BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MATE-INFO UBB Competition 2023
Written test in MATHEMATICS

IMPORTANT NOTE: Problems can have one or more correct answers, which the candidate should indicate on the test form. The grading system of the multiple choice exam can be found in the set of rules of the competition.

Let ABC be a triangle with vertices $A(2,2)$, $B(-3,7)$ and center of gravity (centroid) $G(-2,5)$. The problems **1** and **2** refer to this triangle ABC.

1. The coordinates of the vertex C are

A $C(5,6)$; B $C(-5,6)$; C $C\left(\frac{2}{3},3\right)$; D $C(5,-6)$.

2. The slope of a line perpendicular on the line AG is

A $-\frac{4}{3}$; B $-\frac{3}{4}$; C $\frac{3}{4}$; D $\frac{4}{3}$.

3. The value of the limit $\lim\limits_{x\to 0}\dfrac{\mathrm{e}^x-\sin x-\cos x}{x^2}$ is:

A 0; B $\frac{1}{2}$; C 1; D $+\infty$.

4. The area of a triangle ABC is 6. If the lengths of its sides are $AB=8$ and $AC=3$, then the measure of the angle BAC can be:

A 30°; B 60°; C 120°; D 150°.

5. The area of the planar region enclosed between the graph of the function $f:[1,\mathrm{e}]\to\mathbb{R}$, defined by $f(x)=x^2\ln x$, the Ox axis and the line which has equation $x=\mathrm{e}$ is:

A $\frac{1}{9}$; B $\frac{1}{3}$; C $\frac{1+\mathrm{e}^3}{9}$; D $\frac{1+2\mathrm{e}^3}{9}$.

6. If we denote by S the set of real solutions of the equation

$$\log_2 x+\log_4 x^2+\log_8 x^3+\log_{16} x^4=2,$$

then

A $S\subseteq[0,1)$; B $S\subseteq(1,2]$; C S has exactly two elements; D S has exactly one element.

7. Consider the set

$$S=\{m\in\mathbb{R}\mid x^2-2(m^2+1)x+(m+1)^4\geq 0,\forall x\in\mathbb{R}\}.$$

Which of the following statements are true?

A $1\in S$; B $-1\in S$; C $S=[0,\infty)$; D $S=\mathbb{R}$.

8. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} -x^2, & \text{if } x \in (-\infty, -1) \\ 1, & \text{if } x \in [-1, 1] \\ x^2, & \text{if } x \in (1, \infty). \end{cases}$$

Which of the following statements are true?

A f is injective; B f is surjective; C f is increasing; D f is bijective.

9. Let $ABCD$ be a rectangle with side lengths $AB = 4\sqrt{3}$ and $BC = 4$. Then

A $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$; B $\overrightarrow{BA} \cdot \overrightarrow{BC} = 16\sqrt{3}$; C $\overrightarrow{BA} \cdot \overrightarrow{BD} = 24$; D $\overrightarrow{BA} \cdot \overrightarrow{BD} = 48$.

10. Consider the lines $d_1 : x - 3y + 2 = 0$, $d_2 : (m+1)x - (2m-3)y + 4 = 0$ and $d_3 : mx + y + m + 1 = 0$, where m is a real parameter. Which of the following statements are true?

A The lines d_1 and d_2 are parallel when $m = -6$.
B The lines d_2 and d_3 are not parallel for all possible values of m.
C There exists only one possible value of m such that d_1, d_2 and d_3 are concurrent.
D There are two possible values of m for which the lines d_1, d_2 and d_3 are concurrent.

11. Let S be a point on the side PQ of the triangle PQR such that $\dfrac{PS}{SQ} = \dfrac{1}{2}$ and let T be the midpoint of the segment SR. Which of the following statements are true?

A $\overrightarrow{QT} = \frac{1}{2}(\overrightarrow{QS} - \overrightarrow{QR})$; B $\overrightarrow{QT} = \frac{1}{2}(\overrightarrow{QS} + \overrightarrow{QR})$;

C $\overrightarrow{PS} = \frac{1}{3}\overrightarrow{QP}$; D $\overrightarrow{RT} = \frac{1}{3}\overrightarrow{RP} + \frac{1}{6}\overrightarrow{RQ}$.

12. If the non-zero natural numbers x, y satisfy the equality

$$x(y+1)C_{x+y+1}^{y+1} = 30C_{x+y+1}^{x+1},$$

then

A x is uniquely determined; B y is uniquely determined; C $x < 10$; D $x > 5$.

13. The number of solutions of the equation $\cos(3\pi x) = 0$ which lie in the interval $(0, 2023)$ is

A 2022; B 2023; C 6066; D 6069.

14. Consider the system of equations

$$\begin{cases} 2x + & 2z & = 3 \\ 3x + 2y & & = 0 \\ 2x + ay + & 2z & = a + 3, \end{cases}$$

where a is a real parameter. Which of the following statements are true?

A there exists $a \in \mathbb{R}$ such that the determinant of the system is 0;
B there exists a unique $a \in \mathbb{R}$ such that the system does not have solutions;
C the system has solutions for every $a \in \mathbb{R}$;
D if the system has finitely many solutions, then y does not depend on the parameter a.

15. Consider the matrix $A = \begin{pmatrix} i & 2 \\ 0 & i \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$ and for every $n \in \mathbb{N}^*$ we define the complex numbers a_n, b_n, c_n, d_n by

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = A^n.$$

Which of the following statements are true?

A $a_n = d_n = i^n$ for every $n \in \mathbb{N}^*$;
B $b_n = 2ni^{n-1}$ for every $n \in \mathbb{N}^*$;
C $b_n = 2^n i^{n-1}$ for every $n \in \mathbb{N}^*$;
D there exists $n \in \mathbb{N}^*$ such that a_n, b_n, c_n, d_n are all real numbers.

16. The value of the limit $\lim\limits_{n\to\infty} \ln(1 + \mathrm{e}^n) \cdot \sin\dfrac{1}{n}$ is:

A 0; B 1; C e; D $+\infty$.

17. For every $a > 0$, we denote by $I(a) = \displaystyle\int_0^a \frac{\mathrm{d}x}{(x+2)\sqrt{x+1}}$. Which of the following statements are true?

A $I(2) = \dfrac{\pi}{6}$; B $I(2) = \dfrac{\pi}{3}$; C $\lim\limits_{a\to\infty} I(a) = \dfrac{\pi}{4}$; D $\lim\limits_{a\to\infty} I(a) = \dfrac{\pi}{2}$.

The problems **18**, **19** and **20** refer to the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \dfrac{x^2 + ax + 5}{\sqrt{x^2+1}}$, where a is a fixed real number.

18. The equation of the asymptote to the graph of f towards $+\infty$ is:

A $y = a$; B $y = x + a$; C $y = x - a$; D $y = x + \dfrac{a}{2}$.

19. If d is the tangent line to the graph of f at the point of intersection of this graph with the Oy axis, which of the following statements are true?

A The line d is parallel to the line which has equation $y = ax$;
B For $a \neq 0$ the line d is parallel to the line which has equation $y = \dfrac{1}{a}x$;
C For $a = 0$ the line d is parallel with the Ox axis;
D For $a \neq 0$ the line d is perpendicular on the line which has equation $y = -\dfrac{1}{a}x$.

20. The set consisting of the values of a for which f has three points of local extrema is:

A $(-1, 1)$; B $[-1, 1]$; C $[-2, 2]$; D $(-2, 2)$.

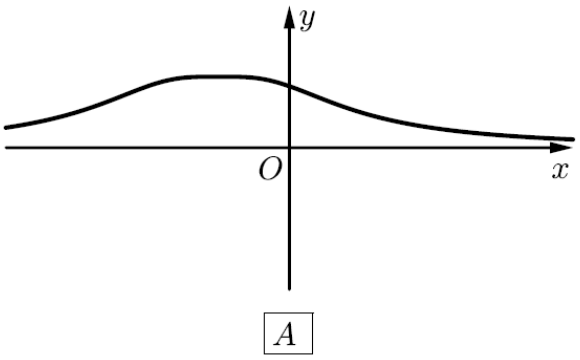
21. Let $a \in \mathbb{R}$ be a real parameter. On the set of real numbers we define the operation "$*$" by

$$x * y = (x - 2023)(y - 2023) + a, \quad \forall x, y \in \mathbb{R}.$$

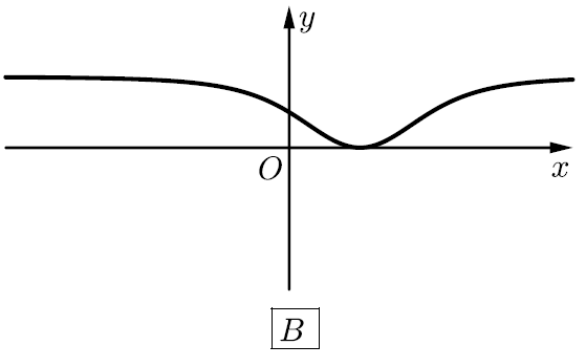
Which of the following statements are true?

A the operation "$*$" is associative if and only if $a = 2023$;
B if $a = 2023$, then 2024 is a neutral element with respect to "$*$";
C if the operation "$*$"is associative, then $(\mathbb{R}, *)$ is a group;
D $((a, +\infty), *)$ is a group for all $a \in \mathbb{R}$.

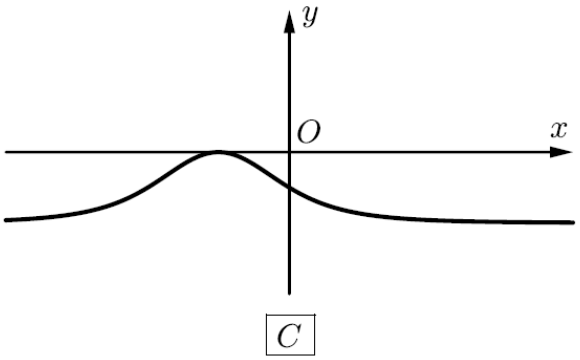
22. Which of the graphs below can be the graph of the function defined by $f(x) = \cos \dfrac{\pi}{x^2 + 2x + 3}$?



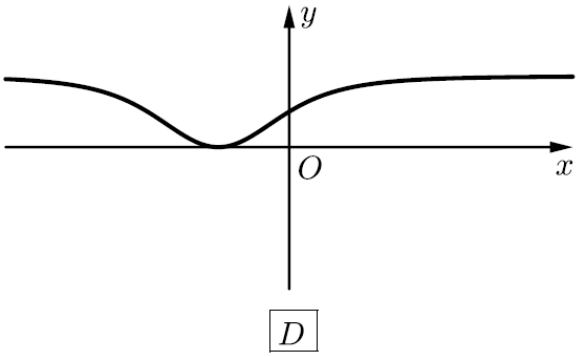
A



B



C



D

23. In a triangle, the angle opposing the largest side is twice the angle opposing the smallest side. If the lengths of the sides of the triangle are consecutive natural numbers, then the perimeter of the triangle is

A 9; B 12; C 15; D 24.

24. The distinct numbers a, b, c are, in this order, in an arithmetic progression of ratio r, and the numbers $a - 1, b, c + 4$ are, in this order, in a geometric progression with the same ratio r. Which of the following statements can be true?

A $b = 0$; B $b = 6$; C $b = -\dfrac{2}{3}$; D $b = -3$.

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Correct Answers

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1. **B**
2. **D**
3. **C**
4. **A**, **D**
5. **D**
6. **B**, **D**
7. **A**, **C**
8. **C**
9. **A**, **D**
10. **A**, **B**, **D**
11. **B**, **D**
12. **A**, **C**
13. **D**
14. **A**, **C**, **D**
15. **A**, **B**
16. **B**
17. **A**, **D**
18. **B**
19. **A**, **C**, **D**
20. **D**
21. **A**, **B**
22. **D**
23. **C**
24. **B**, **C**