BABEȘ-BOLYAI UNIVERSITY IN CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MATE-INFO UBB 2024 COMPETITION
Written test in MATHEMATICS

IMPORTANT NOTE: Problems can have one or more correct answers, which the candidate should indicate on the test form. The grading system of the multiple choice exam can be found in the set of rules of the competition.

1. In the parallelogram $ABCD$ we have that $AB = 1$, $AD = 2$ and $m(\widehat{B}) = 60°$. Which of the following statements are true?

A $\overrightarrow{AB} \cdot \overrightarrow{AD} = 1$; B $\overrightarrow{BA} \cdot \overrightarrow{BC} = 1$; C $\overrightarrow{BA} \cdot \overrightarrow{AD} = -1$; D $\overrightarrow{BA} \cdot \overrightarrow{CD} = 1$.

2. If the points $A(1, 2)$ and $B(4, 6)$ are the vertices of a rectangle $ABCD$, then the equation of the line AD is:

A $4x + 3y - 11 = 0$; B $3x + 4y - 11 = 0$; C $4x - 3y + 2 = 0$; D $4x + 3y + 2 = 0$.

3. Let $\overrightarrow{i}$ and $\overrightarrow{j}$ be the versors of a Cartesian system. If the vectors $\overrightarrow{u} = 2\overrightarrow{i} + b\overrightarrow{j}$ and $\overrightarrow{v} = (b+4)\overrightarrow{i} + 2\overrightarrow{j}$ are perpendicular, then the value of the parameter $b \in \mathbb{R}$ is:

A -2; B -1; C 1; D 2.

4. If the matrix $X \in \mathcal{M}_2(\mathbb{R})$ satisfies the relation $\begin{pmatrix} 2 & 0 \\ 2 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$, then the sum of all the entries of X is:

A -2; B 0; C 2; D 4.

5. Consider the system of equations

$$\begin{cases} x + 2y + 3z &= 1 \\ x - 2y + az &= a \\ 3x + 2y + z &= 2, \end{cases}$$

where a is a real parameter. Which of the following statements are true?

A There exists a unique $a \in \mathbb{R}$ for which the system is incompatible.
B The system is compatible for every $a \in \mathbb{R}$.
C If the determinant of the system is equal to 16, then the solution of the system is $x = \frac{7}{8}$, $y = -\frac{1}{2}$, $z = \frac{3}{8}$.
D If the determinant of the system is equal to 16, then the solution of the system is $x = \frac{1}{4}$, $y = \frac{3}{4}$, $z = -\frac{1}{4}$.

6. The limit $\lim\limits_{x\to 0} \left(1 + x^2 e^x\right)^{\frac{1}{1-\cos x}}$ is equal to:

A $\sqrt{e}$; B 1; C e; D e^2.

7. Let $a, b, c \in \mathbb{R}$ and consider $f : \mathbb{R} \to \mathbb{R}$ the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ a\mathrm{e}^{-x} + b\mathrm{e}^{x} + cx\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right), & \text{if } 0 < x < 1 \\ \mathrm{e}^{2-x}, & \text{if } x \geq 1. \end{cases}$$

If f is continuous on $\mathbb{R}$, then the value of the sum $a + 2b + c$ is:

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ 2; $\boxed{\text{C}}$ 1; $\boxed{\text{D}}$ $\frac{1}{2}$.

8. Let $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ be such that $\sin(x) = \frac{1}{3}$. Which of the following statements are true?

$\boxed{\text{A}}$ $\cos(x) = \frac{2\sqrt{2}}{3}$; $\boxed{\text{B}}$ $\sin(2x) = -\frac{4\sqrt{2}}{9}$; $\boxed{\text{C}}$ $\cos(2x) = \frac{7}{9}$; $\boxed{\text{D}}$ $\mathrm{tg}(x) = -2\sqrt{2}$.

9. In the triangle ABC we have $D \in (AB)$, $DB = 2 \cdot AD$, $E \in (AC)$ and $AC = 3 \cdot EC$. If the points A, D and E have coordinates $A(0, 6)$, $D(4, 4)$ and $E(-4, 2)$, then the coordinates of the centroid (center of gravity) G of the triangle ABC are:

$\boxed{\text{A}}$ $G(2, 2)$; $\boxed{\text{B}}$ $G\left(\frac{20}{9}, \frac{20}{9}\right)$; $\boxed{\text{C}}$ $G(0, 0)$; $\boxed{\text{D}}$ $G\left(-\frac{4}{3}, \frac{2}{3}\right)$.

10. The real numbers $a_1, a_2, a_3, \ldots, a_{98}, a_{99}, a_{100}$ are in arithmetic progression and

$$a_1 + a_2 + a_3 + \ldots + a_{98} + a_{99} + a_{100} = a_2 + a_4 + a_6 + \ldots + a_{96} + a_{98} + a_{100} = 200.$$

Denote by d the ratio of this arithmetic progression. Which of the following statements are true?

$\boxed{\text{A}}$ $d > 0$;
$\boxed{\text{B}}$ $d < 0$;
$\boxed{\text{C}}$ The arithmetic progression is uniquely determined by the given conditions;
$\boxed{\text{D}}$ There is no such arithmetic progression.

11. If $x > 0$ and the third term of the expansion of $\left(\frac{1}{x} + (\sqrt{x})^{1+\lg x}\right)^5$ is equal to 10000, then

$\boxed{\text{A}}$ $x \in \left\{\frac{1}{10}, 10\right\}$; $\boxed{\text{B}}$ $x \in \left\{\frac{1}{1000}, 1000\right\}$; $\boxed{\text{C}}$ $x \in \left\{\frac{1}{10}, 1000\right\}$; $\boxed{\text{D}}$ $x \in \left\{\frac{1}{1000}, 10\right\}$.

12. If we denote by S the set of real solutions of the equation

$$\sqrt{x - 6\sqrt{x+1} + 10} + \sqrt{x + 6\sqrt{x+1} + 10} = 6,$$

then

$\boxed{\text{A}}$ $3 \in S$; $\boxed{\text{B}}$ $15 \in S$; $\boxed{\text{C}}$ the set S is finite; $\boxed{\text{D}}$ the set S is infinite.

13. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 3x + 4\sqrt{1 - x^2}$. Denote by a the smallest value of f and by b the largest value of f. Then the length of the interval $[a, b]$ is:

$\boxed{\text{A}}$ 2; $\boxed{\text{B}}$ 6; $\boxed{\text{C}}$ 8; $\boxed{\text{D}}$ 10.

14. Denote by I the value of the integral $\displaystyle\int_0^1 \frac{\mathrm{d}x}{x^3+x^2+x+1}$. Indicate which of the following statements are true.

A $I > \dfrac{\pi}{8}$; B $I < \dfrac{\pi}{8}$; C $I < \dfrac{1}{4}\ln 2$; D $I > \dfrac{1}{4}\ln 2$.

15. In the triangle ABC we have $A(3,4)$, $B(2,1)$. We also know that $D(0,2)$ is the midpoint of the segment BC. The area of the triangle ABC is equal to:

A 1; B 3; C 5; D 7.

16. If $f\colon \mathbb{R}\to\mathbb{R}$, $f(x)=x^2-x$ and $S=\{x\in\mathbb{R} \mid f(f(x))=0\}$, then

A $1\in S$; B $-1\in S$; C there are exactly two irrational numbers in S;
D there is a unique irrational number in S.

17. If z is a complex number such that $z^2=i$, then $(z^3+\overline{z})^2$ is equal to

A $-2i$; B $2i$; C 2; D 0.

18. The value of the limit $\displaystyle\lim_{n\to\infty} n\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+n)^2}\right)$ is:

A 0; B 1; C $\dfrac{1}{2}$; D $\dfrac{1}{4}$.

19. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\dfrac{x^2}{\sqrt{x^2+1}}$. The area of the set of points in the plane situated between the graph of the function f, the Ox axis and the lines of equations $x=-1$ and $x=1$ is:

A $\sqrt{2}-\ln\left(1+\sqrt{2}\right)$; B $\sqrt{2}$; C $\sqrt{2}+\ln\left(1+\sqrt{2}\right)$; D $2\sqrt{2}-\ln\left(1+\sqrt{2}\right)$.

20. In the rhombus $ABCD$ we have $E\in(BC)$, $BE=2\cdot EC$, $F\in(DC)$ și $FD=3\cdot FC$. Which of the following statements are true?

A $\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}$; B $\overrightarrow{BE}=\dfrac{2}{3}\overrightarrow{AD}$; C $\overrightarrow{DF}=\dfrac{1}{4}\overrightarrow{AB}$; D $\overrightarrow{EF}=\dfrac{1}{3}\overrightarrow{AD}-\dfrac{1}{4}\overrightarrow{AB}$.

21. In the triangle ABC we have $A(2,13)$, $B(-7,1)$ and $C(7,1)$. Knowing that AD is the interior angle bisector with $D\in(BC)$, the length of the segment BD is:

A $\dfrac{13}{2}$; B 7; C $\dfrac{15}{2}$; D 8.

22. If the function $f\colon(\mathbb{Z}_8,+)\to(\mathbb{Z}_{12},+)$ is a group homomorphism and $f(\hat{5})=\hat{9}$, then

A $f(\hat{1})=\hat{0}$; B $f(\hat{1})=\hat{9}$; C $f(\hat{1})$ is not uniquely determined by the given conditions;
D there is no such group homomorphism.

23. Denote by A the set consisting of all pairs of real numbers (x,y) such that $0\le x<y$ and $\dfrac{x}{2024^x}=\dfrac{y}{2024^y}$. Which of the following statements are true?

A The set A is the empty set; B The set A contains a unique element;
C The set A has infinitely many elements; D The set A contains an element of the form $(x,1)$.

24. Let $a\in\mathbb{R}$ and $(x_n)_{n\ge 0}$ the sequence defined by $x_0=a$ and $x_{n+1}=x_n+x_n^2$ for all $n\ge 0$. The set of all values of a for which the sequence is convergent is:

A $\{-1,0\}$; B $[-1,0]$; C $[0,1]$; D $(-\infty,0]$.

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Correct Answers

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1. $\boxed{\textbf{B}}$, $\boxed{\textbf{D}}$
2. $\boxed{\textbf{B}}$
3. $\boxed{\textbf{A}}$
4. $\boxed{\textbf{C}}$
5. $\boxed{\textbf{A}}$, $\boxed{\textbf{D}}$
6. $\boxed{\textbf{D}}$
7. $\boxed{\textbf{B}}$
8. $\boxed{\textbf{B}}$, $\boxed{\textbf{C}}$
9. $\boxed{\textbf{A}}$
10. $\boxed{\textbf{A}}$, $\boxed{\textbf{C}}$
11. $\boxed{\textbf{C}}$
12. $\boxed{\textbf{A}}$, $\boxed{\textbf{D}}$
13. $\boxed{\textbf{C}}$
14. $\boxed{\textbf{A}}$, $\boxed{\textbf{D}}$
15. $\boxed{\textbf{D}}$
16. $\boxed{\textbf{A}}$, $\boxed{\textbf{C}}$
17. $\boxed{\textbf{D}}$
18. $\boxed{\textbf{C}}$
19. $\boxed{\textbf{A}}$
20. $\boxed{\textbf{A}}$, $\boxed{\textbf{B}}$, $\boxed{\textbf{D}}$
21. $\boxed{\textbf{C}}$
22. $\boxed{\textbf{B}}$
23. $\boxed{\textbf{C}}$, $\boxed{\textbf{D}}$
24. $\boxed{\textbf{B}}$