Babeş-Bolyai University
Faculty of Mathematics and Computer Science

GRADUATION EXAM
Written Test - September 2025
Mathematics Computer Science Study Programme

SUBJECT I. Algebra

1. **(4 points)** Let us consider the set $GL_2(\mathbb{Z}_2) = \{A \in M_2(\mathbb{Z}_2) \mid \det A \neq \hat{0}\}$.
   a) Show that $GL_2(\mathbb{Z}_2)$ is closed in $M_2(\mathbb{Z}_2)$ with respect to the matrix multiplication and $GL_2(\mathbb{Z}_2)$ and the induced operation form a group.
   b) How many elements does $GL_2(\mathbb{Z}_2)$ have? Motivate your answer.
   c) Does the group $(GL_2(\mathbb{Z}_2), \cdot)$ have 2-element subgroups? Motivate your answer.
2. **(5 points)** Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map from ${}_{\mathbb{R}}\mathbb{R}^3$ into ${}_{\mathbb{R}}\mathbb{R}^2$ such that
$$f(1,1,1) = (1,-1),\ f(1,1,0) = (1,1) \text{ şi } f(1,0,0) = (0,1).$$
   a) Show that the vectors $(1,1,1),\ (1,1,0),\ (1,0,0)$ form a basis B of the vector space ${}_{\mathbb{R}}\mathbb{R}^3$ and write the matrix of f in the pair of bases (B, E) (where E is the standard basis of the vector space ${}_{\mathbb{R}}\mathbb{R}^2$).
   b) Determine $f(x,y,z)$ (where $(x,y,z) \in \mathbb{R}^3$).
   c) Determine the dimension of the kernel of f and the dimension of the image of f (considered as $\mathbb{R}$-vector spaces).

SUBIECTUL II. Calculus

1. **(3 points)** Study with discussion on the real parameters α and β the nature of the series of real numbers
$$\sum_{n\geq 1} \frac{(1\cdot 5\cdot \ldots \cdot (4n-3))^{\alpha}}{(3\cdot 7\cdot \ldots \cdot (4n-1))^{\beta}}.$$
2. **(3 points)** Determine Taylor's polynomial of random rank n, attached to the function f about the point $a = 0$ for $f : \mathbb{R}\backslash\{-2,-1\} \to \mathbb{R}$, when
$$f(x) = \frac{x+4}{x^2+3x+2}, \quad \forall x \in \mathbb{R}\backslash\{-2,-1\}.$$
3. **(3 points)** Determine
$$\int \frac{\sin x}{3+\cos x+\cos^2 x}dx.$$

SUBJECT III. Geometry

1. (**5 points**) The point $A(3,-2)$ is a vertex of the square $ABCD$, whose diagonals intersect in the point $M(1,1)$.
   a) Write the equation of the line AM and determine the coordinates of the vertex C.
   b) Find the equation of the line BM.
   c) Write the equation of the circumscribed circle of the square and determine the coordinates of the other vertices of the square.
2. **(4 points)** The axis of symmetry of a parabola lies on the Ox axis, and its vertex is at the origin.
   a) Determine the equation of the parabola, given that it passes through the point $A(2,4)$.
   b) Write the equation of the tangent line to the parabola that is parallel to the line $y = 2x$.

SUBJECT IV. Computer Science

Note for the Computer Science subject:

One of the following programming languages C++, Python, Java or C# can be used to solve problems 1 and 2. Indicate the programming language used.

Existing libraries (from Python, C++, Java, C#) can be used in the provided solutions.

1. **(2p)** Write a program that:
   a) Implements a class ***Person*** with the following protected attributes:
      - ***name*** of type string,
      - ***age*** of type integer.

      Add to the class:
      - a ***constructor*** with parameters,
      - ***get/set*** methods for all attributes,
      - ***toString*** method that returns a string containing the name and the age of the person separated by space.

   b) Derive the class ***Student*** from class ***Person*** having all attributes of class ***Person*** and add a new attribute ***faculty*** of type string. Write the methods ***get/set*** for the new attribute. The method ***toString*** from class ***Student*** will return the content of the method ***toString*** from class ***Person*** followed by a space and the ***faculty*** attribute.

2. **(2p)** Create a vector containing two objects of type ***Person*** and one object of type ***Student***. Write a function that receives as parameter the previously created vector and returns the average age of the persons in the vector.

3. **(2p)** Fill in the missing statements from the following function to determine the objects of type ***Student*** from the vector ***stud*** attending the faculty ***specificFaculty***. You can use the function *push_back* which inserts an element at the end of the vector.

```
vector<Student> filter (const vector<Student>& stud, const string& specificFaculty){
    vector<Student> rez;
    ….
   return rez;}
```

4. **(2p)** Write the result of running the following sequence of code. The function *push_back* inserts an element at the end of the vector, the function *pop_back* removes the last element from the vector, and the function *back* returns a reference to the last element of the vector.

```
vector<Person> v;
v.push_back(Person("Alexandru", 23));
v.push_back(Student ("Tudor", 19, "History"));
v.push_back(Person("Ana", 19));
v.push_back(Student("Maria", 20, "Chemistry"));
v.pop_back();
v.pop_back();
cout<<v.back().toString()<<endl;
```

5. **(1p)** Explain the concept of dynamic linking.

NOTE.

All subjects are compulsory and full solutions are requested.

An initial score of **1 point** is awarded to each subject. The minimum passing grade is 5,00.

The working time is 3 hours.

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SUBJECT I. Algebra

Default 1p

1. a) For $A, B \in GL_2(\mathbb{Z}_2)$ we have $\det A \neq \hat{0}$ and $\det B \neq \hat{0}$. By using the fact that $\mathbb{Z}_2$ is a field, and consequently a ring without zero divisors, we have $\det(AB) = \det A \det B \neq \hat{0}$, that is, $AB \in GL_2(\mathbb{Z}_2)$ 0.5p

   The fact that $(GL_2(\mathbb{Z}_2), \cdot)$ is a group follows by the arguments below:
   $GL_2(\mathbb{Z}_2)$, being a closed subset of the monoid $(M_2(\mathbb{Z}_2), \cdot)$,
   inherits associativity of matrix multiplication 0.5p

   Clearly, $I_2 = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix} \in GL_2(\mathbb{Z}_2)$ (since $\det I_2 = \hat{1} \neq \hat{0}$), hence we have identity 0.5p

   Moreover, $\mathbb{Z}_2$ being a field, every matrix $A \in GL_2(\mathbb{Z}_2)$ is invertible in $M_2(\mathbb{Z}_2)$
   and $\det A^{-1} = (\det A)^{-1} \neq \hat{0}$, hence $A^{-1} \in GL_2(\mathbb{Z}_2)$. 0.5p

   b) Clearly, $|M_2(\mathbb{Z}_2)| = 2^4 = 16$. On the other hand, $M_2(\mathbb{Z}_2) \setminus GL_2(\mathbb{Z}_2) = \{A \in M_2(\mathbb{Z}_2) \mid \det A = \hat{0}\} =$
   $\left\{ \begin{pmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{1} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{1} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{0} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix} \right\}$,
   hence $|GL_2(\mathbb{Z}_2)| = 16 - 10 = 6$ 1p

   Alternatively, we may directly look for matrices in $M_2(\mathbb{Z}_2)$ having determinant $\hat{1}$ and we get
   $GL_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{0} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{1} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{1} \end{pmatrix} \right\}$, hence $|GL_2(\mathbb{Z}_2)| = 6$.

   c) $H = \left\{ \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix} \right\}$ is a subgroup with 2 elements of the group $GL_2(\mathbb{Z}_2)$, because $\begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}^2 = I_2$,
   hence we have closure, $I_2 \in H$ and $\begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}^{-1} = \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix} \in H$ 1p

2. a) The transition matrix from the standard basis from ${}_{\mathbb{R}}\mathbb{R}^3$ to the system of vectors B is $T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,
   which is invertible, because $\det T = -1 \neq 0$. Hence B is a basis of ${}_{\mathbb{R}}\mathbb{R}^3$ 1p

   By using the definition of a matrix of a linear map, we have $[f]_{B,E} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ 1p

   b) We determine the coordinates $x', y', z' \in \mathbb{R}$ of (x, y, z) in the basis B:

$$(x, y, z) = x'(1,1,1) + y'(1,1,0) + z'(1,0,0) \;\Leftrightarrow\; \begin{cases} x' + y' + z' = x \\ x' + y' = y \\ x' = z \end{cases}$$

   The solution of the system is $x' = z, y' = y - z, z' = x - y$ 1p

   (*Alternatively*, by using the transition matrix $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = T^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$
   $\begin{pmatrix} z \\ y - z \\ x - y \end{pmatrix}$.)

   Now, by using the linearity of f, we have $f(x, y, z) = f(x'(1,1,1) + y'(1,1,0) + z'(1,0,0)) =$
   $= x'f(1,1,1) + y'f(1,1,0) + z'f(1,0,0) = z(1,-1) + (y - z)(1,1) + (x - y)(0,1) = (y, x - 2z)$ ... 1p

Alternatively for the whole subpoint b), E' being the canonical basis of ${}_{\mathbb{R}}\mathbb{R}^3$, we have

$$[f]_{E',E} = [f]_{B,E}T^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix},$$

hence $f(x,y,z)^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x-2z \end{pmatrix}$.

c) $\operatorname{Ker} f = \{(x,y,z) \in \mathbb{R}^3 | f(x,y,z) = (y, x-2z) = (0,0)\} = \{(2t,0,t) | t \in \mathbb{R}\} = \langle(2,0,1)\rangle$, hence the vector $(2,0,1)$ forms a 1-element basis of ${}_{\mathbb{R}}\operatorname{Ker} f$ and $\dim_{\mathbb{R}} \operatorname{Ker} f = 1$ 0.5p
$\dim_{\mathbb{R}} \operatorname{Im} f = \dim_{\mathbb{R}} \mathbb{R}^3 - \dim_{\mathbb{R}} \operatorname{Ker} f = 3 - 1 = 2$ 0.5p
Alternatively, one may proceed as follows:
Denoting $b_1 = (1,1,1)$, $b_2 = (1,1,0)$, $b_3 = (1,0,0)$, we have $\operatorname{Im} f = f(\langle b_1, b_2, b_3\rangle) = \langle f(b_1), f(b_2), f(b_3)\rangle$,
$\dim_{\mathbb{R}} \operatorname{Im} f = \dim\langle f(b_1), f(b_2), f(b_3)\rangle = \operatorname{rank}[f]_{B,E} = \operatorname{rank} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} = 2$ 0.5p
and $\dim_{\mathbb{R}} \operatorname{Ker} f = \dim_{\mathbb{R}} \mathbb{R}^3 - \dim_{\mathbb{R}} \operatorname{Im} f = 3 - 2 = 1$ 0.5p

NOTE: Any other correct solution will be scored accordingly.

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SUBJECT II. Calculus

Default 1p

1. One can use the Raabe-Duhamel criterion: *Let* $\sum\limits_{n\geq 1} a_n$ *be a series with positive terms. If*

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lambda,$$

then: 1) *pentru* $\lambda>1$ *is convergent;* 2) *for* $\lambda<1$ *the series is divergent.*

.... 0.25p

We denote

$$a_n=\frac{(1\cdot 5\cdot\ldots\cdot(4n-3))^{\alpha}}{(3\cdot 7\cdot\ldots\cdot(4n-1))^{\beta}},\quad n=1,2,3,\ldots$$

Then

$$\begin{aligned}\lambda &= \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(4n+3)^{\beta}}{(4n+1)^{\alpha}}-1\right)=\\ &= \lim_{n\to\infty} n\left(n^{\beta-\alpha}\frac{\left(4+\frac{3}{n}\right)^{\beta}}{\left(4+\frac{1}{n}\right)^{\alpha}}-1\right)=\begin{cases}+\infty, & \text{if} \quad \beta>\alpha\\ -\infty, & \text{if} \quad \beta<\alpha.\end{cases}\end{aligned}$$

.... 0.75p

Thus $\sum\limits_{n\geq 1} a_n$ is convergent, if $\alpha<\beta$, and $\sum\limits_{n\geq 1} a_n$ is divergent, if $\alpha>\beta$.

.... 0.25p

For the case $\alpha=\beta$: by using L'Hopitals rule

$$\begin{aligned}\lambda &= \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(4n+3)^{\alpha}}{(4n+1)^{\alpha}}-1\right)=\lim_{n\to\infty}\frac{\left(\frac{4+\frac{3}{n}}{4+\frac{1}{n}}\right)^{\alpha}-1}{\frac{1}{n}}=\\ &= \lim_{x\searrow 0}\frac{\left(\frac{4+3x}{4+x}\right)^{\alpha}-1}{x}=\lim_{x\searrow 0}\alpha\left(\frac{4+3x}{4+x}\right)^{\alpha-1}\cdot\frac{8}{(4+x)^2}=\frac{8\alpha}{16}=\frac{\alpha}{2}.\end{aligned}$$

.... 0.75p

Therefore, from Raabe-Duhamel: $\sum\limits_{n\geq 1} a_n$ is convergent if $\alpha=\beta>2$, and $\sum\limits_{n\geq 1} a_n$ is divergent if $\alpha=\beta<2$.

.... 0.25p

If $\alpha=\beta=2$, then

$$\frac{a_{n+1}}{a_n}=\left(\frac{4n+1}{4n+3}\right)^2>\frac{n}{n+1},$$

for each $n\geq 1$. Thus the sequence $(na_n)_{n\geq 1}$ is increasing, therefore $na_n\geq a_1$, for each $n\geq 1$, meaning $a_n\geq\frac{1}{9n}$, for each $n\geq 1$.

.... 0.50p

Since the harmonic series is divergent $\sum\limits_{n\geq 1}\frac{1}{n}$ we get that $\sum\limits_{n\geq 1} a_n$ is also divergent.

.... 0.25p

2. Taylor's polynomial of random rank n attached to the function f about the point $a = 0$ is the function

$$T_{n;0}f : \mathbb{R} \to \mathbb{R}$$

with

$$T_{n;0}f(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k, \quad \forall x \in \mathbb{R}$$

.... 0.25p

The function f is indefinite differentiable on $R\backslash\{-2,-1\}$ as a composition of elementary functions. 0.25p

By considering a random $x \in \mathbb{R}\backslash\{-2,-1\}$ and splitting the initial fraction into simple fractions we get

$$\frac{x+1}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1} = \frac{3}{x+1} - \frac{2}{x+2}.$$

.... 0.25p

For a random constant b, we determine by mathematical induction for a random $n \in \mathbb{N}$ the n-th derivative of

$$g : \mathbb{R}\backslash\{b\} \to \mathbb{R} \quad \text{cu} \quad g(x) = \frac{1}{x+b} = (x+b)^{-1}.$$

Thus

$$g'(x) = (-1)(x+b)^{-2}, \quad g"(x) = (-1)(-2)(x+b)^{-3} \quad ... \quad g^{(n)}(x) = (-1)(-2)...(-n)(x+b)^{-(n+1)}$$

$$g^{(n)} = (-1)^n n!(x+b)^{-(n+1)}.$$

.... 0.75p

By applying the result above to f, we get

$$f^{(n)}(x) = 3(-1)^n n!(x+1)^{-(n+1)} - 2\cdot(-1)^n n!(x+2)^{-(n+1)} = (-1)^n n! \cdot \left(\frac{3}{(x+1)^{n+1}} - \frac{2}{(x+2)^{n+1}}\right)$$

Therefore

$$f^{(n)}(0) = (-1)^n n! \cdot \left(\frac{3}{(0+1)^{n+1}} - \frac{2}{(0+2)^{n+1}}\right) = (-1)^n n! \cdot \left(3 - 2^{-n}\right)$$

.... 1p

Consequently

$$T_{n;0}f(x) = \sum_{k=0}^{n} \frac{(-1)^k k! \cdot \left(3-2^{-k}\right)}{k!} x^k = \sum_{k=0}^{n} (-1)^k \cdot \left(3-2^{-k}\right) x^k = \sum_{k=0}^{n} \left(3-2^{-k}\right)(-x)^k.$$

.... 0.5p

3. We use the following change of variable $\cos x = t$, then

$$-\sin x dx = dt$$

.... 1p

We get

$$I = \int \frac{\sin x}{3+\cos x+\cos^2 x} dx = -\int \frac{dt}{3+t+t^2}$$

.... 1p

Consequently

$$I = \int \frac{-dt}{\left(t+\frac{1}{2}\right)^2 + \frac{11}{4}} = -\frac{2}{\sqrt{11}} \arctan \frac{2\left(t+\frac{1}{2}\right)}{\sqrt{11}} + \mathcal{C} = -\frac{2}{\sqrt{11}} \arctan \frac{2t+1}{\sqrt{11}} + \mathcal{C} =$$

$$= -\frac{2}{\sqrt{11}} \arctan \frac{2\cos x + 1}{\sqrt{11}} + \mathcal{C}.$$

.... 1p

NOTE: Any other correct solution will be scored accordingly.

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Grading scheme

SUBJECT III. Geometry

Default 1p

1. a) The equation is $AM : 3x + 2y - 5 = 0$ 1p

   M is the midpoint of the segment $[AC]$ so we find $C(-1, 4)$ 0.5p

   b) The line BM is perpendicular on AM, therefore $m_{BM} = \frac{2}{3}$ 1p

   The equation of the line is $BM : y = \frac{2}{3}x + \frac{1}{3}$ 0.5p

   c) The circle has center $M(1, 1)$ and radius $r = AM = \sqrt{13}$, hence its equation is

$$\mathcal{C} : (x - 1)^2 + (y - 1)^2 = 13.$$

   .... 1p

   The points B and D are at the intersection of the line BM with the circle $\mathcal{C}$.

   Solving the system determined by the equations of the line and circle, we find that the vertices B and D have coordinates $(4, 3)$ and $(-2, -1)$, in some order 1p

2. a) The equation of the parabola is of the form $\mathcal{P} : y^2 = 2px$. 0.5p

   $A(2, 4) \in \mathcal{P} \Leftrightarrow 16 = 2p \cdot 2 \Leftrightarrow p = 4 \Rightarrow y^2 = 8x$ 1p

   b) The tangent is parallel with the line $d \Leftrightarrow m_t = m_d = 2$. 0.5p

   At the same time, the equation of the tangent line is $yy_0 = 4(x + x_0) \Rightarrow m_t = \frac{4}{y_0}$. Hence $m_t = \frac{4}{y_0} = 2 \Rightarrow y_0 = 2$. 1p

   The point of tangency $M_0(x_0, y_0)$ belongs to the parabola, hence $2^2 = 8 \cdot x_0 \Rightarrow x_0 = \frac{1}{2}$............. 0.5p

   The tangent line is: $yy_0 = 4(x + x_0) \Leftrightarrow 2y = 4(x + \frac{1}{2}) \Leftrightarrow t : y = 2x + 1$ 0.5p

NOTE: Any other correct solution will be scored accordingly.

BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND INFORMATICS

Grading scheme for Computer Science Subject
Exam session 1: Assessment of basic and specialist knowledge, license exam
September 2025
Specialization Mathematics Computer Science

Computer Science Subject

1. a) Definition of class Person (constructor, methods, access to data).. b) Definition of derived class Student (inheritance, constructor, method)..	**2p** 1 p 1 p
2. Creating the vector... Computing the average age ....………..	**2p** 1p 1p
3. Iterating the elements... Comparing elements...…… Updating the result..	**2p** 0.5p 1p 0.5p
4. Correct indication of the result that will be displayed …………...	**2p** 2p
5. Theoretical explanation..	**1p** 1p

Remark:
(1p) Default