end{aligned}$$

   f is homogeneous 0.5p

   since for any $(x, y, z) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} f(\alpha(x,y,z)) &= f(\alpha x, \alpha y, \alpha z) \\ &= (\alpha x + \alpha y, 2\alpha x + \alpha y + \alpha z, 2\alpha x + 2\alpha z) = \alpha(x+y, 2x+y+z, 2x+2z) = \alpha f(x,y,z) \end{aligned}$$

   Remark: One can show that f is a linear map by proving that

$$f(\alpha(x,y,z) + \beta(x',y',z')) = \alpha f(x,y,z) + \beta f(x',y',z'),\ \forall \alpha, \beta \in \mathbb{R},\ \forall (x,y,z), (x',y',z') \in \mathbb{R}^3.$$

   (b) The matrix $[f]_e$ of f in the standard basis has the columns $f(e_1) = f(1,0,0) = (1,2,2), f(e_2) = f(0,1,0) = (1,1,0)$ and $f(e_3) = f(0,0,1) = (0,1,2)$, therefore

   $[f]_e = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$ 1p

(c) We have $\operatorname{Ker} f = \{(x,y,z) \in \mathbb{R}^3 | f(x,y,z) = (0,0,0)\} = \{(x,y,z) \in \mathbb{R}^3 | \begin{cases} x+y=0 \\ 2x+y+z=0 \\ 2x+2z=0 \end{cases} \}$

$= \{(x,-x,-x) | x \in \mathbb{R}\} = \langle (1,-1,-1) \rangle$

Obviously, the vector $(1,-1,-1)$ is not zero, hence it is linearly independent, thus a basis for $\operatorname{Ker} f$ is $\{(1,-1,-1)\}$ and $\dim_{\mathbb{R}} \operatorname{Ker} f = 1$ 1.5p

We have $\operatorname{Im} f = \{f(x,y,z) | (x,y,z) \in \mathbb{R}^3\} = \langle f(e_1), f(e_2), f(e_3) \rangle = \langle (1,2,2), (1,1,0), (0,1,2) \rangle$.

Since $3 = \dim_{\mathbb{R}} \operatorname{Ker} f + \dim_{\mathbb{R}} \operatorname{Im} f$, $\dim_{\mathbb{R}} \operatorname{Im} f = 2$, and a basis of $\operatorname{Im} f$ results by choosing two linearly independent vectors from $(1,2,2), (1,1,0), (0,1,2)$. An example is $(1,2,2), (1,1,0)$, because $\operatorname{rank} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 0 \end{pmatrix} = 2$. Thus a basis in $\operatorname{Im} f$ is $\{(1,2,2), (1,1,0)\}$ 1.5p

Remark: One can also use the fact that $2 = \operatorname{rank}[f]_e = \dim_{\mathbb{R}} \operatorname{Im} f$, and deduce that the kernel dimension is 1, and, therefore, a basis in the kernel is given by one of its nonzero elements. Such an element can be found if one notices that in $[f]_e$ we have $c_1 = c_2 + c_3$, because this means $f(e_1) = f(e_2) + f(e_3)$ and implies $f(e_1 - e_2 - e_3) = (0,0,0)$, i.e. $(1,-1,-1) = e_1 - e_2 - e_3 \in \operatorname{Ker} f$.

(d) f is not an isomorphism, since it is not injective because $\dim_{\mathbb{R}} \operatorname{Ker} f = 1 \neq 0$ 1p

Remark: One can also notice that f is not surjective, since $\dim_{\mathbb{R}} \operatorname{Im} f = 2 < 3$, or that f cannot be an isomorphism because $\dim_{\mathbb{R}} \operatorname{Im} f = \operatorname{rank}[f]_e = 2$, i.e. $\det[f]_e = 0$.

NOTE: Any other correct solution will be scored accordingly.

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SUBJECT II. Calculus

Default 1p

1. We have

$$s_n = \sum_{k=1}^{n} \ln\left(\frac{k(k+2)^2}{(k+1)^3}\right) = \ln\left(\prod_{k=1}^{n} \frac{k(k+2)^2}{(k+1)^3}\right) = \ln\left(\frac{1\cdot 3^2}{2^3}\cdot\frac{2\cdot 4^2}{3^3}\cdot\ldots\cdot\frac{n(n+2)^2}{(n+1)^3}\right)$$
$$= \ln\left(\frac{n(n+2)^2}{2^2(n+1)}\right).$$

.... 2p

We calculate the limit:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln\left(\frac{n(n+2)^2}{2^2(n+1)}\right) = \ln\left(\lim_{n\to\infty}\frac{n(n+2)^2}{2^2(n+1)}\right) = \ln(\infty) = +\infty.$$

Thus, the series has a sum of ∞ and is divergent. 1p

2. The Taylor polynomial of degree n associated with a function f and the point $a = 0$ is the function

$$T_{n;0}f : \mathbb{R} \to \mathbb{R}$$

having the expression

$$T_{n;0}f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k, \quad \forall x \in \mathbb{R}$$

The function $f(x) = \sin x \cos(2x)$ is infinitely differentiable on $\mathbb{R}$ because it is a composition and product of elementary functions. 0.5p

To determine the n-th order derivative, we first transform the product into a sum using the trigonometric formula:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

Taking into account that the sine function is odd ($\sin(-x) = -\sin x$), we obtain:

$$f(x) = \frac{1}{2}\left[\sin(x+2x) + \sin(x-2x)\right] = \frac{1}{2}\sin(3x) + \frac{1}{2}\sin(-x) = \frac{1}{2}\sin(3x) - \frac{1}{2}\sin x$$

By induction, it can be proven for any $k \in \mathbb{N}^*$ that:

$$(\sin(kx))^{(n)} = k^n \sin\left(kx + \frac{n\pi}{2}\right)$$

Applying the linearity of the derivative, we obtain the expression for the n-th order derivative:

$$f^{(n)}(x) = \frac{1}{2}\cdot 3^n \sin\left(3x + \frac{n\pi}{2}\right) - \frac{1}{2}\sin\left(x + \frac{n\pi}{2}\right)$$

.... 1.5p

To construct the Taylor polynomial centered at $a = 0$, we evaluate the k-th order derivative at the point $x = 0$:

$$f^{(k)}(0) = \frac{3^k - 1}{2}\sin\left(\frac{k\pi}{2}\right)$$

We observe that the value depends on the parity of k:

- If $k = 2m$ (even), then $\sin(m\pi) = 0 \implies f^{(2m)}(0) = 0$
- If $k = 2m+1$ (odd), then $\sin\left(m\pi + \frac{\pi}{2}\right) = (-1)^m \implies f^{(2m+1)}(0) = (-1)^m \frac{3^{2m+1}-1}{2}$

Therefore, all terms with even powers vanish (including $f(0) = 0$), and the Taylor polynomial contains only odd powers:

$$T_n(x) = x - \frac{13}{3!}x^3 + \frac{121}{5!}x^5 - \cdots + \frac{(-1)^m(3^{2m+1}-1)}{2\cdot(2m+1)!}x^{2m+1}$$

where $2m+1 \le n$. 1p

3. The integral

$$I = \int \frac{x}{\sqrt{x^2-6x+10}}\,dx, \quad \forall x \in \mathbb{R}$$

We notice that $(x^2-6x+10)' = 2x-6$. 0.5p

Then

$$I = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-6x+10}}\,dx = \frac{1}{2}\int \frac{(2x-6)+6}{\sqrt{x^2-6x+10}}\,dx$$

We split the fraction into two distinct integrals:

$$I = \frac{1}{2}\int \frac{2x-6}{\sqrt{x^2-6x+10}}\,dx + 3\int \frac{1}{\sqrt{x^2-6x+10}}\,dx$$

.... 0.5p

Using $\int \frac{u'}{\sqrt{u}}\,dx = 2\sqrt{u}$ and $\int \frac{1}{\sqrt{y^2+a^2}}\,dy = \ln\left|y + \sqrt{y^2+a^2}\right|$, we obtain:

$$I = \sqrt{x^2-6x+10} + 3\ln\left|x-3+\sqrt{x^2-6x+10}\right| + \mathcal{C},$$

....2p 2p

NOTE: Any other correct solution will be scored accordingly.

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SUBJECT III. Geometry

Default 1p

1. a) We have

$$OA = \sqrt{3^2 + 4^2} = 5, \qquad OB = \sqrt{4^2 + (-3)^2} = 5.$$

The circumcircle has center $O(0,0)$ and radius $R = 5$, hence

$$x^2 + y^2 = 25.$$

.... 1p

b) From the condition $C(x,y) \in d$ we obtain $y = x - 1$. Since C belongs to the circle, we have

$$x^2 + (x-1)^2 = 25 \Longleftrightarrow x^2 - x - 12 = 0,$$

hence $x \in \{4, -3\}$. 1p

The points of intersection are $(4,3)$ and $(-3,-4)$. Since C lies in the third quadrant, it follows that

$$C(-3,-4).$$

.... 0,5p

c) The line AB passes through $A(3,4)$ and $B(4,-3)$; by the formula for the line through two points, we obtain

$$AB: \ 7x + y - 25 = 0.$$

.... 0,75p

The line BC passes through $B(4,-3)$ and $C(-3,-4)$; by the formula for the line through two points, we obtain

$$BC: \ x - 7y - 25 = 0.$$

.... 0,75p

d) The area of the triangle is calculated by using the determinant:

$$\text{Area}(ABC) = \frac{1}{2} \begin{vmatrix} 3 & 4 & 1 \\ 4 & -3 & 1 \\ -3 & -4 & 1 \end{vmatrix} = \frac{1}{2} \cdot 50 = 25.$$

.... 1p

2. a) For the hyperbola $\mathcal{H}$ we have

$$a^2 = 9, \qquad b^2 = 16,$$

hence $a = 3$, $b = 4$. Since $c^2 = a^2 + b^2$, it follows that $c^2 = 25$, hence $c = 5$. 0,5p

The foci of the hyperbola are

$$F_1(-5,0), \qquad F_2(5,0).$$

.... 0,5p

The equations of the asymptotes are

$$y = \pm \frac{b}{a} x,$$

that is,

$$y = \pm \frac{4}{3} x.$$

.... 0,5p

b) For $M(x,y) \in d_t \cap \mathcal{H}$ we have $x = t$, hence $y^2 = 16\left(\frac{t^2}{9} - 1\right)$. 0,75p

The points of intersection are

$$M_1\left(t, 4\sqrt{\left(\frac{t^2}{9} - 1\right)}\right) \text{ and } M_2\left(t, -4\sqrt{\left(\frac{t^2}{9} - 1\right)}\right)$$

.... 0,25p

c) Since $6 = 2a$, the condition

$$|MF_1 - MF_2| = 6$$

shows, by the definition of the hyperbola as a locus, that the required points belong to the hyperbola $\mathcal{H}$. 0,5p

Using part b), we have

$$M_1M_2 = 8\sqrt{\left(\frac{t^2}{9} - 1\right)}$$

.... 0,5p

The condition $M_1M_2 = 6$ implies $t^2 = \frac{9 \cdot 25}{16}$, and since $t > 3$ we have $t = \frac{15}{4}$. Optionally, the corresponding points are

$$M_1\left(\frac{15}{4}, 3\right), \text{ and } M_2\left(\frac{15}{4}, -3\right).$$

.... 0,5p

NOTE: Any other correct solution will be scored accordingly.

BABEŞ -BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Rubric for Computer Science Subject
Exam session 1: Assessment of basic and specialist knowledge, license exam June-July 2026
Specialization Mathematics and Informatics

Computer Science

1. a) Definition of the Event class (constructor, methods, data access)................ b) Definition of the derived class CriticalEvent (inheritance, method).............	**2p** 1p 1p
2. Creating the vector... Computing the total duration ……….. Returning the total duration …………..…………………………………..	**2p** 0.5p 1p 0.5p
3. Iterating over elements……………………………………………………. Finding the maximum ……………………………………………………. Updating the result ……………………………………………………….	**2p** 0.5p 1p 0.5p
4. Correct identification of the displayed object…………………………….. Correct indication of the string representation of the displayed object……	**2p** 1p 1p
5. Correct indication of the complexity in both cases	**1p** 1p

Remark:
(1p) Default