Babeş-Bolyai University
Faculty of Mathematics and Computer Science

GRADUATION EXAM
Written Test - June 2025
Mathematics Computer Science Study Programme

SUBJECT I. Algebra

1. **(3 points)** Let us consider the set $\mathbb{Q}(i\sqrt{2}) = \{a + bi\sqrt{2} \mid a, b \in \mathbb{Q}\}$ and the mapping $N : \mathbb{Q}(i\sqrt{2}) \to \mathbb{Q}$, $N(z) = z \cdot \overline{z}$ ($\overline{z}$ denotes the complex conjugate of z).
   a) Show that $\mathbb{Q}(i\sqrt{2})$ is a subfield of $(\mathbb{C}, +, \cdot)$.
   b) Show that $N(z_1 \cdot z_2) = N(z_1) \cdot N(z_2)$ for any $z_1, z_2 \in \mathbb{Q}(i\sqrt{2})$.
   c) Is N a ring homomorphism? Motivate your answer.
2. **(6 points)**
   a) Are the subsets
   $$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 + z = 0\} \text{ and } T = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$$
   subspaces of the $\mathbb{R}$-vector space $\mathbb{R}^3$? Motivate your answer.
   b) In the $\mathbb{R}$-vector space $\mathbb{R}^4$ we consider the vectors $v_1 = (2, 10, -6, 1)$, $v_2 = (4, -1, \alpha, 5)$, $v_3 = (2, \alpha, -1, 2)$. Determine the dimension of the subspace $S \leq_{\mathbb{R}} \mathbb{R}^4$ generated by v_1, v_2, v_3. Discussion on $\alpha \in \mathbb{R}$.

SUBIECTUL II. Calculus

1. **(3 points)** Study with discussion on the real parameters α and β the nature of the series of real numbers:
   $$\sum_{n \geq 1} \frac{(1 \cdot 3 \cdot \ldots \cdot (2n-1))^{\alpha}}{(2 \cdot 4 \cdot \ldots \cdot (2n))^{\beta}}.$$
2. **(3 points)** Determine Taylor's Polynomial of random rank $2n$ attached to the function f about the point $a = 0$ for the function $f : \mathbb{R}\backslash\{-2, 1\} \to \mathbb{R}$, where
   $$f(x) = \frac{x + 8}{x^2 + x - 2}, \quad \forall x \in \mathbb{R}\backslash\{-2, 1\}.$$
3. **(3 points)** Determine
   $$\int \frac{1}{5 \sin x + 2 \cos x + 2} dx, \quad \forall x \in \left(0, \frac{\pi}{2}\right).$$

SUBJECT III. Geometry

1. (**5 points**) In the triangle ABC the coordinates of the vertices $A(1, 1)$ and $B(0, -2)$ are given, as well as the coordinates of its orthocenter $H(2, 2)$.
   a) Write the equations of the lines AH and BH.
   b) Find the equations of the lines AC and BC.
   c) Compute the area of the triangle ABC.
2. **(4 points)** The circle $\mathcal{C} : x^2 + y^2 - 4x + 6y - 12 = 0$ is given.
   a) Find the center and the radius of the circle $\mathcal{C}$.
   b) Compute the distance from the center of the circle $\mathcal{C}$ to the line $d : 3x - 4y + 2 = 0$.
   c) Prove that the line d and the circle $\mathcal{C}$ intersect in two points.

SUBJECT IV. Computer Science

Note for the Computer Science subject:

One of the following programming languages C++, Python, Java or C# can be used to solve problems 1 and 2.
Indicate the programming language used. Existing libraries (from Python, C++, Java, C#) can be used in the provided solutions.

1. **(2p)** Write a program that:
   a) Implements a class ***Book*** with the following protected attributes:
      - ***title*** a string,
      - ***price*** a real number.

      Add to the class:
      - a parameterized constructor,
      - ***get/set*** methods for all the attributes,
      - ***toString*** method which returns a string with the information about an object using the following format: *Title: <title> , Price: <price>.*

   b) Define a subclass of ***Book*** called ***AudioBook***, which has an additional private attribute ***duration*** (a natural number). Implement a constructor with parameters to initialize all attributes. Implement a method ***toString*** that, using the method ***toString*** from the base class, returns a string with the information about an ***AudioBook***: *Title: <title>, Price: <price>, Duration: <duration>*.

2. (**2p**) Create a vector ***v*** that contains three objects, two of which are of type ***Book*** and one of type ***AudioBook***. Write a function that takes as parameters the vector ***v*** and a real number ***p.*** The function displays (using the ***toString*** method) all the objects from the vector ***v*** having the price strictly greater than ***p***.

3. (**2p**) Fill in the missing statements from the function ***insertSorted*** to insert the object ***b*** into the vector ***v*** such that the elements in the vector remain sorted in ascending order of their price. The function ***v.push_back(b)*** inserts the element ***b*** at the end of the vector, and the function ***v.insert(v.begin() + pos, b)*** inserts the element ***b*** at position ***pos*** in the vector ***v***. At the time of the call of the function ***insertSorted***, the elements of vector ***v*** are sorted in ascending order by price.

```
void insertSorted( std::vector<Book>& v, const Book &b) {
   int pos = 0;
   …
   v.insert(v.begin() + pos, b);
}
```

4. (**2p**) Write the result of running the following sequence of code. The *push_back* function inserts an element at the end of the vector, the *pop_back* function deletes the last element of the vector, and the *back* function returns a reference to the last element in the vector. For the ***Audiobook*** class, in the parameterized constructor, the price is given by the first argument.

```
vector<Book> books;

books.push_back(Book{20, "Book1"});
books.push_back(AudioBook{120, "AudioBook1", 40});
books.push_back(Book{20, "Book2"});
books.push_back(AudioBook{40, "Book2", 140});

while(books.back().getPrice() < 100)
   books.pop_back();
std::cout<<books.back().toString()<<endl;
```

5. (**1p**) Define the concepts of method overloading and method overriding.

NOTE.

All subjects are compulsory and full solutions are requested.
An initial score of **1 point** is awarded to each subject. The minimum passing grade is 5,00.
The working time is 3 hours.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science

GRADUATION EXAM
Written Test - June 2025
Mathematics Computer Science Study Programme
Grading scheme

SUBJECT I. Algebra
Default 1p

1. a) We use the characterization theorem of subfields:
$0 = 0 + 0i\sqrt{2}, 1 = 1 + 0i\sqrt{2} \in \mathbb{Q}(i\sqrt{2})$, and for arbitrary $z = a + bi\sqrt{2}, z' = c + di\sqrt{2} \in \mathbb{Q}(i\sqrt{2})$ we have $z - z' = (a-b) + (c-d)i\sqrt{(2)}$ şi $zz' = (ac - 2bd) + (ad + bc)i\sqrt{2} \in \mathbb{Q}(i\sqrt{2})$ 1p
For arbitrary $a + bi\sqrt{2} \in \mathbb{Q}(i\sqrt{2}) \setminus \{0\}$ (so that $a, b \in \mathbb{Q}$ are not both 0), we have, after amplification with the conjugate, $\dfrac{1}{a+bi\sqrt{2}} = \dfrac{a}{a^2+2b^2} - \dfrac{b}{a^2+2b^2} i\sqrt{2} \in \mathbb{Q}(i\sqrt{2})$ 0.5p

b) By using the properties of conjugation and the commutativity of multiplication we obtain
$N(z_1 z_2) = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2 \overline{z}_1 \overline{z}_2 = (z_1 \overline{z}_1)(z_2 \overline{z}_2) = N(z_1)N(z_2)$ 0.5p

c) The mapping N is not a ring homomorphism, because it is not additive.
For instance, $9 = N(3) \neq N(1) + N(2) = 1 + 4 = 5$ ..... 1p.

2. a) S is not a subspace. For instance, $(1,1,0), (0,-1,1) \in S$, but $(1,1,0) + (0,-1,1) = (1,0,1) \notin S$, hence the set S is not closed under addition 1p
By using the characterization theorem of subspaces we obtain below that T is a subspace of ${}_{\mathbb{R}}\mathbb{R}^3$:
$(0,0,0) \in T$ 0.5p
For arbitrary $(x,y,z), (x',y',z') \in T$ and $\alpha, \beta \in \mathbb{R}$, we have
$\alpha(x,y,z) + \beta(x',y',z') = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in T$, because
$(\alpha x + \beta x') - (\alpha y + \beta y') + (\alpha z + \beta z') = \alpha(x - y + z) + \beta(x' - y' + z') = \alpha 0 + \beta 0 = 0$ 1p

b) According to Kronecker's theorem, $d = \dim_{\mathbb{R}} \langle v_1, v_2, v_3 \rangle = \operatorname{rank} \begin{pmatrix} 2 & 10 & -6 & 1 \\ 4 & -1 & \alpha & 5 \\ 2 & \alpha & -1 & 2 \end{pmatrix}$ ..... 1p.
From $\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} = 6 \neq 0$, we have $d \geq 2$, but obviously $d \leq 3$ 1p
If $\begin{vmatrix} 2 & -6 & 1 \\ 4 & \alpha & 5 \\ 2 & -1 & 2 \end{vmatrix} = 2\alpha - 6 = 2(\alpha - 3) = 0$ and $\begin{vmatrix} 2 & 10 & 1 \\ 4 & -1 & 5 \\ 2 & \alpha & 2 \end{vmatrix} = -6\alpha + 18 = -6(\alpha - 3) = 0$, that is, $\alpha = 3$, we have $d = 2$ 1p
For $\alpha \in \mathbb{R} \setminus \{3\}$ we have $d = 3$ 0.5p
Alternatively, we may use elementary operations:

$$\operatorname{rank} \begin{pmatrix} 2 & 10 & -6 & 1 \\ 4 & -1 & \alpha & 5 \\ 2 & \alpha & -1 & 2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 10 & -6 & 1 \\ 2 & -1 & \alpha & 5 \\ 1 & \alpha & -1 & 2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -21 & \alpha + 12 & 3 \\ 0 & \alpha - 10 & 5 & 1 \end{pmatrix} =$$

$$\operatorname{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha - 3 & -3\alpha + 9 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha - 3 & 0 \end{pmatrix} = \begin{cases} 2, \text{ if } \alpha = 3 \\ 3, \text{ if } \alpha \neq 3 \end{cases}.$$

NOTE: Any other correct solution will be scored accordingly.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science

GRADUATION EXAM
Written Test - June 2025
Mathematics Computer Science Study Programme
Grading scheme

SUBJECT II. Calculus

Default 1p

1. One can use the Raabe-Duhamel criterion: *Let* $\sum_{n\geq 1} a_n$ *be a series with positive terms. If*

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lambda,$$

then: 1) *for* $\lambda>1$ *the series is convergent;* 2) *if* $\lambda<1$ *the series is divergent.*

..... 0.25p

We denote

$$a_n=\frac{(1\cdot 3\cdot \ldots\cdot (2n-1))^\alpha}{(2\cdot 4\cdot \ldots\cdot (2n))^\beta},\quad n=1,2,3,\ldots$$

Then

$$\begin{aligned}\lambda &= \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(2n+2)^\beta}{(2n+1)^\alpha}-1\right)= \\ &= \lim_{n\to\infty} n\left(n^{\beta-\alpha}\frac{\left(2+\frac{2}{n}\right)^\beta}{\left(2+\frac{1}{n}\right)^\alpha}-1\right)=\begin{cases}+\infty, & \text{dacă} \quad \beta>\alpha \\ -\infty, & \text{dacă} \quad \beta<\alpha.\end{cases}\end{aligned}$$

..... 0.75p

Thus $\sum_{n\geq 1} a_n$ is convergent if $\alpha<\beta$, and $\sum_{n\geq 1} a_n$ is divergent if $\alpha>\beta$.

..... 0.25p

For the case $\alpha=\beta$ by applying L'Hospital rule we get

$$\begin{aligned}\lambda &= \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(2n+2)^\alpha}{(2n+1)^\alpha}-1\right)=\lim_{n\to\infty}\frac{\left(\frac{2+\frac{2}{n}}{2+\frac{1}{n}}\right)^\alpha-1}{\frac{1}{n}}= \\ &= \lim_{x\searrow 0}\frac{\left(\frac{2+2x}{2+x}\right)^\alpha-1}{x}=\lim_{x\searrow 0}\alpha\left(\frac{2+2x}{2+x}\right)^{\alpha-1}\cdot\frac{2}{(2+x)^2}=\frac{2\alpha}{4}=\frac{\alpha}{2}.\end{aligned}$$

..... 0.75p

From Raabe-Duhamel: $\sum_{n\geq 1} a_n$ is convergent if $\alpha=\beta>2$, and $\sum_{n\geq 1} a_n$ is divergent if $\alpha=\beta<2$.

..... 0.25p

If $\alpha=\beta=2$, then

$$\frac{a_{n+1}}{a_n}=\left(\frac{2n+1}{2n+2}\right)^2>\frac{n}{n+1},$$

for all $n\geq 1$. Thus, the sequence $(na_n)_{n\geq 1}$ is increasing, therefore $na_n\geq a_1$, for all $n\geq 1$, thus $a_n\geq\frac{1}{4n}$, for all $n\geq 1$.

..... 0.50p

Since the harmonic series $\sum_{n\geq 1}\frac{1}{n}$ is divergent, we get that $\sum_{n\geq 1} a_n$ is also divergent.

..... 0.25p

2. Taylor's polynomial of random rank n attached to the function f about the point $a = 0$ is the function

$$T_{n;0}f : \mathbb{R} \to \mathbb{R}$$

with

$$T_{n;0}f(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!}x^k, \quad \forall x \in \mathbb{R}$$

.....0.25p

The function f is indefinite differentiable on $R\backslash\{-2, 1\}$ as a composition of elementary functions. 0.25p

By considering a random $x \in \mathbb{R}\backslash\{-2, 1\}$ we split the initial function into simple fractions

$$\frac{x+8}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{-2}{x+2} + \frac{3}{x-1}.$$

.....0.25p

For a random constant b, we determine by mathematical induction the n-th derivative of the function

$$g : \mathbb{R}\backslash\{b\} \to \mathbb{R} \quad \text{cu} \quad g(x) = \frac{1}{x+b} = (x+b)^{-1}.$$

Therefore

$$g'(x) = (-1)(x+b)^{-2}, \quad g"(x) = (-1)(-2)(x+b)^{-3} \quad ... \quad g^{(n)}(x) = (-1)(-2)...(-n)(x+b)^{-(n+1)}$$

$$g^{(n)} = (-1)^n n!(x+b)^{-(n+1)}.$$

.....0.75p

By applying the result above to f we get

$$f^{(n)}(x) = -2(-1)^n n!(x+2)^{-(n+1)} + 3 \cdot (-1)^n n!(x-1)^{-(n+1)} = (-1)^n n! \cdot \left(\frac{-2}{(x+2)^{n+1}} + \frac{3}{(x-1)^{n+1}}\right)$$

Thus

$$f^{(n)}(0) = (-1)^n n! \cdot \left(\frac{-2}{(0+2)^{n+1}} + \frac{3}{(0-1)^{n+1}}\right) = (-1)^n n! \cdot \left(-2^{-n} + (-1)^{-(n+1)}3\right) =$$

$$= (-1)^{(n+1)} n! \cdot \left(2^{-n} + (-1)^{-n}3\right) = -n! \left((-2)^{-n} + 3\right)$$

.....1p

Consequently

$$T_{n;0}f(x) = \sum_{k=0}^{n} \frac{-k! \cdot \left((-2)^{-k} + 3\right)}{k!}x^k = -\sum_{k=0}^{n} \left((-2)^{-k} + 3\right)x^k.$$

.....0.5p

3. We use the following change of variable $tg\frac{x}{2} = t$, then

$$\frac{x}{2} = arctgt \Longrightarrow \frac{1}{2}dx = \frac{1}{t^2+1}dt \Longrightarrow dx = \frac{2}{t^2+1}dt.$$

Also,

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}.$$

.....1p

Hence

$$I = \int \frac{1}{5\sin x + 2\cos x + 2}dx = \int \frac{\frac{2}{t^2+1}dt}{\frac{10t}{1+t^2} + \frac{2-2t^2}{1+t^2} + 2}$$

. . . . . 1p

Thus

$$I = \int \frac{2}{10t + 2 - 2t^2 + 2 + 2t^2} dt = \int \frac{2}{10t + 4} dt = \int \frac{1}{5t + 2} dt = \frac{1}{5} \ln |5t + 2| + \mathcal{C}$$

$$I = \frac{1}{5} \ln \left| 5tg \frac{x}{2} + 2 \right| + \mathcal{C}.$$

. . . . . 1p

NOTE: Any other correct solution will be scored accordingly.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science

GRADUATION EXAM
Written Test - June 2025
Mathematics Computer Science Study Programme
Grading scheme

SUBJECT III. Geometry

Default 1p

1. a) $AH : y = x$, $BH : y = 2x - 2$ 1p

b) $m_{BH} = 2$, and since AC is perpendicular on BH we have $m_{AC} = -\frac{1}{2}$.

We find $AC : y = -\frac{1}{2}x + \frac{3}{2}$ 1p

Similarly, $m_{AH} = 1$ and hence $m_{BC} = -1$.

We find $BC : y = -x - 2$ 1p

c) The point C is at the intersection of the lines AC and BC. Solving the system consisting of the equations of these lines we find $C(-7, 5)$ 1p

The area of the triangle ABC is

$$\frac{1}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -7 & 5 & 1 \end{vmatrix}\right| = 14$$

..... 1p

2. a)The equation of the circle can be written as: $(x-2)^2 + (y+3)^2 = 25$.

The center of the circle is $M_0(2, -3)$ and its radius is $r = 5$..... 1.5p

b) $d(M_0, d) = \dfrac{|3 \cdot 2 - 4 \cdot (-3) + 2|}{\sqrt{3^2 + 4^2}} = \dfrac{20}{5} = 4$ 1.5p

c) $d(M_0, d) = 4 < r = 5 \Rightarrow$ the line d intersects the circle in two distinct points. 1p

NOTE: Any other correct solution will be scored accordingly.

BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND INFORMATICS

Grading scheme for Computer Science Subject
Exam session 1: Assessment of basic and specialist knowledge, license exam June 2025
Specialization Mathematics Computer Science

Computer Science Subject

1. a) Definition of class Book (constructor, methods, access to data)..... b) Definition of derived class AudioBook (inheritance, constructor, method).....	**2p** 1 p 1 p
2. Creating the vector..... Correct display of objects having the price with the required property	**2p** 1p 1p
3. Iterating the elements..... Comparing elements..... Correct position identification.....	**2p** 0.5p 0.5p 1p
4. Correct indication of the result that will be displayed	**2p** 2p
5. Theoretical explanation of the concepts.....	**1p** 1p

Remark:
(1p) Default